Prof. Dr. Y. Tian
Dr. J. Anschütz **WS 2017/18**

Algebraic Number Theory

10. Exercise sheet

Exercise 1 (4 Points):

Let K be a complete non-archimedean valuation field. Prove that an infinite product $\prod_{n\geq 1}(1+a_n)$ with $a_n \neq -1$ converges in K if and only if $|a_n| \to 0$ as $n \to +\infty$.

Exercise 2 (4 Points):

Let K be a finite extension of $\mathbb{Q}_p$ with ramification index $e = e(K|\mathbb{Q}_p)$. Let π denote a uniformizer of K. Prove that for every element $\alpha \in \mathcal{O}_K$ such that $\alpha \equiv 1 \bmod \pi^{m+1}$ with $m \geq \frac{ep}{p-1}$, the equation $x^p = \alpha$ has solutions in K.

Hint: Write $\alpha = 1+\pi^n a$ with $n \geq m+1$ and $a \in \mathcal{O}_K$ with $\pi \nmid a$. Try to find some $\beta = 1+\pi^k b \in \mathcal{O}_K$ for some k such that $\alpha\beta^{-p} \equiv 1 \bmod \pi^{n+1}$. Repeat the process and use Exercise 1.

Exercise 3 (4 Points):

Let p be a prime. Let $K = \mathbb{F}_p((t))$, $\mathcal{O}_K = \mathbb{F}_p[[t]]$. Consider the polynomial

$$f(x) = x^p - x - t^{-1} \in K[x].$$

1. Prove that $f(x)$ is irreducible in $K[x]$.
2. Let $L = K[x]/(f(x))$, and v_L be the normalized additive valuation on L. Find the different $v_L(\delta_{L/K})$.
3. Is $x^p - x - t$ irreducible in $K[x]$?

Exercise 4 (4 Points):

Let $K = \mathbb{Q}(\sqrt[3]{2})$. Describe the irreducible factors of $K \otimes_{\mathbb{Q}} \mathbb{Q}_p$ for $p = 3, 5, 7$.

To be handed in: Monday, 8. January 2018.