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WS 2017/18

Algebraic Number Theory

8. Exercise sheet

Exercise 1 (4 Points):
Let p be a prime with $p \equiv 1 \bmod 3$, and let $\chi\colon \mathbb{F}_p^\times \to \mathbb{C}^\times$ be a primitive cubic Dirichlet character, i.e., $\chi^3 = 1$. Set $\omega = e^{\frac{2\pi i}{3}}$. Prove that

$$\tau(\chi)^2 = \tau(\overline{\chi})J(\chi) \text{ with } J(\chi) = \sum_{t=1}^{p-2} \chi(t(1+t)) \in \mathbb{Z}[\omega].$$

In particular, if $J(\chi) = x + \omega y$ with $x, y \in \mathbb{Z}$, then $x^2 - xy + y^2 = p$.
Hint: For the last statement use $\tau(\chi)\tau(\overline{\chi}) = \chi(-1)p$ and take the norm of $J(\chi)$.

Exercise 2 (4 Points):
Use the class number formula to compute the class number of $\mathbb{Q}(\sqrt{-23})$ and $\mathbb{Q}(\sqrt{-52})$.

Exercise 3 (4 Points):
Let p be an odd prime, $\zeta_p = e^{2\pi i/p}$. Let $\chi(\cdot) = (\frac{\cdot}{p})\colon (\mathbb{F}_p)^\times \to \mathbb{C}^\times$ denote the Legendre symbol, and let $\tau(\chi) = \sum_{a=1}^{p-1} \chi(a)\zeta_p^a$ be the quadratic Gauss sum. The aim of this exercise is to prove

$$\tau(\chi) = \begin{cases} \sqrt{p} \text{ if } p \equiv 1 \bmod 4 \\ i\sqrt{p} \text{ if } p \equiv 3 \bmod 4. \end{cases}$$

Let S be the $p \times p$ matrix whose (k,l)-th entry is $\zeta_p^{(k-1)(l-1)}$.

1. Prove that $\det(S) = i^{\frac{p(p-1)}{2}} p^{\frac{p}{2}}$.
   Hint: $\det(S)$ is a Vandermonde determinant. Using the derivative of $X^p - 1$ compute $|\det(S)^2|$. Then compute the argument of $\det(S)$.
2. Show that $\mathrm{Tr}(S) = \tau(\chi)$, and $\mathrm{Tr}(S^2) = p$.
3. Show that the possible eigenvalues of S are $\pm\sqrt{p}$, $\pm i\sqrt{p}$.
   Hint: On the space of functions $f\colon \mathbb{F}_p \to \mathbb{C}$ the matrix S represents the "Fourier transform" $f \mapsto \hat{f}$ with $\hat{f}(x) := \sum_{y \in \mathbb{F}_p} f(y)\zeta_p^{xy}$. Prove that $S^4 = p^2\mathrm{Id}$.
4. Prove that if $p \equiv 1 \bmod 4$, the eigenvalue $\sqrt{p}$ has multiplicity $\frac{p+3}{4}$, and the multiplicity of $-\sqrt{p}$, $i\sqrt{p}$,$-i\sqrt{p}$ are $\frac{p-1}{4}$; if $p \equiv 3 \bmod 4$, the eigenvalue $\sqrt{p}$,$-\sqrt{p}$ and $i\sqrt{p}$ have multiplicity $\frac{p+1}{4}$, and $-i\sqrt{p}$ has multiplicity $\frac{p-3}{4}$. Conclude the proof.
   Hint: Show that S^2 has characteristic polynomial $(X-p)^{\frac{p+1}{2}}(X+p)^{\frac{p-1}{2}}$ and use that the eigenvalues of S^2 are the squares of the eigenvalues of S and $\tau(\chi)^2 = \chi(-1)p$.

Exercise 4 (4 Points):

1. Find the 7-adic expansion of $2/5 \in \mathbb{Q}$ up to modulo 7^3.
2. Find the solutions to $x^2 + 2$ in $\mathbb{Z}_7$ up to modulo 7^3.

To be handed in: Monday, 11. Dezember 2017.