Prof. Dr. Y. Tian **WS 2017/18**
Dr. J. Anschütz

Algebraic Number Theory

7. Exercise sheet

Exercise 1 (4 Points):

Let K be a number field with r_1 real embeddings and r_2 non-real complex embeddings. Let H denote the hyperplane $\sum_{i=1}^{r_1+r_2} x_i = 0$ in $\mathbb{R}^{r_1+r_2}$, and $\ell\colon U_K \to H$ be the usual logarithmic map. Let R_K be the regulator of K. Prove that

$$R_K = \frac{\mathrm{Vol}(H/\ell(U_K)}{\sqrt{r_1+r_2}}.$$

Exercise 2 (4 Points):

Let $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character. Let H_χ be its kernel, and K be the subfield of $\mathbb{Q}(\zeta_N)$ fixed by H_χ. Prove that $\chi(-1) = 1$ (i.e. χ is even) if and only if K is a totally real field.

Exercise 3 (4 Points):

Let A be a subset of rational primes. If there exists a number $\rho \in [0,1]$ such that

$$\sum_{p\in A} \frac{1}{p^s} \sim \rho \log \frac{1}{s-1} \text{ for } s \text{ real and } s \to 1^+,$$

(i.e. s approaches 1 along the real line from the right), then we say that A has Dirichlet density ρ. Compute the Dirichlet density for

$$A_n := \{\ p \mid 2 \text{ is an } n\text{th power mod } p\}$$

and $n = 2, 3$.

Exercise 4 (4 Points):

Let A be a subset of primes. For $x > 0$, denote by $\pi(x)$ the number of all primes less than x, and by $\pi_A(x)$ the number of primes in A less than x. The natural density of A is defined as the limit

$$\rho := \lim_{x\to+\infty} \frac{\pi_A(x)}{\pi(x)}$$

whenever it exists.

1) Show that

$$\sum_{p\in A} \frac{1}{p^s} - \rho \sum_p \frac{1}{p^s} = \sum_{n=1}^{\infty} (\pi_A(n) - \rho\pi(n))(\frac{1}{n^s} - \frac{1}{(n+1)^s}).$$

Hint: Use Exercise 4.1) from exercise sheet 6.

2) For every $\varepsilon > 0$, there exists $N > 0$ such that $|\pi_A(n) - \rho\pi(n)| \leq \varepsilon\pi(n)$ for all $n > N$. Use this to prove that for any $\varepsilon' > 0$, there exists $\delta > 0$ such that

$$|\sum_{p\in A} \frac{1}{p^s} - \rho \sum_p \frac{1}{p^s}| \leq \varepsilon' \sum_p \frac{1}{p^s}$$

whenever $1 < s < 1 + \delta$. Conclude that A has Dirichlet density ρ.

To be handed in: Monday, 04. Dezember 2017.