**Prof. Dr. Y. Tian**
**Dr. J. Anschütz** **WS 2017/18**

**Algebraic Number Theory**

**6. Exercise sheet**

**Exercise 1 (4 Points):**

Let $K \subseteq \mathbb{R}$ be a real quadratic field with discriminant $d_K$. Then the fundamental unit of $K$ is defined to be the unique unit $\varepsilon$ of $K$ such that $\varepsilon > 1$ and $U_K = \{\pm 1\} \times \varepsilon^{\mathbb{Z}}$.

1) Let $u > 1$ be a unit of $K$. Show that $u \geq (\sqrt{d_K} + \sqrt{d_K - 4})/2$ if $N_{K/\mathbb{Q}}(u) = -1$ and $u \geq (\sqrt{d_K} + \sqrt{d_K + 4})/2$ if $N_{K/\mathbb{Q}}(u) = 1$.
*Hint: Consider* $\mathrm{Disc}_{K/\mathbb{Q}}(1, u)$ *and use* $\mathrm{Disc}_{K/\mathbb{Q}}(1, u) \geq d_K$.

2) Show that if $d_K$ is divisible by a prime $p$ with $p \equiv 3 \bmod 4$, then $K$ does not contain any units $u$ with $N_{K/\mathbb{Q}}(u) = -1$.

3) Find the fundamental unit of $K = \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{5})$.

**Exercise 2 (4 Points):**

Let $K = \mathbb{Q}(\zeta_p)$ be the $p$-the cyclotomic field, where $p$ is odd, and set $K^+ = K(\zeta_p + \zeta_p^{-1})$. Denote by $U_K$ and $U_{K^+}$ the group of units in $K$ and $K^+$.

1) Let $u$ be a unit of $K$. Show that $u/\bar{u}$ is a root of unity.
*Hint: Use that* $\sigma(u/\bar{u})$ *has absolute value* 1 *for every embedding* $\sigma\colon K \to \mathbb{C}$.

2) Let $u$ be a unit of $K$. Show that $u/\bar{u} = \zeta_p^k$ for some $k \in \mathbb{Z}$ (and not $u/\bar{u} = -\zeta_p^k$).

3) Show that $U_K = U_{K^+} \times \langle \zeta_p \rangle$.
*Hint: Consider the map* $U_K \to \langle \zeta_p \rangle,\ u \mapsto u/\bar{u}$.

**Exercise 3 (4 Points):**

Let $\zeta_5 = e^{2\pi i/5}$, and $u = -(\zeta_5^2 + \zeta_5^3)$.

1) Find a quadratic equation over $\mathbb{Q}$ satisfied by $u$ and prove that $u = (1 + \sqrt{5})/2$.

2) Prove that all the units of $\mathbb{Q}(\zeta_5)$ are given by $\pm \zeta_5^k (1 + \zeta_5)^h$ with $0 \leq k \leq 4$ and $h \in \mathbb{Z}$.

3) Find the regulator of $\mathbb{Q}(\zeta_5)$.

**Exercise 4 (4 Points):**

1) Let $a_n, b_n \in \mathbb{C}$ be two sequences of complex numbers and for $m \leq k$, $m \leq m'$ put

$$A_{m,k} = \sum_{n=m}^{k} a_n \text{ and } S_{m,m'} = \sum_{n=m}^{m'} a_n b_n.$$

Prove

$$S_{m,m'} = \sum_{n=m}^{n=m'-1} A_{m,n}(b_n - b_{n+1}) + A_{m,m'} b_{m'}.$$

2) Let $0 < \alpha < \beta$ real numbers and let $z = x + iy \in \mathbb{C}$, $x, y \in \mathbb{R}$, $x > 0$. Then

$$|e^{-\alpha z} - e^{-\beta z}| \leq |\frac{z}{x}|(e^{-\alpha x} - e^{-\beta x}).$$

*Hint: Write* $e^{-\alpha z} - e^{-\beta z} = z \int_{\alpha}^{\beta} e^{-tz} dt$.

3) Let $f(s) = \sum\limits_{n=1}^{\infty} \frac{a_n}{n^s}$, $a_n \in \mathbb{C}$, be a Dirichlet series. Prove that if $f$ converges for some $s_0 \in \mathbb{C}$, then $f$ converges locally uniformly on $\{ \ s \in \mathbb{C} \mid \mathrm{Re}(s) > \mathrm{Re}(s_0) \ \}$.

To be handed in: Monday, 27. November 2017.