Prof. Dr. Y. Tian **WS 2017/18**
Dr. J. Anschütz

Algebraic Number Theory

5. Exercise sheet

Exercise 1 (4 Points):
Let $K/\mathbb{Q}_p$ be a finite extension and let K^{Gal} be the Galois closure of K. Prove that a prime p is unramified in K if and only if it unramified in K^{Gal}.

Exercise 2 (4 Points):

1) Find the class number if $\mathbb{Q}(\sqrt{m})$ for $m = 5, 6, -5, -7, -13$.
2) Show that the ideal class group of $\mathbb{Q}(\sqrt{-23})$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$, and find explicitly an ideal that generates the ideal class group.

Exercise 3 (4 Points):
Let $K = \mathbb{Q}(\sqrt[3]{m})$.

1) Show that $\mathbb{Z}[\sqrt[3]{m}]$ is the ring of integers of K if m is square free and m is not congruent to ± 1 modulo 9.
2) Prove that $\mathbb{Z}[\sqrt[3]{m}]$ is a principal ideal domain for $m = 3, 5, 6$ and that the class number of $\mathbb{Q}(\sqrt[3]{7})$ is 3.

Exercise 4 (4 Points):
The aim of this exercise is to prove that the pairs $(17, \pm 70)$ are the only solutions in $\mathbb{Z}^2$ to the equation

$$y^2 + 13 = x^3.$$

We denote $A = \mathbb{Z}[\sqrt{-13}]$ and let $(x, y) \in \mathbb{Z}^2$ be a solution.

1) Show that no prime of A contains both $y + \sqrt{-13}$ and $y - \sqrt{-13}$.
2) Show that there exist $(a, b) \in \mathbb{Z}^2$ such that

$$y + \sqrt{-13} = (a + b\sqrt{-13})^3.$$

   Conclude that $(x, y) = (17, \pm 70)$.
   Hint: Use the fact that $\mathbb{Q}(\sqrt{-13})$ has class number 2, cf. Exercise 2.

To be handed in: Monday, 20. November 2017.