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WS 2017/18

Algebraic Number Theory

3. Exercise sheet

Exercise 1 (4 Points):

Let $\zeta_N \in \mathbb{C}$ be a primitive N-th root of unity. Put $\theta = \zeta_N + \zeta_N^{-1}$.

1) Show that $\mathbb{Q}(\theta)$ is the fixed field of $\mathbb{Q}(\zeta_N)$ under the automorphism defined by complex conjugation.
2) Put $n = \varphi(N)/2$. Show that $\{1, \zeta_N, \theta, \theta\zeta_N, \theta^2, \theta^2\zeta_N, \ldots, \theta^{n-1}, \theta^{n-1}\zeta_N\}$ is an integral basis for $\mathbb{Q}(\zeta_N)$.
3) Show that the ring of integers of $\mathbb{Q}(\theta)$ is $\mathbb{Z}[\theta]$.
4) Suppose that $N = p$ is an odd prime number. Prove that the discriminant of $\mathbb{Q}(\theta)$ is $\Delta_{\mathbb{Q}(\theta)} = p^{\frac{p-3}{2}}$.

Exercise 2 (4 Points):

Let A be a local domain with unique maximal ideal $\mathfrak{m} \subseteq A$ such that each non-zero ideal $I \subseteq A$ admits a unique factorization $I = \prod_{i=1}^{n} \mathfrak{p}_i^{e_i}$ into prime ideals $\mathfrak{p}_i \subseteq A$.

1) Show that there exists $x \in \mathfrak{m} \setminus \mathfrak{m}^2$.
2) Let $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ and $y \in \mathfrak{m}$. Prove that $(x, y) \subseteq A$ is prime.
   Hint: Write $(x, y) = \mathfrak{p}_1 \cdots \mathfrak{p}_r$ as a product of prime ideals and use $x \notin \mathfrak{m}^2$.
3) Prove $(x) = \mathfrak{m}$.
   Hint: For $y \in \mathfrak{m}$ show $y \in (x, y^2)$.
4) Conclude that every element $y \in A \setminus \{0\}$ admits a unique expression $y = ux^e$ with $e \geq 0$ and $u \in A^\times$ a unit and that A is a discrete valuation ring, i.e., a local Dedekind domain.
   Hint: First show that A is noetherian.

Exercise 3 (4 Points):

Let A be a Dedekind domain.

1) Prove that, for any nonzero ideal I of A, every ideal of A/I is principal.
   Hint: Reduce to the case, where $I = \mathfrak{p}^e$ with $\mathfrak{p}$ a prime.
2) Prove that every nonzero ideal I of A can be generated by at most two elements.

Exercise 4 (4 Points):

Let k be an algebraically closed field of characteristic $\neq 2$ and consider the extension of rings

$$A = k[x] \subseteq B = k[x, y]/(y^2 - x^3 + x).$$

1) Prove that A and B are Dedekind domains.
2) Prove that $x, y \in B$ are irreducible elements that are not prime.
   Hint: Consider the automorphism $x \mapsto x, y \mapsto -y$ of B and the norm $N\colon B \to A,\ b \mapsto b\sigma(b)$.
3) Prove that the extension $A \subseteq B$ is ramified precisely over the ideals $(x), (x-1), (x+1) \subseteq A$.

To be handed in: Monday, 6. November 2017.