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WS 2017/18

Algebraic Number Theory

1. Exercise sheet

Exercise 1 (4 Points):

i) Let $n \geq 1$ and set $\varphi(n) := \sharp\{m \in \mathbb{Z} \mid 1 \leq m \leq n \text{ and } (n,m) = 1\}$. Prove

$$n = \sum_{d|n} \varphi(d).$$

ii) Let $\zeta(s)$ be the Riemann ζ-function. For $\mathrm{Re}(s) > 2$ prove

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n\geq 1} \frac{\varphi(n)}{n^s}.$$

Exercise 2 (4 Points):

Let p be an odd prime, let q be power of p and let $\mathbb{F}_q$ be the finite field with q elements.

i) Prove that $x \in \mathbb{F}_q^\times$ is a square if and only if $x^{(q-1)/2} = 1$.
Hint: Use that an element $y \in \overline{\mathbb{F}}_q$ lies in $\mathbb{F}_q$ if and only if $y^q = y$.

ii) Prove that $x^2 = 2$ has a solution in $\mathbb{F}_p$ if and only if $p \equiv \pm 1 \mod 8$.
Hint: Let $\alpha \in \overline{\mathbb{F}}_p$ be a primitive 8th root of unity. Show that $(\alpha + \alpha^{-1})^2 = 2$.

iii) Prove that $x^2 = -2$ has a solution in $\mathbb{F}_p$ if and only if $p \equiv 1, 3 \mod 8$.

Exercise 3 (4 Points):

Show $\mathbb{Z}[\sqrt{-2}]^\times = \{1, -1\}$ and that $\mathbb{Z}[\sqrt{-2}]$ is a principal ideal domain. Prove that $\mathbb{Z}[\sqrt{-3}]$ and $\mathbb{Z}[\sqrt{-5}]$ are not principal ideal domains.

Exercise 4 (4 Points):

i) For $n \geq 1$ let $r(n) := \sharp\{(x,y) \in \mathbb{Z}^2 \mid x^2 + 2y^2 = n\}$. Prove that

$$r(n) = 2\sum_{d|n} \chi(d)$$

where $\chi\colon \mathbb{Z}_{\geq 1} \to \{-1, 0, 1\}$ is the multiplicative extension of

$$\chi(p) = \begin{cases} 0 & \text{if } p = 2 \\ 1 & \text{if prime and } p \equiv 1, 3 \mod 8 \\ -1 & \text{if } p \text{ prime and } p \equiv 5, 7 \mod 8 \end{cases}.$$

ii) Deduce that a positive natural number n can be represented by $x^2 + 2y^2$ if and only if in the prime factorization of n each prime p with $p \equiv 5, 7 \mod 8$ has an even exponent.

Hint: Mimick the case $R = \mathbb{Z}[i]$ which was treated in the lecture.

To be handed in: Monday, 23. October 2017.