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SS 2017

Algebraic Geometry II

9. Exercise sheet

Exercise 1 (4 points):
Let $f\colon Y \to X$ be a morphism of schemes and let $\mathcal{F}$ be an abelian sheaf on Y. Prove that the canonical morphism

$$H^n(X, f_*(\mathcal{F})) \to H^n(Y, \mathcal{F})$$

is an isomorphism for any $n \geq 0$ if

i) f is a closed immersion, or

ii) f is affine and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module.

Exercise 2 (4 points):
Let X be a topological space and let $\mathcal{F}$ be a sheaf of abelian groups on X. Let $X = U \cup V$ be an open covering of X. Prove that there exists a (natural) long exact Mayer-Vietoris sequence

$$\ldots \to H^i(X, \mathcal{F}) \to H^i(U, \mathcal{F}) \oplus H^i(V, \mathcal{F}) \to H^i(U \cap V, \mathcal{F}) \to H^{i+1}(X, \mathcal{F}) \to \ldots.$$

Hint: Analyze a suitable spectral sequence.

Exercise 3 (4 points):
Let A be a ring and let $d < 0$ (the case $d \geq 0$ has been handled in the lecture). Prove that

$$H^i(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(d)) \cong \begin{cases} 0 & \text{if } i < n \\ (\frac{1}{x_0 \ldots x_n} A[x_0^{-1}, \ldots, x_n^{-1}])_d & \text{if } i = n \end{cases}$$

for every $n, i \geq 0$. Here the subscript d denotes the space of homogenous polynomials of degree d. Thus in particular, $H^*(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(d)) = 0$ if $-n-1 < d < 0$.
Hint: Use Čech cohomology for the standard covering of $\mathbb{P}^n_A$. To prove the vanishing statement use induction on n and the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n_A}(d-1) \xrightarrow{x_0} \mathcal{O}_{\mathbb{P}^n_A}(d) \to i_*(\mathcal{O}_{\mathbb{P}^{n-1}_A}(d)) \to 0$$

where $i\colon \mathbb{P}^{n-1}_A \to \mathbb{P}^n_A,\ (x_1 : \ldots : x_n) \mapsto (0 : x_1 : \ldots : x_n)$.

Exercise 4 (4 points):
Show that

$$\dim_k H^i(\mathbb{P}^n_k, \Lambda^j \Omega^1_{\mathbb{P}^n_k/k}) = \begin{cases} 1 & \text{if } i = j \leq n \\ 0 & \text{otherwise} \end{cases}$$

and try to find explicit generators. Deduce that $\Omega^1_{\mathbb{P}^n_k/k}$ is not an extension of line bundles if $n \geq 2$.
Hint: Use the Euler sequence from Exercise sheet 7, Exercise 1.

To be handed in on: Monday, 26. June 2017.