Prof. Dr. P. Scholze
Dr. J. Anschütz **SS 2017**

Algebraic Geometry II

5. Exercise sheet

Exercise 1 (4 points):

Let A be a ring and let $a \in A^\times$ be a unit. Let $n \in \mathbb{Z}$ such that n is invertible in A. Prove that $\mathrm{Spec}(A[X]/(X^n - a)) \to \mathrm{Spec}(A)$ is finite and étale.

Exercise 2 (4 points):

Let k be a field and let $f\colon X \to \mathrm{Spec}(k)$ be a morphism locally of finite type. Prove that the following assertions are equivalent:
i) f is étale.
ii) f is unramified.
iii) f is smooth and locally quasi-finite.
iv) X is the disjoint union of schemes $\mathrm{Spec}(l)$ where l/k is a finite separable field extension.
Hint: Let K be an algebraic closure of k. Check that each statement is satisfied for f if and only if it is satisfied for $f_K\colon X_K := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) \to \mathrm{Spec}(K)$ (e.g., that smoothness of f_K implies smoothness of f was proven in the lecture). If $K = k$ use that $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2 \cong \Omega^1_{X/k} \otimes k(x)$ for $x \in X$ closed.

Exercise 3 (4 points):

Let K/k be a finitely generated extension of fields. Prove that the following statements are equivalent:
i) $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$ is formally smooth
ii) K/k is separable, i.e., K contains a purely transcendental subfield $L = k(x_1, \ldots, x_n)$ such that K/L is a separable algebraic extension.
Hint: For i) $\Rightarrow$ ii) pick elements $x_1, \ldots, x_n \in K$ such that the differentials dx_i form a basis of $\Omega^1_{K/k}$. Conclude by formal smoothness that the induced morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(k(x_1, \ldots, x_n))$ is formally étale. Use exercise 2 to deduce that K/L is a finite and separable extension.

Exercise 4 (4 points):

Let $A \to B$ be a morphism of rings such that B is free of finite rank over A. For $b \in B$ let $\mathrm{tr}_{B/A}(b) \in A$ be the trace of the endomorphism

$$b\cdot\colon B \to B,\ b' \to bb'.$$

Prove that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is (finite) étale if and only if the trace bilinear form

$$\mathrm{Tr}_{B/A}\colon B \otimes_A B \to A, (b, b') \mapsto \mathrm{tr}_{B/A}(bb')$$

is non-degenerate, i.e., it induces an isomorphism $B \cong \mathrm{Hom}_A(B, A),\ b \mapsto \mathrm{Tr}_{B/A}(b, -)$.
Remark/Hint: Reduce to the case that A is an algebraically closed field. The same statement, with an appropriate definition of the trace pairing, also holds if B is only assumed to be locally free of finite rank over A.

To be handed in on: Monday, 22. May 2017.