All unspecified references are to the book [Mil80].

The goal of this seminar is to study various advanced topics in étale cohomology, following (mainly) the last two chapters of [Mil80]. One of the main motivations for étale cohomology are the Weil conjectures, and we will come to them (and even prove the easier parts) in the very last talk. The content can roughly slits into the following parts:

- **(Talk 1)** We introduce and discuss properties of the étale fundamental group of a scheme.
- **(Talks 2-5)** We investigate the étale cohomology of curves and surfaces. The main result for curves is the Poincaré duality theorem for constructible sheaves (talk 2). Next, we consider surfaces (talks 3-5). To study their étale cohomology, we fiber them (after blow-up) over \( \mathbb{P}^1 \), introducing the notion of Lefschetz pencils. The aim is basically to describe the cohomology of a Lefschetz pencil. Along the way, we discuss some intersection theory and Chern classes (talk 4).
- **(Talks 6-11)** After curves and surfaces, this is the second main part of the seminar. The idea is to introduce several features in étale cohomology (like smooth pairs, fundamental class of a subvariety, cycle class map, étale Chern classes, ... ) and prove a number of useful properties about them. Among others, we will prove: the smooth base change theorem, the Gysin exact sequence, the weak Lefschetz theorem (which says that the cohomology of a variety agrees with the cohomology of a smooth hyperplane section except in the middle degrees) and, finally, the Poincaré duality theorem.
- **(Talk 12)** Finally, we consider the Weil conjectures over finite fields, and prove two of them by applying the above theory. Namely, we establish a Lefschetz trace formula and deduce the rationality of the zeta function from it, and deduce then the functional equation from the Poincaré duality.

**Talk 1. Fundamental group.** [Mil80, I.5], [SGA1, V.7]

Following the above references (it might also be useful to consult [Stacks, Chap. 57]), define the étale fundamental group of a scheme at a geometric point. Describe \( \mathbb{Z}/n\mathbb{Z} \)-Galois coverings of a scheme \( X \) in terms of \( \pi_1(X, \bar{x}) \). For schemes over \( \mathbb{C} \), discuss the relation to the topological fundamental group of \( X(\mathbb{C}) \). Describe the fundamental exact sequence between the geometric and arithmetic parts of \( \pi_1(X, \bar{x}) \) for a variety \( X \) over a field. Briefly discuss some examples: describe (with more or less details) the fundamental group of (a) \( \mathbb{P}^n \) over a field; (b) nodal rational proper curve over an algebraically closed field; (a) a smooth (not necessarily proper) curve over \( \mathbb{C} \); (c) an elliptic curve (more generally: abelian variety) over a field; (d) Spec \( \mathcal{O}_k \) for a number field \( k \) (in particular \( k = \mathbb{Q} \)); (e) \( \mathbb{A}^1_{\mathbb{F}_p} \).

Recall the definition of constructible sheaves (on Noetherian schemes), cf. Proposition (V.1.8). Prove a criterion for local constancy (V.1.10a). If time permits, briefly discuss (V.1.3). Introduce \( \mathbb{Z}_\ell \) - and \( \mathbb{Q}_\ell \)-sheaves. Relate \( \mathbb{Z}_\ell \)-local systems to representations of \( \pi_1 \) (p. 164).
Talk 2. Cohomology of curves. [Mil80, V.1, V.2]
Briefly discuss the computation of cohomology of a curve with constant or \( \mathbb{G}_m \) coefficients (III.2.22, a-e). Prove the Poincaré duality for curves Theorem (V.2.1). Therefore, first introduce the various pairings via Ext-groups and via the cup product, and trace maps (pp.168-174). Also prove Proposition (V.2.2) and Corollary (V.2.3).

Talk 3. Cohomology of Surfaces I. [Mil80, V.3]
Sketch the proof of the existence of Lefschetz pencils (V.3.1). State (V.3.3) and briefly sketch its proof. Study \( R^i \pi_* \mathbb{Q} \) for a fibration \( \pi : X \to S \) with good properties (V.3.5), (V.3.6). For (V.3.5), recall the arithmetic genus of a curve (e.g. [Stacks, Tag 0BY7]) along with some geometric intuition for it, and its constancy (e.g. [Har, III.9.10]). Before dealing with (V.3.6), recall the Jacobian variety of a curve [Mil21], and explain briefly the idea of the generalized Jacobian [Ser, IV.4.V.13] (which is needed in a very particular special case). Deduce the corollaries (V.3.8) and (V.3.9). If time permits, discuss (V.3.10).

Talk 4. Cohomology of Surfaces II. [Har, App.A] and [Mil80, V.3]
Define Chern classes of a vector bundle on a scheme [Har, App. A§1-§3]. Discuss the statement of the (Grothendieck–)Hirzebruch–Riemann–Roch Theorem [Har, App. A§1-§3] and the 1- and 2-dimensional special cases [Har, App.A, Examples 4.1.1, 4.1.2]. (It might also be useful to consult [Ful].) Finally, relate the \( \ell \)-adic Euler characteristic to the classical one and the Chern classes (V.3.12). For that it will be necessary to investigate how cohomology behaves under blow-up (V.3.11).

Talk 5. Cohomology of Surfaces III. [Mil80, V.3]
Study vanishing cycles for a given Lefschtz pencil as on p. 205. Prove the Picard–Lefschetz formula (V.3.14) and the conjugacy of vanishing cycles (V.3.17). (If time permits, mention the irreducibility of monodromy action (V.3.20) and its consequence, the theorem of Kazhdan–Margulis (V.3.21).) Compute then the cohomology of \( X \) with constant coefficients, where \( X \to \mathbb{P}^1 \) is a Lefschetz pencil with irreducible fibers (V.3.22), (V.3.23).

Talk 6. Smooth base change and acyclic morphisms. [Mil80, VI.4]
State and prove the smooth base change theorem (VI.4.1). In particular, introduce (universally, locally) \( n \)-acyclic morphisms and discuss their properties. Altogether, explain (VI.4.6)-(VI.4.20). It would be good to mention the counterexample in (VI.4.4). If time permits, discuss some immediate consequences of the smooth base change (VI.4.2), (VI.4.3), (VI.4.5).

Talk 7. Purity and Gysin sequence. [Mil80, VI.5]
State and prove the theorem on cohomological purity (VI.5.1) and the Gysin exact sequence (VI.5.3). Deduce the result on finiteness of cohomology (VI.5.5) and mention Remark (VI.5.7).

Talk 8. Fundamental class and the cycle class map. [Mil80, VI.6, VI.9]
Introduce the fundamental class of a variety (VI.6.1). Prove that it behaves as expected under the Gysin map (VI.6.5). Following (VI.9) define the cycle class map \( c_Y \) with values in the (étale) cohomology ring \( H^*(X) \) of a variety \( X \), and describe some of its basic properties. Discuss the case of \( \mathbb{P}^n \) (VI.9.7). (It might also be useful to consult [Ful].)

Talk 9. Weak Lefschetz theorem. [Mil80, VI.7]
State and prove the weak Lefschetz theorem (VI.7.1). Also state the Künneth formula (VI.8.5). If time permits, give a rough outline of its proof.

**Talk 10. Étale Chern classes.** [Mil80, VI.10]
Compute the cohomology ring of a projective bundle (VI.10.1). Using Grothendieck’s formalism (without proof) and a result from talk 8 deduce that there is a theory of Chern classes with values in étale cohomology, satisfying nice properties (VI.10.3). Follow the rest of (VI.10) to prove that when taking $\mathbb{Q}_\ell$ as coefficients, $c_1_X$ gets an homomorphism of graded rings $CH^* (X) \to H^* (X)$, where $CH$ denotes the Chow ring (VI.10.8). (It might also be useful to consult [Ful].)

**Talk 11. Poincaré duality.** [Mil80, VI.11]
State the Poincaré duality theorem (VI.11.1) (and its corollary (VI.11.2)), and give as many details of the proof as possible.

**Talk 12. Lefschetz trace formula. Rationality and functional equation.** [Mil80, VI.12]
State the Weil conjectures (VI.12). Prove the Lefschetz trace formula (VI.12.3). Deduce the rationality of the zeta function from it (VI.12.4) and (V.2.6). Also explain Remark (VI.12.5). Finally, deduce the functional equation (VI.12.6) from Poincaré duality.

**References**


