A SURVEY ON VECTOR BUNDLES OVER $\mathbf{P}^1_{\mathbb{Z}}$

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1. Historical remarks, motivation, and notation

My work is mostly concerned with the study of vector bundles on projective lines over Dedekind domains, the ring of rational integers is of particular interest. First of all, let me explain the motivation of our work and recall some related results.

Date: June 2017.

There are several ways which led us to the question of classification of bundles over higher-dimensional arithmetic schemes.

1.1. **Lattices.** Firstly, vector bundles generalize naturally the notion of a lattice (a finitely generated torsion free module over integral domain with an embedding into the vector space over the field of fractions). Despite the fact that the Grothendieck group $K(S)$ of unimodular lattices is quite simple, in fact, it is identified with the subgroup of $\mathbb{Z} \times \mathbb{Z}$ consisting of such elements (a, b) that $a \equiv b \pmod 2$, the lattices are among the most interesting objects in mathematics, and give rise to modular forms, theta functions, Hecke operators and so on. In the case of higher dimensional schemes, we have to define what exactly "embedding" should mean. The possible answer is provided by Arakelov theory. Due to Arakelov, Soulé, Gillet et al. we should study metrized bundles.

1.2. **Amalgams.** Second source of motivation goes back to Serre's book "Trees", where he deals with vector bundles over curves and amalgamated decompositions of certain "arithmetic" groups. Let me remind some results from this work.

Theorem 1 (Nagao)**.** *The group* $\mathbf{GL}_2(k[t])$ *is the sum of the subgroups* $\mathbf{GL}_2(k)$ *and* $B(k[t])$ *amalgamated along their intersection* $B(k)$*:*

$$\mathbf{GL}_2(k[t]) = \mathbf{GL}_2(k) *_{B(k)} B(k[t]). \tag{1.1}$$

For any commutative ring R, we define $B(R)$ as an intersection of upper triangular matrices with $\mathbf{GL}_2(R)$.

Theorem 2 (Ihara)**.** *The group* $\mathbf{SL}_2(\mathbb{Q}_p)$ *is an amalgam of two copies of* $\mathbf{SL}_2(\mathbb{Z}_p)$*, the sum is taken over congruence subgroup* Γ*, first injection is identical, and second is conjugate.*

1.3. **Serre's amplitudes for bundles on curves.** Assume for simplicity that k is finite or algebraically closed field. Let C be a smooth projective curve over k, geometrically connected and of genus g. Let E be a vector bundle of rank two on C, and $F \subset E$ be a subbundle of E of rank 1, so that the quotient sheaf E/F is a bundle. We put

$$N(E, F) = \deg(F) - \deg(E/F), \quad \text{and } N(E) = \sup_F N(E, F). \tag{1.2}$$

It follows from Riemann-Roch theorem that

$$-2g \leq N(E) < -\infty.$$

Note that if $N(E, F) > 2(g-1)$ then F is a direct factor of E.

1.4. **Complex geometry, line bundles over arbitrary base.** The last source I would like to mention is a classification of vector bundles over complex projective spaces. Every vector bundle over projective line over a field is decomposable (isomorphic to the sum of line bundles), due to Grothendieck-Birkhoff theorem, which is no longer the case over $\mathbb{Z}$, but is nevertheless useful when studying fibres of bundles over closed points. In addition, line bundles are easy to describe even over locally noetherian base. Any line bundle over locally noetherian connected base S is of the form $\pi^*L \otimes \mathcal{O}(d)$, where π is a structure morphism $\mathbf{P}^n_S \to S$, and $\mathcal{O}(d)$ is the dth tensor power of a Serre twisting sheaf.

The classification problems over complex projective spaces are proved to be quite hard. Fortunately, the complex setting also provides us an approach to the arithmetic problem. Namely, we have two Beilinson spectral sequences that help us to reduce classification to the questions of linear algebra (often quite difficult even over algebraically closed fields). Nevertheless, we have applied this technique to obtain the classification in several particular cases.

1.5. **Non-Euclidean PID.** It is natural to ask how one can construct bundles on $\mathbf{P}^1_A$, where A is a PID. There are three 'regular ways' to do that.

1.6. **Beilinson.** As it was mentioned earlier, we can use the Beilinson spectral sequences and classify bundles with some fixed cohomological data.

1.7. **Quillen–Suslin.** In the case when A is a PID, we can apply Quillen–Suslin theorem to see that the set of isomorphism classes of framed vector bundles on the projective line can be represented as the following double quotient

$$\mathbf{Bun}_r(\mathbf{P}^1_A) = \mathbf{GL}_r(A[x]) \backslash \mathbf{GL}_r(A[x, x^{-1}]) / \mathbf{GL}_r(A[x^{-1}]. \tag{1.3}$$

Let E be a vector bundle, and let $r = \mathrm{rk}(E)$. By frame for E we mean two chosen trivialisations over the standard open sets U_0 and U_1 glued together over U_{01} by the invertible matrix over $A[x, x^{-1}]$. Also, every such a matrix defines a vector bundle.

1.8. **Hanna's theorem.** Of course, the latter approach is very explicit but the groups above are quite complicated, especially in the cases of low rank. Due to first theorem of Hanna, we can reduce the study of bundles to the case of rank two. Moreover, he obtained a surprising result in the case where A is a Euclidean domain.

Theorem 3 (Hanna)**.** *Let A be a Euclidean domain, and let F be any vector bundle on $\mathbf{P}^1_A$. Then F has a filtration*

$$0 = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n = F$$

such that F_i/F_{i-1} is a line bundle ($1 \leq i \leq n = \operatorname{rk} F$).

In particular, every bundle on $\mathbf{P}^1_{\mathbb{Z}}$ admits a filtration with linear bundles as quotients.

Remark 1. It is worth mentioning that even in the case when A is a non-euclidean PID, we do not know if the analogous statement still holds. Since straightforward methods of proving statements of that spirit lead us to questions about integral points on rational varieties of high degree (actually, Fano varieties), we have no easy way of computing an invariant of A given by $\sup_E(N(E)_{loc} - N(E))$. So our strategy, roughly speaking, is to find obstructions in the arithmetic of extensions of $\mathrm{Frac}(A)$.

Remark 2. Given a vector bundle E, we want to study its sub**bundles** (not only subsheaves), this means that $L \subset E$ if and only if E/L is a locally free sheaf, or, equivalently, a section $s : L \to E$ has no zeroes on $\mathbf{P}^1_A$.

We let $H^0(E)^\times$ denote the subset of all nowhere zero sections in $H^0(\mathbf{P}^1, E)$.

1.9. **Interpretation of Hanna's theorem $r = 2$.** Let E be a bundle of rank two on $\mathbf{P}^1_{\mathbb{Z}}$. We have seen that there exist line bundles L and M, such that E fits into the exact sequence

$$0 \to L \to E \to M \to 0. \tag{1.4}$$

Thus, we get an element in $\mathrm{Ext}^1(M, L)$, and the latter group is naturally isomorphic to $H^0(M \otimes L^* \otimes \omega_{\mathbf{P}^1})^\vee$ (Serre duality). We want to identify the corresponnding element with a binary form.

Note that $\mathbf{P}^1_A = \mathrm{Proj}(V)$, where V is a free A-module. There is a natural isomorphism $H^0(\mathcal{O}(k)) \simeq \mathrm{Sym}^k(V^\vee)$, and

$$Ext^1(\mathcal{O}, \mathcal{O}(-k-2)) = H^1(\mathbf{P}^1, \mathcal{O}(-k-2)) \simeq \widetilde{\mathrm{Sym}}^k(V),$$

here $\widetilde{\mathrm{Sym}}^k(V)$ means a submodule of symmetric tensors in $V^{\otimes k}$, and we have an obvious map to $\mathrm{Sym}^k(V)$ (e.g. $v \otimes u + u \otimes v \mapsto 2uv$).

We choose a basis of V^*, so that $\mathbf{P}^1_A = \mathrm{Proj}(V) = \mathrm{Proj}(A[t_0, t_1])$, $x = t_1/t_0$, and $y = t_0/t_1$. We let v_0 and v_1 denote the dual basis (of $V^\vee$).

1.10. **Filtration on the space of binary forms.** Let Φ_n denote the space of all integral binary forms such that ith coefficient is divisible by $\binom{n}{i}$ where n is a degree.

It is easy to see that if $L \subset E$ then $L(-k) \subset E$ for every $k > 0$. So one can define a space of 'newforms' of degree n. In particular, this is closely related to the notion of minimal filtrations discussed above. It could be interesting to understand this structure.

For example, starting with the element of $\mathrm{Sym}^4(V)$: $F = 12(v_0^4 - 3v_0^3v_1 - 7v_0^2v_1^2 + v_1^4)$ we obtain a bundle E sitting in the exact sequence $\mathcal{O}(-3) \to E \to \mathcal{O}(3)$. We can check that the fibres of E are either of the form $\mathcal{O} \oplus \mathcal{O}$ or $\mathcal{O}(1) \oplus \mathcal{O}(-1)$, so the classification theorem says that $E \simeq V_0(m, e)$ for some m and e. But m could be recovered from the catalecticant of the polynomial, which is $244 = 4 \cdot 61$. We are left to somehow find e. Brutal computation shows that E admits also a filtration of lower amplitude; namely $\mathcal{O}(-2) \to E \to \mathcal{O}(2)$, and there is a corresponding quadratic form of discriminant 244: $16v_0^2 + 4v_0v_1 - 15v_1^2$, which represents 1 modulo 244. This proves that E is isomorphic to the simplest bundle with conductor 244, the bundle $V_0(244, 1)$. Thus, F is an old form of degree-'level' 4 coming from the level 0 form, which is simply 244.

1.11. **Notation, and first invariants.** To continue I should introduce a bit of notation and several invariants of bundles.

1.12. **Subscheme of jumps.** Let E be a rank 2 bundle on $\mathbf{P}^1_{\mathbb{Z}}$, then its fibre over the closed point $P \in \operatorname{Spec}\mathbb{Z}$ is a decomposable bundle. We define finite subscheme of "jumps" $J(E) \subset \operatorname{Spec}\mathbb{Z}$ as a set of points, where this decomposition differs from the decomposition obtained by restriction to the general fiber. We can also define a more subtle invariant working with p-adic fibers of E, which are naturally defined for every prime p. We write E_p for such a bundle.

1.13. **Conductor.** Let P be a point in $J(E)$, then restricting to the corresponding special fibre we obtain a decomposition $E\mid_P \simeq \mathcal{O}(d_{1,P}) \oplus \mathcal{O}(d_{2,P})$, and $d_{1,P} \le d_{2,P}$. Then the base change theorem implies that E_p has $\mathcal{O}(d_{1,P})$ as a subbundle (this means that $E_p/\mathcal{O}(d_{1,P})$ is also locally free). It is crucial that we have only one closed point in this situation. We can define local conductor of E at the point P as a p^{k_P}, where k_P – is the largest natural k such that $E_p \times \mathbb{Z}_p/p^k\mathbb{Z}_p$ decomposes as a sum of linear bundles.

The global conductor is, as usual, defined to be the product of local ones, and denoted $\Delta(E)$.

1.14. **Amplitudes.** It is natural to generalize Serre's amplitude to our situation. Note that we have local one, which we denote $N_{loc}(E)$ defined as a minimum over all points in $J(E)$. There is also a global amplitude $N(E)$, defined the same way as above. Thus, we have an obvious inequality $N(E) \leq N_{loc}(E)$.

It is easy to see that there is no lower bound on $N_{loc}(E)$ even in the case of bundles over $\mathbf{P}^1_{\mathbb{Z}_p}$, to see this one combines the existence of jumps and semicontinuity theorem. Thus $N(E)$ is unbounded. This means that we have many indecomposable bundles. The following question seems interesting

Question 1. If there exists an upper bound for $N_{loc}(E) - N(E)$?

As above (cf. 1.3), if $N(E) \geq 0$ then E is decomposable.

1.15. **Simple jumps.** We say that E has simple jumps if its local fibers differs from the generic fiber not too much. Namely, let $E_{\mathbb{Q}} \simeq \mathcal{O}(d_1) \oplus \mathcal{O}(d_2)$ with $d_1 \leq d_2$, so that $d_1 - d_2 \leq 0$. Then simple jumps correspond to the case $(d_1 - d_2) - N_{loc}(E) = 2$.

E.g. $E \otimes \mathbb{F}_p \simeq \mathcal{O}(-1) \oplus \mathcal{O}(1)$ for only one prime p, and $E \otimes \mathbb{Q} \simeq \mathcal{O}^2$.

2. Classification

The information required to apply Beilinson spectral sequences agrees well with the notion of jumps, and we have classified vector bundles in two cases:

– Trivial generic fibre and simple jumps. Smirnov.
– Generic fibre isomorphic to $\mathcal{O} \oplus \mathcal{O}(1)$ and simple jumps. Me.

Although the statements of this theorems are a bit cumbersome, I would like to formulate both in detail. Because they are related to reciprocity laws (namely, quadratic and cubic), and further results rely on these statements.

2.1. **Cubic reciprocity.** Let p and q be primary numbers in the ring $\mathbb{Z}[\omega]$, both coprime to 3, where ω is a primitive cube root of unity (rational integer is prime in $\mathbb{Z}[\omega]$ iff it's congruent to 2 modulo 3). The congruence $x^3 \equiv p \pmod{q}$ is solvable if and only if $x^3 \equiv q \pmod{p}$ is solvable.

2.2. **Standard examples.** We let $V_0(m,e)$ denote the sheaf $\mathrm{Coker}[\mathcal{O}(-2)^2 \xrightarrow{\varphi} \mathcal{O}(-1)^4]$, where the map φ has the following form

$$\varphi = \begin{pmatrix} t_1 & 0 \\ et_0 & t_1 \\ mt_0 & 0 \\ 0 & t_0 \end{pmatrix}. \tag{2.1}$$

We set $V_1'(m,e) = \mathrm{Coker}[\mathcal{O}(-2)^3 \xrightarrow{\psi'} \mathcal{O}(-1)^5]$, where ψ' is defined by the matrix

$$\psi' = \begin{pmatrix} t_1 & et_0 & 0 \\ 0 & t_1 & 0 \\ t_0 & 0 & t_1 \\ 0 & mt_0 & 0 \\ 0 & 0 & t_0 \end{pmatrix}. \tag{2.2}$$

Theorem 4. *Let $m \neq 0$ and $\gcd(m,e) = 1$, then $V_1'(m,e)$ is a vector bundle. Moreover, $V_1'(m,e)$ is generically isomorphic to $\mathcal{O} \oplus \mathcal{O}(1)$ and has simple jumps if m is not invertible, and $\Delta(E)(V_1'(m,e)) = |m|$.*

(1) *Every vector bundle with the generic fibre $\mathcal{O} \oplus \mathcal{O}(1)$ and simple jumps is isomorphic to $V_1'(m,e)$ up to the action on $\mathbf{Bun}_2$ of the involution W.*
(2) *$V_1'(m,e) \simeq V_1'(m',e')$ if and only if $(m) = (m')$, and $e \equiv \pm e' \pmod{m}$.*
(3) *$V_1'(m,e)$ can be defined by the gluing matrix $\begin{pmatrix} ax^{-1} & b \\ cx & dx^2 \end{pmatrix}$, where где $ad - bc = 1$, $(b) = (m)$, and $a \equiv \pm e \pmod{m}$.*
(4) *If e is not a perfect cube modulo m then $\mathcal{O}(-1) \not\subset V_1'(m,e)$.*

Remark 3. It is likely that the latter statement can be reversed. It was checked for $m \leq 30$

Theorem 5 (Smirnov)**.** *Let m be a non-zero integer, and e be an integer prime to m, then $V_0(m,e)$ is a vector bundle. Moreover, every rank two vector bundle with trivial generic fibre and simple jumps is isomorphic to $V_0(m,e)$ for some m and e such that $\gcd(m,e) = 1$, and*

(1) *$\Delta(E) = |m|$.*
(2) *$V_0(m,e) \simeq V_0(m',e')$ if and only if $(m) = (m')$, and there exists $\lambda \in \mathbb{Z}$, such that $e\lambda^2 = \pm e' \pmod{m}$.*
(3) *$\mathcal{O}(-1) \subset V_0(m,e)$ if and only if at least one of the numbers $\pm e$ is a quadratic residue modulo m.*
(4) *$\mathcal{O}(-2)$ is a subbundle of $V_0(m,e)$ for each pair (m,e) such that $\gcd(m,e) = 1$. $\Leftarrow$ **Gauss Reciprocity!***

(5) *The bundle* $V_0(m,e)$ *can be identified with its gluing matrix* $\begin{pmatrix} ax^{-1} & b \\ c & dx \end{pmatrix}$*, where* $ad-bc=1$*,* $(b)=(m)$*, and there exists* $\lambda\in\mathbb{Z}$ *such that* $a\lambda^2\equiv\pm e \pmod m$*.*

This provides us an example of the situation when $N_{loc}(E)-N(E)=2$, the 'first' such a bundle is $V_0(5,2)$, due to the third statement above, since neither 2 or -2 is a square modulo 5.

2.3. **Translation into the language of quadratic forms.** As we have seen earlier, every bundle with trivial fiber and simple jumps admits a filtration

$$0\to\mathcal{O}(-2)\to E\to\mathcal{O}(2)\to 0. \tag{2.3}$$

Associated gluing matrix will have the form $\begin{pmatrix} x^{-2} & a_0x^{-1}+a_1+a_2x \\ 0 & x^2 \end{pmatrix}$.

Thus we can find an element in $\mathrm{Ext}^1(\mathcal{O}(2),\mathcal{O}(-2))$. Applying duality, we obtain a form $f=a_0v_0^2+2a_1v_0v_1+a_2v_1^2$, called associated to E. Let $d(f)=a_1^2-a_0a_2$ denote the determinant of an obvious bilinear form. We can state the classification as follows.

Theorem 6. *Let* E *and* E' *be vector bundles of rank two with trivial generic fibre and simple jumps, and let* $f=(a_0,2a_1,a_2)$*,* $f'=(a_0',2a_1',a_2')$ *be the forms associated to* E *and* E'*, respectively. Then*

(1) f *and* f *are almost primitive (outside* 2*).*
(2) $\Delta(E)=|d(f)|$*,* $\Delta(E')=|d(f')|$
(3) $E\simeq E'$ *if and only if* $|d(f)|=|d(f')|=D$*, and* $f,f'\in\Phi_2(D,e)$*.*

Where $\Phi_2(D,e)$ *is the set of binary forms* q *with even middle term,* $|d(q)|=D$*, and* $q\equiv\pm e$ *modulo squares in* $(\mathbb{Z}/D\mathbb{Z})^\times$*.*

Remark 4. Given any upper-triangular matrix in $\mathbf{GL}_2(\mathbb{Z}[x,x^{-1}])$, we can describe the fibres of an associated vector bundle in terms of certain invariants of the Laurent polynomial sitting in the obvious position. This will be illustrated below for the binary cubic forms.

2.4. **Compactified projective line.** As mentioned earlier, there is an analogy between lattices and metrized vector bundles. Unfortunately, the operation of arithmetic compactification doesn't seem canonical. However, we can get some interesting information about vector bundles over $\mathbf{P}^1_{\mathbb{Z}}$.

Let us start with introducing notation and necessary definitions. Arakelov model X of the projective line over the integers consists of the following data

– $X_0=\mathbf{P}^1_{\mathbb{Z}}$ – a finite part of X.

– Kähler metric on the tangent bundle of $X_\infty = \mathbf{P}^1_{\mathbb{C}}$, and associated Kähler form ω_0 is normalized $\int_{X_\infty} \omega_0 = 1$. It is simply Fubini-Study form for $\mathbf{P}^1_{\mathbb{C}}$.

We are going to define vector bundles on X by gluing pieces over X_0 and X_∞. By vector bundle E on X we mean a triple (E_0, E_∞, α) where

– E_0 is a bundle on X_0.
– $E_\infty = \bigoplus \overline{\mathcal{O}(d_i)}$, where $\overline{\mathcal{O}(d_i)}$ is a line bundle with Fubini-Study metric induced from $\mathcal{O}(-1)$.
– $\alpha : E_0 \otimes \mathbb{R} \to E_\infty$, an isomorphism.

Since we have Kähler form ω_0 on X_∞, and hermitian metrics $(\ ,\ )$ on E_∞, we can measure the length of sections of E_0. More precisely, given a global section $s \in H^0(E)$, we set

$$\langle s,\, s\rangle_2 = \int_{X_\infty} (\alpha(s), \alpha(s))\omega_0. \tag{2.4}$$

We restrict ourselves to the case, where E is a semi-stable bundle over the generic fibre. In our setting, this simply means that $E_{\mathbb{Q}}$ is isomorphic to $\mathcal{O}^{\oplus r}$ up to the twist. Moreover, we assume that E is isomorphic to the trivial bundle of rank r over the generic fibre. To avoid difficulties with constants, we consider only normalized α. More accurately, we have a morphism

$$\mathrm{Det}(\alpha) : \mathrm{Det}(E) \otimes \mathbb{R} \to \mathrm{Det}(E_\infty), \tag{2.5}$$

this induces a map on cohomology, and we require that the length of $1 \in H^0(\mathbf{P}^1_{\mathbb{Z}}, \mathrm{Det}(E))$ is equal to one.

We define

$$\widehat{\varepsilon}_1(E, n) = \inf_{\alpha} \inf_{s \in H^0(E(n))^\times} \langle \alpha(s),\, \alpha(s)\rangle_2. \tag{2.6}$$

In the case of the bundles with trivial generic fibre and simple jumps this invariant can be computed explicitly.

Theorem 7. *Let $V_0(m, e)$ and $\Phi_2(D, e)$ be the same as above. Then*

$$\widehat{\varepsilon}_1(V_0(m, e), 2) = \frac{1}{3m} \min_{f \in \Phi_2(m,e)} \sqrt{2(\mathrm{tr}(f)^2 + 2m)}. \tag{2.7}$$

In particular, 'short' sections correspond to reduced binary forms.

So this justifies the idea that sections of vector bundles on $\mathbf{P}^1$ are of particular interest.

2.5. **Cubic extensions and sections.** In conclusion, we explain how sections of $V_1(m, e)$ are related to binary forms. It is not known yet, if such a bundle admits $\mathcal{O}(-2)$ as a subbundle. However, we briefly describe properties of the associated cubic forms in terms of sections and invariants of $E = V_1(m, e)$.

Since we have computed gluing matrices for E, it is easy to describe spaces of global sections of $E(2)$. Each global section can be represented by a pair of polynomials

$$s_1 = u_1 x + u_0 + mwx^2, \quad s_2 = w_0 + w_1 x + w_2 x^2 + w_3 x^3 + dwx^4, \tag{2.8}$$

where $de \equiv \pm 1 \pmod{m}$.

Theorem 8. *Let f be a cubic form associated to the section $s \in H^0(V_1'(m, e) \otimes \mathcal{O}(2))^\times$. Then*

(1) $\operatorname{disc}(f/3) = -m^2(u_1^2 - 4mu_0 w)$.

(2) *The Hessian of f is completely defined by m and s_1, namely*

$$H_{f/3} = m(u_0 v_0^2 + u_1 v_0 v_1 + mwv_1^2).$$

(3) $\operatorname{disc}(f) \neq 0$ *if* $m \neq \pm 1$.

Roughly speaking, this theorem is about Galois closures $\widetilde{K}$ of cubic extensions $K/\mathbb{Q}$ defined by adjoining a root of the equation $f(v_0, 1) = 0$. We know that $\deg \widetilde{K} \leq 6$. Let me focus on the case $\deg \widetilde{K} = 6$. By standard Galois theory, there is a quadratic subfield L in $\widetilde{K}$, such that K/L is abelian, of degree 3. The Hessian of a cubic form contains the information about the quadratic subfield, square-free part of its discriminant is a discriminant of a quadratic subfield. Content of the Hessian (in our situation, it is m) says what primes are ramified completely in the ring of integers of K (those dividing m).