The Standard Conjectures for the Variety of Lines of a Cubic Hypersurface

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Einführung

Weil's Vermutungen, formuliert im Jahre 1949, verbinden die Anzahl an Punkten auf einer glatten projektiven Varietät über einem endlichen Körper mit der Zeta-Funktion der Varietät. Seine ursprüngliche Idee für einen Beweis bestand darin, ein Analogon zu dem Fixpunktsatz von Lefschetz zu finden. Dies brachte ihn zur Einführung von, nach ihm benannten *Weil Kohomologietheorien*, welche all die Eigenschaften axiomatisieren, die die singuläre Kohomologie von kompakten komplexen Mannigfaltigkeiten besitzt, wie z.B. die Poincaré Dualität, den Künneth Isomorphismus und die beiden Lefschetz Theoreme.
Zwei der drei Weil-Vermutungen wurden Mitte der 1960er Jahre bewiesen: die Rationalität der Zeta-Funktion von Dwork im Jahre 1960 [Dwo60] und 5 Jahre später die Vermutung über die Funktionalgleichung von Grothendieck [Gro66]. Zwar bewies Grothendieck nie den dritten Teil der Weil-Vermutung, jedoch veranlasste dieses Problem ihn dazu, die *Standardvermutungen* zu formulieren [Gro68]. Grothendieck's Standardvermutungen implizieren nicht nur die Weil-Vermutungen, sondern ihr Beweis hätte auch viel tiefere Konsequenzen. Grothendieck hat beispielsweise bereits darauf hingewiesen, dass aus seinen Standardvermutungen in Charakteristik 0 folgt, dass numerische und homologische Äquivalenz übereinstimmen. Im Jahr 1974 bewies Deligne den letzten Teil der Weil-Vermutungen. Grothendieck's Standardvermutungen bleiben jedoch unbewiesen und sind bis heute nur in einigen Spezialfällen bestätigt.
In einer der Originalarbeiten zu dem Thema [Kle94a] präsentiert Kleiman einen Beweis der Standardvermutungen für Abelsche Varietäten. Des Weiteren sind die Standardvermutungen auch für vollständige Schnitte im projektiven Raum, spezielle 3- und 4- dimensionale Mannigfaltigkeiten [Ara19], sowie ein paar andere Klassen von Varietäten bekannt [CM13; Lie68]. Vor kurzem publizierte Diaz [Dia17] einen Beweis für einen anderen Spezialfall der Standardvermutungen. Das Ziel dieser Bachelorarbeit ist es, Diaz's Arbeit zu verstehen und einen vollständigen Beweis des Hauptergebniss zu liefern:

Theorem *Die Standardvermutungen gelten für die Fano-Varietät von Geraden* $F(Y, 1)$ *von einer komplexen, glatten, kubischen Hyperfläche* Y *mit Betti-Kohomologie.*

In Kapitel 1 der Arbeit beschreiben wir die Fano-Varietät einer projektiven Hyperfläche. Diese Fano-Varietät parametrisiert die Hyperebenen in der projektiven Hyperfläche. Zum Beispiel wird die klassische Tatsache, dass eine glatte kubische Hyperfläche 27 Geraden enthält, darin reflektiert, dass die entsprechende Fano-Varietät von Geraden aus 27 Punkten besteht.
In Kapitel 2 etablieren wir das notwendige Hintergrundwissen für den Beweis des Hauptergebnisses. Nach der Einführung der (rationalen) Betti-Kohomologie glatter komplexer projektiver Varietäten besprechen wir die Zykelklassenabbildung und *algebraische Korrespondenzen*. Damit sind wir in der Lage, die Standardvermutungen zu formulieren und drei äquivalente Umformulierungen davon zu beweisen [Kle94a; Kle94b]. Zwar geben wir diese Vermutungen nur für Betti-Kohomologie an, jedoch lässt sich diese einfach auf andere Weil-Kohomologietheorien übertragen. Anschließend besprechen wir (polarisierte) Hodgestrukturen, die beim Beweis eine entscheidende Rolle spielen.
In Kapitel 3 beginnt der Beweis des Hauptresultats, Galkin–Shinder folgend [GS14] beschreiben wir die Klasse von $F(Y, 1)$ im *Grothendieck Ring der Varietäten* $K_0(\mathrm{Var}_{\mathbb{C}})$. Dank dem *Hodge realisations Mor-*

phismus $\mu_{\mathrm{Hdg}} \colon K_0(\mathrm{Var}_{\mathbb{C}}) \to K_0\left(\mathfrak{hs}_{\mathbb{Q}}\right)$ beschreiben wir die Kohomologie von $F(Y,1)$ bezüglich der Kohomologie von Y.

In Kapitel 4 folgen wir Diaz [Dia17] und erhalten eine andere Beschreibung der Kohomologie von $F(Y,1)$ bezüglich der Kohomologie von Y und einer Grassmanschen Varietät.

In Kapitel 5 kombinieren wir diese Beschreibung der Kohomologie mit der Zylinderkorrespondenz, um eine algebraische Korrespondenz zu konstruieren, die den Lefschetzoperator induziert.

Introduction

Weil's conjectures, formulated in 1949, relate the number of points on a smooth projective variety X over a finite field to its associated zeta function. Weil's original idea for a proof of his conjectures was to find an analogue to the Lefschetz fixed-point theorem for varieties over finite fields. This led him to the definition of, what is now known as, *Weil cohomology theories*. These axiomatize all the well known properties which singular cohomology of compact complex manifolds inherits, e.g. Poincaré duality, the Künneth isomorphism, the existence of a cycle class map as well as the weak and hard Lefschetz theorems.

Two of the three Weil conjectures were proven by the mid 1960s: the rationality of the zeta function was proven by Dwork in 1960 [Dwo60] and Grothendieck proved the conjecture on the functional equation in 1965 [Gro66]. While Grothendieck did not prove the third part of Weil's conjectures, this problem prompted him to formulate his farther reaching *standard conjectures* [Gro68]. Not only do these conjectures imply the Weil conjectures but their proof would have much deeper consequences. For instance, Grothendieck already noted that in characteristic 0, his standard conjectures imply that numerical and homological equivalence of algebraic cycles coincides. In 1974, Deligne famously proved the remaining part of the Weil conjectures [Del74]. However Grothendieck's standard conjectures remain unproven to this day.

Grothendieck's standard conjectues have been shown to hold in a few special cases. In one of the original papers on the subject, Kleiman [Kle94a] presents a proof of the standard conjectures for abelian varieties. The standard conjectures are also known for complete intersections in projective space, uniruled three- and fourfolds [Ara19] as well as a few other classes of varieties [CM13; Lie68]. Recently, Diaz [Dia17] published a proof of another special case for which the standard conjectures hold. The goal of this bachelor thesis is to understand Diaz's paper and to provide a detailed proof of its main result:

Theorem *The standard conjectures hold for the Fano variety of lines* $F(Y, 1)$ *of a complex smooth cubic hypersurface* Y *with Betti cohomology.*

In the first part of this thesis we explore the Fano variety of a projective hypersurface. The Fano variety of a projective variety X is a scheme that parametrizes planes contained in X, for instance, the fact that a smooth cubic surface contains 27 lines is reflected in the fact that its Fano variety of lines consists of 27 reduced points.

In section 2, we introduce the necessary background for the proof of the main result. After introducing the (rational) Betti cohomology of smooth complex projective varieties, we introduce the cycle class map and discuss *algebraic correspondences*. We are then able to formulate the standard conjectures and prove three equivalent reformulations of these, following [Kle94a; Kle94b]. While we only state the standard conjectures for Betti cohomology, the formulation we give is easily adapted to any Weil cohomology theory. In a last part, we give a brief account of (polarized) Hodge structures, which play a decisive role in the proof.

In section 3, we start with the proof of the standard conjectures for $F(Y, 1)$, this proof illustrates why a general proof is difficult. Following [GS14], we start by investigating the class of the Fano variety of a smooth cubic hypersurface Y in the *Grothendieck ring of varieties* $K_0\left(\mathrm{Var}_{\mathbb{C}}\right)$ and prove a formula relating

it to the class of Y.
Using the *Hodge realization morphism* $\mu_{\mathrm{Hdg}}\colon K_0(\mathrm{Var}_{\mathbb{C}}) \to K_0(\mathfrak{hs}_{\mathbb{Q}})$, we obtain an isomorphism of Hodge structures which allows us to fully describe the cohomology of the Fano variety $F(Y,1)$ in terms of the cohomology of Y.
In section 4, following [Dia17], we analyse the consequences of this description and prove isomorphisms relating the cohomology of $F(Y,1)$ to the cohomology of Y and of a Grassmanian, see lemma 4.10.
Finally, in section 5, we use the *cylinder correspondence* to construct an algebraic correspondence inducing the Lefschetz operator.

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1 The Fano Variety of a Hypersurface

In this first section we recall the definition of the Grassmanian, then we give a quick introduction to, and prove the existence of, the Fano variety of m-planes $F(X, m)$ of a projective variety X. The Fano variety is a scheme which parametrises m-dimensional planes contained in X. Later, we will specialise to the case $m = 1$ of lines contained in a cubic hypersurface. Of particular importance is diagram 1.1 which will allow us to relate the cohomology of $F(X, 1)$ to the cohomology of X and of the Grassmanian in sections 4 and 5.

Throughout this section, we work over an arbitrary field k, all schemes are taken of finite type over k. We fix an $n+2$ dimensional vector space V over k.

1.1 Grassmanians

First, recall the definition of the Grassmanian and of the Grassmanian functor of V.

Definition 1.1 (Grassmanian) Let $0 \leqslant m \leqslant n+1$, the Grassmanian functor $\underline{G}(m+1, V)$ is defined as

$$\begin{aligned} \underline{G}(m+1, V) \colon (\mathrm{Sch}_k)^{op} &\to \mathrm{Set} \\ T &\mapsto \{L \subset T \times V |\ L \text{ a rank } m+1 \text{ subvector bundle of } T \times V\} \\ \{f \colon S \to T\} &\mapsto \{f^* \colon L \subset T \times V \to f^*L \subset S \times V\} \end{aligned}$$

It is a classical fact that $\underline{G}(m+1, V)$ is represented by a scheme $G(m+1, V)$ which is smooth, projective and of dimension $(m+1)(n+1-m)$ over k, see [Stacks, 089R] for a proof.
The identity map $G(m+1, V) \to G(m+1, V)$ now corresponds to an $m+1$ dimensional vector bundle $\mathcal{S}$ over $G(m+1, V)$ called the tautological bundle. It comes with a canonical inclusion $\mathcal{S} \hookrightarrow \mathcal{V}$, where $\mathcal{V}$ is the trivial vector bundle on $G(m+1, V)$ with fiber V.

One may also introduce the functor $\underline{G}(m, \mathbb{P}(V)) \colon (\mathrm{Sch}_k)^{op} \to \mathrm{Set}$ sending a scheme T to the set of T-flat subschemes $L \subset T \times \mathbb{P}(V)$ such that , for all $t \in T$ the fiber $L_t \subset \mathbb{P}(V)_{k(t)}$ is an m-dimensional linear subspace. Thinking of $m+1$ dimensional subvector spaces of V as m-dimensional planes in $\mathbb{P}(V)$, one expects these two functors to be isomorphic, let us spell out this isomorphism.

Proposition 1.2 *The functors $\underline{G}(m+1, V)$ and $\underline{G}(m, \mathbb{P}(V))$ are naturally isomorphic.*

Proof Let $T \in \mathrm{Sch}_k$ and let $L \in \underline{G}(m+1, V)(T)$, then the relative projectivisation $\mathbb{P}(L)$ of L over T gives a subbundle of $\mathbb{P}(V) \times T$. Clearly every fiber of $\mathbb{P}(L) \to T$ is an m-dimensional linear subspace and this map is flat by [Stacks, 0D4C].
Conversely, given $L' \in \underline{G}(m, \mathbb{P}(V))(T)$, let $p : \mathbb{P}(V) \times T \to T$ be the projection. For all d, there is an inclusion of $\mathcal{O}_T$-algebras $p_*(\mathcal{O}_{L'}(d)) \subset p_* \left(\mathcal{O}_{\mathbb{P}(V) \times T}(d)\right)$. Both sheaves are locally free as L' and $\mathbb{P}(V) \times T$ are flat over T. Let $\mathcal{L} = \bigoplus_{d \geqslant 0} p_* (\mathcal{O}_{L'}(d))$ be the corresponding graded algebra. Taking relative spec gives an inclusion $\underline{\mathrm{Spec}}_T(\mathcal{L}) \to V \times T$ which realises $\underline{\mathrm{Spec}}_T(\mathcal{L})$ as an $m+1$ dimensional subbundle of $V \times T$. $\square$

1.2 The Fano scheme of m-planes

Let X be a closed subvariety of $\mathbb{P}(V)$. We will define a subfunctor of $\underline{G}(m, \mathbb{P}(V))$ called the Fano functor of m-planes of X, the k-rational points of this subfunctor will correspond to m-dimensional hyperplanes contained in X.
We will then show that this functor is a closed subfunctor of $\underline{G}(m, V)$ and is representable by a scheme called the Fano scheme of m-planes

Definition 1.3 (Fano functor of m-planes) The Fano functor of m-planes is defined as

$$\underline{F}(X, m) \colon (\mathrm{Sch}_k)^{op} \to \mathrm{Set}$$

$$T \mapsto \left\{ L \subset T \times X \;\middle|\; \begin{array}{l} L \text{ is } T\text{-flat and} \\ \forall t \in T, L_t \subset X_{k(t)} \subset \mathbb{P}_{k(t)} \\ \text{is an } m\text{-dimensional linear subspace} \end{array} \right\}$$

$$\{f \colon S \to T\} \mapsto \{f^* \colon L \subset T \times X \to f^* L \subset S \times X\}$$

Theorem 1.4 *The functor $\underline{F}(X, m)$ is representable by a scheme $F(X, m)$ called the Fano scheme of m-planes of X.*

Proof Let us highlight two different constructions of $F(X, m)$.

Construction 1 We can prove the existence of the Fano scheme assuming existence of the Hilbert scheme. Since m-dimensional linear subspaces of $\mathbb{P}(V)$ are precisely the subschemes with Hilbert polynomial $P_m(l) = \binom{m+l}{m}$, we deduce that $F(X, m) \simeq \mathrm{Hilb}^{P_m}(X)$.

Construction 2 Let $\mathcal{S}$ be the tautological bundle on $G \coloneqq G(m, \mathbb{P}(V))$, let $\mathcal{V}$ be the trivial bundle with fiber V and let $\iota \colon \mathcal{S} \hookrightarrow \mathcal{V}$ denote the canonical inclusion. Since X is a closed subvariety, we may write it as the zero locus of ℓ homogeneous polynomials $F_1, \ldots, F_\ell$, with each $F_i \in S^{d_i}(V^\vee)$. Note that each F_i determines a global section of $S^{d_i}(\mathcal{V}^\vee)$ on G which we also call F_i.
Dualizing ι and taking d_i-th symmetric powers, we obtain a restriction map $\iota^\vee \colon H^0(G, S^{d_i}(\mathcal{V}^\vee)) \to H^0(G, S^{d_i}(\mathcal{S}^\vee))$. Define $F = V(\iota^\vee F_1, \ldots, \iota^\vee F_\ell)$, we claim that F represents the Fano scheme of m planes of X.
Indeed, let $T \to F$ be a morphism of schemes, postcomposing with $F \hookrightarrow G$, we get a projective m-bundle L on T. It suffices to check that for all $t \in T$, $L_t \subset \mathbb{P}(V)_{k(t)}$ factors through X. Let $\mathcal{E} = \iota^* \mathcal{S}$ be the pullback of the universal bundle to F. Viewing the $\iota^\vee F_i$ as functions on $\mathbb{P}(\mathcal{S})$, we see that $\mathbb{P}(\mathcal{E})$ is the vanishing locus of the $\iota^\vee F_i$. Thus, $\mathbb{P}(\mathcal{E})_{k(t)} \subset X_{k(t)}$ for all $t \in F$, and the general result follows by taking fibers.

Remark 1.5 In particular, the second construction gives an embedding $F(X, m) \hookrightarrow \mathbb{G}(m, \mathbb{P}(V))$ and the construction shows that the pullback of the universal bundle $\mathcal{S}$ on $\mathbb{G}(m, \mathbb{P}(V))$ gives the universal bundle $\mathcal{E}$ on $F(X, m)$.

There are two natural maps

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}) & \xrightarrow{q} & X \\ {\scriptstyle p}\downarrow & & \\ F(X,m) & & \end{array}$$

Here, p is the projection and the map $\mathbb{P}(\mathcal{E}) \to X$ is the restriction of the composition $\mathbb{P}(\mathcal{E}) \to \mathbb{P}(V) \times \mathbb{G}(m, \mathbb{P}(V)) \to \mathbb{P}(V)$.
These two maps fit into a commutative diagram as follows:

$$\begin{array}{ccccc} F & \xleftarrow{p} & \mathbb{P}(\mathcal{E}) & \xrightarrow{q} & X \\ {\scriptstyle \iota}\downarrow & & {\scriptstyle \bar{\iota}}\downarrow & & \downarrow \\ G(m, \mathbb{P}(V)) & \xleftarrow[\bar{p}]{} & \mathbb{P}(\mathcal{S}) & \xrightarrow[\bar{q}]{} & \mathbb{P}(V) \end{array} \tag{1.1}$$

Here, all the horizontal maps are the natural projections and the vertical maps are all the natural inclusions.
Later on, we will specialise to the case where $k = \mathbb{C}$ and $m = 1$ and investigate the actions of p and q on Betti cohomology. The commutativity of this diagram will come in useful.
In the case $m = 1$, there is the following important theorem

Theorem 1.6 ([Huy23, cor. 2.1.14]) *Let $X \subset \mathbb{P}(V)$ be a smooth cubic hypersurface, then the Fano scheme $F(X, 1)$ is a smooth projective variety of dimension $2n - 4$.*

2 Grothendieck's Standard Conjectures

Betti cohomology is a cohomology theory which is particularly well suited for complex varieties, in particular when restricted to smooth projective complex varieties. In this section, we introduce Betti cohomology and deduce some of its main properties from the analoguous properties in singular cohomology. It turns out that when restricted to this class of varieties, Betti cohomology enjoys two special properties coming from complex geometry: the weak and hard Lefschetz theorems. These properties imply that Betti cohomology is an example of a *Weil cohomology theory*.
In the second part of this section, we state and prove different formulations of the *standard conjectures* for Betti cohomology. In particular, we will see the form in which we will prove the standard conjectures for F in section 5.

2.1 Betti Cohomology

We will mostly follow [Kle94b] and [Kle94a] for the following exposition.

2.1.1 Basic Properties. We first recollect some basic facts about Betti cohomology.

Definition 2.1 (Betti Cohomology) Given X a scheme of finite type over $\mathbb{C}$, we define the n*-th Betti cohomology group with coefficients in* G as

$$H^n_{\text{Betti}}(X; G) \coloneqq H^n_{\text{sing}}(X^{\text{an}}; G)$$

where X^{an} is the associated analytic space of X.
By $H^n(X)$, we will always mean Betti cohomology of X with coefficients in $\mathbb{Q}$.

Let us explore some properties of $H^n(X)$ when X is smooth and projective over $\mathbb{C}$.
Recall that the direct sum over all cohomology groups gets the structure of a graded commutative algebra over $\mathbb{Q}$ via the cup product pairing.
As the analytification of a smooth complex variety is a complex manifold (in particular, it is oriented) and the analytification of a projective variety is compact, we recover Poincaré duality as well as the Künneth formula from the classical theory:

Theorem 2.2 (Poincaré Duality) *Let X be a smooth projective variety over $\mathbb{C}$ of dimension n, then there is a trace isomorphism $t\colon H^{2n}(X) \to \mathbb{Q}$ such that for all $0 \leqslant i \leqslant 2n$, the cup product induces a non-degenerate pairing*

$$H^{2n-i}(X) \times H^i(X) \to H^{2n}(X) \xrightarrow{t} \mathbb{Q}.$$

Theorem 2.3 (Künneth isomorphism) *Let X, Y be smooth projective varieties over $\mathbb{C}$, then there is an isomorphism of graded commutative $\mathbb{Q}$-algebras*

$$H^*(X \times Y) \simeq H^*(X) \otimes H^*(Y),$$

where $H^*(-)$ denotes the full cohomology ring.

In particular, Poincaré duality gives a pushforward map in cohomology: if $f\colon X \to Y$ is a morphism of smooth projective varieties with $\dim X = n, \dim Y = m$, there is an induced map $f_*\colon H^i(X) \to H^{2(m-n)+i}(Y)$ defined as the dual of $f^*\colon H^{2n-i}(Y) \to H^{2n-i}(X)$. More explicitly, f_* is the unique map making the following diagram commute

$$\begin{array}{ccc} H^k(X) & \xrightarrow{f_*} & H^{2(m-n)+k}(Y) \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ H_{2n-k}(X) & \xrightarrow[f_*]{} & H_{2n-k}(Y). \end{array}$$

2.1.2 Cycle class map. We now come to the existence of a cycle class map for Betti cohomology. Let $Z^p(X)$ be the $\mathbb{Q}$-vector space with basis given by codimension p cycles.
For each p, there is a cycle class map $\gamma_X\colon Z^p(X) \to H^{2p}(X)$ which is

- Compatible with the pushforward and pullback maps of cycles (resp. in cohomology), given $f\colon X \to Y$, we have
$$f_*\gamma_X = \gamma_Y f_* \text{ and } f^*\gamma_Y = \gamma_X f^*.$$
See [EH16, sec. 1.3.6] for details on pushforwards and pullbacks of cycles, because of these equalities, we will not differentiate notation for these.

- Compatible with the Künneth decomposition in the sense that, for $W \subset X, Z \subset Y$ two subvarieties
$$\gamma_{X\times Y}(W \times Z) = \gamma_X(W) \otimes \gamma_Y(Z).$$

The construction of the cycle class map goes as follows, let $i\colon Z \subset X$ be a subvariety of codimension p, let $p\colon \tilde{Z} \to Z$ be a resolution of Z and let $[Z] \in H_{2n-p}(\tilde{Z})$ be the fundamental class. Then $\gamma_X([Z]) \in H^p(X)$ is the Poincaré dual of the pushforward $(i \circ p)_*([Z])$.
The maps γ_X factor through the Chow groups of X and induce a morphism $CH(X) \to H^*(X)$ which is compatible with the ring structures on both.

2.1.3 Weak and hard Lefschetz theorems. Finally, there are two special properties that the cycle class of a hyperplane section enjoys inside the cohomology ring of a smooth projective variety:

Theorem 2.4 (Weak Lefschetz theorem) *Let X be a smooth projective variety of dimension n over $\mathbb{C}$ and let $h\colon W \to X$ be the inclusion of a smooth hyperplane section, then*

$$h^*\colon H^i(X) \to H^i(W) \textit{ is an isomorphism when } i \leqslant \dim X - 2$$

and

$$h^*\colon H^{n-1}(X) \to H^{n-1}(W)$$

is injective.

Theorem 2.5 (Hard Lefschetz theorem) *Let $W \subset X$ be a smooth hyperplane section and let*

$$L\colon H^i(X) \to H^{i+2}(X)$$
$$x \mapsto \gamma_X(W) \cup x.$$

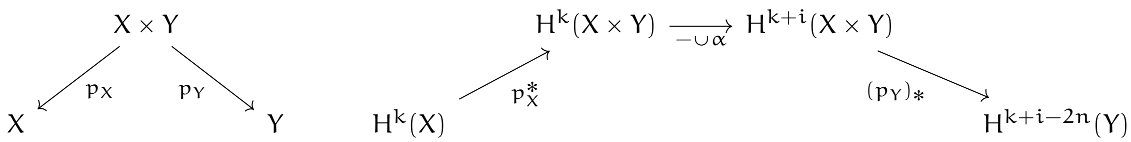
Then, for all $i \leqslant \dim X$, the composite

$$L^{\dim X - i}\colon H^i(X) \to H^{2\dim X - i}(X)$$

is an isomorphism.

Proofs of these theorems are found in [Huy04, prop. 5.2.6] and [Voi02, thm. 6.4] respectively.

2.1.4 Correspondences. We first describe how an element $\alpha \in H^i(X \times Y)$ induces a map $H^k(X) \to H^{k+i-2n}(Y)$.
Let $p_X\colon X \times Y \to X, p_Y\colon X \times Y \to Y$ be the two projections, then $\alpha_* \coloneqq (p_Y)_*(p_X^*(-) \cup \alpha)$ induces a linear map $\alpha_*\colon H^k(X) \to H^{k+i-2n}(Y)$.



The element α is called a *topological correspondence*. If α is in the image of the cycle class map $Z^{\frac{i}{2}}(X \times Y) \to H^i(X \times Y)$, then α is an *algebraic correspondence*, the corresponding map α_* is called *algebraic*. We notice in particular that, given an element $\alpha \in Z^{\frac{i}{2}}(X)$, the map $- \cup \alpha\colon H^k(X) \to H^{k+i}(X)$ is algebraic. Indeed, it is induced by the correspondence $\alpha \times X \subset X \times X$.

There is another, more explicit, description of the action of cycles in $X \times X$ on cohomology groups of X. Let $\langle -, - \rangle\colon H^k(X) \times H^{2n-k}(X) \to \mathbb{Q}$ denote the cup product pairing and let $\alpha \otimes \beta \in H^k(X) \otimes H^{2n-k}(X)$, thought of as a correspondence in $H^{2n}(X \times X)$ via the Künneth isomorphism. Then $\alpha \otimes \beta$ acts as 0 on all cohomology groups except for the $2n-k$-th one where it sends $x \in H^{2n-k}(X) \mapsto \langle x, \alpha \rangle \beta$, this follows directly from the fact that the Künneth isomorphism is a ring isomorphism.

Furthermore, algebraic correspondences may be composed to yield algebraic correspondences. More precisely, let $\alpha \in Z^a(X \times Y), \beta \in Z^b(Y \times Z)$ and let $p_{XZ} : X \times Y \times Z \to X \times Z$ be the projection. Then the formula $\alpha \circ \beta \coloneqq p_{XZ*}(\alpha \times Z \cap X \times \beta)$ defines a correspondence $(\alpha \circ \beta)_*\colon H^k(X) \to H^{k+2a+2b-2n}(Y)$ which is equal to $\alpha_* \circ \beta_*$. Thus, the composition of algebraic maps is algebraic.

2.2 Lefschetz Standard Conjectures

Let X, Y and Z be smooth projective varieties over $\mathbb{C}$ of respective dimensions n, m and l.

2.2.1 Standard Conjecture $B(X)$**.** Fix a smooth hyperplane section H of X and let $L \coloneqq - \cup H \colon H^i(X) \to H^{i+2}(X)$ be the associated Lefschetz operator. By the hard Lefschetz theorem, the maps $L^{n-i} \colon H^i(X) \to H^{2n-i}(X)$ are isomorphisms when $i \leqslant n$, hence, there is a unique map Λ making the following diagram commute

$$\begin{array}{ccc} H^i(X) & \xrightarrow{L^{n-i}} & H^{2n-i}(X) \\ \Lambda \downarrow & & \downarrow L \\ H^{i-2}(X) & \xrightarrow[L^{n-i+2}]{} & H^{2n-i+2}(X) \end{array}$$

Similarly, if $i \geqslant n$, there is a unique map making the following diagram commute

$$\begin{array}{ccc} H^{2n-i+2}(X) & \xleftarrow{L^{n-i+2}} & H^{i-2}(X) \\ \Lambda \downarrow & & \downarrow L \\ H^{2n-i}(X) & \xleftarrow[L^{n-i}]{} & H^i(X) \end{array}$$

The Lefschetz standard conjecture $B(X)$ states that, for X a smooth projective variety, the map Λ is algebraic for $0 \leqslant i \leqslant 2n$.

It is worth noting here that by 2.1.4, the map $L^{n-i+1} \colon H^i(X) \to H^{2n-i+2}(X)$ is algebraic as it is the composition of algebraic maps, however, it is not a priori clear whether the inverse $\left(L^{n-i+2}\right)^{-1}$ is induced by an algebraic correspondence. It is clear that the inverse of L^{n+i-2} being algebraic implies $B(X)$, we will see that it is in fact equivalent. Even more is true, it is sufficient to find any algebraic cycle inducing an isomorphism $H^{2n-i+2}(X) \to H^{i-2}(X)$, not necessarily inverse to L^{n-i+2}.

Remark 2.6 Let $\star$ denote the Hodge star operator, classically, the Lefschetz operator is defined by the formula $\tilde{\Lambda} = \star^{-1} \circ L \circ \star$, see for instance [Huy04, p. 115]. It is not in general true that this definition of $\tilde{\Lambda}$ coincides with the one above, however, algebraicity of $\tilde{\Lambda}$ is still equivalent to algebraicity of Λ, see [Kle94a, prop. 2.3].

The standard conjectures can be stated in larger generality than just Betti cohomology, in fact they may be stated for any *Weil cohomology theory*, see [Stacks, Tag 0FGS]. Weil cohomology theories axiomatize the properties of Betti cohomology we enunciated in subsection 2.1 and formalize what a suitable cohomology theory for smooth projective varieties over *algebraically closed* fields should look like. There are numerous other examples of Weil cohomologies, for instance ℓ-adic cohomology, algebraic de-Rham cohomology or crystalline cohomology.

As we saw in theorem 1.6, the Fano variety of lines F of a smooth cubic is a smooth projective variety, hence it makes sense to ask whether the standard conjectures hold for F with Betti cohomology. Proving this is the goal of this thesis.

2.2.2 Primitive cohomology. The hard Lefschetz theorem implies in particular that, for $i \leqslant n$, the map $L \colon H^i(X) \to H^{i+2}(X)$ is injective, however the map $L^{n-i+1} \colon H^i(X) \to H^{2n-i+2}(X)$ is not in general, hence it makes sense to make the following definition

Definition 2.7 (Primitive cohomology) For $0 \leqslant i \leqslant n$, the i-th primitive cohomology of X is defined as

$$H^i_{prim}(X) \coloneqq \ker\left(L^{n-i+1}\colon H^i(X) \to H^{2n-i+2}(X)\right).$$

In fact, there is a natural way to define a projection $H^i(X) \to H^i_{prim}(X)$. Since L is injective, the kernel of the map $L \circ \Lambda\colon H^i(X) \to H^i(X)$ is $H^i_{prim}(X)$, furthermore

$$\begin{aligned}\Lambda \circ L \circ \Lambda &= (L^{n-i+2})^{-1} \circ L \circ L^{n-i} \circ L \circ (L^{n-i+2})^{-1} \circ L \circ L^{n-i} \\ &= \Lambda.\end{aligned}$$

Hence, the map $\mathrm{Id} - L \circ \Lambda\colon H^i(X) \to H^i(X)$ defines a projector onto the i-th primitive cohomology of X.

2.2.3 Reformulations of $B(X)$**.** Let $i \leqslant n$, we introduce the two following conjectures:

- $\theta(X)$: There exists an algebraic correspondence θ^i inducing $\left(L^{n-i}\right)^{-1} : H^{2n-i}(X) \to H^i(X)$
- $\nu(X)$: There is an algebraic correspondence ν^i inducing an isomorphism $\nu^i_*\colon H^{2n-i}(X) \to H^i(X)$.

Our main goal in this section will be to prove the following equivalence:

Theorem 2.8 *Conjectures* $B(X), \theta(X)$ *and* $\nu(X)$ *are all equivalent.*

Before proving this, we need a bit more theory, let $\pi^i_*\colon H^*(X) \to H^*(X)$ denote the composition $H^*(X) \to H^i(X) \hookrightarrow H^*(X)$ where the first map is the natural projection and the second one is the inclusion. We notice the following: let $\Delta \in H^{2n}(X \times X)$ be the cycle class of the diagonal subvariety of $X \times X$. We may write $\Delta = \pi^1 + \ldots + \pi^{2n}$ for the decomposition under the Künneth isomorphism. Then the π^i act as π^i_* on cohomology, indeed, for all $k \in \mathbb{N}$, if $x \in H^k(X)$, then $\Delta_*(x) = \left(\pi^k\right)_*(x) = x$.
It is natural to ask whether the π^i are algebraic, this is in fact implied by $B(X)$ as we will see later.

Though we will only need the trace formula in what follows, we also prove slightly more general statements.

Lemma 2.9 ([Kle94a, prop. 1.3.6]) *Let* $u \in H^{2n}(X \times X)$ *and let* $\mathrm{Tr}_i(u)$ *denote the trace of the endomorphism* $u_*\colon H^i(X) \to H^i(X)$*, the following equalities hold*

1. *The* ***Trace formula***

$$\mathrm{Tr}_i(u) = (-1)^i \langle u, \pi^{2n-i}\rangle.$$

2. *The* ***Lefschetz fixed-point formula***

$$\langle u.\Delta\rangle = \sum_{i=0}^{2n} (-1)^i \,\mathrm{Tr}_i(u).$$

3. *Given* $v \in H^{4n-k}(X \times X), w \in H^k(X \times X)$*, we have*

$$\langle v.{}^t w\rangle = \sum_{i=0}^{2n} (-1)^i \,\mathrm{Tr}_i(w \circ v).$$

Proof We first prove the third part: by linearity, we may suppose that $v \in H^{2n-i}(X) \otimes H^j(X), w \in H^{2n-i}(X) \otimes H^j(X)$ and it suffices to show that $\langle v.{}^t w \rangle = (-1)^i \operatorname{Tr}_i(w \circ v)$. Let a_ℓ be a basis for $H^{2n-i}(X)$ and let $\alpha_\ell \in H^i(X)$ be the dual basis for the cup product pairing, we may write $v = \sum_\ell a_\ell \otimes b_\ell, w = \sum_\ell c_\ell \otimes \alpha_\ell$ with $b_\ell \in H^j(X)$ and $c_\ell \in H^{2n-j}(X)$. Then we find that

$$\begin{aligned}\langle v.{}^t w \rangle &= \left\langle \sum_\ell a_\ell \otimes b_\ell.{}^t(\sum_\ell c_\ell \otimes \alpha_\ell) \right\rangle \\ &= \sum_\ell \langle b_\ell . c_\ell \rangle.\end{aligned}$$

On the other hand

$$\begin{aligned}(w \circ v)(\alpha_p) &= (-1)^i w\left(\left\langle \sum_\ell a_\ell \otimes b_\ell, \alpha_p \right\rangle\right) \\ &= (-1)^i w(b_p) \\ &= (-1)^i \left(\langle b_p . c_p \rangle a_p + \sum_{\ell \neq p} \langle b_p . c_\ell \rangle a_\ell \right).\end{aligned}$$

Hence, with respect to the basis a_ℓ, the trace of the endomorphism $(w \circ v)_* : H^i(X) \to H^i(X)$ is given by

$$\operatorname{Tr}_i(w \circ v) = (-1)^i \sum_\ell \langle b_\ell . c_\ell \rangle = (-1)^i \langle v.{}^t w \rangle.$$

As was to be shown.
The first formula now follows by noticing that $(\pi^i \circ u)_*|_{H^i(X)} = u_*|_{H^i(X)}$ and the second follows from the first and the decomposition $\Delta = \pi^1 + \ldots + \pi^{2n}$. ☐

Lemma 2.10 ([Kle94b, thm. 3.1]) *Suppose π^{2n-i} is algebraic and let $u \in H^{2n}(X \times X)$ be an integral sum of cycle classes of subvarieties. Then the characteristic polynomial of $u_* : H^i(X) \to H^i(X)$ is a polynomial with integer coefficients.*

Proof Let $u^{(n)}$ denote the n-fold composition of u with itself and set $s_n := \langle u^{(n)} . \pi^{2n-i} \rangle$. By assumption, there exists $m \in \mathbb{N}$ such that $m\pi^{2n-i}$ is also an integral combination of classes of algebraic cycles, hence, ms_n is an integer. Let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of $u_* : H^i(X) \to H^i(X)$, then by the trace formula 2.9

$$s_n = (-1)^i (\lambda_1^n + \cdots + \lambda_k^n).$$

Proposition 2.11 will imply that the λ_i are algebraic integers. Hence, the coefficients of the characteristic polynomial p_u of $u_* : H^i(X) \to H^i(X)$ are also algebraic integers. But the coefficients of the characteristic polynomial satisfy the Newton identities for s_n, which have integer coefficients. Hence the coefficients of the characteristic polynomial are actually integers. ☐

Proposition 2.11 ([Kle94a, lemma 2.8]) *Let A be a subring of $\overline{\mathbb{Q}}$, let $\alpha_1, \ldots, \alpha_\ell \in \overline{\mathbb{Q}}$ be distinct elements and $p_1, \ldots, p_\ell \in \mathbb{Z}$. Define $s_m = p_1 \alpha_1^m + \cdots + p_\ell \alpha_\ell^m$. Suppose there exists a non-zero element $a \in A$ such that $as_m \in A$ for all $m \geqslant 1$. Then the α_i are integral over A.*

Proof Consider the matrix equation

$$\begin{pmatrix} \alpha_1 & \dots & \alpha_\ell \\ \vdots & \ddots & \vdots \\ \alpha_1^\ell & \dots & \alpha_\ell^\ell \end{pmatrix} \cdot \begin{pmatrix} p_1 \alpha_1^m \\ \vdots \\ p_\ell \alpha_\ell^m \end{pmatrix} = \begin{pmatrix} s_{m+1} \\ \vdots \\ s_{m+l} \end{pmatrix}$$

By general linear algebra, since the α_i are distinct, the matrix on the left hand side is invertible and hence

$$p_i \alpha_i^m = \beta_{i1} s_{m+1} + \ldots + \beta_{i\ell} s_{m+\ell} \quad 1 \leqslant i \leqslant \ell$$

where the coefficients β_{ik} are independent of m. Hence, the ring $A[\alpha_1, \ldots, \alpha_\ell]$ is finite over A, as was to be shown. □

We are now ready to prove the equivalence of these conjectures

Proof (of theorem 2.8) The implications $\theta(X) \implies \nu(X)$ and $\theta(X) \implies B(X)$ are immediate, we show the two other implications.

- $B(X)$ **implies** $\theta(X)$**.** Assume Λ is algebraic and define $\theta^i = \Lambda^{n-i}$. Then θ^i is algebraic as it is a composite of algebraic maps. Furthermore, we have

  $$L^{n-i} \circ \Lambda^{n-i} = \mathrm{Id} \text{ and } \Lambda^{n-i} \circ L^{n-i} = \mathrm{Id}$$

  as follows by an immediate computation, so $\theta^i = \left(L^{n-i}\right)^{-1}$ on $H^{2n-i}(X)$.

- $\nu(X)$ **implies** $\theta(X)$ Let $\nu^i \in H^{2i}(X \times X)$ be an algebraic cycle inducing an isomorphism $\nu^i_*\colon H^{2n-i}(X) \to H^i(X)$ and define $u \coloneqq \nu^i \cdot L^{n-i}$, this is still an algebraic correspondence. By lemma 2.10, its characteristic polynomial $P(t)$ has rational coefficients. By Cayley–Hamilton, $P(u) = 0$, and hence u^{-1} is rational linear combination of powers of u, in particular, it is itself algebraic. Now define $\theta^i \coloneqq u^{-1}\nu^i$, it is clear that this element is algebraic and that θ^i_* is an inverse to L^{n-i}. □

As promised, we now show the following

Theorem 2.12 *If $B(X)$ holds, then the projections π^i_* are algebraic.*

Proof Since $B(X)$ implies $\theta(X)$, let θ^i be correspondences inducing inverses to the Lefschetz operators and notice that for all i

$$
\begin{aligned}
&\theta^i\left(1-\sum_{j>2r-i}\pi^j\right)L^{r-i}\left(1-\sum_{j<i}\pi^j\right)\\
=&\left(\theta^i-\sum_{j>2r-i}\theta^i\pi^j\right)\left(L^{r-i}-\sum_{j<i}L^{r-i}\pi^j\right)\\
=&\theta^iL^{r-i}-\sum_{j<i}\theta^iL^{r-i}\pi^j-\sum_{j>2r-i}\theta^i\pi^jL^{r-i}+\left(\sum_{j'>2r-i}\theta^i\pi^{j'}\right)\left(\sum_{j<i}L^{r-i}\pi^j\right) \qquad (2.1)\\
=&\Delta-\sum_{j<i}\pi^j-\sum_{j>i}\pi^j=\pi^i.
\end{aligned}
$$

Where in (2.1), we use that for all $j' > 2r-i, j < i$ we have $\pi^{j'}L^{r-i}\pi^j = 0$ and hence the last term vanishes.
The above formula implies that π^0 is algebraic, proceeding inductively on i, we deduce that all π^i are algebraic. □

One should also note that $B(X)$ implies that numerical and homological equivalence of cycles coincides. This is true for any Weil cohomology theory over a field of characteristic 0, see [Kle94a, cor. 3.9].

2.3 Hodge Structures

In this subsection, we give a quick summary of Hodge structures. The main goal is to define polarizable Hodge structures and prove that a short exact sequence of polarizable Hodge structures always splits. This will play a decisive role in the last part of section 3.

2.3.1 Pure Hodge structures. As a motivation, recall the following well-known theorem from complex geometry

Theorem 2.13 (Hodge decomposition theorem) *Let X be a compact complex manifold and let $H^{p,q}(X)$ be the space of (p,q)-forms, then there is a decomposition*

$$H^n(X)\otimes\mathbb{C}=\bigoplus_{p+q=n}H^{p,q}(X),$$

With $H^{p,q}(X)=\overline{H^{q,p}(X)}$.

The goal is to axiomatize the properties of this decomposition, this leads to the following definition

Definition 2.14 (Hodge structures) Let V be a finite dimensional $\mathbb{Q}$-vector space.

- A *Hodge structure of weight* n on V is a decomposition

$$V\otimes\mathbb{C}=\bigoplus_{p+q=n}V^{p,q} \text{ with } V^{p,q}=\overline{V^{q,p}}.$$

such that $V^{p,q} = \overline{V^{q,p}}$.
If V, W are two Hodge structures of weight n, a *morphism of Hodge structures* $f\colon V \to W$ is a linear map such that $f_{\mathbb{C}} \coloneqq f \otimes \mathbb{C}$ respects the filtration, ie. such that $f_{\mathbb{C}}(V^{p,q}) = W^{p,q}$.

- A *pure Hodge structure* on V is a decomposition

$$V \otimes \mathbb{C} = \bigoplus_{p+q} V^{p,q} \text{ with } V^{p,q} = \overline{V^{q,p}}.$$

A morphism of pure Hodge structures is defined in the same way as above.

Lemma 2.15 *The category $\mathfrak{hs}_{\mathbb{Q}}$ of pure Hodge structures is abelian.*

We omit the proof of this result, details about the constructions of kernels and cokernels are given in [Pet08, section 2.1]. It is noteworthy that the underlying $\mathbb{Q}$-vector space of kernels (resp. cokernels) of morphisms of Hodge structures coincide with the usual kernels (resp. cokernels) of $\mathbb{Q}$-vector spaces.

The Hodge decomposition theorem, together with the de Rham isomorphism shows that any smooth map of complex manifolds $f\colon X \to Y$ induces a morphism of weight n Hodge structures $f^*\colon H^n(Y) \to H^n(X)$.
This also shows that f induces a morphism of pure Hodge structures $H^*(Y) \to H^*(X)$.

Definition 2.16 Let V, W be two pure Hodge structures, then there is a pure Hodge structure on $V \otimes W$ given by the direct sum decomposition

$$(V \otimes W)^{p,q} = \bigoplus_{\substack{r+r'=p \\ s+s'=q}} V^{r,s} \otimes W^{r',s'}$$

It is easily checked that this indeed defines a Hodge structure on the $\mathbb{Q}$-vector space $V \otimes W$.
We notice that if V and W carry Hodge structures of weight n and m, then $V \otimes W$ carries a pure Hodge structure of weight $n+m$.

This definition of the tensor product of Hodge structures is compatible with the Künneth isomorphism in the sense that it makes the canonical isomorphism $H^*(X \times Y) \simeq H^*(X) \otimes H^*(Y)$ into an isomorphism of pure Hodge structures.

We define the *Tate Hodge structures of weight* n to be $\mathbb{Q}(-n) = H^{2n}(\mathbb{P}^n)$. Given a pure Hodge structure V, notice that $V(-n) \coloneqq V \otimes \mathbb{Q}(-n)$ is isomorphic as a vector space to V but the terms in the direct sum decomposition are shifted.

2.3.2 Polarized Hodge structures. We refer to [Huy04, section 3.1] for details on Kähler manifolds.
Let X be an n-dimensional compact complex manifold and let ω be a Kähler form on X. Notice that ω

induces a pairing on $H^k(X)$ for all $k \leqslant n$

$$\begin{aligned} Q\colon H^k(X) \times H^k(X) &\to \mathbb{Q} \\ (\xi,\eta) &\mapsto \int_X \xi \wedge \eta \wedge \omega^{n-k}. \end{aligned}$$

We may extend this pairing by linearity via the de Rham isomorphism to obtain a pairing $Q_{\mathbb{C}}\colon H^k(X;\mathbb{C}) \times H^k(X;\mathbb{C}) \to \mathbb{C}$ which satisfies the three following key properties

1. $Q_{\mathbb{C}}$ is $(-1)^k$-symmetric.
2. $Q_{\mathbb{C}}(\xi,\eta) = 0$ if $\xi \in H^{p,q}(X)$ and $\eta \in H^{p',q'}(X)$ with $q' \neq p$.
3. $Q_{\mathbb{C}}(C(u),\bar{v})$ is positive definite, where C is the *Weil operator* defined by $C|_{H^{p,q}} = i^{p-q}$.

This motivates the following definition

Definition 2.17 (Polarized Hodge Structure) A polarized Hodge structure of weight k is a Hodge structure V of pure weight k together with a bilinear form $Q\colon V \otimes V \to \mathbb{Q}$ satisfying properties (1)-(3) above.

We call a Hodge structure admitting a polarization *polarizable*.
Since any smooth projective variety X admits a Kähler form, the vector spaces $H^k(X)$ are all polarizable Hodge structures. In particular, the Tate Hodge structures are polarizable.

Proposition 2.18 ([Voi02, lemma 7.26]) *Let $W \subset V$ be an inclusion of Hodge structures, if V is polarizable, then W is too and there is an isomorphism $V \simeq W \oplus W^{\perp}$.*

Proof Let Q be a polarization on V, since $W^{p,q} = V^{p,q} \cap W_{\mathbb{C}}$ it is clear that $Q|_W$ is a polarization on W.
Notice that Q induces a Hermitian form on $V_{\mathbb{C}}$ via the formula $H(\alpha,\beta) := i^k Q(\alpha,\overline{\beta})$.
We want to show that the orthogonal complement of W in V also has a Hodge structure. Define $(W^{\perp})^{p,q}$ to be the orthogonal complement of $W^{p,q}$ inside $V^{p,q}$ with respect to H, by condition (3), there is a direct sum decomposition

$$W_{\mathbb{C}}^{\perp} = \bigoplus_{p+q=k} \left(W^{\perp}\right)^{p,q}$$

Furthermore, if $v \in \left(W^{\perp}\right)^{p,q}$, then for all $w \in W^{q,p}$ we find

$$H(w,\bar{v}) = i^k Q_{\mathbb{C}}(w,v) = 0.$$

The last equality follows either by property (2) if $p \neq q$ or by construction if $p = q$. By condition (3), we see that $V \simeq W \oplus W^{\perp}$. □

3 The Grothendieck Ring of Varieties

Throughout, let Y be a smooth cubic hypersurface of dimension $n > 2$ over $\mathbb{C}$ and let $F \coloneqq F(Y, 1)$ denote its Fano variety of lines.
In this section, we introduce the main technical tool on which the proof of the standard conjectures for F relies: *the Grothendieck ring of varieties* $K_0(\mathrm{Var}_{\mathbb{C}})$. After giving a quick introduction and citing a few important results, we investigate the class of F in $K_0(\mathrm{Var}_{\mathbb{C}})$, following [GS14]. In the last part, we use the *Hodge realization morphism* to fully describe the cohomology of F in terms of the cohomology of Y.

3.1 Preliminaries about $K_0(\mathrm{Var}_{\mathbb{C}})$

Most results in this section hold over an arbitrary field of characteristic 0, for the sake of simplicity, we state everything over $\mathbb{C}$.

Definition 3.1 (Grothendieck ring of varieties) The Grothendieck ring of varieties $K_0(\mathrm{Var}_{\mathbb{C}})$ is the free abelian group generated by isomorphism classes of varieties over $\mathbb{C}$ modulo the scissor relation: given a variety X over $\mathbb{C}$ and a closed subvariety Z, we impose the relation $[X] = [X - Z] + [Z]$.
This group inherits a ring structure via the usual product of varieties, it is easy to check that this product preserves the scissor relations.

We first recall a few classical facts about $K_0(\mathrm{Var}_{\mathbb{C}})$, for this, we need a slight variation of the above definition

Definition 3.2 (Grothendieck ring of smooth varieties) Let $K_0(\mathrm{SmProj}_{\mathbb{C}})$ be the free abelian group generated by isomorphism classes of smooth projective varieties modulo the relation

$$\text{For all } Z \subset X \text{ smooth closed subvariety } : [\mathrm{Bl}_Z X] - [E] = [X] - [Z]$$

Here, E is the exceptional divisor of the blowup.
$K_0(\mathrm{SmProj}_{\mathbb{C}})$ gets a ring structure via the usual product of varieties as above.

Theorem 3.3 (Bittner's Theorem [Bit01]) *The natural map* $K_0(\mathrm{SmProj}_{\mathbb{C}}) \to K_0(\mathrm{Var}_{\mathbb{C}})$ *sending the class of a smooth projective variety to itself in* $K_0(\mathrm{Var}_{\mathbb{C}})$ *is an isomorphism.*

In this lemma, we collect some known results about $K_0(\mathrm{Var}_{\mathbb{C}})$.

Lemma 3.4 1. *Let* $p\colon X \to S$ *be a Zariski-locally trivial fibration with fiber* F, *then* $[X] = [F][S]$ *in* $K_0(\mathrm{Var}_{\mathbb{C}})$.

2. *Taking symmetric powers of varieties gives a well defined operation on* $K_0(\mathrm{Var}_{\mathbb{C}})$*, furthermore, we have the equalities*

$$\mathrm{Sym}^n(\alpha + \beta) = \sum_{i+j=n} \mathrm{Sym}^i(\alpha)\,\mathrm{Sym}^j(\beta) \tag{3.1}$$

$$\mathrm{Sym}^n(\mathbb{L}\alpha) = \mathbb{L}^n\,\mathrm{Sym}^n(\alpha) \tag{3.2}$$

To avoid cluttering, we will sometimes also denote the n*-th symmetric of a variety* Y *by* $Y^{(n)}$.

Proof 1. We proceed inductively on the dimension of S. If $\dim S = 0$, then the result is trivial. For the inductive step $\dim S = n+1$, first suppose that S is irreducible. Let $U \subset S$ be a trivializing open set for $X \to S$, then because S is irreducible $\dim(S \setminus U) \leqslant n$ and we obtain

$$[S] \cdot [F] = [U] \cdot [F] + [S \setminus U] \cdot [F] = [p^{-1}(U)] + [p^{-1}(S \setminus U)] = [X].$$

Proving the result for S irreducible. In general, write $S = \bigcup_{j=1}^{\ell} S_j$ as a union of its irreducible components. From the inclusion-exclusion formula, we obtain

$$[S] = \sum_{1 \leqslant j \leqslant \ell} [S_j] - \sum_{1 \leqslant j_1 \leqslant j_2 \leqslant \ell} [S_{j_1} \cap S_{j_2}] + \dots$$

multiplying by $[F]$ on both sides, we obtain the result.

2. Let $W, W' \in \mathrm{Var}_{\mathbb{C}}$, it is clear that $\mathrm{Sym}^n\left(W \coprod W'\right) = \coprod_{i+j=n} \mathrm{Sym}^i(W) \times \mathrm{Sym}^j(W')$ as schemes. Hence, it suffices to check that (3.1) still holds for the scissor relation. Let $X \in \mathrm{Var}_{\mathbb{C}}$ and let $Y \subset X$ be a closed subscheme, we first show that $\mathrm{Sym}^n(Y)$ still is a closed subscheme of $\mathrm{Sym}^n(X)$.
Since the map $p \colon X^n \to \mathrm{Sym}^n X$ is an affine morphism, it suffices to show the result for X an affine variety.
Let $Y = \operatorname{Spec} A/I \hookrightarrow X = \operatorname{Spec} A$ be a closed subscheme, let $q \colon A \to A/I$ be the corresponding quotient map. The abelian group $I^{\otimes n}$ is uniquely divisible as it is a $\mathbb{C}$-vector space, thus $H^1\left(\mathfrak{S}_n, I^{\otimes n}\right) = 0$ and it follows that the induced map $\left(A^{\otimes n}\right)^{\mathfrak{S}_n} \to \left((A/I)^{\otimes n}\right)^{\mathfrak{S}_n}$ is surjective. Thus $\mathrm{Sym}^n Y$ is a locally closed subscheme of $\mathrm{Sym}^n X$, to see that it is closed, it suffices to notice that the quotient map is closed, then $\mathrm{Sym}^n Y$ is the image under p of $Y^n \subset X^n$. The equality $\mathrm{Sym}^n([X]) = \sum_{i+j=n} \mathrm{Sym}^i([Y]) \mathrm{Sym}^j([X \setminus Y])$ now follows easily and equality (3.1) follows.
To prove equality (3.2), we reduce to the affine case as above. Let A be a $\mathbb{C}$-algebra, it is easily seen that
$$\left(\mathbb{C}[x_1, \dots, x_n] \otimes A^{\otimes n}\right)^{\mathfrak{S}_n} = (\mathbb{C}[x_1, \dots, x_n])^{\mathfrak{S}_n} \otimes \left(A^{\otimes n}\right)^{\mathfrak{S}_n}.$$
Since $(\mathbb{C}[x_1, \dots, x_n])^{\mathfrak{S}_n} \simeq \mathbb{C}[y_1, \dots, y_n]$, we deduce equality (3.2). □

3.2 The $Y - F$ relation

We can start investigating the class of F in $K_0(\mathrm{Var}_{\mathbb{C}})$, throughout, we will let $\mathbb{L} = \left[\mathbb{A}^1\right]$. We first need the following result

Lemma 3.5 *Let Y be a smooth cubic hypersurface of dimension d, let $Y^{[2]}$ be the Hilbert scheme of length 2 susbchemes of Y and F the Fano scheme of Y, then*

$$\left[Y^{[2]}\right] = \left[\mathbb{P}^d\right][Y] + \mathbb{L}^2[F].$$

Proof Let $T_{\mathbb{P}^{d+1}}$ be the tangent bundle of $\mathbb{P}^{d+1}$ and let $W = \mathbb{P}\left(T_{\mathbb{P}^{d+1}}|_Y\right)$, this is a projective bundle on Y with fiber $\mathbb{P}^d$, we think of its points as pairs $(y \in L)$ corresponding to a point $y \in Y$ together with a line L through it.

As before, we let $\mathbb{P}(\mathcal{E})$ be the universal bundle over F. There is an obvious inclusion map $\mathbb{P}(\mathcal{E}) \to W$. We let Z be the closed subset of $Y^{[2]}$ consisting of pairs $\{x, y\}$ such that $\overline{\{x, y\}} \subset Y$.
There is a map $Z \to F$ sending a pair of points $\{x, y\}$ to the corresponding line through them. We now define a map $\phi \colon Y^{[2]} \setminus Z \dashrightarrow W \setminus \mathbb{P}(\mathcal{E})$ which sends a pair $\{x, y\}$ to the third intersection point with the cubic $\overline{\{x, y\}} \cap Y$.
This map is in fact an isomorphism. An inverse is given by sending a pair $(y \in L)$ to the residual two points of intersection $(L \cap Y) \setminus y$.
With this in mind, we notice that $Z \to F$ is a $\mathrm{Sym}^2(\mathbb{P}^1) = \mathbb{P}^2$-bundle over F and $\mathbb{P}(\mathcal{E})$ is a $\mathbb{P}^1$-bundle over F. Hence, in $K_0(\mathrm{Var}_{\mathbb{C}})$, we have

$$\begin{aligned} [W] - [\mathbb{P}(\mathcal{E})] &= [Y^{[2]}] - [Z], \\ [\mathbb{P}^d][Y] - [\mathbb{P}^1][F] &= [Y^{[2]}] - [\mathbb{P}^2][F]. \end{aligned}$$

Hence, $[Y^{[2]}] = [\mathbb{P}^d][Y] + \mathbb{L}^2[F]$. □

Our next goal is to describe the class of F in $K_0(\mathrm{Var}_{\mathbb{C}})$, for this we first need a technical lemma

Lemma 3.6 *In* $K_0(\mathrm{Var}_{\mathbb{C}})$,

$$\mathrm{Sym}^2\left(\left[\mathbb{P}^d\right]\right) - (1+\mathbb{L}^2)\left[\mathbb{P}^2\right] = \mathbb{L}^2\,\mathrm{Sym}^2\left(\left[\mathbb{P}^{d-2}\right]\right) - \mathbb{L}^d$$

for all $d \geqslant 2$.

Proof We proceed by induction on d, for $d = 2$, the right hand side of the equality is simply 0. The left hand side becomes

$$\begin{aligned} \left[\mathbb{P}^2\right]^{(2)} - (1+\mathbb{L}^2)\left[\mathbb{P}^2\right] &= \left(\mathbb{L}^2 + \left[\mathbb{P}^1\right]\right)^{(2)} - \left[\mathbb{P}^2\right] - \mathbb{L}^2\left[\mathbb{P}^2\right] \\ &= \mathbb{L}^4 + \mathbb{L}^2\left[\mathbb{P}^1\right] + \left(\mathbb{L}^2 + \mathbb{L} + *\right) - \mathbb{L}^2 - \mathbb{L} - * - \mathbb{L}^4 - \mathbb{L}^3 - \mathbb{L}^2 \\ &= 0 \end{aligned}$$

Now suppose the result holds for $d-1$, then

$$\begin{aligned} \mathbb{L}^2\left(\left[\mathbb{P}^{d-2}\right]\right)^{(2)} - \mathbb{L}^d &= \mathbb{L}^2\left(\mathbb{L}^{d-2} + \left[\mathbb{P}^{d-3}\right]\right)^{(2)} - \mathbb{L}^d \\ &= \left[\mathbb{P}^{d-1}\right]^{(2)} + \mathbb{L}^d\left[\mathbb{P}^{d-3}\right] - \mathbb{L}^{d-1}\left[\mathbb{P}^{d-2}\right] + \mathbb{L}^{d-1} - \left[\mathbb{P}^d\right] \\ &= \left[\mathbb{P}^{d-1}\right]^{(2)} - \left[\mathbb{P}^d\right] \\ &= \left[\mathbb{P}^{d-1}\right]^{(2)} + \mathbb{L}^{2d} + \mathbb{L}^d\left[\mathbb{P}^{d-1}\right] - \mathbb{L}^d(\mathbb{L}^d + \left[\mathbb{P}^{d-1}\right]) - \left[\mathbb{P}^d\right] \\ &= \left[\mathbb{P}^d\right]^{(2)} - \left(1+\mathbb{L}^d\right)\left[\mathbb{P}^d\right] \end{aligned}$$

as was to be shown. □

Theorem 3.7 *Let* Y *be a smooth cubic hypersurface of dimension* d.
In $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$, *we have*

$$[F] = \left(\mathcal{M}_Y + \left[\mathbb{P}^{d-2}\right]\right)^{(2)} - \mathbb{L}^{d-2},$$

where $\mathcal{M}_Y := \frac{Y - [\mathbb{P}^d]}{\mathbb{L}}$.

Proof To prove this, we recall that given a smooth projective variety Y of dimension d, its Hilbert scheme of length 2 subschemes is the blowup of $\mathrm{Sym}^2(Y)$ along the diagonal.
Hence, in $K_0(\mathrm{Var}_{\mathbb{C}})$, we have the relation $\left[Y^{[2]}\right] - [E] = \left[Y^{(2)}\right] - [Y]$, where E is the exceptional divisor.
But E is a $\mathbb{P}^{d-1}$ bundle over Y, hence we get the relation $Y^{[2]} = [Y^{(2)}] - \left(\left[\mathbb{P}^{d-1}\right] - 1\right)[Y]$ by lemma 3.4.
Plugging this into lemma 3.5

$$\begin{aligned}
\mathbb{L}^2[F] &= [Y^{(2)}] - \left(1 + \mathbb{L}^d\right)[Y] \\
&= \left[\mathbb{P}^d\right]^{(2)} + \mathbb{L}\left[\mathbb{P}^d\right]\mathcal{M}_Y + \mathbb{L}^2\mathcal{M}_Y^{(2)} - (1 + \mathbb{L}^d)[Y] \\
&= \mathbb{L}\left(\left[\mathbb{P}^d\right] - 1 - \mathbb{L}^d\right)\mathcal{M}_Y + \mathbb{L}^2\mathcal{M}_Y^{(2)} + \left(\left[\mathbb{P}^d\right]^{(2)} - (1 + \mathbb{L}^d)\left[\mathbb{P}^d\right]\right) \\
&= \mathbb{L}\left(\left[\mathbb{P}^{d-1}\right] - 1\right) + \mathbb{L}^2\mathcal{M}_Y^{(2)} + \left[\mathbb{P}^d\right]^{(2)} - (1 + \mathbb{L}^d)\left[\mathbb{P}^d\right] \\
&= \mathbb{L}^2\left[\mathbb{P}^{d-2}\right] + \mathbb{L}^2\mathcal{M}_Y^{(2)} + \mathbb{L}^2\left[\mathbb{P}^{d-2}\right]^{(2)} - \mathbb{L}^2 \\
&= \mathbb{L}^2\left(\mathcal{M}_Y + \left[\mathbb{P}^{d-2}\right]\right)^{(2)} - \mathbb{L}^d.
\end{aligned}$$

Dividing by $\mathbb{L}^2$, we get the claim. □

Corollary 3.8 *In* $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$, *we have*

$$[F] = \mathcal{M}_Y^{(2)} + [\mathbb{P}^{d-2}]\mathcal{M}_Y + \sum_{k=0}^{2d-4} a_k\mathbb{L}^k,$$

with

$$a_k = \begin{cases} \left\lfloor \frac{k+2}{2} \right\rfloor & \text{if } k < d-2 \\ \left\lfloor \frac{d-2}{2} \right\rfloor & \text{if } k = d-2 \\ \left\lfloor \frac{2d-2-k}{2} \right\rfloor & \text{if } k > d-2. \end{cases}$$

Proof This follows from a combinatorial argument applied to the decomposition

$$\begin{aligned}
[F] &= \mathcal{M}_Y^{(2)} + \mathbb{P}^{d-2}\mathcal{M}_Y + \left[\mathbb{P}^{d-2}\right]^{(2)} - \mathbb{L}^{d-2} \\
&= \mathcal{M}_Y^{(2)} + \mathbb{P}^{d-2}\mathcal{M}_Y + \left(\mathbb{L}^{d-2} + \ldots + *\right)^{(2)} - \mathbb{L}^{d-2}.
\end{aligned}$$

□

Remark 3.9 In particular, notice that $a_{\frac{4(n-2)-2}{2}} = a_{\frac{k}{2}}$.

3.3 Cohomology of F

The final goal of this section is to deduce from corollary 3.8 a relation between the cohomology of F and the cohomology of Y. To obtain this relation, Bittner's theorem 3.3 will be fundamental, it implies that there is a well defined ring morphism $K_0(\mathrm{Var}_{\mathbb{C}}) \to K_0\left(\mathfrak{hs}_{\mathbb{Q}}\right)$ and we will be able to lift an equality in $K_0\left(\mathfrak{hs}_{\mathbb{Q}}\right)$ to an isomorphism in $\mathfrak{hs}_{\mathbb{Q}}$.

First, we recall the following classical description of the cohomology of a blowup

Theorem 3.10 ([Voi02, sec. 7.3.3]) *Let X be a smooth projective variety over $\mathbb{C}$ and let $Z \subset X$ be a closed smooth projective subvariety of codimension r. There is an isomorphism of polarized rational Hodge structures*

$$H^*(\mathrm{Bl}_Z X) \oplus H^*(Z)(-r) \simeq H^*(Y) \oplus H^*(E)(-1).$$

Since the Künneth isomorphism is an isomorphism of Hodge structures, there is a well defined ring morphism

$$\begin{aligned} \mu_{\mathrm{Hdg}} \colon K_0(\mathrm{SmProj}_{\mathbb{C}}) &\to K_0\left(\mathfrak{hs}_{\mathbb{Q}}\right) \\ [X] &\mapsto [H^*(X)(\dim X)] \end{aligned}$$

Under the isomorphism $K_0(\mathrm{Var}_{\mathbb{C}}) \simeq K_0(\mathrm{SmProj}_{\mathbb{C}})$, the class of the affine line $\mathbb{L}$ gets mapped to $[\mathbb{P}^1] - [*] \in K_0\left(\mathrm{SmProj}_{\mathbb{C}}\right)$, see [Bit01] for the explicit construction. Applying the Hodge realization and recalling the Hodge structure on the cohomology of projective space, we obtain $\mu_{\mathrm{Hdg}}\left(\mathbb{P}^1 - [*]\right) = [\mathbb{Q}(-1)]$. Since $[\mathbb{Q}(-1)] \in K_0\left(\mathfrak{hs}_{\mathbb{Q}}\right)$ is invertible, the map μ extends to a map from $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$:

$$\begin{array}{ccc} K_0(\mathrm{Var}_{\mathbb{C}}) & \xrightarrow{\mu_{\mathrm{Hdg}}} & K_0\left(\mathfrak{hs}_{\mathbb{Q}}\right) \\ \downarrow & \nearrow_{\mu_{\mathrm{Hdg}}} & \\ K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}] & & \end{array}$$

Our goal is to apply μ_{Hdg} to the equality in corollary 3.8 and to lift the corresponding equality to an isomorphism in $\mathfrak{hs}_{\mathbb{Q}}$.

First, we need some preliminary notions about the cohomology of hypersurfaces and of their symmetric products.

Lemma 3.11 *Let X be a smooth n-dimensional hypersurface, then there is an isomorphism of Hodge structures*

$$H^*(X) = \bigoplus_{k=0}^{n} \mathbb{Q}(-k) \oplus H^n_{\mathrm{prim}}(X).$$

Proof Let $i\colon X \hookrightarrow \mathbb{P}^n$ be an embedding, by weak Lefschetz, for $k \neq n$, the pullback $H^k(\mathbb{P}^n) \to H^k(X)$ is injective. It follows that for all $0 \leqslant k \leqslant 2n, k \neq n$

$$H^k(X) = \begin{cases} \mathbb{Q}(-\frac{k}{2}) & \text{if } k \text{ even} \\ 0 & \text{if } k \text{ odd} . \end{cases}$$

Here, we use Poincaré duality to determine the cohomology when $k > n$. Furthermore, since the cohomology of $\mathbb{P}^n$ is generated by a hyperplane section, we find that

$$H^n(X) = \begin{cases} H^n_{\text{prim}}(X) & \text{if } n \text{ odd} \\ H^n_{\text{prim}}(X) \oplus \mathbb{Q}(-\frac{n}{2}) & \text{if } n \text{ even} . \end{cases}$$

□

Lemma 3.12 ([Mot19, lemma 1.1.5]) *Let X be a smooth projective n-dimensional surface, then there is an isomorphism of polarized Hodge structures*

$$H^k(\operatorname{Sym}^\ell X) = \operatorname{Sym}^\ell(H^k(X))$$

Proof Let $f\colon X^\ell \to \operatorname{Sym}^\ell X$ be the natural quotient map by the $\mathfrak{S}_l$ action on X^ℓ. Since f is finite, $f_* f^* = \deg f \cdot \mathrm{Id}\colon H^i(X^\ell) \to H^i(X^\ell)$ for all i and hence $f^*\colon H^i(\operatorname{Sym}^\ell X) \to H^i(X^\ell)$ is injective of Hodge structures. Furthermore, it is clear that f^* factors through the $\mathfrak{S}_\ell$-invariant part of $H^*(X^\ell)$:

$$\begin{array}{ccc} H^*(\operatorname{Sym}^\ell(X)) & \xrightarrow{\quad f^* \quad} & H^*(X^\ell) \\ & \searrow \qquad \nearrow & \\ & H^*(X^\ell)^{\mathfrak{S}_\ell} & \end{array} .$$

We claim that the induced map $f^*\colon H^i(\operatorname{Sym}^\ell(X)) \to H^i(X^\ell)^{\mathfrak{S}_\ell}$ is an isomorphism for all i. It suffices to show that both sides have the same dimension. From [Gro57, theorem 5.2.2], there are two spectral sequences $I_2^{p,q}(\underline{\mathbb{Q}}) = H^p\left(X^{(\ell)}, R^q p_* \underline{\mathbb{Q}}\right)$ and $II_2^{p,q}(\underline{\mathbb{Q}}) = H^p\left(\mathfrak{S}_\ell, H^q(X^\ell, \underline{\mathbb{Q}})\right)$ which both converge to $R^{p+q}\left(\Gamma(X^\ell, -)^{\mathfrak{S}_\ell}\right)(\underline{\mathbb{Q}})$.
Since f is a finite map, $R^q f_*(\underline{\mathbb{Q}}) = 0$ for all $q > 0$. Furthermore, since $H^q(X^\ell, \underline{\mathbb{Q}})$ is a $\mathbb{Q}$-vector space, it is uniquely divisible and thus $H^p\left(\mathfrak{S}_\ell, H^q(X^\ell, \underline{\mathbb{Q}})\right) = 0$ for all $p > 0$.
Hence, $H^i(X^\ell)^{\mathfrak{S}_\ell} \simeq H^i(X^{(\ell)})$, concluding the proof. □

Theorem 3.13 ([GS14, thm. 6.1]) *Let Y be a smooth cubic hypersurface of dimension d and let F be its Fano variety of lines. We have the following isomorphism of Hodge structures*

$$H^*(F) \simeq \operatorname{Sym}^2(\mathcal{H}_Y) \oplus \bigoplus_{k=0}^{d-2} \mathcal{H}_Y(-k) \oplus \bigoplus_{k=0}^{2d-4} \mathbb{Q}(-k)^{\oplus a_k},$$

where $\mathcal{H}_Y = H^d_{\text{prim}}(Y)(1)$.

Proof We apply μ_{Hdg} to the equality of corollary 3.8, then combine lemmas 3.11 and 3.12 to obtain the equality

$$[H^*(F)] = \left[\mathrm{Sym}^2(\mathcal{H}_Y)\right] + \left[\bigoplus_{k=0}^{d-2} \mathcal{H}_Y(-k)\right] + \left[\bigoplus_{k=0}^{2d-4} \mathbb{Q}(-k)^{\oplus a_k}\right] \qquad \in K_0\left(\mathfrak{hs}_{\mathbb{Q}}\right).$$

Now, notice that every direct summand in this equality is polarizable, in particular $\mathrm{Sym}^2(\mathcal{H}_Y)$ is polarizable as it is a sub Hodge structure of $H^d(Y^n)(1)$. By proposition 2.18, it lifts to the desired equality.□

4 Preliminary Lemmas

As before Y is a smooth cubic hypersurface of dimension $n > 2$ over $\mathbb{C}$ and $F := F(Y, 1)$ will denote its Fano variety of lines.
In this technical section we collect results about the cohomology of F which will help us prove the standard conjectures in section 5. We will mainly use the description of theorem 3.13 to relate the cohomology of F to the cohomology of Y and of the Grassmanian of lines. Throughout this section, we follow [Dia17]. All Chow groups are taken with coefficents in $\mathbb{Q}$.

4.1 Cohomology of the Grassmanian

By remark 1.5, there is an embedding $i\colon F \hookrightarrow G := \mathbb{G}(1, \mathbb{P}(V))$ with V a $n+2$ dimensional complex vector space. We let $\mathcal{S}$ be the universal bundle of G and $\mathcal{E} = i^*\mathcal{S}$ the universal bundle on F. Recall that G is a variety of dimension $2n$ and $F \hookrightarrow G$ is a closed embedding of codimension 4 by theorem 1.6.

Lemma 4.1 *For $d \leqslant n+2$, the vector space $H^{2d}(G)$ has*

$$\left\{c_1(\mathcal{S})^d, c_1(\mathcal{S})^{d-2}c_2(\mathcal{S}), \dots, c_1(\mathcal{S})^{d-2[\frac{d}{2}]}c_2(\mathcal{S})^{[\frac{d}{2}]}\right\} \tag{4.1}$$

as a basis, where $c_i(\mathcal{S})$ is the i-th Chern class of $\mathcal{S}$. In particular, $\dim\left(H^{2d}(G)\right) = \left\lfloor \frac{d+2}{2} \right\rfloor$ *and there are no relations among the Chern classes.*

Proof From [EH16, thm. 5.26] it follows that $CH^d(G)$ is generated by the set (4.1). By [Ful98, example 19.1.11], the cycle class map induces an isomorphism $CH^d(G) \to H^{2d}(G)$.
Recall that a basis for $CH^d(G)$ (and hence $H^{2d}(G)$) is given by the Schubert classes of dimension d. The number of Schubert classes of codimension d is given by the cardinality of the set

$$\left\{(a_1, a_2) \in \mathbb{Z}^2 \middle| n \geqslant a_1 \geqslant a_2 \geqslant 0 \text{ and } a_1 + a_2 = d\right\}.$$

Rewriting this, we see that $a_1 + a_2 = 2a_2 + (a_1 - a_2) = d$, so the number of Schubert cells of codimension d is the set of positive integer solutions to $2n_1 + n_2 = d$ such that $n_1 + n_2 \leqslant n$.
Under the hypothesis that $d \leqslant n+2$, the number of such solutions is given by $\left\lfloor \frac{d+2}{2} \right\rfloor$. We deduce that the set in (4.1) is a basis for $CH^d(G)$ and in particular linearly independent. □

Lemma 4.2 *Let $i\colon X \hookrightarrow G$ be a smooth closed subvariety of codimension r. For $k \leqslant 2(n+2) - 2r$, the pullback map on cohomology*

$$i^*\colon H^k(G) \to H^k(X)$$

is injective. In particular, when $X = F$, the inclusion map is injective for $k \leqslant 2(n+2) - 8 = 2(n-2)$.

Proof Let $\alpha \in H^k(G)$ and suppose $i^*\alpha = 0$, then the projection formula ([EH16, theorem 1.23.b]) implies that $0 = i_*i^*\alpha = \alpha \cdot X \in H^{k+2r}(G)$.
Since α and X are both non-zero homogeneous polynomials in the Chern classes their product $\alpha \cdot Y$ is a non-zero homogeneous polynomial in the Chern classes of degree $k + 2r \leqslant 2(n+2)$, contradicting lemma 4.1. □

Corollary 4.3 *Let $i^*\colon H^k(G) \to H^k(F)$ be the map induced by the inclusion. For $0 \leqslant k \leqslant 4(n-2), k \neq 2(n-2)$, we have*

$$\dim(i^*H^k(G_{1,n+1})) = \begin{cases} \geqslant a_{\frac{k}{2}} \text{ for } k \text{ even} \\ 0 \text{ for } k \text{ odd.} \end{cases}$$

Proof The statement for k odd is immediate as the cohomology of the Grassmanian is concentrated in even degrees.
If k is even, first suppose that $k \leqslant 2(n-2)$, then by lemma 4.2

$$\dim(i^*H^k(G)) = a_{\frac{k}{2}}$$

Now, we turn to the case where $k > 2(n-2)$, then

$$\begin{aligned}\dim(i^*H^k(G)) &\geqslant \dim(i^*H^{4(n-2)-k}(G) \cdot c_1(\mathcal{S})^{k-2(n-2)}) \\ &= \dim(i^*H^{4(n-2)-k}(G)) \\ &= a_{\frac{4(n-2)-2}{2}} = a_{\frac{k}{2}}.\end{aligned}$$

Where we use the hard Lefschetz theorem for the first equality. Note here that $-c_1(\mathcal{S}) = c_1(\det \mathcal{S}^\vee)$ is the first Chern class of the Plücker polarization, see [Huy23, Chapter 2.1]. □

4.2 The Cylinder Correspondence

Recall the two maps

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}) & \xrightarrow{p} & F \\ {\scriptstyle q}\downarrow & & \\ Y & & \end{array}$$

from diagram 1.1, they induce an *algebraic* correspondence called the *cylinder correspondence* $\Gamma_* := p_*q^*$. The cylinder correspondence will be the main tool we will use to relate the cohomology of F to the cohomology of Y.

Lemma 4.4 ([Stacks, Tag 0FHW]) *Let $q\colon S \to T$ be a dominant morphism of irreducible smooth projective complex varieties. Then $q^*\colon H^i(T) \to H^i(S)$ is injective.*

Proof Let $Z \subset S$ be an integral subscheme of the same dimension as T mapping onto T. One may always find such a subscheme by embedding S into some projective space and taking generic hyperplane sections. Now, $f_*(\gamma_S(Z)) = m \cdot [T]$ for some positive integer m. Hence, by the projection formula $f_*(f^*(a) \cdot \gamma_S(Z)) = m \cdot a$ and f^* is injective. □

Lemma 4.5 *The cylinder correspondence induces an injective map*

$$\Gamma_*\colon H^n_{\text{prim}}(Y) \to H^{n-2}(F).$$

Proof First, by Leray–Hirsch, there is a direct sum decomposition

$$H^n(\mathbb{P}(\mathcal{E})) = p^*H^n(F) \oplus p^*H^{n-2}(F) \cdot h \quad \text{with } h := c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)).$$

Since $n > 2$, any closed point $y \in Y(\mathbb{C})$ is contained in a line in Y. Hence $q\colon \mathbb{P}(\mathcal{E})(\mathbb{C}) \to Y(\mathbb{C})$ is surjective. From this, it follows by lemma 4.4 and the fact that q is a projective morphism that q^* is injective and we claim that it suffices to show that

$$p^*H^n(F) \cap q^*H^n_{\mathrm{prim}}(Y) = 0. \tag{4.2}$$

Suppose (4.2) holds, and let $\alpha \in H^n(Y)$, then we may write $q^*(\alpha) = p^*(\gamma_{n-2}) \cdot h$ with $\gamma_{n-2} \in H^{n-2}(F)$ unique. Since $p_*(p^*\gamma_{n-2} \cdot h) = \gamma_{n-2}$, we see that $p_*(q^*(\alpha)) = \gamma_{n-2}$ and Γ_* is injective.
So it suffices to prove (4.2). Suppose there exists $\gamma \in H^n_{\mathrm{prim}}(Y)$ and $\gamma_n \in H^n(F)$ such that $p^*\gamma_n = q^*\gamma \in H^n(\mathbb{P}(\mathcal{E}))$. We may choose a hyperplane section H of Y such that $h = q^*H$, then

$$p^*\gamma_n \cdot h = q^*\gamma \cdot q^*H = q^*(\gamma \cdot H) = 0$$

where in the last equality we use that $\gamma \in H^n_{\mathrm{prim}}(X)$. □

The following is a technical lemma relating the ranks of the cohomology groups of F and X, we will use it in the following corollaries

Lemma 4.6 *We have the following three equalities*

1. $\dim H^{n-2}(F) = \dim H^n_{\mathrm{prim}}(Y)$ *when n is odd.*
2. $\dim H^k(F) = \dim H^n_{\mathrm{prim}}(Y)$ *when n is odd and $k = n-2+2s$ with $0 \leqslant s \leqslant n-2$.*
3. $\dim H^k(F) = \dim H^n_{\mathrm{prim}}(Y) + a_{\frac{k}{2}}$ *for n even and $k = n-2+2s$ with $0 \leqslant s \leqslant n-2$.*

Proof In all cases we use the isomorphism of mixed Hodge structures of theorem 3.13, writing it out more explicitly, we get

$$H^*(F) = \operatorname{Sym}^2 H^n_{\mathrm{prim}}(Y)(2) \oplus \bigoplus_{k=0}^{n-2} H^n_{\mathrm{prim}}(Y)(-k+1) \oplus \bigoplus_{k=0}^{2(n-2)} \mathbb{Q}(-k)^{\oplus a_{\frac{k}{2}}}.$$

1. Since n is odd, the $n-2$ graded part of the Hodge structures of $\operatorname{Sym}^2 H^n_{\mathrm{prim}}(Y)(2)$ and of $\bigoplus_{k=0}^{2(n-2)} \mathbb{Q}(-k)^{\oplus a_{\frac{k}{2}}}$ are 0. The $n-2$ graded part of $\bigoplus_{k=0}^{n-2} H^n_{\mathrm{prim}}(Y)(-k+1)$ corresponds to the summand $k = 0$ and the equality follows.
2. This is the same argument as above, the k-th graded parts of $\operatorname{Sym}^2 H^n_{\mathrm{prim}}(Y)(2)$ and of $\bigoplus_{k=0}^{2(n-2)} \mathbb{Q}(-k)^{\oplus a_{\frac{k}{2}}}$ are 0 while the k-th graded part of $\bigoplus_{k=0}^{n-2} H^n_{\mathrm{prim}}(Y)(-k+1)$ corresponds to the summand $k = s$.
3. The k-th graded part of $\operatorname{Sym}^2 H^n_{\mathrm{prim}}(Y)(2)$ is 0, the k-th graded part of $\bigoplus_{k=0}^{n-2} H^n_{\mathrm{prim}}(Y)(-k+1)$ is $H^n_{\mathrm{prim}}(Y)$ and the k-th graded part of $\bigoplus_{k=0}^{2(n-2)} \mathbb{Q}(-k)^{\oplus a_{\frac{k}{2}}}$ now is $\mathbb{Q}^{\oplus a_{\frac{k}{2}}}$ (because k is even). □

Lemma 4.7 *The bundles* $\mathbb{P}_G(\mathcal{S}) \to \mathbb{P}(V)$ *and* $\mathbb{P}(T\mathbb{P}(V)) \to \mathbb{P}(V)$ *are isomorphic.*

Proof Let $p\colon \mathbb{P}(T\mathbb{P}(V)) \to \mathbb{P}(V)$ and $q_G\colon \mathbb{P}_G(\mathcal{S}) \to G$ be the natural projections, recall that the map $q_V\colon \mathbb{P}_G(\mathcal{S}) \to \mathbb{P}(V)$ is obtained from the factorisation $\mathbb{P}_G(\mathcal{S}) \hookrightarrow \mathbb{P}(V) \times G \to \mathbb{P}(V)$. Since p is flat, the pullback of the Euler sequence on $\mathbb{P}(V)$ gives

$$0 \to \mathcal{O}_{\mathbb{P}(T\mathbb{P}(V))}(-1) \to V \xrightarrow{\phi} p^*T\mathbb{P}(V)(-1) \to 0.$$

The relative Euler sequence on $\mathbb{P}(T\mathbb{P}(V))$ gives

$$0 \to \mathcal{O}_{\mathbb{P}(T\mathbb{P}(V))}(-2) \xrightarrow{\iota} p^*T\mathbb{P}(V)(-1) \to \mathcal{T}(-2) \to 0$$

where $\mathcal{T}$ is the relative tangent bundle of p. Let $F := \phi^{-1}\left(\iota\left(\mathcal{O}_{\mathbb{P}(T\mathbb{P}(V))}(-2)\right)\right) \subset V$. By definition this is a rank 2 subbundle and hence it induces a morphism $\mathbb{P}(T\mathbb{P}(V)) \to G$. The inclusion $\mathcal{O}_{\mathbb{P}(T\mathbb{P}(V))}(-1) \subset F$ induces the morphism $\mathbb{P}(T\mathbb{P}(V)) \to \mathbb{P}_G(\mathcal{S})$.
To construct an inverse, we notice that $q_V^*(\mathcal{O}_{\mathbb{P}(V)}(-1)) = \mathcal{O}_{\mathbb{P}_G(\mathcal{S})}(-1)$ and we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}_G(\mathcal{S})} & \longrightarrow & p^*\mathcal{S}(1) & \longrightarrow & T\mathbb{P}_G(\mathcal{S}) & \longrightarrow & 0 \\
 & & {\scriptstyle =}\downarrow & & \downarrow & & \vdots & & \\
0 & \longrightarrow & q^*(\mathcal{O}_{\mathbb{P}(V)}) & \longrightarrow & V(1) & \longrightarrow & q^*(T\mathbb{P}(V)) & \longrightarrow & 0
\end{array}$$

where $T\mathbb{P}_G(\mathcal{S})$ denotes the relative tangent bundle of $p\colon \mathbb{P}_G(\mathcal{S}) \to G$. A simple diagram chase shows that the map $T\mathbb{P}_G(\mathcal{S}) \to q^*(T\mathbb{P}(V))$ is injective, hence $q_*T\mathbb{P}_G(\mathcal{S}) \to T\mathbb{P}(V)$ induces a morphism $\mathbb{P}_G(\mathcal{S}) \to \mathbb{P}(T\mathbb{P}(V))$. In the last step, we used that the fibers of q are connected and reduced to identify $q_*q^*(T\mathbb{P}(V)) = T\mathbb{P}(V)$ and to see that $q_*T\mathbb{P}_G(\mathcal{S})$ is invertible. It is now readily checked that the morphisms $\mathbb{P}_G(\mathcal{S}) \to \mathbb{P}(T\mathbb{P}(V))$ and $\mathbb{P}(T\mathbb{P}(V)) \to \mathbb{P}_G(\mathcal{S})$ are mutually inverse. □

Corollary 4.8 *When* n *is odd, the cylinder correspondence* $\Gamma_*\colon H^n_{prim}(Y) \to H^{n-2}(F)$ *is an isomorphism. When* n *is even,* $\Gamma_*H^n_{prim}(Y)$ *and* $\iota^*H^{3(n-2)}(G)$ *are orthogonal with respect to the cup product. Here* ι *is the embedding* $F(Y,1) \to G$ *inside the Grassmanian of lines of diagram 1.1.*

Proof If n is odd, the statement follows from the injectivity of Γ_* together with the first part of lemma 4.6.
Now suppose n is even, keeping with the naming of diagram 1.1, we have

$$\begin{aligned}
\iota^*H^{3(n-2)}(G) \cdot \Gamma_*H^n_{prim}(Y) &= p_*(p^*\iota^*H^{3(n-2)}(G) \cdot q^*H^n_{prim}(Y)) \\
&= p_*\left(\bar{\iota}^*\bar{p}^*H^{3(n-2)}(G) \cdot q^*H^n_{prim}(Y)\right).
\end{aligned}$$

Since $\bar{\iota}^*\bar{p}^*H^{3(n-2)}(G) \subset \bar{\iota}^*H^{3(n-2)}(\mathbb{P}_G(\mathcal{E}))$, it is sufficient to show that

$$p_*\left(\bar{\iota}^*H^{3(n-2)}(\mathbb{P}_G(\mathcal{E})) \cdot q^*H^n_{prim}(Y)\right) = 0.$$

Recall that $\mathbb{P}_G(\mathcal{S}) \to \mathbb{P}(V)$ is the projective bundle bundle $\mathbb{P}(T\mathbb{P}(V)) \to \mathbb{P}(V)$.
We consider the diagram

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}) & \xrightarrow{q} & Y \\ \iota\downarrow & & \downarrow j \\ \mathbb{P}(T\mathbb{P}(V)) & \xrightarrow[\overline{q}]{} & \mathbb{P}(V) \end{array}$$

From Leray–Hirsch, we again get a decomposition

$$H^{3(n-2)}(\mathbb{P}(T\mathbb{P}(V))) = \bigoplus_{k=0}^{n+1} \overline{q}^* H^{3(n-2)-2k}(\mathbb{P}(V)) \cdot c_1(\mathcal{O}_{T\mathbb{P}(V)}(1))^k.$$

And for each k in the direct sum above, we find that

$$\begin{aligned} &\bar{\iota}^* \left(\overline{q}^* H^{3(n-2)-2k}(\mathbb{P}(V)) \cdot c_1(\mathcal{O}_{T\mathbb{P}(V)}(1))^k \right) \cdot q^* \left(H^n_{\text{prim}}(Y) \right) \\ =&q^* \left(j^* H^{3(n-2)-2k}(\mathbb{P}(V)) \cdot \bar{\iota}^* c_1(\mathcal{O}_{T\mathbb{P}(V)}(1))^k \cdot H^n_{\text{prim}}(Y) \right), \end{aligned} \tag{4.3}$$

where we use the commutativity of the diagram above. Now, recall that the cohomology ring of projective space is generated in degree 2 by a hyperplane section. Hence, for all k, any element in $j^* H^{3(n-2)-2k}(\mathbb{P}(V)) \cdot \bar{\iota}^* c_1(\mathcal{O}_{T\mathbb{P}(V)}(1))^k$ is divisible by the class of a hyperplane section and thus (4.3) is 0. □

4.3 Cohomology of F

Using the results from the two previous sections, we are now able to explicitly relate the cohomology of F to the cohomology of Y via algebraic maps.
In what follows, we let $L := - \cup c_1(\mathcal{E})\colon H^i(F) \to H^{i+2}(F)$ denote the Lefschetz operator. We introduce, for all $0 \leqslant k \leqslant 2(n-2)$ a bilinear pairing $(-,-)_k\colon H^k(F) \times H^k(F) \to \mathbb{Q}$ defined by

$$(\alpha, \alpha')_k := \alpha \cdot \alpha' \cdot c_1(\mathcal{E})^{2(n-2)-k}.$$

Proposition 4.9 *The pairing $(-,-)_k$ is perfect.*

Proof By the hard Lefschetz theorem, the map $H^k(F) \to H^{4(n-2)-k}(F)$ sending $\alpha \mapsto \alpha \cdot c_1(\mathcal{E})^{2(n-2)-k}$ is an isomorphism. By Poincaré duality, the pairing $H^k(F) \times H^{4(n-2)-k}(F) \to \mathbb{Q}$ is perfect. □

Lemma 4.10 *Let s be an integer $0 \leqslant s \leqslant n-2$. When $0 \leqslant k \neq 4(n-2)$ with $k \neq 2(n-2)$, then*

1.

$$H^k(F) := \begin{cases} L^s \Gamma_* H^n_{\text{prim}}(Y) & \text{for } n \text{ odd, } k = n-2+2s \\ L^s \Gamma_* H^n_{\text{prim}}(Y) \oplus \iota^* H^k(G) & \text{for } n \text{ even, } k = n-2+2s \\ \iota^* H^k(G) & \text{else.} \end{cases}$$

2. *When n is even and $k = n-2+2s$, the decomposition $H^k(F) = L^s\Gamma_* H^n_{prim}(Y) \oplus \iota^* H^k(G)$ is orthogonal with respect to $(-,-)_k$.*

Proof • **If n is odd and $k = n-2+2s$**
Since $s \leqslant n-2$, the map $L^s\colon H^{n-2}(F) \to H^k(F)$ is injective by hard Lefschetz, furthermore, by lemma 4.5, Γ_* is injective. So it is sufficient to show that $\dim H^k(F) = \dim H^n_{prim}(Y)$, which follows from the second part of lemma 4.6.

- **If n is even and $k = n-2+2s$**
We claim it is sufficient to show:

$$\iota^* H^k(G) \text{ is orthogonal to } L^s\Gamma_* H^n_{prim}(Y) \text{ with respect to } (-,-)_k \tag{4.4}$$

Indeed, suppose that (4.4) holds then

$$\begin{aligned}\dim(L^s\Gamma_* H^n_{prim}(Y) \oplus \iota^* H^k(G)) &= \dim(L^s\Gamma_* H^n_{prim}(Y)) + \dim(\iota^* H^k(G)) \\ &\leqslant \dim(H^n_{prim}(Y)) + a_{\frac{k}{2}}\end{aligned}$$

But the third part of lemma 4.6 shows that

$$\dim(H^k(F)) = \dim(H^n_{prim}(Y)) + a_{\frac{k}{2}}$$

So it suffices to prove (4.4). Let $\alpha \cdot c_1(\mathcal{E})^s \in L^s\Gamma_* H^n_{prim}(Y)$ with $\alpha \in \Gamma_* H^n_{prim}(Y)$ and $\beta \in \iota^* H^k(G)$, then

$$(\alpha \cdot c_1(\mathcal{E})^s, \beta)_k = \alpha \cdot \beta \cdot c_1(\mathcal{E})^{n-2-s}$$

□

Now $\beta \cdot c_1(\mathcal{E})^{(n-2)-s} \in \iota^* H^{3(n-2)}(G) \subset H^{3(n-2)}(F)$ because the pullback of a hyperplane section is a hyperplane section. Hence it suffices to show that $\iota^* H^{3(n-2)}(G)$ is orthogonal to $\Gamma_* H^n_{prim}(Y)$ with respect to the cup product, this is the statement of corollary 4.8.

- **For all other n**
This is corollary 4.3.

Corollary 4.11 *For all $0 \leqslant k \leqslant 4(n-2), k \neq 2(n-2)$:*

1. $a_{\frac{k}{2}} = \dim\left(\iota^* H^k(G)\right) = \dim\left(\iota^* H^{4(n-2)-k}\right) = a_{\frac{4(n-2)-k}{2}}$.

2. *The cup product pairing on $H^*(F)$ restricts to a perfect pairing*

$$\iota^* H^k(G) \times \iota^* H^{4(n-2)-k}(G) \to \mathbb{Q}$$

Proof 1. If n is even, by the second part of lemma 4.10

$$\begin{aligned}\dim\left(\iota^*H^k(G)\right) &= \dim\left(H^k(F)\right) - \dim\left(L^s\Gamma_* H^n_{\text{prim}}(Y)\right)\\ &= \dim\left(H^k(F)\right) - \dim\left(H^n_{\text{prim}}(Y)\right)\\ &= a_{\frac{k}{2}}\end{aligned}$$

Combining the same reasoning for $\iota^*H^{4(n-2)-k}(G)$ and the observation that $a_{\frac{k}{2}} = a_{\frac{4(n-2)-k}{2}}$ gives the statement.

2. By symmetry, we may suppose that $k < 2(n-2)$. Let $\alpha \neq H^k(G)$, it suffices to find $\beta \in H^{4(n-2)-k}(G)$ such that $\iota^*\alpha \cdot \iota^*\beta \neq 0$.
By lemma 4.1, since α and F are non-zero polynomials in the Chern classes, we deduce that $\alpha \cdot F \neq 0 \in H^{k+8}(G)$. By Poincaré duality on G, there exists $\beta \in H^{4n-(k+8)}(G)$ such that $\alpha \cdot \beta \cdot F \neq 0$. Finally, by the projection formula $\iota_*\left(\iota^*\alpha \cdot \iota^*\beta\right) = \alpha \cdot \beta \cdot F \neq 0$ and hence $\iota^*\alpha \cdot \iota^*\beta \neq 0$.
□

5 Proof of the Standard Conjectures for F

Keeping with the notations of section 4, in this section we prove the standard conjectures for F, following [Dia17]. By theorem 2.8 it will be sufficient to find, for every $0 \leqslant k \leqslant 2(n-2)$ a correspondence $\Gamma_k \in CH^k(F \times F)$ such that

$$\Gamma_{k*}\colon H^{4(n-2)-k}(F) \to H^k(F)$$

is an isomorphism.

5.1 Construction of Γ_k

Throughout, we let $L \in CH^{2(n-2)+1}(F \times F)$ denote the correspondence associated to $c_1(\mathcal{E})$ and define

$$\delta_{\mathrm{prim}} := \Delta_Y - \sum_{0 \leqslant r \leqslant n} \frac{1}{3} H^r \times H^{n-r} \in CH^n(Y \times Y)$$

where H is a smooth hyperplane section of Y.
It will turn out that the desired correspondence will be

$$\Gamma_k := \Gamma_k' + \Gamma_k'' \in CH^k(F \times F)$$

where

$$\Gamma_k' := \begin{cases} L^s \circ \Gamma \circ \delta_{\mathrm{prim}} \circ {}^t\Gamma \circ L^s & \text{for } k = n-2+2s, 0 \leqslant s \leqslant \left\lfloor \frac{n-2}{2} \right\rfloor \\ 0 & \text{else} \end{cases}$$

and

$$\Gamma_k'' := \begin{cases} \sum_{m=1}^{N_{\frac{k}{2}}} \alpha_{\frac{k}{2},m} \times \alpha_{\frac{k}{2},m} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Here $\alpha_{\ell,1}, \dots, \alpha_{\ell,N_\ell} \in CH^\ell(F)$ is the pullback of a basis of $CH^\ell(G)$ along the inclusion $\iota\colon F \hookrightarrow G$.
The main result of this section is

Theorem 5.1 *The correspondence $\Gamma_{k*}\colon H^{4(n-2)-k}(F) \to H^k(F)$ is an isomorphism for $0 \leqslant k \leqslant 2(n-2)$.*

Before proving this, we need a few lemmas describing the correspondences Γ_{k*}' and Γ_{k*}''

Lemma 5.2 *We have $(\delta_{\mathrm{prim}})_* H^*(Y) = H^n_{\mathrm{prim}}(Y)$.*

Proof Notice that for all $k \geqslant 0$, we have

$$H^k \cdot \left(\sum_{0 \leqslant r \leqslant n} \frac{1}{3} H^r \times H^{n-r} \right) = H^k \cdot \left(\frac{1}{3} H^{n-k} \times H^k \right)$$
$$= \frac{1}{3} \langle H^k, H^{n-k} \rangle H^k$$
$$= H^k.$$

Where the last equality follows because Y is a cubic of dimension n and hence the intersection of n generic hyperplanes gives 3 distinct points.
Thus, $\sum_{0 \leqslant r \leqslant n} \frac{1}{3} H^r \times H^{n-r}$ acts as the identity on the part of cohomology generated by hyperplane sections and thus necessarily $(\delta_{\text{prim}})_* H^*(Y) = H^n_{\text{prim}}(Y)$. □

5.2 The correspondences Γ'_k and Γ''_k

Before we prove theorem 5.1, we need to understand how Γ'_k and Γ''_k act on the summands appearing in lemma 4.10.

Lemma 5.3 *When $\Gamma'_k \neq 0$, we have*

1. $\Gamma'_{k*}(H^k(F)) = L^s \Gamma_* H^n_{\text{prim}}(Y)$
2. $\Gamma'_{k*}(\iota^* H^{4(n-2)-k}(G)) = 0.$

Proof 1. It is sufficient to show that the correspondence

$$\delta_{\text{prim}} \circ {}^t\Gamma \circ L^s \colon H^{4(n-2)-k}(F) \to H^n_{\text{prim}}(Y)$$

is surjective, since then

$$\Gamma'_{k*}(H^k(F)) = L^s \circ \Gamma_* \circ \delta_{\text{prim}} \circ {}^t\Gamma \circ L^s \left(H^k(F) \right)$$
$$= L^s \circ \Gamma_* \left(H^n_{\text{prim}}(Y) \right).$$

To see this, we show that L^s and $\delta_{\text{prim}} \circ {}^t\Gamma_*$ are surjective. By hard Lefschetz, the map L^s is an isomorphism. To see surjectivity of $\delta_{\text{prim}} \circ {}^t\Gamma_*$, notice that from the definition of δ_{prim}, it is immediate that ${}^t\delta_{\text{prim}} = \delta_{\text{prim}}$. Hence, the Poincaré dual of $\delta_{\text{prim}} \circ {}^t\Gamma : H^{3(n-2)}(F) \to H^n_{\text{prim}}(Y)$ is

$${}^t \left(\delta_{\text{prim}} \circ {}^t\Gamma \right) = \Gamma \circ \delta^{\text{prim}} \colon H^n_{\text{prim}}(Y) \to H^{n-2}(F).$$

This map is injective by lemma 4.5 and the fact that δ_{prim} acts trivially on primitive cohomology.

2. We show that
$$\delta_{\text{prim}} \circ {}^t\Gamma \circ L^s \left(\iota^* H^{4(n-2)-k}(G) \right) = 0.$$

Since the Lefschetz operator commutes with ι^* (the pullback of a hyperplane section is a hyperplane section), this is the same as showing that

$$\delta_{\mathrm{prim}} \circ {}^t\Gamma\left(\iota^* H^{3(n-2)}(G)\right) = 0.$$

In fact, it is sufficient to show that

$$\ker\left\{\delta_{\mathrm{prim}} \circ {}^t\Gamma \colon H^{3(n-2)}(F) \to H^n_{\mathrm{prim}}(Y)\right\} = \left(\Gamma_* H^n_{\mathrm{prim}}(Y)\right)^{\perp}, \tag{5.1}$$

since by the second part of lemma 4.8, we have $\iota^* H^{3(n-2)}(G) \subseteq \left(\Gamma_* H^n_{\mathrm{prim}}(Y)\right)^{\perp}$.

We now show the equality in (5.1). Let $\alpha \in \ker\left\{\delta_{\mathrm{prim}\,*} \circ {}^t\Gamma_* \colon H^{3(n-2)}(F) \to H^n_{\mathrm{prim}}(Y)\right\}$ and $\Gamma_*\beta \in \Gamma_* H^n_{\mathrm{prim}}(Y)$, then

$$\langle \alpha, \Gamma_*\beta\rangle = \langle {}^t\Gamma_*\alpha, \beta\rangle = 0.$$

The last equality follows from the fact that $\delta_{\mathrm{prim}\,*}{}^t\Gamma_*\alpha = 0$, hence a hyperplane section divides ${}^t\Gamma_*\alpha$. □

Lemma 5.4 *For $0 \leqslant k \leqslant 2(n-2)$,*

1. $\Gamma''_{k*}\left(\iota^* H^{4(n-2)-k}(G)\right) = \iota^* H^k(G)$
2. *If n is even and $k = n-2+2s$,* $\Gamma''_{k*}\left(L^{n-2-s}\Gamma_* H^n_{\mathrm{prim}}(Y)\right) = 0.$

Proof 1. If k is odd, this is immediate by our definition of Γ''_{k*} and the fact that the cohomology of the Grassmanian is concentrated in even degree. So suppose k is even. By corollary 4.11, we may choose a dual basis $\{\beta_{k,m}\} \subset \iota^* H^{4(n-2)-k}(G)$ to $\{\alpha_{k,m}\}$ with respect to the cup product, then one sees that

$$\Gamma''_{k*}(\beta_{k,m}) = \delta_{m,n} \cdot \alpha_{k,m} \in \iota^* H^k(G).$$

This implies the result.

2. By lemma 4.10, $\iota^* H^k(G)$ is orthogonal to $L^s\Gamma_* H^n_{\mathrm{prim}}(Y)$ with respect to $(-,-)_k$ and hence $\iota^* H^k(G)$ is orthogonal to $L^{n-2-s}\Gamma_* H^n_{\mathrm{prim}}(Y)$ with respect to the cup product. Letting $p : F \times F \to F$ denote the projection, we find that for all $\gamma \in L^{n-2-s}\Gamma_* H^n_{\mathrm{prim}}(Y)$

$$\begin{aligned}\Gamma''_{k*}(\gamma) &= p_*\left(\sum_{m=1}^{N_{\frac{k}{2}}} \tilde{\alpha}_{\frac{k}{2},m} \times \tilde{\alpha}_{\frac{k}{2},m} \cdot p^*\gamma\right)\\ &= p_*\left(\sum_{m=1}^{N_{\frac{k}{2}}} \left\langle \tilde{\alpha}_{\frac{k}{2},m}, \gamma\right\rangle \tilde{\alpha}_{\frac{k}{2},m}\right) = 0.\end{aligned}$$

□

5.3 Γ_{k*} is an isomorphism

We are now ready to prove the main result

Proof (of theorem 5.1) In all cases, it suffices to show that Γ_{k*} is surjective as the ranks of $H^{4(n-2)-k}(F)$ and $H^k(F)$ are the same by Poincaré duality. We distinguish the cases appearing in lemma 4.10.

- **If n is odd and $k = n-2+2s$**
  Then

$$\Gamma_{k*}\left(H^{4(n-2)-k}(F)\right) = \Gamma'_{k*}\left(H^{4(n-2)-k}(F)\right) = L^s\Gamma_* H^n_{\text{prim}}(Y) = H^k(F),$$

  where the first equality is because $\Gamma''_{k*} = 0$, the second one follows from lemma 5.3 and the third from 4.10.

- **If n is even and $k = n-2+2s$**
  Notice the following four equalities

$$\begin{aligned}\Gamma'_{k*}\left(L^{n-2-s}\Gamma_* H^n_{\text{prim}}(Y)\right) &= L^s\Gamma_* H^n_{\text{prim}}(Y)\\ \Gamma'_{k*}\left(\iota^* H^{4(n-2)-k}(G)\right) &= 0\\ \Gamma''_{k*}(L^{n-2-s}\Gamma_* H^n_{\text{prim}}(Y)) &= 0\\ \Gamma''_{k*}\left(\iota^* H^{4(n-2)-k}(G)\right) &= \iota^* H^k(G).\end{aligned}$$

  From this, it follows that

$$\begin{aligned}\Gamma_{k*}\left(H^{4(n-2)-k}(F)\right) &= \Gamma_{k*}\left(L^{n-2-s}\Gamma_* H^n_{\text{prim}}(Y) \oplus \iota^* H^{4(n-2)-k}(G)\right)\\ &= L^s\Gamma_* H^n_{\text{prim}}(Y) \oplus \iota^* H^k(G)\\ &= H^k(F).\end{aligned}$$

- **For other n** Then $\Gamma'_k = 0$ and we have the following chain of equalities

$$\Gamma_{k*}\left(H^{4(n-2)-k}(F)\right) = \Gamma''_{k*}\left(H^{4(n-2)-k}(F)\right) = \iota^* H^k(G) = H^k(F).$$

  The second equality follows by lemma 5.4. □

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