Lefschetz Classes on Projective Varieties

Maximilian Schimpf

Geboren am 18. Juni 1998 in Ravensburg

4. Juli 2019

Bachelorarbeit Mathematik

Betreuer: Prof. Dr. Daniel Huybrechts

Zweitgutachter: Dr. Gebhard Martin

Mathematisches Institut

Mathematisch-Naturwissenschaftliche Fakultät der

Rheinischen Friedrich-Wilhelms-Universität Bonn

Zusammenfassung

Alexander Grothendieck stellte 1969 in seiner Arbeit[Gr68] zwei sogenannte "Standardvermutungen" auf, die versuchen das Zusammenspiel einer Weil Kohomologie Theorie mit den durch sie definierten algebraischen Klassen zu erfassen. Die "Lefschetz-Standardvermutung" besitzt mehrere Formulierungen und ist im Wesentlichen eine Existenzaussage für algebraische Klassen, die dem Schweren Lefschetz-Satz nachempfunden ist. Die "Hodge-Standardvermutung" ist eine Positivitätsaussage, die für komplexe Varietäten mit Betti Kohomologie aus der Hodge-Theorie folgt. Grothendieck bemerkte, dass diese zusammen die letzte damals noch offene Weil-Vermutung implizieren, das Analogon zur Riemannvermutung. Obwohl diese von Pierre Deligne inzwischen vollständig bewiesen wurde, sind Grothendiecks Standardvermutungen noch bis heute offen. Es scheinen immernoch keine Werkzeuge vorhanden zu sein, um sie zu beweisen.
June Huh und Botong Wang [HW17] betrachten für komplexe Varietäten eine Variante der Lefschetz-Standardvermutung. Die algebraischen Klassen ersetzen sie hierbei durch sogenannte Lefschetz Klassen. Nach einer Reihe von Gültigkeitskriterien zeigen sie, dass diese Variante jedoch im Allgemeinen fehl schlägt. Ziel der vorliegenden Bachelorarbeit ist es, eine kurze Einführung in die Standardvermutungen zu geben und für den Fall der Lefschetz Klassen einige der Gegenbeispiele und Gültigkeitskriterien in [HW17] zu präsentieren, sowie neue zu entwickeln. Im ersten Kapitel werden die Standardvermutungen gesammelt und die gültigen Implikationen zwischen ihnen erwähnt, ohne dabei ins Detail der relavanten Beweise zu gehen. Im zweiten Kapitel wird die Lefschetz-Algebra einer komplexen projektiven Varietät eingeführt. Nachdem kurz erläutert wird, welche der Implikationen für Lefschetz Klassen gültig bleiben, schränken wir uns auf eine Variante der Lefschetz-Standardvermutung ein und beweisen einige Gültigkeitskriterien. Schließlich rechnen wir drei Gegenbeispiele zu dieser Variante explizit nach - zwei davon stammen aus der Arbeit [HW17]. Ein Appendix stellt mehrere verwendete technische Resultate bereit.

Danksagungen

Zuerst möchte ich mich bei all denen bedanken, die mir bei der Anfertigung dieser Arbeit geholfen haben. Besonders bedanke ich mich bei Herrn Prof. Huybrechts für seine Geduld und all die hilfreichen Ideen und Ratschläge. Außerdem danke ich ganz herzlich meinen Eltern für Ihre Unterstützung über meine ganze schulische und universitäre Laufbahn hinweg.

Contents

Introduction

In 1969, Alexander Grothendieck proposed the two so-called standard conjectures, with which he tries to capture the behaviour of the algebraic classes defined on a Weil cohomology theory [Gr68]. The "Lefschetz standard conjecture" is essentially an existence claim for algebraic classes and is modelled after the Hard Lefschetz theorem. The "Hodge standard conjecture" is a positivity assertion, which follows from Hodge theory in the case of complex varieties with betti cohomology. They were intended to aid in proving the most difficult part of the Weil conjectures - an analogue of the Riemann hypothesis, which was still open at the time. Even though the last of the Weil conjectures has finally been proven in 1974 by Pierre Deligne, Grothendieck's standard conjectures are still open to this day. There seem to be no tools available for proving them.
In [HW17], June Huh and Botong Wang consider a variant of the Lefschetz standard conjecture in the case of complex varieties. They replace the algebraic classes in the formulation by so called Lefschetz classes. After a series of validity criteria, they show however that this variant fails in general.
The aim of this thesis is to give a short introduction to the standard conjectures, to discuss some of the counterexamples and validity criteria in the case of Lefschetz classes which are presented in their paper and to develop new ones. In the first chapter, the standard conjectures are formulated and the implications between them discussed, without going into much detail about the proofs. In the second chapter, we introduce the Lefschetz algebra of a complex projective variety. After briefly explaining which of the implications remain valid for Lefschetz classes we restrict ourselves to a variant of the Lefschetz standard conjecture and prove some criteria for its validity. Finally, we present three counterexamples to this variant - two of which are taken from [HW17]. The Appendix provides details on the cohomology of the blowup and collects some facts on the Grassmannian.

Acknowledgements

First of all, I would like to offer special thanks to my advisor Prof. Daniel Huybrechts for his patience and all the helpful advice he gave me. Also, I want to thank my family for their unfailing support and encouragement throughout the entirety of my school and university career.

1. Algebraic cycles and Grothendiecks standard conjectures

Throughout this chapter we assume k to be an algebraically closed field. By a *variety* we mean an integral scheme which is of finite type over k. All subvarieties are assumed to be closed.

1.1. Algebraic cycles

In order to state the standard conjectures, we are going to introduce the machinery of algebraic cycles. This section largely follows the first two sections of [EH16].

Definition 1.1. Let X be a scheme of finite type over k.
Let $Z^*(X)$ be the free abelian group generated by the set of all subvarieties of X graded by codimension. The elements of $Z^*(X)$ are called *algebraic cycles.*

Cycles can be viewed as approximations to subschemes of X which are just coarse enough so that intersection theory can be developed. Originally singular homology was used for this (see for example [GH94] or [Br93, Section VI.11] for intersections of real submanifolds). But with the help of algebraic cycles, this can be carried out over arbitrary algebraically closed fields without assuming the existence of a homology or cohomology theory over k. In fact, one has the following:

Definition 1.2. Let $Z \subseteq X$ be a closed subscheme. The algebraic cycle in X associated to Z is defined as:

$$[Z] := \sum_{i=1}^{k} \mathrm{length}_{\mathcal{O}_{Z,Z_i}}(\mathcal{O}_{Z,Z_i}) Z_i$$

where $Z_1, ..., Z_k$ are the irreducible components of Z. Note that $\mathcal{O}_{Z,Z_i}$ is zero-dimensional and Noetherian hence of finite length over itself.

Definition 1.3. Let $\mathrm{Rat}^*(X) \subseteq Z^*(X)$ be the subgroup generated by expressions of the form:

$$\sum_{i=1}^{k} a_i \bigg([i_0^{-1} Z_i] - [i_\infty^{-1} Z_i] \bigg),$$

where $Z_i \subset X \times \mathbb{P}^1$ are subvarieties each not contained in any fiber over $\mathbb{P}^1$ and i_0, i_∞ are the identifications of X with the fiber over $[0 : 1]$ and $[1 : 0]$.
The quotient $Z^*_{rat}(X) := Z^*(X)/\mathrm{Rat}^*(X)$ is called the *Chow group* of X. Two cycles in $Z^*(X)$ are called *rationally equivalent* if their classes in $Z^*_{rat}(X)$ agree.

Using Krull's Principal ideal theorem, one shows that the schemes $i_0^{-1}Z_i$ and $i_\infty^{-1}Z_i$ are of pure dimension $\dim Z_i - 1$ hence the Chow group does indeed have an induced grading. From now on, we restrict ourselves to smooth projective varieties. In this case one can define an intersection product on the Chow group giving it a ring structure(see [Fu84] for details of this). One might ask whether the Chow ring is functorial in X. Indeed:

Definition 1.4. Let $f : X \to Y$ be a morphism of smooth projective varieties over k.
A subvariety $Z \subseteq X$ is called *generically transverse to* f if $f^{-1}(Z) \subseteq Y$ is generically reduced and $\operatorname{codim}(f^{-1}(Z), Y) = \operatorname{codim}(Z, X)$. There is a unique ring homomorphism

$$f^* : Z^*_{rat}(Y) \to Z^*_{rat}(X)$$

such that $f^*([Z]) = [f^{-1}(Z)]$ if $Z \subseteq X$ is generically transverse to f and Cohen–Macaulay. It is called the *pullback* along f. The *pushforward* f_* is given by

$$\begin{aligned} f_* : Z^*_{rat}(X) &\longrightarrow Z^{*+c}_{rat}(Y) \\ [Z] &\longmapsto \begin{cases} [k(Z) : k(f(Z))][f(Z)], & \text{if } \dim f(Z) = \dim Z \\ 0, & \text{if } \dim f(Z) < \dim Z \end{cases} \end{aligned}$$

with $c = \dim X - \dim Y$. Note that $f(Z) \subseteq Y$ is a subvariety, since f is proper.

If f is flat, f^* can be described simply as $f^*[Z] = [f^{-1}Z]$. One also has the following useful *projection formula*:

$$f_*(f^*(\alpha).\beta) = \alpha.f_*(\beta) \text{ for any } \alpha \in Z^*_{rat}(Y),\ \beta \in Z^*_{rat}(X)$$

There is yet another useful equivalence relation on $Z^*(X)$:

Definition 1.5. The degree map on $Z^*_{rat}(X)$ is given by the composite

$$\langle\ \rangle \colon Z^*_{rat}(X) \xrightarrow{p_*} Z^{*-\dim X}_{rat}(\text{point}) = \mathbb{Z},$$

where p is the canonical morphism to the point. The *intersection pairing* on X is now defined as:

$$\begin{aligned} \langle\ .\ \rangle \colon Z^*_{rat}(X) \times Z^*_{rat}(X) &\longrightarrow \mathbb{Z} \\ (\alpha, \beta) &\longmapsto \langle \alpha.\beta \rangle \end{aligned}$$

Two cycles $a, b \in Z^*(X)$ are called *numerically equivalent* if

$$\langle [a].\gamma \rangle = \langle [b].\gamma \rangle \text{ for any cycle } \gamma \in Z^*_{rat}(X)$$

Remark 1.6. The pairing

$$Z^*_{num}(X) \times Z^*_{num}(X) \to \mathbb{Z}$$

induced by the intersection pairing is non-degenerate (essentially by definition)

1.2. Weil Cohomology theories and the standard conjectures

We now discuss the standard conjectures which are a series of statements concerning algebraic cycles and their relations with a fixed Weil Cohomology. We mostly follow the approach in [Kl68] and occasionally [Kl91].

Definition 1.7. Let K be a field of characteristic zero. A *Weil cohomology theory* with coefficients in K is a functor

$$H^* : \{\text{smooth proj. varieties over k}\} \longrightarrow \{\text{fin. dim. graded-commutative } K\text{-Algebras}\}$$

satisfying the following axioms:

1. **Poincaré duality:**
   Let X be a smooth projective variety over k of dimension d. Then
   a) $H^i(X) = 0$ for $i \notin [0, 2d]$
   b) We are given an orientation $\langle\ \rangle : H^{2d}(X) \xrightarrow{\cong} K$
   c) The pairing
   $$H^i(X) \times H^{2d-i}(X) \longrightarrow H^{2d}(X) \xrightarrow{\cong} K, \qquad (\alpha, \beta) \mapsto \langle \alpha.\beta \rangle$$
   induced by multiplication is non-degenerate.

Using this, one can define the *pushforward*: Given a morphism $f : X \to Y$ of smooth projective varieties of dimension d and e, we define f_* as the composite

$$f_* : H^*(X) \xrightarrow{\cong} \left(H^{2d-*}(X)\right)^\vee \xrightarrow{(f^*)^\vee} \left(H^{2d-*}(Y)\right)^\vee \xrightarrow{\cong} H^{2e-2d+*}(Y)$$

where the two isomorphisms come from Poincaré duality. It follows that:

$$f_*(f^*(\alpha).\beta) = \alpha.f_*(\beta) \text{ for } \alpha \in H^*(Y),\ \beta \in H^*(X)$$

2. **Künneth:**
   Let X and Y be smooth projective varieties. The cross-product
   $$\begin{aligned} \times : H^*(X) \otimes_K H^*(Y) &\longrightarrow H^*(X \times Y) \\ \alpha \otimes \beta &\longmapsto \pi_1^*(\alpha)\ .\ \pi_2^*(\beta) \end{aligned}$$
   is an Isomorphism of graded K-algebras

3. **Cycle map:**
   For any smooth projective variety X one is given a group homomorphism
   $$\eta_X : Z^*(X) \longrightarrow H^{2*}(X)$$
   which satisfies:
   a) the morphism
   $$\mathbb{Z} = Z^*(\text{point}) \longrightarrow H^*(\text{point}) = K$$
   is the canonical inclusion.

b) if $f : X \to Y$ is a morphism of smooth projective varieties, then

$$f_* \eta_X = \eta_Y f_* \text{ and } f^* \eta_Y = \eta_X f^*,$$

where f^* is defined on cycles which are generically transverse to f and Cohen–Macaulay.

c) for any two subvarieties $Z \subseteq X$ and $W \subseteq Y$ of smooth projective varieties X and Y, one has:

$$\eta_{X \times Y}(Z \times W) = \pi_1^*(\eta_X(Z)) \,.\, \pi_2^*(\eta_Y(W))$$

4. **Weak Lefschetz:**
Let $\iota : Z \hookrightarrow X$ be a smooth hyperplane section and $d = \dim X$. Then

$$\iota^* : H^i(X) \longrightarrow H^i(Z)$$

is bijective for $i \leq d-2$ and injective for $i \leq d-1$

5. **Hard Lefschetz:**
Let $Z \subset X$ be any hyperplane section and $\omega = \eta_X(W) \in H^2(X)$ its cohomology class. Then

$$H^i(X) \longrightarrow H^{2n-i}(X), \qquad \alpha \longmapsto \omega^{d-2i}\alpha$$

is bijective for all $0 \leq i \leq d$

Remark 1.8. Using Axiom 3 c), one can show that η_X descends to a ring homomorphism $\eta_X \colon Z^*_{rat}(X) \to H^{2*}(X)$ for all X.

Example 1.9. Betti cohomology, which is singular cohomology with coefficients in $\mathbb{Q}$ on complex varieties, is the prototypical example of a Weil cohomology theory. The cycle map is most easily defined as follows: Let $\iota \colon Z \hookrightarrow X$ be a subvariety and $\pi \colon \hat{Z} \to Z$ a resolution of singularities [Hi64], meaning a birational morphism whith $\hat{Z}$ a smooth projective variety. Then we define $\eta_X(Z) \coloneqq (\iota \circ \pi)_*(1)$. One can show that $\eta_X(Z) \in H^{2i}(X, \mathbb{Q}) \cap H^{i,i}(X) =: H^{i,i}(X, \mathbb{Q})$. See [GH94, p.61] and [Vo02, Chapter IV] for the details as well as other constructions.

For the rest of this section, let H^* be a fixed Weil cohomology theory.

Definition 1.10. Let X be a smooth projective variety. Two cycles $a, b \in Z^*(X)$ are called *homologically equivalent*, if $\eta_X(a) = \eta_X(b)$.

Remark 1.11. One can prove the following implications:

rational equivalence $\Longrightarrow$ homological equivalence $\Longrightarrow$ numerical equivalence

Definition 1.12. Let X be smooth projective. The image of

$$\eta_X \otimes \mathbb{Q} : Z^*(X) \otimes \mathbb{Q} \longrightarrow H^{2*}(X)$$

is denoted $Alg^*(X)$ and its elements are called *algebraic cohomology classes.*

Remark 1.13. Note that $Alg^*(X)$ is *only* a $\mathbb{Q}$-vector space. In particular, if $K \neq \mathbb{Q}$, then there is no guarantee that it is finite dimensional!

We can now formulate one of the standard conjectures for H^*

Conjecture D. Let X be a smooth projective variety of dimension d and $0 \leq i \leq \frac{d}{2}$ an integer, the following equivalent conditions hold:

1. homological equivalence and algebraic equivalence agree for cycles in $Z^i(X)$.
2. the pairing $Alg^i(X) \times Alg^{d-i}(X) \to \mathbb{Q}$ induced by the pairing on $H^*(X)$ is non-degenerate for all $0 \leq i \leq d$

Remark 1.14. (a) Using Axiom 3 and Remark 1.8 we see that the intersection pairings on Chow groups and Weil cohomology are compatible, i.e.

$$\begin{tikzcd} Z^i_{rat}(X) \times Z^{n-i}_{rat}(X) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d, hook] \\ H^{2i}(X) \times H^{2n-2i}(X) \arrow[r] & K \end{tikzcd}$$

commutes hence the above pairing does indeed have image in $\mathbb{Q}$.

(b) The equivalence follows from Remark 1.6 and the above diagram. In fact, the equivalence holds for each degree individually.

From now on, we will write $X \subseteq \mathbb{P}^N$ in order to indicate a choice of projective embedding.

Definition 1.15. Let $X \subseteq \mathbb{P}^N$ be smooth projective of dimension d and $\omega \in H^2(X)$ the class of the hyperplane section. The *Lefschetz operator* $L : H^*(X) \to H^{*+2}(X)$ is defined as multiplication with ω. A class $\alpha \in H^i(X)$ for $0 \leq i \leq n$ is called *primitive* if $L^{d-i+1}\alpha = 0$. We denote $P^i(X) := \left\{\alpha \in H^i(X) \mid L^{d-i+1}\alpha = 0\right\}$

Remark 1.16. (a) All hyperplanes in $\mathbb{P}^N$ are rationally equivalent. One can see this by letting the coefficients of the defining linear equation vary. It follows that the class ω is indeed well-defined. However the classes corresponding to different projective embeddings might not be the same. This is, why we have to fix a projective embedding.

(b) Analogous to the Lefschetz decomposition in complex geometry, one has a unique decomposition for every class $\alpha \in H^i(X)$:

$$\alpha = \sum_{j \geq i_0} L^j \alpha_j \text{ with } \alpha_j \in P^{i-2j}(X) \text{ primitive and } i_0 = \max(i-d, 0)$$

the proof uses the Hard Lefschetz axiom and is purely formal.

Definition 1.17. For any class $\alpha \in H^i(X)$ with primitive decomposition $\alpha = \sum_{j \geq i_0} L^j \alpha_j$ we set

$$\Lambda\alpha := \sum_{j \geq i_0} L^{j-1}\alpha_j$$

The operator Λ is called the *dual Lefschetz operator*.

It turns out that indeed any operator (i.e. any K-linear function) on cohomology groups has a corresponding cohomology class

Lemma 1.18. Let X and Y be smooth projective. The map

$$\begin{aligned} H^*(X \times Y) &\longrightarrow Hom_K(H^*(X), H^*(Y)) \\ \alpha &\longmapsto \alpha^* \end{aligned}$$

sending α to the composite

$$H^*(X) \xrightarrow{\pi_1^*} H^*(X \times Y) \xrightarrow{\alpha.-} H^*(X \times Y) \xrightarrow{(\pi_2)_*} H^*(Y)$$

is an isomorphism and when $\alpha \in H^*(X \times Y)$, then α^* has degree $i - 2\dim X$.

Proof. One can check that $(\ )^*$ is just the composite

$$H^i(X \times Y) \xrightarrow{\cong} \bigoplus_{j=0}^{i} H^{i-j}(X) \otimes_K H^j(Y) \xrightarrow{\cong} \bigoplus_{j=0}^{i} \Big(H^{2d-i+j}(X)^\vee \otimes_K H^j(Y)\Big)$$

and the right hand side equals $\bigoplus_{j=0}^{i} Hom_K(H^{2d-i+j}(X), H^j(Y))$. □

Example 1.19. Let $\Delta \in H^{2n}(X \times X)$ be the cycle of the diagonal. By Künneth, we can write

$$\Delta = \sum_{i=0}^{2n} \pi^i \text{ with } \pi^i \in H^{2n-i}(X) \otimes H^i(X)$$

and one can see that π^i corresponds to the projection

$$H^*(X) \twoheadrightarrow H^i(X) \hookrightarrow H^*(X)$$

to the i-th component.

Definition 1.20. An operator $F : H^*(X) \to H^*(Y)$ is called *algebraic* if its corresponding cohomology class in $H^*(X \times Y)$ is algebraic.

Remark 1.21. Let X, Y and Z be smooth projective and $\alpha \in H^*(X \times Y), \beta \in H^*(Y \times Z)$ cohomology classes. One can show that the operator $\beta^* \circ \alpha^*$ is represented by

$$(\pi_{1,3})_*(\pi_{1,2}^*\alpha.\pi_{1,3}^*\beta) \in H^*(X \times Z)$$

In particular, the composition of two algebraic operators is also algebraic.

Remark 1.22. Since pushforward, pullback and multiplication with an algebraic cycle all preserve algebraic cycles, it follows that any algebraic operator does too.

We can now state the rest of the standard conjectures.

Conjecture B. For any smooth projective variety $X \subseteq \mathbb{P}^N$ with fixed projective embedding: Λ is algebraic

Using the previous remark we see that Conjecture B implies:

Conjecture A. For any smooth projective variety $X \subseteq \mathbb{P}^N$ of dimension d, $0 \leq i \leq \frac{d}{2}$ the following equivalent statements hold:

1. $L^{d-2i} : Alg^i(X) \to Alg^{d-i}(X)$ is an isomorphism
2. $\Lambda^{d-2i} : H^{2d-2i}(X) \to H^{2i}(X)$ preserves algebraic cycles

Remark 1.23. (a) Kleiman [Kl91] shows that $A(X \times X)$ implies $A(X)$. The projective embedding of $X \times X$ used here comes from the embedding of X and the Segre embedding.

(b) Using our Weil Cohomology theory H^* it is easy to see that $Z^i_{num}(X)$ is free of rank $\leq \dim H^{2i}(X)$. If Conjecture D holds for X, this implies that $Alg^*(X)$ is finite-dimensional and $\dim Alg^i(X) = \dim Alg^{d-i}(X)$ for all i, hence Conjecture A holds for X.

Conjecture B also can be shown to imply:

Conjecture C. For any smooth projective variety $X \subseteq \mathbb{P}^N$, the projections

$$H^*(X) \twoheadrightarrow H^i(X) \hookrightarrow H^*(X)$$

are algebraic

To summarize, we have the following implications

$$D \implies A \iff B \implies C$$

In fact, the conjectures A, B and C are usually put under the umbrella term *Conjecture 1* or *Lefschetz standard conjecture.* The second of Grothendiecks conjectures is the Hodge Standard Conjecture:

Hodge Standard Conjecture. Let $X \subset \mathbb{P}^N$ be a smooth projective variety and $\omega \in H^2(X)$ the class of the hyperplane section. Let $P^i_{alg}(X) := P^{2i}(X) \cap Alg^i(X)$. Then the pairing:

$$\begin{aligned} P^i_{alg}(X) \times P^i_{alg}(X) &\longrightarrow \mathbb{Q} \\ (\alpha, \beta) &\longmapsto (-1)^i \langle \alpha.\beta.\omega^{n-2i} \rangle \end{aligned}$$

is positive definite.

Lemma 1.24. If $k = \mathbb{C}$ and H^* is betti cohomology, the Hodge standard Conjecture holds.

Proof. Indeed, let $\alpha \in Alg^i(X) \subseteq H^{i,i}(X, \mathbb{Q})$ be a primitive algebraic class. The Hodge–Riemann bilinear relation [Hu05, Proposition 3.3.15] proves that

$$(-1)^i \langle \alpha^2.\omega^{n-2i} \rangle = \frac{1}{Vol(X)} (-1)^{\frac{2i(2i-1)}{2}} \int_X \alpha \wedge \bar{\alpha} \wedge \omega^{n-2i} > 0$$

□

Theorem 1.25. [Kl68, Proposition 3.8] Assuming the Hodge standard conjecure: If Conjecture A holds for $X \subseteq \mathbb{P}^N$ in degree up to i, then Conjecture D holds for X in degree up to i as well.

Thus, assuming the Hodge standard conjecture, we have

$$D \iff A \iff B \implies C$$

1.3. The case $k = \mathbb{C}$

From now on, we will only deal with varieties over $\mathbb{C}$ and assume our Weil cohomology to be singular cohomology with coefficients in $\mathbb{Q}$. The Hodge standard conjecture holds in this case, so the conjectures A, B and D are equivalent and all imply C. We have noted before that $Alg^i(X) \subseteq H^{2i}(,\mathbb{Q}) \cap H^{i,i}(X) =: H^{i,i}(X,\mathbb{Q})$. The famous Hodge Conjecture asserts the converse:

Hodge Conjecture. For any smooth complex projective variety of dimension d:

$$Alg^i(X) = H^{i,i}(X,\mathbb{Q}) \text{ for all } 0 \leq i \leq d$$

Remark 1.26. (a) the case $i = 0$ is of course trivial. The case $i = 1$ can be proved using the exponential sequence and is called the *Lefschetz (1,1)-theorem* (Proposition 3.3.2 in [Hu05]). However all other cases are far from being proved.

(b) The conjecture is known to fail if one replaces $\mathbb{Q}$ with $\mathbb{Z}$ coefficients. Even if one considers both groups modulo torsion. See [Ko92] for a counterexample.

Proposition 1.27. Let X be smooth projective. If the Hodge conjecture holds in degree i, then Conjecture A also holds in degree i.

Proof. Let $\omega \in H^{1,1}(X,\mathbb{Q})$ be an ample class. Indeed,

$$L^{n-2i} : H^{2i}(X,\mathbb{Q}) \longrightarrow H^{2n-2i}(X,\mathbb{Q}) \text{ and } L^{n-2i} : H^{2i}(X,\mathbb{C}) \longrightarrow H^{2n-2i}(X,\mathbb{C})$$

are isomorphisms because of Hard Lefschetz. Since ω has bidegree $(1,1)$, the second map is even an isomorphism of Hodge structures. Thus

$$L^{n-2i} : Alg^i(X) = H^{i,i}(X,\mathbb{Q}) \cap H^{i,i}(X) \longrightarrow H^{2n-2i}(X,\mathbb{Q}) \cap H^{n-i,n-i}(X) = Alg^{n-i}(X)$$

is an isomorphism. □

Since the Hodge conjecture always holds in degree up to 1, Theorem 1.25 and Remark 1.26 yield the following

Theorem 1.28. Let X be smooth projective of dimension n and $\alpha, \beta \in Z^1(X)$ divisors on X. Then α and β are numerically equivalent if and only if they are homologically equivalent.

2. Lefschetz algebras

Throughout this chapter let X be a smooth complex projective variety of dimension d with a hyperplane class $\omega \in H^2(X, \mathbb{Q})$.

Lemma 2.1. The following subspaces of $H^2(X, \mathbb{Q})$ are equal

1. the space of divisor classes $Alg^1(X)$
2. $H^{1,1}(X, \mathbb{Q}) = H^2(X, \mathbb{Q}) \cap H^{1,1}(X)$
3. $NS(X) \otimes_{\mathbb{Z}} \mathbb{Q}$,

where $NS(X)$ is the image of $c_1 \colon Pic(X) \to H^2(X, \mathbb{Z})$.

Proof. The equality $NS(X) \otimes_{\mathbb{Z}} \mathbb{Q} = H^{1,1}(X, \mathbb{Q})$ results from the Lefschetz $(1,1)$-theorem [Hu05, Prop 3.3.2]. For any divisor $D \subset X$, we have $c_1(\mathcal{O}(D)) = [D]$ [Hu05, Prop 4.4.13], where $\mathcal{O}(D)$ is constructed in [Hu05, Prop 2.3.10]. Finally, [Hu05, Corollary 5.3.7] proves $Alg^1(X) = NS(X)$. □

Definition 2.2. The *Lefschetz algebra* $L^*(X) \subset H^*(X, \mathbb{Q})$ of X is the $\mathbb{Q}$-subalgebra generated by any one of these spaces. Its elements are called *Lefschetz classes*. We write $l^k(X) = \dim L^k(X)$

Remark 2.3. (a) The morphism $c_1 \colon Pic(X) \to H^2(X, \mathbb{Z})$ is the first chern class. One way to obtain it is to look at the *exponential sequence*

$$0 \longrightarrow \mathbb{Z}_X \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{exp} \mathcal{O}_X^{\times} \longrightarrow 0$$

where $\mathbb{Z}_X$ is the locally constant sheaf associated to $\mathbb{Z}$. This induces a long exact sequence in sheaf cohomology. Then, c_1 is just the boundary homomorphism:

$$c_1 \colon Pic(X) \cong H^1(X, \mathcal{O}_X^{\times}) \longrightarrow H^2(X, \mathbb{Z}_X) \cong H^2(X, \mathbb{Z}).$$

The group $NS(X)$ is called the *Neron–Severi group* and its rank $\rho(X) = \mathrm{rk} NS(X) = \dim L^1(X)$ is the *Picard number* of X.

(b) Let $f \colon X \to Y$ be a morphism of smooth projective varieties, then f^* sends Lefschetz classes to Lefschetz classes. This also follows from the commutativity of:

$$\begin{tikzcd} Pic(Y) \arrow[r, "f^*"] \arrow[d, "c_1"] & Pic(X) \arrow[d, "c_1"] \\ H^2(Y, \mathbb{Z}) \arrow[r, "f^*"] & H^2(X, \mathbb{Z}) \end{tikzcd}$$

However f_* does *not* preserve Lefschetz classes in general. Indeed, if $[Z] \in H^*(X, \mathbb{Q})$ is the cohomology class of a subvariety $\iota\colon Z \hookrightarrow X$ and $\pi\colon \hat{Z} \to Z$ is a Hironaka resolution, then $[Z] = (\iota \circ \pi)_*(1)$. Or slightly less trivially, if Z is smooth of codimension k in X, $\pi\colon Y \to X$ the blow-up of X at Z and $e \in L^1(Y)$ the class of the exceptional divisor, then $\pi_*((-e)^n) = [Z]$ as is shown in the proof of Theorem A.2.

In general, $L^*(X)$ is a proper subalgebra of $Alg^*(X)$. Since conjecture A always holds in dimension 1, we have $Alg^{d-1}(X) = \omega^{d-2} Alg^1(X) = L^{d-1}(X)$. Hence any possible example of an X with $L^*(X) \subsetneqq Alg^*(X)$ has to be of dimension at least four. Indeed, we have an example in dimension four:

Example 2.4. The Grassmannnian $\mathcal{G}(2,4)$ is a smooth projective variety of dimension four. Its cohomology is given as

$$\begin{aligned}
H^0(\mathcal{G}(2,4), \mathbb{Q}) &= \mathbb{Q}1 \\
H^2(\mathcal{G}(2,4), \mathbb{Q}) &= \mathbb{Q}\sigma_{1,0} \\
H^4(\mathcal{G}(2,4), \mathbb{Q}) &= \mathbb{Q}\sigma_{2,0} \oplus \mathbb{Q}\sigma_{1,1} \\
H^6(\mathcal{G}(2,4), \mathbb{Q}) &= \mathbb{Q}\sigma_{2,1} \\
H^8(\mathcal{G}(2,4), \mathbb{Q}) &= \mathbb{Q}\sigma_{2,2},
\end{aligned}$$

where σ_{a_1,a_2} is the Schubert cycle associated to (a_1, a_2) (see Appendix B). $L^*(\mathcal{G}(2,4))$ is generated by $\sigma_{1,0}$, so Proposition B.4 implies that $\sigma_{1,0}^2 = \sigma_{2,0} + \sigma_{1,1}$. Thus neither $\sigma_{2,0}$ nor $\sigma_{1,1}$ are Lefschetz classes. Indeed for any $0 \leq k \leq n$ we have $\rho(\mathcal{G}(k,n)) = 1$ and thus if $0 < k < n$, then $L^*(\mathcal{G}(k,n)) \subsetneqq Alg^*(\mathcal{G}(k,n))$.

We can now ask, what happens if the standard conjectures are reformulated and $Alg^*(X)$ is replaced by $L^*(X)$. More specifically, we consider the following:

$A_{\text{Lef}}(X)$: $L^{d-2k}\colon L^k(X) \to L^{d-k}(X)$ is an isomorphism for all $0 \leq k \leq \frac{d}{2}$

$B_{\text{Lef}}(X)$: The associated class of Λ is a Lefschetz class

$C_{\text{Lef}}(X)$: $\pi^k \in L^*(X)$ for all $0 \leq k \leq 2d$

$D_{\text{Lef}}(X)$: The pairing $L^k(X) \times L^{d-k}(X) \to \mathbb{Q}$ is nondegenerate for all $0 \leq k \leq \frac{d}{2}$

Here, we dropped the first equivalent version of Conjecture D since the equivalence fails.
Of the implications

$$A(X \times X) \implies B(X), \quad B(X) \implies A(X), \quad B(X) \implies C(X) \text{ and } D(X) \iff A(X)$$

only the last carries over to Lefschetz classes. In fact, $B(X) \implies A(X)$ uses the fact that the pushforward preserves algebraic classes. The same does not hold for Lefschetz classes. The two left ones depend on the fact that the composition of algebraic operators is still algebraic. Looking at the Remark 1.21, this fails for Lefschetz classes for the same reason. Hence, we will only continue with conjectures A_{Lef} and B_{Lef}. It is worth mentioning however that all of the above conjectures hold if X is abelian [Mi99]. In fact, Milne also shows that the pushforward along any morphism of abelian varieties preserves Lefschetz classes.

2.1. Theorems for Lefschetz algebras

For completeness sake, we give a proof of the equivalence

$$A_{\text{Lef}}(X) \iff D_{\text{Lef}}(X)$$

Theorem 2.5. Let $X \subseteq \mathbb{P}^N$ be a smooth complex projective variety, $\omega \in L^1(X)$ the cohomology class of a hyperplane section and $n \leq \frac{d}{2}$ a nonzero integer. Then the following conditions on $L^*(X)$ are equivalent:

$A_{\textbf{Lef}}(X, \leq n):$ $L^*(X)$ also satisfies the hard Lefschetz axiom, i.e. the map

$$L^k(X) \xrightarrow{\omega^{d-2k}} L^{d-k}(X)$$

is an isomorphism for every integer $0 \leq k \leq n$.

$D_{\textbf{Lef}}(X, \leq n):$ The multiplication map

$$L^k(X) \times L^{d-k}(X) \to L^d(X) \cong \mathbb{Q}$$

is a nondegenerate form for $0 \leq k \leq n$

$LD(X, \leq n):$ For any $0 \leq k \leq n$, we have a Lefschetz decomposition for $L^*(X)$:

$$L^k(X) = \bigoplus_{i=0}^{k} \omega^{k-i} PL^i(X)$$

where $PL^i(X) := PH^i(X) \cap L^i(X) = \operatorname{Ker}\left(\omega^{d-2i+1} : L^i(X) \to L^{d-i+1}(X)\right)$

$SD(X, \leq n):$ The Lefschetz algebra has symmetric dimensions, i.e.:

$$\dim L^k(X) = \dim L^{d-k}(X)$$

for all $0 \leq k \leq n$.

Proof. By the Hard Lefschetz theorem, we know that the map in $A_{\text{Lef}}(X, \leq n)$ is injective between finite dimensional $\mathbb{Q}$-vector spaces, so $A_{\text{Lef}}(X, \leq n)$ is equivalent to $SD(X, \leq n)$. Also, $D_{\text{Lef}}(X, \leq n)$ implies $SD(X, \leq n)$, so it suffices to check that $SD(X, \leq n)$ implies $LD(X, \leq n)$ and $LD(X, \leq n)$ implies $D_{\text{Lef}}(X, \leq n)$:
Let us consider the implication $SD(X, \leq n) \implies LD(X, \leq n)$. The inclusion "$\supseteq$" holds because of the ordinary Lefschetz decomposition [Hu05, Proposition 3.3.13], so it suffices to check that the dimensions of both spaces agree:

$$\begin{aligned}
&\dim \bigoplus_{i=0}^{k} \omega^{k-i} PL^i(X) = \sum_{i=0}^{k} \dim \omega^{k-i} PL^i(X) = \sum_{i=0}^{k} \dim PL^i(X) \\
&\geqslant \sum_{i=0}^{k} \left[\dim L^i(X) - \dim L^{d-i+1}(X)\right] = \sum_{i=0}^{k} \dim L^i(X) - \sum_{i=0}^{k} \dim L^{d-i+1}(X) \\
&\overset{SD}{=} \sum_{i=0}^{k} \dim L^i(X) - \sum_{i=0}^{k} \dim L^{i-1}(X) = \dim L^k(X),
\end{aligned}$$

hence equality holds as well.
Next, we consider the implication $LD(X, \leq n) \implies D_{\mathrm{Lef}}(X, \leq n)$. Let $x \in L^k(X)$ be nonzero and consider the Lefschetz decomposition $x = \sum_{i=0}^{k} \omega^{k-i} x_i$ with $x_i \in L^i(X)$. Some summand $\omega^{k-j} x_j \in H^{k,k}(X)$ has to be nonzero as well. Hence, the Hodge–Riemann bilinear relations tell us:

$$(-1)^j \int_X \omega^{d-j-k} x_j x = (-1)^j \int_X \omega^{d-2j} {x_j}^2 > 0$$

The first equality follows because for any $i \neq j$: $\omega^{d-j-i} x_j x_i = 0$ since x_i and x_j are primitive. Thus, only one summand remains. Hence, $\omega^{d-j-k} x_j x \neq 0$ and we have $D_{\mathrm{Lef}}(X, \leq n)$.

□

Remark 2.6. The dimension argument that we used in the implication $\mathrm{SD}(X, \leq n) \implies \mathrm{LD}(X, \leq n)$ is the reason, why this proof cannot show equivalence in each separate degree. There seems to be no way to remedy this. In fact, the author of this thesis knows of no variety X satisfying $\mathrm{SD}(X, n)$ but not $\mathrm{SD}(X, \leq n)$ for some integer $n < \frac{d}{2}$! For more on this, see Remark 2.25. From now on, we will only consider SD.

Proposition 2.7. For any nonzero integer $k \leq \frac{d}{2}$, the map

$$L^k(X) \longrightarrow L^{d-k}(X), \qquad x \longmapsto \omega^{d-2k} x$$

is injective and for $k = 0, 1$ even bijective.

Proof. The first part follows from the Hard Lefschetz theorem for cohomology [Hu05, Proposition 3.3.13]. For the second part, we use $L^1(X) = H^{1,1}(X, \mathbb{Q})$ from which the bijectivity follows as in the proof of Proposition 1.27. □

Corollary 2.8. Let X be smooth projective. $SD(X)$ holds if one of the following is satisfied

1. X is abelian
2. X has dimension $d \leq 4$
3. $L^*(X) = H^*(X)$
4. X has Picard number $\rho(X) = 1$

In particular, all Grassmannians satisfy SD.

Proof. 1. was proved by Milne [Mi99, Proposition 5.2]. If X has dimension at most four, then this follows from the previous Proposition. The second point follows from the Hard Lefschetz Theorem and the third is trivial. For Grassmannians, this is Example 2.4. □

2.2. Validity criteria for SD(X)

We now examine various ways of obtaining new varieties out of old ones and see what we can say about the Lefschetz algebra in this case. One of the surprisingly useful facts about the Lefschetz algebra in contrast to $Alg^*(X)$ is the fact that it is generated in degree one. This fact will be exploited throughout this section. This section for the most part follows [HW17].

Proposition 2.9. Let $\iota\colon D \hookrightarrow X$ be a smooth ample hypersurface.

1. If $d \geq 4$, then $\iota^*\colon L^*(X) \to L^*(D)$ is surjective and $\iota_*\colon H^*(D,\mathbb{Q}) \to H^*(X,\mathbb{Q})$ sends Lefschetz classes to Lefschetz classes.
2. For any nonnegative integer $n < \frac{d}{2}$:

$$SD(X,n) \implies SD(D,n)$$

Proof. Let us first consider 1. The exponential sequence is natural in the sense that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_X \arrow[r, "2\pi i"] \arrow[d, "\iota^*"] & \mathcal{O}_X \arrow[r, "exp"] \arrow[d, "\iota^*"] & \mathcal{O}_X^\times \arrow[r] \arrow[d, "\iota^*"] & 0 \\
0 \arrow[r] & \iota_*\mathbb{Z}_X \arrow[r, "2\pi i"] & \iota_*\mathcal{O}_X \arrow[r, "exp"] & \iota_*\mathcal{O}_X^\times \arrow[r] & 0
\end{tikzcd}$$

commutes. This induces a morphism of long exact cohomology sequences. Part of which is

$$\begin{tikzcd}
... \arrow[r] & Pic(X) = H^1(X,\mathcal{O}_X^\times) \arrow[r, "c_1"] \arrow[d, "\iota^*"] & H^2(X,\mathbb{Z}) \arrow[r] \arrow[d, "\iota^*", "\cong"'] & H^2(X,\mathcal{O}_X) \arrow[r] \arrow[d, "\iota^*"] & ... \\
... \arrow[r] & Pic(D) = H^1(D,\mathcal{O}_D^\times) \arrow[r, "c_1"] & H^2(D,\mathbb{Z}) \arrow[r] & H^2(D,\mathcal{O}_D) \arrow[r] & ...
\end{tikzcd}$$

The middle ι^* is an isomorphism because of $d \geq 4$ and the weak Lefschetz theorem [Hu05, Prop. 5.2.6]. Since $L^1(D)$ is generated by the image of the lower c_1, it suffices that ι^* maps $NS(X)$ surjectively onto $NS(D)$. By exactness, this reduces to $\iota^*\colon H^2(X,\mathcal{O}_X) \to H^2(D,\mathcal{O}_D)$ being injective. Indeed, we have the structure sheaf sequence [Hu05, Lemma 2.3.22]

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \xrightarrow{\iota^*} \iota_*\mathcal{O}_D \longrightarrow 0$$

which induces a long exact sequence

$$... \longrightarrow H^2(X,\mathcal{O}_X(-D)) \longrightarrow H^2(X,\mathcal{O}_X) \xrightarrow{\iota^*} H^2(X,\iota_*\mathcal{O}_D) = H^2(D,\mathcal{O}_D) \longrightarrow ...$$

Hence if we can show $H^2(X,\mathcal{O}_X(-D)) = 0$, the surjectivity part follows. And in fact, by Serre duality [Hu05, Cor. 4.1.16] we have $H^2(X,\mathcal{O}_X(-D)) \cong H^{d-2}(X,\Omega_X^d \otimes \mathcal{O}_X(D))^\vee$ and since $\mathcal{O}_X(D)$ is ample, the right hand side is zero by Kodaira vanishing [Hu05, Prop. 5.2.2] and the fact that $d > 2$. ι_* sends Lefschetz classes to Lefschetz classes since any class in $L^*(D)$ is $\iota^*\alpha$ for some class $\alpha \in L^*(X)$ and

$$\iota_*\iota^*\alpha = \iota_*(\iota^*(\alpha).1) = \alpha.\iota_*(1) = \alpha.\omega, \tag{2.1}$$

where $\omega \in L^1(X)$ is the hyperplane class associated to D. Let us now turn towards 2. If $d < 4$, then $SD(D)$ holds because of Proposition 2.8, so we may assume $d \geq 4$. Since the claim is obvious if $n = \frac{d-1}{2}$, we may even assume $n \leq \frac{d}{2} - 1$. It follows from weak Lefschetz and 1. that $\iota^*\colon L^n(X) \to L^n(D)$ is an isomorphism. Also, the pushforward

$$\iota_*\colon H^{2(d-1-n)}(D) \xrightarrow{\cong} \left(H^{2n}(D,\mathbb{Q})\right)^\vee \xrightarrow{\cong} \left(H^{2n}(X,\mathbb{Q})\right)^\vee \xrightarrow{\cong} H^{2d-2n}(X)$$

is an isomorphism. Equation (2.1) now yields

$$L^{d-1-n}(D) \cong \omega.L^{d-1-n}(X) = L^{d-n}(X) \cong L^n(X) \cong L^n(D)$$

The equality $\omega.L^{d-1-n}(X) = L^{d-n}(X)$ comes from the fact that $\omega^{d-2n}\colon L^n(X) \to L^{d-n}(X)$ and hence $\omega\colon L^{d-1-n}(X) \to L^{d-n}(X)$ is surjective. $\square$

Remark 2.10. The assumption $d \geq 4$ is indeed necessary for 1. Indeed if we take $X = \mathbb{P}^3$ and $D = V(x_0x_1 - x_2x_3) \subset \mathbb{P}^3$ the quadric surface, then D is smooth and ample since it is a hyperplane section of the Veronese embedding. $\mathbb{P}^3 \hookrightarrow \mathbb{P}^{\binom{5}{2}-1} = \mathbb{P}^9$. As $D \cong \mathbb{P}^1 \times \mathbb{P}^1$, $L^1(D)$ is a two-dimensional vector space while $L^1(\mathbb{P}^3)$ is one-dimensional.

Proposition 2.11. Let X_1 and X_2 be smooth projective varieties of dimension d_1 and d_2 with $H^1(X,\mathbb{Q}) = 0$. Then

$$L^*(X_1 \times X_2) \cong L^*(X_1) \otimes_{\mathbb{Q}} L^*(X_2).$$

For any $n \geq 0$, we have:

1. If $n \leq min(d_1, d_2)$, then $SD(X_1, \leq n)$ and $SD(X_2, \leq n)$ imply $SD(X_1 \times X_2, \leq n)$
2. If $n \leq d_i$, then $SD(X_i, \leq n)$ and $SD(X_{2-i})$ imply $SD(X_1 \times X_2, \leq n)$
3. $SD(X_1)$ and $SD(X_2)$ imply $SD(X_1 \times X_2)$

Proof. The cross product

$$H^*(X_1,\mathbb{Q}) \otimes_{\mathbb{Q}} H^*(X_2,\mathbb{Q}) \underset{\cong}{\xrightarrow{\times}} H^*(X_1 \times X_2,\mathbb{Q}), \qquad (x_1,x_2) \mapsto \pi_1^*(x_1) \cup \pi_2^*(x_2)$$

is an isomorphism by the Künneth theorem (see for example [Hat01, Theorem 3.15, Corollary A.8 and A.9]), so it remains to show that $L^1(X_1) \oplus L^1(X_2) = L^1(X_1 \times X_2)$. Because of the assumption, we have $H^2(X_1,\mathbb{Q}) \oplus H^2(X_2,\mathbb{Q}) = H^2(X_1 \times X_2,\mathbb{Q})$ and since this isomorphism is compatible with the Hodge structures, the Lefschetz $(1,1)$ theorem yields $L^1(X_1) \oplus L^1(X_2) = L^1(X_1 \times X_2)$. For all of the other assertions, the proof goes as follows: Let d_1 and d_2 be the dimensions of X_1 and X_2 and let $k \geq 0$ be such that $l^i(X_1) = l^{d_1-i}(X_1)$ and $l^i(X_2) = l^{d_2-i}(X_2)$ for all $i \leq k$. Then

$$\begin{aligned} l^{d_1+d_2-k}(X_1 \times X_2) &= \sum_{i=0}^{d_1+d_2-k} l^i(X_1) l^{d_1+d_2-k-i}(X_2) \\ &= \sum_{i=d_1-k}^{d_1} l^i(X_1) l^{d_1+d_2-k-i}(X_2) = \sum_{i=0}^{k} l^{d_1-k+i}(X_1) l^{d_2-i}(X_2) \\ &= \sum_{i=0}^{k} l^{k-i}(X_1) l^i(X_2) = l^k(X_1 \times X_2) \end{aligned}$$

□

The assumption $H^1(X, \mathbb{Q}) = 0$ is indeed necessary. At least for the first part of the Proposition, as the following example shows.

Example 2.12. Let C be an elliptic curve over $\mathbb{C}$ i.e. $C \cong \mathbb{C}/\Lambda$ for some full lattice $\Lambda \subset \mathbb{C}$. Then $H^1(X, \mathbb{Q})$ is of dimension 2 since C is diffeomorphic to $S^1 \times S^1$.

Claim. The cohomology class of the diagonal $\Delta \subset C \times C$ is not contained in $L^1(C) \oplus L^1(C)$.

Indeed, let

$$\begin{aligned} \mu \colon C \times C &\longrightarrow C \\ (a, b) &\longmapsto a - b \end{aligned}$$

It follows that $[\Delta] = \mu^*([0])$, hence $[\Delta]^2 = \mu^*([0]^2) = 0$. On the other hand, if we had $[\Delta] = \pi_1^* \alpha + \pi_2 * \beta$ for some $\alpha, \beta \in L^1(C)$, then it would follow that

$$\alpha = i_1^* \pi_1^*(\alpha) = i_1^*[\Delta] = i_1^* \mu^*[0] = [0],$$

where

$$\begin{aligned} i_1 \colon C &\longrightarrow C \times C \\ x &\longmapsto (x, 0) \end{aligned}$$

is the inclusion into the first factor.. Similarly, one shows $\beta = [0]$, hence $[\Delta]^2 = 2[0] \times [0]$ which is nonzero by Künneth. We arrive at a contradiction.

Remark 2.13.

It can be shown somewhat similarly that $L^1(A) \oplus L^1(A) \subsetneqq L^1(A \times A)$ for any abelian variety A and thus $L^*(A \times A) \ncong L^*(A) \otimes_{\mathbb{Q}} L^*(X)$. But since $A \times A$ is an abelian variety, we still have $SD(A \times A)$.

Proposition 2.14. [GH94, p. 606] Let $E \xrightarrow{\pi} X$ be a complex vector bundle of rank n and

$$p \colon \mathbb{P}(E) \longrightarrow X$$

its associated projective bundle. The Cohomology $H^*(\mathbb{P}(E), \mathbb{Q})$ is generated by $\zeta = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ as an $H^*(X, \mathbb{Q})$-algebra with the only relation

$$\zeta^n = -\sum_{i=0}^{n-1} p^* c_{n-i}(E) \zeta^i$$

Corollary 2.15. We have $L^1(\mathbb{P}(E)) = L^1(X) \oplus \mathbb{Q}\zeta$ and if the total chern class

$$c(E) = \sum_{i=0}^{n} c_i(E) \in H^{2*}(X, \mathbb{Q})$$

is a Lefschetz class, then we have

$$L^*(\mathbb{P}(E)) = \bigoplus_{i=0}^{n-1} L^{*-i}(X) \otimes \mathbb{Q}\zeta^i$$

In this case, $SD(X, \leq k)$ implies $SD(\mathbb{P}(E), \leq n)$ for any nonnegative $k \leq \frac{d}{2}$ and $SD(X)$ implies $SD(\mathbb{P}(E))$

Proof. The inclusion $L^1(\mathbb{P}(E)) \supseteq L^1(X) \oplus \mathbb{Q}\zeta$ is trivial. On the other hand, let $\alpha \in L^1(\mathbb{P}(E))$. By Proposition 2.14 we can write $\alpha = p^*\beta + a\zeta$ for some $\beta \in L^2(\mathbb{P}(E))$ and $a \in \mathbb{Q}$. Since p_* preserves algebraic classes, we get

$$p_*(\zeta^{n-1}p^*\alpha + a\zeta^n) = p_*(\zeta^{n-1})\alpha - ap_*\big(\sum_{i=0}^{n-1} p^*c_{n-i}(E)\zeta^i\big) = \alpha - ac_1(N_{Z/X}) \in Alg^1(X) = L^1(X)$$

For the last equality we used Lemma A.3. For the second claim note that the right subspace is closed under multiplication and is generated by $L^1(X) \oplus \mathbb{Q}\zeta = L^1(\mathbb{P}(E))$. The third claim follows as in Proposition 2.11. In fact, $L^*(\mathbb{P}(E)) \cong L^*(X \times \mathbb{P}^{n-1})$ as graded vector spaces. $\square$

Proposition 2.16. For any pair of integers $1 \leq k \leq n$, the flag variety of the form:

$$X_{k,n} = \left\{ 0 \subset V_1 \subset ... \subset V_k \subset \mathbb{C}^n \mid V_i \subset \mathbb{C}^n \text{ linear subspace with } \dim V_i = i \right\}$$

satisfies SD.

Proof. We show that $L^*(X_{k,n}) = H^*(X_{k,n}, \mathbb{Q})$. For this, we fix n and do induction on $0 \leq k \leq n$. The case $k = 0$ is trivial. Assume $k < n$ and $L^*(X_{k,n}) = H^*(X_{k,n}, \mathbb{Q})$. Let $E \coloneqq X_{k,n} \times \mathbb{C}^n/T$, where T is the subbundle $T \coloneqq \big\{(V_1, ..., V_k, v) \in X_{k,n} \times \mathbb{C}^n \mid v \in V_k\big\}$. We now have a map

$$\begin{aligned} \mathbb{P}(E) &\xrightarrow{\cong} X_{k+1,n} \\ (V_1, ..., V_k, \mathbb{C}[v]) &\longmapsto (V_1, ..., V_k, V_k + \mathbb{C}v) \end{aligned}$$

which is easily seen to be well-defined and an isomorphism of algebraic varieties. It follows from Proposition 2.14 that $L^*(X_{k+1,n}) = H^*(X_{k+1,n}, \mathbb{Q})$. $\square$

Next, we consider the Lefschetz algebra of the blowup $Bl_Z X$ of a smooth subvariety $\iota\colon Z \hookrightarrow X$ of codimension k.

Proposition 2.17. Let $N_{Z/X}$ denote the normal bundle of Z in X and $e \in L^1(Bl_Z X)$ the cohomology class of the exceptional divisor. We have $L^1(Bl_Z X) = \pi^*L^1(X) \oplus \mathbb{Q}e$. Further, if

1. $c_i(NZ/X) \in \iota^* L^i(X)$ for all integers i with $1 \leq i \leq k-1$
2. and $[Z] \in L^k(X)$,

then we have a decomposition of graded vector spaces:

$$L^*(Bl_Z X) \cong L^*(X) \oplus \bigoplus_{i=1}^{k-1} \iota^* L^{*-i}(X) \otimes \mathbb{Q}e^i$$

In particular, if $\iota^*\colon L^*(X) \to L^*(Z)$ is surjective, then $SD(X,n)$ and $SD(Z, \leq i)$ for all $n-k+1 \leq i \leq n-1$ together imply $SD(Bl_Z X, n)$.

Proof. We use the decomposition in Proposition A.2 as well as its proof. The inclusion $L^1(Bl_Z X) \supseteq \pi^* L^1(X) \oplus \mathbb{Q}e$ is clear. For the other inclusion, let $\alpha \in L^1(Bl_Z X)$ and $\alpha = \pi^*\beta + ae$ for $\beta \in L^1(X)$ and $a \in \mathbb{Q}$. Then

$$\pi_*\alpha = \pi_*(\pi^*\beta + ae) = \beta$$

is in $Alg^1(X) = L^1(X)$. The second assertion follows from the decomposition in Proposition A.2 and the relations given there. The third is trivial. $\square$

Perhaps more concisely, we have the following consequence:

Corollary 2.18. Let $d \geq 4$ and $Z \subset X$ be obtained by $k \leq d-3$ repeated smooth hyperplane sections. We have

$$L^*(Bl_Z X) \cong L^*(X) \oplus \bigoplus_{i=1}^{k-1} L^{*-i}(Z) \otimes \mathbb{Q}e^i$$

and $SD(X, \leq n)$ implies $SD(Bl_Z X, \leq n)$ for all nonnegative $k \leq \frac{d}{2}$

Proof. Let $Z = Z_k \overset{i_k}{\hookrightarrow} Z_{k-1} \overset{i_{k-1}}{\hookrightarrow} ... \overset{i_2}{\hookrightarrow} Z_1 \overset{i_1}{\hookrightarrow} Z_0 = X$ be a sequence of subvarieties each one a smooth hyperplane section of the next. In order to show the first condition in Proposition 2.17, we consider the normal bundle sequences:

$$0 \longrightarrow T_Z \longrightarrow T_X|_Z \longrightarrow N_{Z/X} \longrightarrow 0$$

and

$$0 \longrightarrow T_{Z_s} \longrightarrow T_{Z_{s-1}}|_{Z_s} \longrightarrow N_{Z_s/Z_{s-1}} \longrightarrow 0$$

for $1 \leq s \leq k$. Whitney's formula [EH16, Theorem 5.3(c)] proves

$$c(N_{Z/X}) = \iota^* c(T_X) c(T_Z)^{-1} \text{ and } c(N_{Z_s/Z_{s-1}}) = i_s c(T_{Z_{s-1}}) c(T_{Z_s})^{-1} \text{ for } 1 \leq s \leq k.$$

And thus

$$c(N_{Z/X}) = \iota^* c(T_{Z_0}) c(T_{Z_k})^{-1} = (i_2 \circ ... \circ i_k)^* c(N_{Z_1/Z_0}) ... i_{k-1}^* c(N_{Z_{k-1}/Z_{k-2}}) c(N_{Z_k/Z_{k-1}}) \in L^*(Z)$$

since the normal bundle of i_s is a line bundle, $c(N_{Z_s/Z_{s-1}}) = 1 + c_1(N_{Z_s/Z_{s-1}}) \in L^*(Z_s)$. Note that a total chern class is always an invertible element since the sum of the unit and a nilpotent element is invertible in any commutative ring. The first part of Proposition 2.9 is applicable to each hyperplane section, so we obtain the surjectivity of $\iota^*\colon L^*(X) \to L^*(Z)$. Since each $(i_s)_*$ sends Lefschetz classes to Lefschetz classes, we finally obtain

$$[Z] = \iota_*(1) = i_{1*}i_{2*}...i_{k*}(1) \in L^k(X)$$

□

Example 2.19. Let $X \subset \mathbb{P}^3$ be a cubic surface. One can show that X is the blowup of $\mathbb{P}^2$ at six points [GH94, p.480-489]. Using the very first assertion in Proposition 2.17, we see that

$$l^0(X) = 1 \quad , \quad l^1(X) = 7 \quad , \quad l^2(X) = 1$$

and thus, X has Picard number $\rho(X) = 7$.

Example 2.20. Recall the *Veronese embedding* for $1 \leq d \leq n$, $N = \binom{n+d}{d} - 1$:

$$\begin{aligned} \iota\colon Z = \mathbb{P}^n &\longrightarrow \mathbb{P}^N = X \\ [x_i] &\longmapsto [x_1^{n_1}...x_d^{n_d}]_{n_1+...+n_d=d} \end{aligned}$$

Since $\iota^*\mathcal{O}_{\mathbb{P}^N}(1) \cong \mathcal{O}_{\mathbb{P}^n}(d)$, ι^* is surjective. Further, we have $L^*(Z) = H^*(Z)$ and $L^*(X) = H^*(X)$. Thus Proposition 2.17 gives us $L^*(Bl_Z X) = H^*(Bl_Z X)$. In particular, $SD(Bl_Z X)$ holds true.

2.3. Counterexamples

SD holds trivially up to dimension 4, but not for higher dimensions. In fact, here is a 5-dimensional example:

Let $C \subset \mathbb{P}^2$ be a smooth plane cubic curve. Then we can embedd $C \times \mathbb{P}^1 \subset \mathbb{P}^5$ via

$$Z = C \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5 = Y,$$

where the second map is the Segre embedding. Let X be the blow-up of Y along Z.

Proposition 2.21. We have $l^2(X) = 3$ and $l^3(X) = 4$.

Proof. By the Künneth formula, we have $H^2(Z;\mathbb{Q}) \simeq H^2(C;\mathbb{Q}) \oplus H^2(\mathbb{P}^1;\mathbb{Q}) = \mathbb{Q}\,a \oplus \mathbb{Q}\,b$, where $a = [C \times \text{point}]$ and $b = [\text{point} \times \mathbb{P}^1]$ come from the canonical generators in top degree of the two factors. Let $c = [H]$ be the cohomology class of a hyperplane $H \subset \mathbb{P}^5$. Then $H^*(\mathbb{P}^5;\mathbb{Q}) = \mathbb{Q}[c] \simeq \mathbb{Q}[T]/(T^6)$. So by Proposition A.2, we have:

$$\begin{aligned} H^0(X;\mathbb{Q}) &= \mathbb{Q}\,1, \\ H^2(X;\mathbb{Q}) &= \mathbb{Q}\,c \oplus \big(\mathbb{Q}\,1\big)e, \\ H^4(X;\mathbb{Q}) &= \mathbb{Q}\,c^2 \oplus \big(\mathbb{Q}\,a \oplus \mathbb{Q}\,b\big)e \oplus \big(\mathbb{Q}\,1\big)e^2, \\ H^6(X;\mathbb{Q}) &= \mathbb{Q}\,c^3 \oplus \big(\mathbb{Q}\,ab\big)e \oplus \big(\mathbb{Q}\,a \oplus \mathbb{Q}\,b\big)e^2, \\ H^8(X;\mathbb{Q}) &= \mathbb{Q}\,c^4 \oplus \big(\mathbb{Q}\,ab\big)e^2, \\ H^{10}(X;\mathbb{Q}) &= \mathbb{Q}\,c^5, \end{aligned}$$

where e is the cohomology class of the exceptional divisor in X.

The algebra $L^*(X)$ is generated by c and e since $H^2(X;\mathbb{Q})$ is. The restriction of $c = [H] = [V(x_0)] = [H_0]$ to $\mathbb{P}^2 \times \mathbb{P}^1$ is $[H_0 \times \mathbb{P}^1 \cup \mathbb{P}^2 \times H_0] = [H_0 \times \mathbb{P}^1] + [\mathbb{P}^2 \times \mathit{point}]$ and this in turn restricts to $[H_0 \cap C \times \mathbb{P}^1] + [C \times \mathit{point}] = 3b + a$. Hence

$$L^2(X) = \mathbb{Q}\, c^2 \oplus \mathbb{Q}\, ce \oplus \mathbb{Q}\, e^2 = \mathbb{Q}\, c^2 \oplus \mathbb{Q}(a+3b)e \oplus \mathbb{Q}\, e^2.$$

This proves the first claim.

We next show $L^3(X) = H^6(X;\mathbb{Q})$. It is enough to check that e^3 is not in the subspace

$$V = \mathbb{Q}\, c^3 \oplus \mathbb{Q}\, c^2 e \oplus \mathbb{Q}\, ce^2 = \mathbb{Q}\, c^3 \oplus \mathbb{Q}\,(ab)e \oplus \mathbb{Q}\,(a+3b)e^2 \subseteq H^6(X;\mathbb{Q}).$$

According to our description of the cohomology ring of the blow-up, the following relation holds in the cohomology of X:

$$e^3 = -6c^3 - c_2(N_{Z/Y})\, e + c_1(N_{Z/Y})\, e^2 = -6c^3 - c_2(N_{Z/Y})\, e + c_1(T_Y)\, e^2 - c_1(T_Z)\, e^2.$$

Since $c_1(T_Y)\, e^2$ is a multiple of ce^2 and $c_2(N_{Z/Y})\, e$ is a multiple of abe, we have

$$e^3 = -c_1(T_Z)\, e^2 \mod V$$

where we are using the normal bundle sequence [Hu05, Definition 2.2.16]

$$0 \longrightarrow T_Z \longrightarrow T_Y|_Z \longrightarrow N_{Z/Y} \longrightarrow 0$$

and Whitney's formula[EH16, Theorem 5.3(c)] to see that $c_1(N_{Z/Y}) = c_1(T_Y|_Z) - c_1(T_Z)$. Since C is a 1-dimensional complex torus, its tangent bundle is trivial, so $c_1(T_Z)\, e^2$ must be a multiple of ae^2. It follows that e^3 is not contained in V. This proves the second claim. □

In fact, a counterexample of dimension six can easily be constructed from the above X. Using Proposition 2.11 we get

$$l^2(X \times \mathbb{P}^1) = l^2(X) + l^1(X), l^4(X \times \mathbb{P}^1) = l^4(X) + l^3(X)$$

and since $l^1(X) = l^4(X)$, it follows that $l^2(X \times \mathbb{P}^1) < l^4(X \times \mathbb{P}^1)$. Using this inductively, we arrive at the following Theorem:

Theorem 2.22. For any integer $d \geq 5$ there is a smooth complex projective variety of dimension d with

$$l^2(X) < l^{d-2}(X)$$

In fact, in higher dimensions, there somewhat simpler constructions of counterexamples. For the next one, it turns out that $l^2(X) = l^{d-2}(X)$ but $l^3(X) \neq l^{d-3}(X)$

Proposition 2.23. Let X be the blow-up at the Segre embedding

$$Z = \mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8 = Y$$

Then $l^3(X) = 4$ and $l^5(X) = 5$.

Proof. Let $c \in L^1(\mathbb{P}^2)$ and $c_3 \in L^1(\mathbb{P}^8)$ be the hyperplane classes and $c_i = \pi_i^* c \in L^1(\mathbb{P}^2 \times \mathbb{P}^2)$ for $i = 1, 2$. Z has codimension 4 in Y. Again, using our description of the cohomology of the blow-up, we obtain:

$$
\begin{aligned}
H^6(X) &= \mathbb{Q}\, c_3^3 \oplus \mathbb{Q}\, e^3 \oplus \mathbb{Q}\, c_1 e^2 \oplus \mathbb{Q}\, c_2 e^2 \oplus \mathbb{Q}\, c_1^2 e \oplus \mathbb{Q}\, c_1 c_2 e \oplus \mathbb{Q}\, c_2^2 e \\
H^{10}(X) &= \mathbb{Q}\, c_3^5 \oplus \mathbb{Q}\, c_1^2 e^3 \oplus \mathbb{Q}\, c_1 c_2 e^3 \oplus \mathbb{Q}\, c_2^2 e^3 \oplus \mathbb{Q}\, c_1^2 c_2 e^2 \oplus \mathbb{Q}\, c_1 c_2^2 e^2 \oplus \mathbb{Q}\, c_1^2 c_2^2 e
\end{aligned}
$$

We have $L^1(X) = \mathbb{Q}c_3 \oplus \mathbb{Q}e$. Thus $L^3(X)$ is generated by

$$c_3^3, c_3^2 e, c_3 e^2 \text{ and } e^3,$$

which are linearly independent. Hence the first part follows. On the other hand, $L^5(X)$ is generated by

$$c_3^5, c_3^4 e, c_3^3 e^2, c_3^2 e^3, c_3 e^4, e^5.$$

The first four elements are linearly independent. Let V denote their linear span. In order to determine the other two elements we have to calculate the chern classes $c_1(N_{Z/Y})$ and $c_2(N_{Z/Y})$. Because of the normal bundle sequence, we have $c(N_{Z/Y}) = \iota^* c(T_Y) c(T_Z)^{-1}$. Since $c(T_{\mathbb{P}^n}) = (1 + c)^{n+1}$ [EH16, 5.7.1] and $T_Z \cong \pi_1^* T_{\mathbb{P}^2} \oplus \pi_2^* T_{\mathbb{P}^2}$, we get

$$
\begin{aligned}
c(N_{Z/Y}) &= \iota^*(1 + c_3)^9 (1 + c_1)^{-3} (1 + c_2)^{-3} = (1 + c_1 + c_2)^9 (1 - 3c_1 + 6c_1^2)(1 - 3c_1 + 6c_2^2) \\
&= 1 + 6(c_1 + c_2) + (15c_1^2 + 27c_1 c_2 + 15c_2^2) + ...
\end{aligned}
$$

In particular, the chern classes are all symmetric in c_1 and c_2. Now, we consider $c_3 e^4$:

$$c_3 e^4 = c_3(\pi^*(\iota_*(1)) + c_3(N_{Z/Y})e - c_2(N_{Z/Y}) + c_1(N_{Z/Y})e^3)$$

In fact, the e^3-term is $c_3 c_1(N_{Z/Y})e^3 = 6c_3^2 e^3$ and all other coefficients of e^i are symmetric in c_1 and c_2. Without having to compute them, it now follows that $c_3 e^4$ is in V. On the other hand, the coefficient of e^3 in e^5 is

$$c_1(N_{Z/Y})^2 - c_2(N_{Z/Y}) = 21\iota^* c_3^2 - 3c_1 c_2$$

And thus, e^5 is not in V, finally establishing the claim. $\square$

In fact, this can be generalized:

Proposition 2.24. Let $1 \leq r \leq s$ be natural numbers with $(r, s) \neq (1, 1), (1, 2)$.
The blow-up Y of $\mathbb{P}^{rs+r+s}$ at the Segre subvariety

$$
\begin{aligned}
\iota \colon Z = \mathbb{P}^r \times \mathbb{P}^s &\hookrightarrow \mathbb{P}^{(r+1)(s+1)-1} = X \\
([x_i], [y_j]) &\longmapsto [x_i y_j]
\end{aligned}
$$

does not satisfy SD.

Proof. First, let $c_1 \in L^1(\mathbb{P}^r)$, $c_2 \in L^1(\mathbb{P}^s)$ and $c_3 \in L^1(\mathbb{P}^{rs+r+s})$ be the hyperplane classes. We have

$$L^1(Y) = \pi^* L^1(X) \oplus \mathbb{Q}\, e = \mathbb{Q}\, c_3 \oplus \mathbb{Q}\, e$$

As before, we get $\iota^*(c_3) = c_1 + c_2$ in $L^1(\mathbb{P}^r \times \mathbb{P}^s) = L^1(\mathbb{P}^r) \oplus L^1(\mathbb{P}^s)$. In order to study $L^*(Y)$ we will need the first chern class of the normal bundle $N_{Z/X}$. Because of the normal bundle sequence

$$0 \longrightarrow T_Z \longrightarrow T_X|_Z \longrightarrow N_{Z/X} \longrightarrow 0$$

we know that $c_1(N_{Z/X}) = \iota^* c_1(T_X) - c_1(T_Z)$. Since $c(T_{\mathbb{P}^n}) = (1+c)^{n+1}$ it follows in particular that $c_1(T_{\mathbb{P}^n}) = (n+1)c$, where $c \in L^1(\mathbb{P}^n)$ is the hyperplane class. Since $T_Z = \pi_1^* T_{\mathbb{P}^r} \oplus \pi_2^* T_{\mathbb{P}^s}$, we get

$$c_1(N_{Z/X}) = \iota^* c_1(T_X) - c_1(T_Z) = (r+1)(s+1)(c_1+c_2) - (r+1)c_1 - (s+1)c_2 = s(r+1)c_1 + r(s+1)c_2.$$

Now, we can subdivide the proposition into three cases

Case 1. If $r \neq s$, then $\dim L^2(Y) = 3$ and $\dim L^{d-2}(Y) = 4$.

Indeed, since Z has codimensin $rs > 2$ in X, our description of the cohomology of Y reads

$$H^4(Y) = \mathbb{Q}\, c_3^2 \oplus \mathbb{Q}\, c_1 e \oplus \mathbb{Q}\, c_2 e \oplus \mathbb{Q}\, e^2$$
$$H^{2(d-2)}(Y) = H^{2rs+2r+2s-4}(Y) = \mathbb{Q}\, c_3^{2(d-2)} \oplus \mathbb{Q}\, e^{rs-1} c_1^{r-1} c_2^s \oplus \mathbb{Q}\, e^{rs-1} c_1^r c_2^{s-1} \oplus \mathbb{Q}\, e^{rs-1} c_1^r c_2^s.$$

The first claim follows from the fact that $L^2(Y)$ is generated by

$$c_3^2, c_3 e = c_1 e + c_2 e, e^2$$

and these elements are linearly independent. We are left to show that $L^{d-2}(Y) = H^{2(d-2)}(Y)$. The three elements of $L^{d-2}(Y)$

$$c_3^{d-2}, c_3^{r+s-1} e^{rs-1}, c_3^{r+s} e^{rs-2}$$

are also linearly independent. It suffices to check that $c_3^{r+s-2} e^{rs}$ is not in the linear span. More specifically, we will show that the coefficient of e^{rs-1} in $c_3^{r+s-2} e^{rs}$ is not a multiple of $(c_1 + c_2)^{r+s-1}$. Indeed, using the formula

$$(-1)^{rs} e^{rs} = \pi^* \iota_*(1) - \sum_{i=1}^{rs-1} c_{r-i}(N_{Z/X})(-e)^i$$

one sees that this coefficient is

$$\begin{aligned}(-1)^{rs}(c_1+c_2)^{r+s-2} c_1(N_{Z/X}) &= (-1)^{rs}(c_1+c_2)^{r+s-2}(s(r+1)c_1 + r(s+1)c_2) \\ &= (-1)^{rs} s(r+1)(c_1+c_2)^{r+s-1} + (-1)^{rs}(r-s)(c_1+c_2)^{r+s-2} c_2\end{aligned}$$

Since $r \neq s$ it suffices to show that $(c_1+c_2)^{r+s-1} = (c_1+c_2)^{r+s-2}(c_1+c_2)$ and $(c_1+c_2)^{r+s-2} c_2$ are linearly independent in $L^*(Z)$. This can either be seen through explicit computation or using the Hard Lefschetz theorem and the fact that $c_1 + c_2 = \iota^*(c_3)$ is the class of a hyperplane section.

Case 2. If $r = s \geq 3$, then $\dim L^{2r+1}(Y) = 2r+2$ and $\dim L^{r^2-1}(Y) = r^2$

Indeed, Z has codimension r^2 in X, so for any $k < r^2$, the elements

$$c_3^k, c_3^{k-1} e, ..., c_3 e^{k-1}, e^k$$

form a basis of $L^k(Y)$, thus $\dim L^k(Y) = k+1$. To conclude, we need $2r+1 < r^2 - 1$ which is equivalent to $(r-1)^2 > 3$. The only remaining case is $(r,s) = (2,2)$, which was examined before. □

Remark 2.25. 1. As stated in remark 2.6 the author knows of no counterexample to the conjecture that SD in dimension n implies SD in all dimensions lower than n, but using the above examples, we can at least show that the function

$$\begin{aligned}\{0,1,...,\left\lfloor\frac{d-1}{2}\right\rfloor\} &\longrightarrow \mathbb{N}\\ k &\longmapsto \dim L^{d-k}(Y) - \dim L^k(Y)\end{aligned}$$

is not always monotonic. Indeed, the argument for case 3 above shows that this is false whenever $(r-1)(s-1) > 5$

However, one can also construct counterexamples through other means. The following example shows that the assumption in Proposition 2.16 was indeed necessary. For the relevant facts about Grassmannians, see Appendix B or [EH16, Chapter 3].

Proposition 2.26. Let X be the partial flag variety of the form

$$X = \left\{0 \subset V_2 \subset V_3 \subset \mathbb{C}^5 \mid \dim V_2 = 2,\ \dim V_3 = 3\right\}$$

which has dimension 8. Then we have $\dim L^2(X) = 3$ and $\dim L^6(X) = 4$

Proof. Let Y be the Grassmann variety $G(2,5)$ of two-dimensional subspaces of $\mathbb{C}^5$. Then the cohomology of Y is given by:

$$H^*(Y,\mathbb{Q}) = \mathbb{Q}\sigma_{3,3} \oplus \mathbb{Q}\sigma_{3,2} \oplus \mathbb{Q}\sigma_{3,0} \oplus \mathbb{Q}\sigma_{2,1} \oplus \mathbb{Q}\sigma_{2,0} \oplus \mathbb{Q}\sigma_{1,1} \oplus \mathbb{Q}\sigma_{1,0} \oplus \mathbb{Q}1$$

And one has $X \cong \mathbb{P}(\mathcal{Q})$, where $\mathcal{Q}$ is the universal quotient bundle [EH16, 3.2.3] Proposition now gives us:

$$H^4(X,\mathbb{Q}) = \mathbb{Q}\sigma_{2,0} \oplus \mathbb{Q}\sigma 1,1 \oplus \mathbb{Q}\sigma_{1,0}\zeta \oplus \mathbb{Q}\zeta^2 H^{12}(X,\mathbb{Q}) = \mathbb{Q}\sigma_{3,3} \oplus \mathbb{Q}\sigma_{3,2}\zeta \oplus \mathbb{Q}\sigma_{3,1}\zeta^2 \oplus \mathbb{Q}\sigma_{2,2}\zeta^2$$

We know that $L^2(X)$ is spanned by $\sigma_{2,0} + \sigma_{1,1} = \sigma_{1,0}^2, \sigma_{1,0}\zeta, \zeta^2$, which are all linearly independent, which shows the first claim. Note that we have used Pieri's formula here, which asserts for example if $n > a > b$, then $\sigma_{1,0}\sigma a, b = \sigma_{a+1,b} + \sigma_{a,b+1}$ and in the other cases one omitts the terms that do not make sense. We show that $H^{12}(X,\mathbb{Q}) = L^6(X)$. Since $\sigma_{1,0}^5\zeta = 5\sigma_{3,2}\zeta, \sigma_{1,0}^6 = 5\sigma_{3,3}$ and $\sigma_{1,0}^4\zeta^2 = (3\sigma_{3,1} + 2\sigma_{3,2})\zeta^2$ are linearly independent, it suffices to show that $\sigma_{1,0}^3\zeta^3$ is not contained in the subspace V generated by these three elements. First recall the identity from Proposition 2.14:

$$\zeta^3 = -c_1(\mathcal{Q})\zeta^2 - c_2(\mathcal{Q})\zeta - c_3(\mathcal{Q})$$

The total Chern class of $\mathcal{Q}$ is [EH16, 5.6.2]

$$c(\mathcal{Q}) = 1 + \sigma_{1,0} + \sigma_{2,0} + \sigma_{3,0}$$

which gives us

$$\sigma_{1,0}^3\zeta^3 = -\sigma_{1,0}^4\zeta^2 - \sigma_{1,0}^3\sigma_{2,0}\zeta - \sigma_{1,0}^3\sigma_{3,0} = \sigma_{1,0}^4\zeta^2 - 3\sigma_{3,2}\zeta - \sigma_{3,3} = 3\sigma_{3,2}\zeta mod V$$

and thus the Proposition. □

A. The Blow-up and its Cohomology

In this section, let X be a smooth projective variety of over $\mathbb{C}$ of dimension d and $\iota\colon Z \hookrightarrow X$ a smooth subvariety of codimension r.

Proposition A.1. There is a smooth projective variety Y together with a morphism $\pi\colon Y \to X$ satisfying the following properties

1. If $U = X - Z$, then $\pi\colon \pi^{-1}(U) \to U$ is an isomorphism

2. the subvariety $E \coloneqq \pi^{-1}Z \subset Y$ is of codimension 1 and there is an isomorphism $E \xrightarrow{\phi} \mathbb{P}(N_{Z/X})$ such that the diagram

$$\begin{tikzcd} E \arrow[rr, "\cong", "\phi"'] \arrow[dr] & & \mathbb{P}(N_{Z/X}) \arrow[dl] \\ & Z & \end{tikzcd}$$

commutes

Y is called the *blow-up of X at Z* and E is the *exceptional divisor.* See [Ha77, Chapter II.7] and [Hu05, Section 2.5] for details.

Theorem A.2. Let $e \in H^2(X, \mathbb{Q})$ be the cohomology class of the exceptional divisor. As a $\mathbb{Q}$-vector space, the cohomology of the blow-up can be described as follows:

$$H^*(Y, \mathbb{Q}) \cong H^*(X, \mathbb{Q}) \oplus \left(\sum_{i=1}^{r-1} H^{*-2i}(Z, \mathbb{Q}) \otimes \mathbb{Q}e^i \right)$$

for the cup product, we have the following two relations:

$$\alpha.(\beta \otimes e^i) = (\iota^*(\alpha).\beta) \otimes e^i \text{ for any } \alpha \in H^*(X, \mathbb{Q}) \text{ and } \beta \in H^*(Z, \mathbb{Q})$$

and

$$(-1)^k e^k = \pi^*[Z] - \sum_{i=1}^{n-1} c_{n-i}(N_{Z/X}) \otimes (-e)^i,$$

where $N_{Z/X}$ is the normal bundle of $Z \subset X$ and c_i is its i-th *chern class.*

Proof. First, we consider the following claim:
Claim. the pushfoward π_* is a left inverse of π^*. In particular, π^* is injective.

Indeed, since $\pi\colon Bl_Z X \to X$ is birational, it induces an isomorphism on function fields. Thus, $\pi_*(1) = \pi_*[Bl_Z X] = [X] = 1$. Now, if $\alpha \in H^*(X, \mathbb{Q})$, then the projection formula gives us $\pi_*\pi^*\alpha = \alpha.\pi_*(1) = \alpha$. Next, let us name the relevant maps

$$\begin{tikzcd}
E \arrow[r, hook, "i"] \arrow[d, "p"] & Bl_Z X \arrow[d, "\pi"] \\
Z \arrow[r, hook, "\iota"] & X
\end{tikzcd}$$

Let $Z \subset U$ be a tubular neighborhood of Z in X, i.e. an open neighborhood that deformation retracts onto Z. We get two Mayer-Vietoris sequences associated to the covering $X = U \cup (X - Z)$ and $Bl_Z X = \pi^{-1}U \cup (Bl_Z X - E)$. Since $H^*(U, \mathbb{Q}) \cong H^*(Z, \mathbb{Q})$ and $H^*(E, \mathbb{Q}) \cong H^*(\pi^{-1}U, \mathbb{Q})$, we have the following diagram

$$\begin{tikzcd}
\ldots \arrow[r] & H^{*-1}(\pi^{-1}U - E, \mathbb{Q}) \arrow[r] & H^*(Bl_Z X, \mathbb{Q}) \arrow[r] & H^*(E, \mathbb{Q}) \oplus H^*(Bl_Z X - E, \mathbb{Q}) \arrow[r] & H^*(\pi^{-1}U - E, \mathbb{Q}) \arrow[r] & \ldots \\
\ldots \arrow[r] & H^{*-1}(U - Z, \mathbb{Q}) \arrow[r] \arrow[u, "\cong"] & H^*(X, \mathbb{Q}) \arrow[r] \arrow[u, hook, "\pi^*"] & H^*(Z, \mathbb{Q}) \oplus H^*(X - Z, \mathbb{Q}) \arrow[r] \arrow[u, hook, "p"] \arrow[u, "\cong"'] & H^*(U - Z, \mathbb{Q}) \arrow[r] \arrow[u, "\cong"] & \ldots
\end{tikzcd}$$

Using a diagram chase, one can see that the sequence

$$0 \longrightarrow H^*(X) \xrightarrow{\pi^*} H^*(Bl_Z X) \xrightarrow{i^*} H^*(E)/p^* H^*(Z) \longrightarrow 0$$

is exact. Since π_* is a splitting homomorphism, we have an isomorphism

$$H^*(Bl_Z X) \xrightarrow[\cong]{\pi_* \oplus i^*} H^*(X) \oplus H^*(E)/p^* H^*(Z) \longrightarrow 0$$

From Proposition 2.14 it follows that

$$H^*(E)/p^* H^*(Z) \cong \bigoplus_{i=1}^{r-1} H^{*-2i}(Z) \otimes \mathbb{Q}\zeta^i$$

where $\zeta = c_1(\mathcal{O}_{\mathbb{P}(N_{Z/X})}(1))$. In order to obtain the decomposition, it remains to see that $i^* e = -\zeta$. Indeed Proposition 2.4.7 in [Hu05] proves that $i^* e = c_1(N_{E/Bl_Z X})$ and in fact it is easy to see that $N_{E/Bl_Z X}$ is precisely the tautological bundle $\mathcal{O}_{\mathbb{P}(N_{Z/X})}(-1)$ on E. Of the two relations, the first is trivial. We show the second. In fact, $i * (-e)^r = \zeta^r = -\sum_{i=0}^{r-1} c_{r-i}(N_{Z/X})\zeta^i$, which accounts for the second summand. For the first, we need to show $\pi_*((-e)^r) = [Z] = \iota_*(1)$. We get

$$\pi_*((-e)^r) = \pi_*((-i_*(1))^r) = \pi_* i_*(i^*(-i_*(1))^{r-1}) = \iota_* p_*(\zeta^{r-1})$$

The rest follows from the following lemma. $\square$

Lemma A.3. Let $\pi \colon E \longrightarrow X$ be a holomorphic vector bundle on X of rank k, $p \colon \mathbb{P}(E) \longrightarrow X$ its associated projective bundle and $\zeta = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$. Then we have

$$p_*(\zeta^{k-1}) = 1$$

Proof. First note that the map p has codimension $2(k-1)$ and $\zeta^{k-1} \in H^{2(k-1)}(\mathbb{P}(E), \mathbb{Q})$, so $p_*(\zeta^{k-1})$ is indeed in degree 0. Consider the following diagram:

$$\begin{tikzcd}
\mathbb{P}^{k-1} \arrow[r, hook, "i"] \arrow[d, "q"] & \mathbb{P}(E) \arrow[d, "p"] \\
* \arrow[r, hook, "j"] & X
\end{tikzcd}$$

where $*$ is any point in X and $\mathbb{P}^{k-1}$ is the fiber of $\mathbb{P}(E)$ over it. Since $*$ is generically transverse to p and obviously Cohen–Macaulay, Definition 1.4 yields $p^*j_*(1) = i_*(1)$. Also, since $i^*\mathcal{O}_{\mathbb{P}(E)}(1) = \mathcal{O}_{\mathbb{P}^{k-1}}(1)$, we have $i^*\zeta = c$ with $c \in H^2(\mathbb{P}^{k-1}, \mathbb{Q})$ being the hyperplane class. Since $j_*(1)$ is the generator of $H^{2d}(X, \mathbb{Q})$, it suffices to show that $p_*\zeta^{k-1}j_*(1) = j_*(1)$. Indeed:

$$\begin{aligned} p_*(\zeta^{k-1})j_*(1) &= p_*(\zeta^{k-1}p^*j_*(1)) = p_*(\zeta^{k-1}i_*(1)) = p_*i_*i^*(\zeta^{k-1}) \\ &= j_*q_*i^*(\zeta^{k-1}) = j_*p_*(c^{k-1}) = j_*(1), \end{aligned}$$

where we have used the projection formula in the first and third equality. $\square$

B. The Cohomology of the Grassmannian

This section collects all the needed facts about Grassmannians. All details and proofs can be found in Chapter 1 Section 5 in the book [GH94] by Griffiths and Harris and in [EH16]. First recall the definition of the Grassmann manifold

Definition B.1. Let k and n be integers with $0 \leq k \leq n$. The *Grassmannian* $\mathcal{G}(k, n)$ is defined to be

$$\mathcal{G}(k,n) := \left\{ V \subset \mathbb{C}^n \mid V \subset \mathbb{C}^n \text{ linear subspace, } \dim V = k \right\}$$

It can be shown that $\mathcal{G}(k, n)$ is a smooth projective variety of dimension $k(n - k)$.

Definition B.2. Let $e_1, ..., e_n \in \mathbb{C}^n$ be the standard basis and $V_i = \mathbb{C}e_1 + ... + \mathbb{C}e_i$. To any sequence $a = (a_1, ..., a_k)$ of integers with $0 \leq a_k \leq a_{k-1} \leq ... \leq a_1 \leq n - k$ we associate its *Schubert variety*

$$\Sigma_a := \left\{ V \subset \mathbb{C}^n \mid \dim(V \cap V_{n-k+i-a_i}) \geq i \right\}.$$

Its cohomology class $\sigma_a = [\Sigma_a] \in H^*(\mathcal{G}(k, n), \mathbb{Q})$ is called the associated *Schubert cycle*

It can be shown that $\Sigma_{aa} \subset \mathcal{G}(k, n)$ is an algebraic subvariety of codimension $|a| = a_1 + ... + a_k$ and that the Schubert varieties are exactly the closed cells of a CW decomposition of $\mathcal{G}(k, n)$. It results that

Theorem B.3. The cohomology of the Grassmannian is freely generated by the set of all Schubert cycles $\sigma_a \in H^{2|a|}(\mathcal{G}(k, n), \mathbb{Q})$ with $a = (a_1, ..., a_k)$ and $0 \leq a_1 \leq ... \leq a_k \leq n - k$

Indeed, when looking at the cellular chain complex of $\mathcal{G}(k, n)$, one sees that there are only cells in even degree, hence the boundary maps are zero. It is easy to check that the classes $\sigma_a \in H^{2|a|}(\mathcal{G}(k, n), \mathbb{Q})$ and $[\Sigma_a] \in H^{2|a|}_{cell}(\mathcal{G}(k, n), \mathbb{Q})$ correspond under the isomorphism of singular with cellular cohomology. For the cup product on the cohomology ring, we have the following:

Proposition B.4 (Pieri's Formula)**.** If $a = (a_1, 0, ..., 0)$, then for any b

$$\sigma_a \sigma_b = \sum_{\substack{b_i \leq c_i b_{i-1} \\ |c| = a_1 + |b|}} \sigma_c$$

In fact, it can be shown that the Schubert cycles σ_a with $a = (a_1, 0, ..., 0)$ generate $H^*(\mathcal{G}(k, n), \mathbb{Q})$ as an algebra, thus the cup product is at least in theory already determined by this formula.

Bibliography

[Br93] Glen E. Bredon. *Topology and Geometry.* Springer Verlag, 1993.

[EH16] David Eisenbud, Joseph Harris. *3264 & All That: A Second Course in Algebraic Geometry.* Camebridge University Press, 2016.

[Fu84] William Fulton. *Intersection Theory.* Springer Verlag, 1984.

[GH94] Phillip Griffiths, Joseph Harris. *Principles of Algebraic Geometry.* Wiley Classics Library, 1994.

[Gr68] Alexander Grothendieck. *Standard Conjectures on Algebraic Cycles.* Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), Oxford University Press, pp. 193-–199

[Ha77] Robin Hartshorne. *Algebraic Geometry.* Springer Verlag, 1977.

[Hat01] Allen Hatcher *Algebraic Topology.* Cambridge University Press, 2002.

[Hi64] Heisuke Hironaka. *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. of Math., 2, **79** (1): pp. 109-–203

[Hu05] Daniel Huybrechts. *Complex Geometry.* Springer Verlag, 2005.

[HW17] June Huh, Botong Wang. *Lefschetz Classes on Projective Varieties.* Proceedings of the American Mathematical Society **145** (2017), pp. 4629–4637.

[Mi99] James Milne, *Lefschetz classes on abelian varieties.* Duke Math. J. **96** (1999), pp. 639–675.

[Kl68] Steven L. Kleiman. *Algebraic Cycles and the Weil Conjectures.* In: Dix exposés sur la cohomologie des schémas, Amsterdam: North-Holland(1968), pp. 359-–386.

[Kl91] Steven L. Kleiman. *The standard conjectures.* In: Proceedings of Symposia in Pure Mathematics, **55**, American Mathematical Society(1991), pp. 3-–20.

[Ko92] János Kollár. *Trento examples.* In: Classification of irregular varieties, Springer Verlag(1992) p.134.

[Vo02] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry I.* Cambridge University Press, 2002.