BEAUTIFUL FORMULAE FOR SMOOTH CUBIC SURFACES: QUADRUPLES OF POINTS AND TWISTED CUBICS

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ZUSAMMENFASSUNG. Motiviert durch die in [GS14] bewiesene X-$F(X)$ Formel, untersuchen wir die Existenz und mögliche Form von Formeln bis zum Grad vier im Grothendieck-Ring der Varietäten für Körper der Charakteristik 0, die glatte kubische Flächen mit ihrem Hilbert-Schemata verallgemeinerter getwisteter Kubiken in Beziehung setzt. Dafür wenden wir die stabil-birationale Realisation und die motivische Realisation nach Gillet–Soulé. Insbesondere beweisen wir in Anlehnung an [Pop18], dass es keine „beautiful" Formel für glatte kubische Flächen mit dem zugehörigen Hilbert-Schemata verallgemeinerter getwisteter Kubiken gibt und dass die X-$F(X)$ Formel die einzige „beautiful" Formel vom Grad zwei für glatte kubische Flächen mit ihrer Fano Varietät von Geraden ist.

CONTENTS

1. Introduction

The aim of this thesis is to study the "beautiful" formulae in the Grothendieck ring of varieties relating the classes of symmetric powers of smooth cubic surfaces $S^{(n)}$, their Fano variety of lines $F(S)$, and their associated LLSvS varieties $Z(S)$, which parameterises the generalised twisted cubic curves in the cubic surface S, [Leh+17]. We study the existence and the possible form of such formulae up to degree 4 and for fields of characteristic 0, following [Pop18].

The Grothendieck ring of K-varieties is the free abelian group generated by isomorphism classes of varieties modulo the scissor relation $[X] = [W] + [X - W]$, for $W \subset X$ a closed subscheme, with the ring structure induced by $[X] \cdot [Y] = [X \times Y]$. We can consider formulae in this ring, for instance the so called X-$F(X)$ relation $[X^{[2]}] + [\mathbb{P}^d][X] + \mathbb{L}^2[F(X)]$ proved in [GS14], where $X \subset \mathbb{P}^{n+1}$ is a cubic hypersurface and $F(X)$ its associated Fano variety of lines. This formula is not only beautiful in the sense that it allows us to relate the geometry of a cubic hypersurface and the geometry of its Fano variety of lines, for example by relating the Hodge numbers of $F(X)$ and X [Huy23] or permitting us to calculate the zeta functions of $F(X)$ for smooth cubic threefolds and fourfolds [DLR17], but it is also beautiful in a more precise sense introduced by Galkin in [Gal17], see Definition 2.15.

There are different geometric objects that can be considered in a cubic surface to shed light on its geometry. In this thesis we will be interested mainly in configurations of points and twisted cubics. Since the Hilbert scheme of points parameterises configurations of points and the LLSvS variety parameterises generalised twisted cubics in the cubic surface [Leh+17], our main goal is to address the following question: What types of formulae exist in the Grothendieck ring of varieties for smooth cubic surfaces S with its LLSvS variety and symmetric powers of S? In order to deal with this question, we will employ realisations of the Grothendieck ring of varieties, more specifically the stable birational realisation [LL01] and the Gillet–Soulé motivic realisation [GS96]. This will allow us to apply results from birational geometry and representation theory to find obstructions to the possible formulae holding in the Grothendieck ring of varieties.

The first chapter briefly introduces the notion of the Grothendieck ring of varieties and explores basic results about this ring. In this chapter will be presented a proof of the X-$F(X)$ relation, which is an example of a beautiful formula of degree 2 in the Grothendieck ring of varieties. Additionally, we will study two realisations of this ring, namely the stable birational realisation and the Gillet–Soulé motivic realisation, which will be our main tools to study the Grothendieck ring of varieties and beautiful formulae.
In the second chapter we will explore obstructions to the possible form of beautiful formulae coming from the Gillet–Soulé motivic realisation and representation theory. In particular, we will prove, following [Pop18], that the only possible form of a beautiful formula of degree 4 for

smooth cubic surfaces S, with their LLSvS variety $Z(S)$, is

$$\begin{aligned}\mathbb{L}^4[Z(S)] = [S^{(4)}] - (1-\mathbb{L}+\mathbb{L}^2)[S^{(3)}] - \mathbb{L}[S][S^{(2)}] + (\mathbb{L}+\mathbb{L}^2+\mathbb{L}^3)[S^2] - 2\mathbb{L}^2[S^{(2)}] \\ - (\mathbb{L}-\mathbb{L}^2+\mathbb{L}^3-\mathbb{L}^4+\mathbb{L}^5)[S] + (\mathbb{L}^2+\mathbb{L}^4+\mathbb{L}^6)\end{aligned} \tag{1.1}$$

and the only possible form of a beautiful formula of degree 2 for S and its Fano variety of lines, $F(S)$, is given by the X-$F(X)$ relation.
Finally, the third chapter will be dedicated to obstructions to the form of beautiful formulae coming from the stable birational realisation. In particular, we will prove, following [Pop18], that the formula (1.1) is not a beautiful formula and, in consequence, there is no formula of degree 4 relating smooth cubic surfaces and their associated LLSvS variety.

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2. Preliminaries

2.1. **Grothendieck ring of varieties.** This subsection is based on [CNS18].

Definition 2.1 (Category $\mathcal{V}$)**.** A K-variety is for us a reduced and separated K-scheme of finite type. The K-varieties form the category $\mathcal{V}$ with morphisms of K-schemes as morphisms.[1]

Convention: In this thesis we let K be a field of characteristic zero, unless otherwise indicated, and we will refer to the K-varieties as varieties.

Definition 2.2 (The Grothendieck ring of varieties)**.** The free abelian group of isomorphism classes of $\mathcal{V}$ modulo the subgroup generated by $\langle [X]-[W]-[X-W]\rangle$, for $W \subset X$ a closed subscheme with the reduced induced subscheme structure, carries a unique ring structure via $[X]\cdot[Y] := [X\times Y]$ with neutral element $[\mathrm{Spec}(K)]$. This ring is called the Grothendieck ring of varieties $\mathrm{K}_0(\mathcal{V})$.

Example 2.3. The class of the affine line and of the projective space will play a crucial role.

1. Define $\mathbb{L} := [\mathbb{A}^1]$. Since $\mathbb{A}^n \simeq (\mathbb{A}^1)^n$, we obtain $[\mathbb{A}^n] = \mathbb{L}^n$.
2. Consider $D_+(x_0) \subset \mathbb{P}^n$. Since $D_+(x_0) \simeq \mathbb{A}^n$ and $\mathbb{P}^n - D_+(x_0) \simeq \mathbb{P}^{n-1}$, we have $[\mathbb{P}^n] = 1+\mathbb{L}+\cdots+\mathbb{L}^n$.

Definition 2.4 (Pointwise trivial fibration)**.** Let X, Y, F be varieties. A morphism $f\colon X \longrightarrow Y$ is a piecewise trivial fibration with fibre F if there exists a finite partition $(Y_i)_{i\in I}$ of Y into locally closed subsets such that $X\times_Y Y_i \simeq F\times Y_i$ are isomorphic as Y_i schemes for every $i \in I$, where Y_i are endowed with the reduced induced subscheme structure.

Example 2.5. Any geometric vector bundle is a pointwise trivial fibration by definition.

[1] Here we are using the convention used in [Bit04], since Bittner's presentation of the Grothendieck ring of varieties plays an important role in this thesis.

Definition 2.6 (Piecewise isomorphism)**.** Let X, Y be varieties. They are said to be piecewise isomorphic if there exists an integer n and finite partitions $(X_1, \ldots, X_n)$, $(Y_1, \ldots, Y_n)$ into locally closed subsets of X and Y respectively endowed with the reduced induced subscheme structure, such that for every i holds $X_i \simeq Y_i$ as varieties.

Lemma 2.7. *Let X be a variety with finite partition $(X_1, \ldots, X_n)$ of locally closed subsets endowed with the reduced induced subscheme structure. Then*

$$[X] = \sum_{i \geqslant 1} [X_i].$$

Proof. Since X_1 is a locally closed subset, there exist closed subsets $X_1' \subset X_1''$ with $X_1 = X_1'' - X_1'$. Let $U = X - X_1''$, then $[X] = [X_1''] + [U]$. Additionally, by $X_1 = X_1'' - X_1'$, we get $[X_1''] = [X_1'] + [X_1]$, hence $[X] = [X_1'] + [X_1] + [U]$.
Now, for $i \in 2, \ldots, n$ we have $X_1' \cap X_i \subset X_i$ is a closed subset and $U \cap X_i$ is its complement, therefore $[X_i] = [X_1' \cap X_i] + [U \cap X_i]$. Additionally, $\{X_1' \cap X_i\}_{i \geqslant 2}$ and $\{U \cap X_i\}_{i \geqslant 2}$ form a partition of X_1 and U respectively, which implies by induction $[X_1'] = \sum_{i \geqslant 2} [X_1' \cap X_i]$ and $[U] = \sum_{i \geqslant 2} [U \cap X_i]$. Thus, we have $[X] = \sum_{i \geqslant 2} [X_1' \cap X_i] + [X_1] + \sum_{i \geqslant 2} [U \cap X_i] = \sum_{i \geqslant 1} [X_i]$. □

Corollary 2.8. *Let X, Y be piecewise isomorphic varieties. Then $[X] = [Y]$.* □

Lemma 2.9. *Let X, Y, F be varieties and let $f \colon X \longrightarrow Y$ be a piecewise trivial fibration with fibre F. Then X and $F \times Y$ are piecewise isomorphic and*

$$[X] = [F][Y].$$

Proof. Since f is a piecewise trivial fibration, there exists a finite partition into locally closed subsets of Y, let $(Y_i)_i$ be such partition. Then $(X \times_Y Y_i)_i$ is a partition of X and $F \times Y$ has partition $(F \times Y_i)_i$. By definition of pointwise trivial fibration we know $[X \times_Y Y_i] = [F \times Y_i]$, hence $[X] = [F \times Y] = [F][Y]$ by Corollary 2.8. □

Corollary 2.10. *Let X be a smooth variety and $W \subset X$ a smooth closed subvariety of codimension n. Then*

$$[\mathrm{Bl}_W(X)] - [\mathbb{P}^{n-1}][W] = [X] - [W].$$

Proof. By the construction of blow-ups we have the relation $[\mathrm{Bl}_W(X)] - [\mathbb{P}(N_{W/X})] = [X] - [W]$. For $W \subset X$ smooth closed subvariety of codimension n, $\mathbb{P}(N_{W/X}) \longrightarrow W$ is a projective bundle of dimension $n-1$, which means that it is a Zariski locally trivial fibration with fibre $\mathbb{P}^{n-1}$ and therefore the result follows from Lemma 2.9. □

Definition 2.11 (pre-λ-structure)**.** Let R be a commutative ring with 1. A pre-λ-structure on R is an operation $\lambda \colon R \times \mathbb{N} \longrightarrow R$ such that for all $x, y \in R$ hold:
 (i) $\lambda^0(x) = 1$,
 (ii) $\lambda^1(x) = x$,
 (iii) $\lambda^n(x + y) = \sum_{i+j=n} \lambda^i(x) \lambda^j(y)$.

Remark 2.12. For $x \in R$ let $\Lambda(x,t) := \sum_{n=0}^{\infty} \lambda^n(x)t^n$. Note that the three conditions of a pre-λ-structure are equivalent to the following two conditions:

(i) $\Lambda(x,t) = 1 + xt + \sum_{n=2}^{\infty} \lambda^n(x)t^n$,

(ii) $\Lambda(x+y,t) = \Lambda(x,t)\Lambda(y,t)$.

Lemma 2.13. *The symmetric product* $X^{(n)} = \mathrm{Sym}^n(X)$ *defines a pre-*λ*-structure on* $K_0(\mathcal{V})$ *with the property that* $[\mathrm{Sym}^n(\mathbb{A}^m \times X)] = \mathbb{L}^{nm}[\mathrm{Sym}^n(X)]$.

Proof. The first two conditions of Definition 2.11 are clear. By the last remark, the third condition is equivalent to the fact that the Kapranov's Zeta function $Z_{\mathrm{Kap}}([X],t) = \sum_{i=0}^{\infty}[X^{(n)}]t^n$ is multiplicative, which was proved by Totaro [Göt00]. The property $[\mathrm{Sym}^n(\mathbb{A}^m \times X)] = \mathbb{L}^{nm}[\mathrm{Sym}^n(X)]$ was proved in [Göt00]. $\square$

2.2. **Beautiful Formulae in the Grothendieck ring of varieties.** We have enough information to make precise what we mean by a formula relating the geometry of two varieties. For instance, we could ask if it is possible to find a formula in the Grothendieck ring of varieties encoding the geometric relations between smooth cubic hypersurfaces and their associated Fano variety of lines. The following discussion is based on [Huy23] and [GS14].

Let V be a K vector space of dimension $n+2$. Consider the Grassmannian of $m+1$ planes in V, denoted by $G(m+1,V)$, or equivalently $\mathbb{G} := \mathbb{G}(m,\mathbb{P}^{n+1})$, where $\mathbb{P}^{n+1}$ is the projectivisation of V. Let $\mathcal{E}$ be the universal bundle of the Grassmannian $\mathbb{G}$, which fits in the following short exact sequence:

$$0 \longrightarrow \mathcal{E} \longrightarrow V \otimes \mathcal{O}_{\mathbb{G}} \longrightarrow \mathcal{Q} \longrightarrow 0\ .$$

The universal bundle $\mathcal{E}$ can be thought of as $\{(\Gamma,v) : \Gamma \in \mathbb{G}, v \in \Gamma\}$, from which it is clear that it is of rank $m+1$. After taking the projectivisation of the universal bundle we obtain, [GW10]:

$$\mathcal{L}_{\mathbb{G}} := \mathbb{P}(\mathcal{E}) \hookrightarrow \mathrm{Proj}(\mathrm{Sym}(V^* \otimes \mathcal{O}_{\mathbb{G}})) = \mathbb{G}(m,\mathbb{P}^{n+1}) \times \mathbb{P}^{n+1} \longrightarrow \mathbb{G}(m,\mathbb{P}^{n+1}) = \mathbb{G},$$

which is a $\mathbb{P}^m$-bundle, since the fibre at $\Gamma \in \mathbb{G}$ corresponds exactly to elements of the form (Γ,v), where $v \in \Gamma$.

Let X be a smooth cubic hypersurface, consider the case $m=1$ and restrict the bundle to $F(X) \subset \mathbb{G}(1,\mathbb{P}^{n+1})$ to obtain the $\mathbb{P}^1$-bundle

$$\mathcal{L} := \mathcal{L}_{\mathbb{G}}|_{F(X)} \longrightarrow F(X). \tag{2.1}$$

Consider now

$$\mathcal{L}_{\mathbb{G}}|_X := \{(x,L) : x \in X \cap L, L \subseteq \mathbb{P}^{n+1} \text{ a line}\} \longrightarrow X, \tag{2.2}$$

which is a $\mathbb{P}^n$-bundle because the fibre at $x \in X$ corresponds to lines in $\mathbb{P}^{n+1}$ passing through x.

We still need one ingredient in order to prove the X-$F(X)$ relation, namely the isomorphism

$$X^{[2]} - \mathcal{L}^{[2]} \simeq \mathcal{L}_{\mathbb{G}}|_X - \mathcal{L},$$

where $X^{[2]}$ is the Hilbert scheme of subschemes of length 2 of X. A length 2 subscheme τ of X is either a pair of K-points, a pair of Galois conjugate points or one K-point and a tangent direction to it. In any of these cases there is a unique K-rational line l_τ passing trough τ. Define $X^{[2]} - \mathcal{L}^{[2]} \longrightarrow \mathcal{L}_{\mathbb{G}}|_X$ sending $\tau \in X^{[2]}$ to $(x, l_\tau) \in \mathcal{L}_{\mathbb{G}}|_X$, where l_τ is the unique line containing τ and x is the residual intersection of l_τ and X. Note that l_τ is not contained in X because we are restricted to elements in $X^{[2]} - \mathcal{L}^{[2]}$. The inverse of the latter morphism is $\mathcal{L}_{\mathbb{G}}|_X - \mathcal{L} \longrightarrow X^{[2]}$ defined by sending (x, l) to the residual intersection of l and X.

Theorem 2.14 (X-$F(X)$ relation, [GS14]). *Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface. Then in $K_0(\mathcal{V})$ the following equation holds*

$$[X^{[2]}] = [\mathbb{P}^n][X] + \mathbb{L}^2[F(X)],$$

where $\mathbb{L} = [\mathbb{A}^1]$.

Proof. The $\mathbb{P}^n$-bundle $\mathcal{L}_{\mathbb{G}}|_X \longrightarrow X$ in (2.2) implies by Lemma 2.9 the following formula in the Grothendieck ring of varieties

$$[\mathcal{L}_{\mathbb{G}}|_X] = [\mathbb{P}^n][X].$$

Similarly, the $\mathbb{P}^1$-bundle $\mathcal{L} \longrightarrow F(X)$ in (2.1) implies

$$[\mathcal{L}] = [\mathbb{P}^1][F(X)].$$

We have furthermore the isomorphism $X^{[2]} - \mathcal{L}^{[2]} \simeq \mathcal{L}_{\mathbb{G}}|_X - \mathcal{L}$ constructed above and the $\mathbb{P}^2$-bundle $\mathcal{L}^{[2]} \longrightarrow F(X)$, yielding

$$[X^{[2]}] = [\mathcal{L}^{[2]}] + [\mathcal{L}_{\mathbb{G}}|_X] - [\mathcal{L}] = [\mathbb{P}^2][F(X)] + [\mathbb{P}^n][X] - [\mathbb{P}^1][F(X)] = [\mathbb{P}^n][X] + \mathbb{L}^2[F(X)],$$

where $\mathbb{L}$ denotes the class of the affine space $\mathbb{A}^1$ in the Grothendieck ring of varieties. $\square$

The last result is known as the X-$F(X)$ relation and allows us to relate the geometry of X and the geometry of $F(X)$ via a formula in the Grothendieck ring of varieties, for instance this relation was used in [GS14] to study the Hodge structure of $F(X)$. Motivated by this relation we will define what we mean by a beautiful formula and this will permit us to state the main objective of this thesis.

Definition 2.15 (Beautiful formulae, [Pop18]). A polynomial expression with formal symbols $[X] = [X^{(1)}]$, $[X^{(n)}]$, $[X^n]$, $[F(X)]$, $[Z(X)]$ and $\mathbb{L}$ vanishing at any smooth cubic surface $X = S$, where $F(X)$ and $Z(X)$ are the Fano variety of lines and the LLSvS variety associated to X respectively, cf. Theorem 2.39, will be called a *beautiful formula for smooth cubic surfaces with $F(X)$ and/or $Z(X)$*. We assign a degree to such a formula by assigning degree n to the classes $[X^{(n)}]$ and $[X^n]$ and 0 to the classes $[F(X)]$, $[Z(X)]$ and $\mathbb{L}$. If the formula does not contain the symbols $[F(X)]$ or $[Z(X)]$, we call the formula *homogeneous*.

Remark 2.16. If S is a smooth surface, the Göttsche formula, [Göt00]:

$$\sum_{n=0}^{\infty}[S^{[n]}]t^n = \prod_{i=1}^{\infty} Z_{\mathrm{Kap}}([S], \mathbb{L}^{i-1}t^i),$$

where $Z_{\mathrm{Kap}}([S], t) = \sum_{i=0}^{\infty}[S^{(n)}]t^n$ is the Kapranov's Zeta function, allows us to express any beautiful formula for a smooth surface S involving symmetric powers $[S^{(n)}]$ in terms of their Hilbert scheme of points $[S^{[n]}]$ and vice versa.

Example 2.17. The X-$F(X)$ relation can be written in the following form

$$[X^{(2)}] = (1 + \mathbb{L}^n)[X] + \mathbb{L}^2[F(X)],$$

which is an example of a beautiful formula of degree 2.

2.3. **Realisations of the Grothendieck ring of varieties.** The Grothendieck ring of varieties and therefore formulae in this ring are still poorly understood and, in order to understand it better the so called realisations or motivic measures are useful tools. More specifically, we will use the stable birational realisation and the Gillet–Soulé motivic realisation to determine obstructions to the possible form that beautiful formulae can possibly have.

Definition 2.18 (Realisation)**.** A realisation of the Grothendieck ring of varieties with values in the ring R is a ring homomorphism $\mathrm{K}_0(\mathcal{V}) \longrightarrow R$.

We will first consider the stable birational realisation, which will play a crucial role in determining obstructions to the form of beautiful formulae via stable birational geometry.

Definition 2.19 (Stable birational equivalence)**.** Two varieties X, Y are said to be stably birationally equivalent if the varieties $X \times \mathbb{P}^n$ and $Y \times \mathbb{P}^m$ are birational for $n, m \in \mathbb{Z}_{\geqslant 0}$. Let SB denote the multiplicative monoid of classes of stable birational equivalence of smooth varieties with product $[X] \cdot [Y] := [X \times Y]$.

We will proceed to prove an interesting result by Larsen–Lunts [LL01] that ensures the existence of a realisation of the Grothendieck ring of varieties onto the monoid ring $\mathbb{Z}[\mathrm{SB}]$, which induces an isomorphism $\mathrm{K}_0(\mathcal{V})/(\mathbb{L}) \simeq \mathbb{Z}[\mathrm{SB}]$. In order to prove it, we need the so called Bittner's description of the Grothendieck ring of varieties [Bit04].

Theorem 2.20 (Bittner's description of the Grothendieck ring of varieties, [Bit04])**.** *The Grothendieck ring of varieties has the following alternative presentations:*

(sm) Let $\mathcal{N}$ be the multiplicative monoid of isomorphism classes of smooth varieties. Then the Grothendieck ring of varieties is isomorphic to the free abelian group $\mathbb{Z}[\mathcal{N}]$ subject to the relations $[X] = [X - W] + [W]$, where X is smooth and $W \subset X$ is a smooth closed subvariety.

(bl) Let $\mathcal{M}$ be the multiplicative monoid of isomorphism classes of smooth complete varieties. Then the Grothendieck ring of varieties is isomorphic to the abelian group $\mathbb{Z}[\mathcal{M}]$ subject to the relations $[\varnothing] = 0$ and $[\mathrm{Bl}_W(X)] = [X] + [E] - [W]$, where X is smooth and complete, $W \subset X$ is a smooth closed subvariety and E is the exceptional divisor of the blow-up $\mathrm{Bl}_W(X)$.

Remark 2.21 ([Bit04])**.** We get the same group in Theorem 2.20 if in (sm) we restrict to quasi-projective smooth varieties and if in (bl) we restrict to smooth projective complete varieties. We can also restrict to the connected case in both presentations.

Corollary 2.22 ([LL01])**.** *Let G be a commutative monoid and $\mathbb{Z}[G]$ its monoid ring. Let $\mathcal{M}$ be the multiplicative monoid of isomorphism classes of smooth connected complete varieties, see Remark 2.21, and let $\Psi\colon \mathcal{M} \longrightarrow G$ be a homomorphism of monoids such that:*
(i) $\Psi([X]) = \Psi([Y])$ *if X and Y are birational,*
(ii) $\Psi([\mathbb{P}^n]) = 1$ *for all $n \geqslant 0$.*
Then there exists a unique ring homomorphism

$$\Phi\colon \mathrm{K}_0(\mathcal{V}) \longrightarrow \mathbb{Z}[G],$$

such that $\Phi([X]) = \Psi([X])$ for all $[X] \in \mathcal{M}$.

Proof. The morphism Ψ induces a morphism $\Psi'\colon \mathbb{Z}[\mathcal{M}] \longrightarrow \mathbb{Z}[G]$. Additionally, consider the canonical projection $\pi\colon \mathbb{Z}[\mathcal{M}] \longrightarrow \mathbb{Z}[\mathcal{M}]/\sim$, where $\sim$ represents the relation $[\mathrm{Bl}_W(X)] - [E] = [X] - [W]$ for $\mathrm{Bl}_W(X)$ the blow-up of X with centre W and exceptional divisor E.
By Theorem 2.20, we have $\mathrm{K}_0(\mathcal{V}) \simeq \mathbb{Z}[\mathcal{M}]/\sim$, thus we need the morphism Ψ' to factor through $\mathbb{Z}[\mathcal{M}]/\sim$, which happens if $\Psi'([\mathrm{Bl}_W X] - [E]) = \Psi'([X] - [W])$. For this we will prove $\Psi(\mathrm{Bl}_W(X)) = \Psi(X)$ and $\Psi(E) = \Psi(W)$.
Since the blow-up morphism is a birational map, we have by the first condition $\Psi(\mathrm{Bl}_W(X)) = \Psi(X)$. Furthermore, we know that $\Psi(E) = \Psi(W)\Psi(\mathbb{P}^r)$ holds for some r, since $[E] = [W][\mathbb{P}^r]$, then by the second condition we have $\Psi(E) = \Psi(W)$. Hence, we have a morphism $\mathrm{K}_0(\mathcal{V}) \longrightarrow \mathbb{Z}[G]$. □

Corollary 2.23 (Stable birational realisation, [LL01])**.** *There exists a realisation*

$$\Phi_{\mathrm{SB}}\colon \mathrm{K}_0(\mathcal{V}) \longrightarrow \mathbb{Z}[\mathrm{SB}],$$

which induces an isomorphism $\mathrm{K}_0(\mathcal{V})/(\mathbb{L}) \longrightarrow \mathbb{Z}[\mathrm{SB}]$.

Proof. Two isomorphic smooth complete varieties are stably birationally equivalent, therefore we have a natural surjection $\Psi\colon \mathcal{M} \longrightarrow \mathrm{SB}$ satisfying by definition the first condition in Corollary 2.22. The second condition follows from the fact that $[\mathbb{P}^n] = [\mathrm{Spec}(K)]$ in SB. This implies by Corollary 2.22 the existence of a realisation $\Phi_{\mathrm{SB}}\colon \mathrm{K}_0(\mathcal{V}) \longrightarrow \mathbb{Z}[\mathrm{SB}]$, which is surjective by the surjectivity of the morphism $\Psi\colon \mathcal{M} \longrightarrow \mathrm{SB}$. We need now to prove that the kernel of the morphism Φ_{SB} is exactly $[\mathbb{L}]$.
We have in particular by Corollary 2.22 that $\Phi_{\mathrm{SB}}(\mathbb{P}^n) = 1$, then $\Phi_{\mathrm{SB}}(1 + \mathbb{L}) = \Phi_{\mathrm{SB}}(\mathbb{P}^1) = 1$, therefore $(\mathbb{L}) \subset \mathrm{Kern}(\Phi_{\mathrm{SB}})$.
Let $[X] \in \mathrm{Kern}(\Phi_{\mathrm{SB}})$, then by Theorem 2.20 we can write $[X] = \sum_{i=1}^{k} n_i[X_i] - \sum_{j=1}^{l} m_j[Y_j]$ as sum of smooth complete varieties. Apply the realisation Φ_{SB} to get

$$\sum_{i}^{k} m_i \Phi_{\mathrm{SB}}(X_i) = \sum_{j}^{l} n_j \Phi_{\mathrm{SB}}(Y_j)$$

in $\mathbb{Z}[\mathrm{SB}]$, which implies by the structure of the monoid ring, after renumbering, that $k = l$, $m_i = n_i$ and X_i and Y_i are stably birationally equivalent. By this result, it suffices to prove

that $X - Y \in (\mathbb{L})$ for X, Y stably birationally equivalent.
Note that in $\mathrm{K}_0(\mathcal{V})$ one has

$$[X \times \mathbb{P}^k] = [X][\mathbb{P}^k] = [X](1 + \mathbb{L} + \cdots + \mathbb{L}^k),$$

hence $[X \times \mathbb{P}^k] = [X] \mod \mathbb{L}$. Thus, it suffices to prove that for X, Y being birational, we have $X - Y \in (\mathbb{L})$. Let X, Y be birational varieties, then by Theorem 5.20 we can factor the birational map by a sequence of blow-ups and blow-downs, which implies that we can assume X to be a blow-up of Y with smooth centre Z and exceptional divisor E. Therefore for some t we have

$$[X] - [Y] = [Z] - [E] = [Z] - [\mathbb{P}^t][Z] = [Z](\mathbb{L} + \cdots + \mathbb{L}^t),$$

which implies the claimed result. □

Corollary 2.24. *Let $X, Y_1, \ldots, Y_n$ be smooth, complete varieties such that the following equality holds in the Grothendieck ring of varieties:*

$$[X] = \sum_{i=1}^{n} n_i [Y_i],$$

for some $n_i \in \mathbb{Z}$. Then X is stably birationally equivalent to Y_i for some $1 \leqslant i \leqslant n$.

Proof. It follows directly after applying the stable birational realisation and considering the formula in $\mathbb{Z}[\mathrm{SB}]$. □

Another realisation that will allow us to determine obstructions to the form of beautiful formulae is the Gillet–Soulé motivic realisation.

Theorem 2.25 (Gillet–Soulé motivic realisation, [GS96])**.** *There exists a unique ring homomorphism*

$$\mu_{mot}\colon \mathrm{K}_0(\mathcal{V}) \longrightarrow \mathrm{K}_0(\mathrm{Chow}_{\mathbb{Q}}).$$

If $[X]$ is the class of a smooth projective variety, its image is $[\mathfrak{h}(X)]$, where $\mathfrak{h}(X) := (X, id_X, 0)$.

2.4. **The 27 lines.** The symmetries of the lines lying in a smooth cubic surface $S \subset \mathbb{P}^3$ will give us restrictions to the possible formulae that we can construct for smooth cubic surfaces in the Grothendieck ring of varieties. For this reason, we will review in this subsection some results about them. We will be using mainly results from [Har77] and [Dol12].

Lemma 2.26. *For any smooth cubic surface $S \subset \mathbb{P}^3$ over an algebraically closed field K holds $\omega_S \simeq \mathcal{O}_S(-1)$.*

Proof. Since S is the zero locus of a cubic polynomial, say f, multiplication by f induces an isomorphism $\mathcal{I}_S \simeq \mathcal{O}_{\mathbb{P}^n}(-3)$. Hence, $\omega_{\mathbb{P}^3} \simeq \mathcal{O}_{\mathbb{P}^3}(-4)$ implies via the adjunction formula $\omega_S \simeq (\omega_{\mathbb{P}^3} \otimes \mathcal{O}(S))|_S \simeq \mathcal{O}_S(-1)$. □

Lemma 2.27. *Let $S \subset \mathbb{P}^3$ be a smooth cubic surface over an algebraically closed field. Then S contains six pairwise disjoint lines $l_1, \ldots, l_6$.*

Proof. This follows from the following classical facts. One can find at least two skew lines in S, say $L_1.L_2$, [Sha13]. Given $L_1 \subset S$, it meets exactly ten other lines, which come in pairs of intersecting lines $\{l_i, l_i'\}_{1 \leqslant i \leqslant 5}$ such that $l_i \cap l_j = l_i \cap l_j' = \varnothing$ for $i \neq j$, [Sha13]. Since L_2 does not meet L_1, then L_2 meets at most one of the lines l_i, l_i' for each $1 \leqslant i \leqslant 5$, otherwise L_1 and L_2 would be coplanar and therefore not disjoint. Without loss of generality, L_2 does not meet l_i for $1 \leqslant i \leqslant 5$, which are by hypothesis disjoint. □

Any line in a smooth cubic surface S is a (-1)-curve, cf. Remark 2.30. Moreover, Table 8.3. in [Dol12] shows that there is no set of 7 disjoint (-1)-curves in a smooth cubic surface. This result can also be proved by showing that $\mathrm{Pic}(S) \simeq \mathbb{Z}^7$ as in [Huy23], however we will deduce the form of the Picard group as a consequence of the description of a smooth cubic surface as a blow-up.

Lemma 2.28. *Let S be a smooth cubic surface over an algebraically closed field. Then S is isomorphic to the blow-up of $\mathbb{P}^2$ in six distinct points $x_i \in \mathbb{P}^2$ for $i = 1, \ldots, 6$.*

Proof. Lemma 2.27 allows us to apply successively Castelnuovo's Theorem 5.17 six times, hence S is the blow-up of a smooth surface S_0 at 6 distinct points. Since S does not have more than 6 skew lines, by Corollary 5.18 we have that S_0 is minimal. The classification of minimal smooth surfaces via the Kodaira dimension in Theorem 5.19 implies that $S_0 \simeq \mathbb{P}^2$. □

The last result provide us with substantial geometric information about smooth cubic surfaces as we can appreciate from the following lemma. Let $S \subset \mathbb{P}^3$ be a smooth cubic surface over an algebraically closed field. As proved in Lemma 2.28, S is isomorphic to the blow-up of $\mathbb{P}^2$ at six distinct points $P_1, \ldots, P_6$. Let $\pi \colon S \longrightarrow \mathbb{P}^2$ be the composition of the blow-ups, $E_1, \ldots, E_6 \in \mathrm{Pic}(S)$ the corresponding exceptional curves and let $E_0 \in \mathrm{Pic}(S)$ be the class of $\pi^* l_0$ for a line $l_0 \subset \mathbb{P}^2$.

Lemma 2.29. *Let $S \subset \mathbb{P}^3$ be a smooth cubic surface. The following statements hold:*

(i) $\mathrm{NS}(S) := \mathrm{Pic}(S_{\bar{K}}) \simeq \mathbb{Z}^7$,

(ii) *The canonical divisor is* $K_{S_{\bar{K}}} = -3E_0 + E_1 + \cdots + E_6$,

(iii) *The intersection form* $\mathrm{NS}(S) \times \mathrm{NS}(S) \longrightarrow \mathbb{Z}$ *is given by* $E_0.E_0 = 1$, $E_i.E_j = -\delta_{ij}$ *for* $j \neq 0$.

Proof. Since $S_{\bar{K}}$ is a blow-up of 6 points over $\mathbb{P}^2$, we can blow-up point by point and apply Lemma 5.16 successively. Hence, $\mathrm{NS}(S) = \mathbb{Z}^7$.

The canonical divisor of $\mathbb{P}^2$ is given by $K_{\mathbb{P}^2} = -3E_0$. By applying Lemma 5.13, we obtain $K_{S_{\bar{K}}} = -3E_0 + E_1 + \cdots + E_6$.

Finally, the description of the intersection form follows from Lemma 5.16. □

It can be checked that the intersection form defined above is a bilinear form with signature $(1, 6)$, which allows us to interpret the *Néron–Severi group* of S, $\mathrm{NS}(S)$, as the lattice $\mathrm{I}_{1,6}$. The orthogonal complement of the canonical divisor $K_S^{\perp}$ is a root lattice of type $\mathbf{E_6}$, with group of isometries $O(\mathbf{E_6})$, [Dol12].

Remark 2.30. Consider the set $I \coloneqq \{L \in \mathrm{NS}(S) : L.\mathrm{K}_S = -1, L.L = -1\}$. A curve C has linear Hilbert polynomial $\mathcal{H}_C(d) = ad + b$, where $p_a(C) = 1 - b$ is the arithmetic genus of C and $\deg(C) = a$ is the degree of C. Additionally, via intersection theory we have $p_a(C) = \frac{1}{2}(C.K_S + C^2 + 2)$ and $\deg(C) = C.(-K_S)$, [Har77]. This implies that $L \in \mathrm{NS}(S)$ maps to a line under the closed immersion $S_{\bar{K}} \hookrightarrow \mathbb{P}^3_{\bar{K}}$ if and only if $L \in I$, since the Hilbert polynomial of a line is $1 + d$. In consequence, I consists of the classes of lines in the smooth cubic surface $S_{\bar{K}}$. Then, by Lemma 2.29 we obtain 27 classes of lines in $S_{\bar{K}}$ which are characterised as formal sums of the exceptional curves $E_0, \dots, E_6 \in \mathrm{NS}(S)$ as follows:

(i) Six lines: E_i for $i \neq 0$,
(ii) Fifteen lines: $E_0 - E_i - E_j$ for $i \neq j \neq 0$,
(iii) Six lines: $2E_0 + E_i - \sum_{j=1}^{6} E_j$ for $i \neq 0$.

Remark 2.31. Consider the set of roots $R \coloneqq \{r \in \mathrm{NS}(S) : r.K_S = 0, r.r = -2\}$. The reflections[2] associated to roots generate a subgroup of the group of isometries of $\mathbf{E_6}$, denoted by $\mathbb{W}$, which is called Weyl group of type $\mathbf{E_6}$. This group stabilises I and preserves the intersection form. By Lemma 2.29 we get following description of the vectors in R:

(i) One vector: $2E_0 - \sum_{i=1}^{6} E_i$,
(ii) Twenty vectors: $E_0 - E_i - E_j - E_k$ for different $i, j, k \neq 0$,
(iii) Thirty vectors: $E_i - E_j$ for $i \neq j$,
(iv) Twenty vectors: $-E_0 + E_i + E_j + E_k$ for different $i, j, k \neq 0$,
(v) One vector: $-2E_0 + \sum_{i=1}^{6} E_i$.

2.5. **Twisted cubics in cubic surfaces.** In analogy to the X-$F(X)$ relation, we could ask if studying a specific family of curves in a variety gives us useful geometric information of the original variety. We could for instance consider the geometric relations between smooth cubic hypersurfaces and a variety parameterising the twisted cubics lying in them. To make this notion precise we need some definitions and results. This subsection is based on [Leh+17].

Definition 2.32 (Hilbert scheme with Hilbert polynomial P)**.** Let $X \subset \mathbb{P}^{n+1}$ be closed subscheme and let $P(d)$ be a polynomial in the variable d. There exists a scheme $\mathcal{H}^P(X)$ called the Hilbert scheme of X for the Hilbert polynomial $P(d)$, with a flat family of subschemes of X

$$\mathcal{F} \subset \mathcal{H}^P(X) \times X \xrightarrow{\pi} \mathcal{H}^P(X),$$

having the following properties:

(i) All the fibres of π have Hilbert polynomial $P(d)$.

(ii) For any flat family $\mathcal{F}' \subset B \times X \xrightarrow{\pi'} B$ whose fibres have Hilbert polynomial $P(d)$, there is a unique morphism $\alpha \colon B \longrightarrow \mathcal{H}^P(X)$, such that $\mathcal{F}'$ is equal to the pullback of $\mathcal{F}$.

Recall that a *twisted cubic* is a smooth curve $C \subset \mathbb{P}^3$, which is projective equivalent to the image of $\mathbb{P}^1$ under the Veronese embedding of degree 3, $[x_0 : x_1] \longmapsto [x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3]$. We want to consider a Hilbert scheme as in Definition 2.32 parameterising the twisted cubics, therefore a natural question is: What is the Hilbert polynomial of a twisted cubic?

[2]For $\alpha \in R$ is defined the reflection associated to α as $r_\alpha \colon \mathrm{I}_{1,6} \longrightarrow \mathrm{I}_{1,6}$, via $\nu \longmapsto \nu + (\nu, \alpha)\alpha$, [Dol12]

Lemma 2.33. *A twisted cubic has Hilbert polynomial* $3d + 1$.

Proof. Let $C \subset \mathbb{P}^3$ be a twisted cubic, hence C is isomorphic to $\mathbb{P}^1$ via the Veronese embedding of degree 3, which implies that C is embedded with degree 3, therefore $\mathcal{O}_C(1) \simeq \mathcal{O}_{\mathbb{P}^1}(3)$ and $\dim_K H^0(C, \mathcal{O}_C(d)) = \dim_K H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3d)) = 3d+1$. Additionally, we have $H^i(C, \mathcal{O}_C(d)) = 0$ for d large enough and $i \geqslant 1$. $\square$

Let $S \subset \mathbb{P}^3$ be a smooth cubic surface and consider the Hilbert schemes $\mathcal{H}^{3d+1}(\mathbb{P}^3), \mathcal{H}^{3d+1}(S)$ parameterising curves with Hilbert polynomial $3d+1$ in $\mathbb{P}^3$ and S, respectively. By Lemma 2.33, the twisted cubics are part of this schemes and it was proved by Piene and Schlessinger [PS85] that the Hilbert scheme $\mathcal{H}^{3d+1}(\mathbb{P}^3)$ consists of two irreducible components H_0, H_1 of dimensions 12 and 15 respectively. The irreducible component of dimension 12 contains all the twisted cubics of $\mathbb{P}^3$. From now on we denote $\mathcal{H}^{gtc}(\mathbb{P}^3) = H_0$ and we will call any element in $\mathcal{H}^{gtc}(\mathbb{P}^3)$ a *generalised twisted cubic*. For a smooth cubic surface S, let $\mathcal{H}^{gtc}(S)$ be the scheme theoretic intersection of $\mathcal{H}^{3d+1}(S)$ and $\mathcal{H}^{gtc}(\mathbb{P}^3)$ in $\mathcal{H}^{3d+1}(\mathbb{P}^3)$, which is the variety parameterising the generalised twisted cubics contained in S. Finally, following the notation of Popov [Pop18], let the LLSvS variety $Z(S)$ be $\mathcal{H}^{gtc}(S)$ modulo linear equivalence.

Question 2.34. *Does a beautiful formula of degree* 4 *for smooth cubic surfaces* S *with its LLSvS variety* $Z(S)$ *exist? If so, what are the possible beautiful formulae of this form?*

With the aim of answering this question, it is necessary to understand the structure of the LLSvS variety $Z(S)$. The following discussion follows closely [Leh+17].

Let $S \subset \mathbb{P}^3$ be a normal cubic surface with at most rational double point singularities over an algebraically closed field K and let $\sigma \colon \tilde{S} \longrightarrow S$ be its minimal resolution. In particular, $\tilde{S}$ is a weak Del Pezzo surface, σ is crepant, i.e. it preserves the canonical bundle, and as proved in Lemma 5.23 we have $\sigma_* \mathcal{O}_{\tilde{S}} = \mathcal{O}_S$, [Dol12].

Since $\tilde{S} \in \mathrm{SmProj}$ is a surface, the Picard group of $\tilde{S}$ equipped with the intersecting form, cf. Definition 5.15, can be treated as a lattice, see [Dol12]. Analogously to Remark 2.31, we define the root system R and the Weyl group $\mathbb{W}$ associated to such lattice. The irreducible components $E_1, \ldots, E_m$ of the exceptional curves $\pi^{-1}(p)$ over all singularities $p \in S$ are isomorphic to $\mathbb{P}^1$ and form a basis of a subset of the root system $R_0 \subset R$, we call them *effective roots*, see Lemma 5.27.

Let $\mathbb{W}(R_0)$ denote the subgroup of the Weyl group $\mathbb{W}$ generated by the reflections associated to the effective roots. The root system R decomposes into finitely many orbits with respect to the action of $\mathbb{W}(R_0)$, [Dol12]. Additionally, in every orbit $B \subset R$ can be found unique roots α_B^+, α_B^- characterised by the property $\pm\alpha_B^{\pm}.E_i \leqslant 0$ for all i, which we will refer to as the *maximal* respectively *minimal* root of the orbit B, [Dol12]. Given a singularity $p \in S$, let $R_p \subset R$ be the irreducible subsystem generated by the exceptional curves in the fibre of p. It can be proved that R_p is an orbit under the action of $\mathbb{W}(R_0)$ on R, [Dol12].

Using this information, we will relate the orbits of R under the action of $\mathbb{W}(R_0)$ with $\mathbb{P}^2$-families of generalised twisted cubics in S. In particular, we will prove, following [Leh+17], that for smooth cubic surfaces S we have $Z(S)_{\mathrm{red}} = R \times \mathbb{P}^2$, with R as in Remark 2.31.

Lemma 2.35. *Let $C \subset S$ be a generalised twisted cubic, and let $\tilde{C} = \sigma^{-1}(C) \subset \tilde{S}$ denote the scheme theoretic inverse image. Then $\tilde{C}$ is an effective divisor such that the class of $\tilde{C} + K_{\tilde{S}}$ is a root in R. Moreover, $\sigma_*\mathcal{O}_{\tilde{C}} = \mathcal{O}_C$.*

Proof. Let $I \subset \mathcal{O}_S$ and $\tilde{I} \subset \mathcal{O}_{\tilde{S}}$ be the ideal sheaves of C and $\tilde{C}$ respectively, so that we have $\sigma^*I \twoheadrightarrow \tilde{I}$ and $I \hookrightarrow \sigma_*\tilde{I}$ by definition of inverse image. For any singular point $p \in S$, there is an open neighbourhood U and an epimorphism $\mathcal{O}_U^n \twoheadrightarrow I|_U$. This induces a surjective morphism $\mathcal{O}_V^n \rightarrow \sigma^*I|_V \rightarrow \tilde{I}|_V$ on a neighbourhood $V = \sigma^{-1}(U)$ of the fibre $\sigma^{-1}(p)$. Since fibres of σ are at most 1-dimensional, by the theorem on formal functions all second and higher direct images of coherent sheaves on $\tilde{S}$ vanish, hence pushing down the epimorphism $\mathcal{O}^n|_V \rightarrow \tilde{I}|_V$ along σ we get the epimorphism $(R^1\sigma_*\mathcal{O}_{\tilde{S}})^n|_U \rightarrow R^1\sigma_*\tilde{I}|_U$. By definition of rational singularities, $R^1\sigma_*\mathcal{O}_{\tilde{S}} = 0$ and so $R^1\sigma_*\tilde{I} = 0$. In consequence, the rows of the following commutative diagram are exact, α is injective and β is surjective.

$$\begin{tikzcd}
0 \arrow[r] & \sigma_*\tilde{I} \arrow[r] & \sigma_*\mathcal{O}_{\tilde{S}} \arrow[r] & \sigma_*\mathcal{O}_{\tilde{C}} \arrow[r] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[u, "\alpha"] & \mathcal{O}_S \arrow[r] \arrow[u, equal] & \mathcal{O}_C \arrow[r] \arrow[u, "\beta"'] & 0.
\end{tikzcd}$$

If C has no embedded points, β is an isomorphism everywhere. If β is an isomorphism, then $\tilde{C}$ cannot have embedded points, otherwise they would show up in $\sigma_*\mathcal{O}_{\tilde{C}}$. Hence, $\tilde{C}$ is an effective divisor. Assume now that C has an embedded point p, then C is a non-Cohen–Macaulay curve, since being Cohen–Macaulay and having no embedded points are equivalent for locally Noetherian schemes of dimension $\leqslant 1$. We also have that p is a singular point of S because C is non-Cohen–Macaulay [Leh+17], say with ideal sheaf $\mathfrak{m}$, and there exists a hyperplane section H through p such that $I = \mathcal{O}_S(-H)$. Let Z_p be the fundamental cycle supported on the exceptional fibre $\sigma^{-1}(p)$, see Appendix 5.4. By Artin's Theorem 4 in [Art66], $(\sigma^*\mathfrak{m})\mathcal{O}_{\tilde{S}} = \mathcal{O}_{\tilde{S}}(-Z_p)$ and $\sigma_*\mathcal{O}_{\tilde{S}}(-Z_p) = \mathfrak{m}$, so that $\tilde{I} = \mathcal{O}_{\tilde{S}}(-Z_p - \sigma^*H)$ and $\sigma_*\tilde{I} = I$. Then $\tilde{C}$ is always an effective divisor and $\sigma_*\mathcal{O}_{\tilde{C}} = \mathcal{O}_C$.
Since $R^i\sigma_*\mathcal{O}_{\tilde{S}} = R^i\sigma_*\tilde{I} = 0$ for $i \geqslant 1$, one gets $R^i\sigma_*\mathcal{O}_{\tilde{C}} = 0$ for $i \geqslant 1$. We also have $\chi(\mathcal{O}_{\tilde{C}}) = \chi(\mathcal{O}_C) = 1$ because C has arithmetic genus 0 by definition. Furthermore, we have $\tilde{C}.(-K_{\tilde{S}}) = C.(-K_S) = 3$, which implies via the adjunction formula $\tilde{C}^2 = 1$. Therefore, $(\tilde{C} + K_{\tilde{S}}).K_{\tilde{S}} = 0$ and $(\tilde{C} + K_{\tilde{S}})^2 = -2$, hence $\tilde{C} + K_{\tilde{S}}$ is a root by definition. □

Lemma 2.36. *Let α be a maximal root and let $\tilde{C} \in |\alpha - K_{\tilde{S}}|$. Then $C = \sigma(\tilde{C}) \subset S$ is a subscheme with Hilbert polynomial $3n + 1$.*

Proof. Take direct images of the short exact sequence $0 \rightarrow \mathcal{O}_{\tilde{S}}(-\tilde{C}) \rightarrow \mathcal{O}_{\tilde{S}} \rightarrow \mathcal{O}_{\tilde{C}} \rightarrow 0$ to get $0 \rightarrow I \rightarrow \mathcal{O}_S \rightarrow \sigma_*\mathcal{O}_{\tilde{C}} \rightarrow R^1\sigma_*\mathcal{O}_{\tilde{S}}(-\tilde{C}) \rightarrow 0$, where I is the ideal sheaf of C, and all other higher direct image sheaves vanish. Since $\tilde{C} \in |\alpha - K_{\tilde{S}}|$ and α is maximal, then $E_i.(-\tilde{C}) = E_i.(-\alpha + K_{\tilde{S}}) \geqslant 0$. Hence, the restriction of $\mathcal{O}_{\tilde{S}}(-\tilde{C})$ to any exceptional curve has non-negative degree. For a singularity $p \in S$, let Z_p be the fundamental cycle supported on $\sigma^{-1}(p)$, see Appendix 5.4. Theorem 4 and Lemma 5 in [Art66] yield $H^1(Z_p, \mathcal{O}_{\tilde{S}}(-\tilde{C} - mZ_p)) = 0$ for all $m \geqslant 0$ and for all singularities $p \in S$. Thus, by applying the theorem on formal functions and using that non-singular points have finite fibres, we obtain $R^1\sigma_*(\mathcal{O}_{\tilde{S}}(-\tilde{C})) = 0$, hence $\sigma_*\mathcal{O}_{\tilde{C}} = \mathcal{O}_C$. This implies

$$\chi(\mathcal{O}_C(-nK_S)) = \chi(\mathcal{O}_{\tilde{C}}(-nK_{\tilde{S}})) = \chi(\mathcal{O}_{\tilde{S}}(-nK_{\tilde{S}})) - \chi(\mathcal{O}_{\tilde{S}}(-\tilde{C} - nK_{\tilde{S}})),$$

hence by applying the Riemann–Roch formula twice we have

$$\chi(\mathcal{O}_C(-nK_S)) = \frac{1}{2}\big(n(n+1)K_{\tilde{S}}^2 - (-\tilde{C} - nK_{\tilde{S}})(-\tilde{C} - (n+1)K_{\tilde{S}})\big) = 3n + 1.$$

Hence, the Hilbert polynomial of $\tilde{C}$ is $3n + 1$ as claimed. □

Lemma 2.37 ([Leh+17])**.** *Let α^- be a minimal root. Then the linear system $|\alpha^- - K_{\tilde{S}}|$ is two dimensional and base point free, i.e. $|\alpha^- - K_{\tilde{S}}| \simeq \mathbb{P}^2$.*

Lemma 2.38. *Let $\alpha \in R - R_0$ and let α^+ and α^- denote the maximal and the minimal root of its orbit respectively. Then,*

(i) The linear system $|\alpha - K_{\tilde{S}}|$ is independent of the choice of α in its $\mathbb{W}(R_0)$-orbit.
(ii) The image $C = \sigma(\tilde{C})$ of a generic curve $\tilde{C} \in |\alpha - K_{\tilde{S}}|$ is smooth.
(iii) For every curve $\tilde{C} \in |\alpha^- - K_{\tilde{S}}|$, we have $\sigma(\tilde{C})$ is a generalised twisted cubic.

Proof. Assume that $\alpha^- \neq \alpha^+$ and let β be any root from the orbit of α. Since β is not a minimal root, there exists an effective root E_i with $\beta.E_i < 0$. Hence, E_i is one of the irreducible components of β, which leave us only with two possible intersection numbers, either $\beta.E_i = -1$ or $\beta.E_i = -2$. Since $\beta.E_i = -2$ implies $\beta = E_i$, then we must have $\beta.E_i = -1$. Let $\beta' = \beta - E_i$ be the root obtained by reflecting β in E_i. We have following short exact sequence:

$$0 \longrightarrow \mathcal{O}_{\tilde{S}}(\beta' - K_{\tilde{S}}) \longrightarrow \mathcal{O}_{\tilde{S}}(\beta - K_{\tilde{S}}) \longrightarrow \mathcal{O}_{\tilde{S}}(\beta - K_{\tilde{S}})|_{E_i} \longrightarrow 0.$$

In Lemma 5.27 we proved $E_i \simeq \mathbb{P}^1$. Since $(\beta - K_{\tilde{S}}).E_i = -1$, we have $\mathcal{O}(\beta - K_{\tilde{S}})|_{E_i} = \mathcal{O}_{E_i}(-1)$, which has trivial cohomology. This implies $h^i(\mathcal{O}(\beta' - K_{\tilde{S}})) = h^i(\mathcal{O}(\beta - K_{\tilde{S}}))$, hence we have an isomorphism of linear systems $|\mathcal{O}(\beta' - K_{\tilde{S}})| \simeq |\mathcal{O}(\beta - K_{\tilde{S}})|$. Thus, given $\tilde{C} \in |\mathcal{O}(\beta - K_{\tilde{S}})|$, we have the short exact sequence:

$$0 \longrightarrow \mathcal{O}_{\tilde{S}}(-\tilde{C} - E_i) \longrightarrow \mathcal{O}_{\tilde{S}}(-\tilde{C}) \longrightarrow \mathcal{O}_{E_i}(-1) \longrightarrow 0.$$

In particular, since $\mathcal{O}_{E_i}(-1)$ has no non-trivial cohomology, $\sigma_*\mathcal{O}_{\tilde{S}}(-\tilde{C} - E_i) = \sigma_*\mathcal{O}_{\tilde{S}}(-\tilde{C}) \subset \mathcal{O}_S$ define the same image curve $\sigma(\tilde{C} - E_i) = \sigma(\tilde{C})$. Replacing β by β' subtracts a fixed component from the linear system $|\mathcal{O}_{\tilde{S}}(\beta - K_{\tilde{S}})|$. Iterations of this step lead in finitely many steps to the minimal root $-\alpha$. Hence, all roots in the $\mathbb{W}(R_0)$-orbit of α define isomorphic linear systems and the same family of subschemes in S. The same procedure works for $\alpha^- = \alpha^+$.

Take $\alpha = \alpha^-$, then a generic curve in $\tilde{C} \in |\alpha^- - K_{\tilde{S}}|$ is smooth by Lemma 2.37. Let $p \in S$ be a singularity and recall that $R_p \subset R_0 \subset R$ is the irreducible subsystem generated by the exceptional curves in the fibre of p, which is an orbit under the action of $\mathbb{W}(R_0)$ on R. The

preimage $\sigma^{-1}(p)$ corresponds to the maximal root $\alpha^+_{R_p}$ with $\alpha^-.\alpha^+_{R_p} \in \{0,1\}$. Hence, $C = \sigma(\tilde{C})$ does not contain p or is smooth at p. Since σ is birational outside of the singular locus of S, the curve $C = \sigma(\tilde{C})$ is smooth.
Finally, taking $\alpha = \alpha^+$, Lemma 2.36 implies that for any $\tilde{C} \in |\alpha^- - K_{\tilde{S}}|$ we have $\sigma(\tilde{C}) \subset S$ is a generalised twisted cubic. □

Theorem 2.39. *Let S be a normal cubic surface with at most rational double point singularities over an algebraically closed field K. Then, we have*

$$\mathcal{H}^{gtc}(S)_{\mathrm{red}} \simeq \bigsqcup_{B \in R/\mathbb{W}(R_0)} |\mathcal{O}_{\tilde{S}}(\alpha_B^- - K_{\tilde{S}})| \simeq (R/\mathbb{W}(R_0)) \times \mathbb{P}^2.$$

In particular, if S is a smooth cubic surface over a field of characteristic zero, the geometric points of $Z(S)$ correspond to the roots $R \subset \mathrm{NS}(S)$ as in Remark 2.31. □

3. Obstructions from representation theory via the Gillet–Soulé motivic realisation

We are interested in finding conditions for the possible beautiful formulae for smooth cubic surfaces $S \subset \mathbb{P}^3$. In order to achieve this, we will consider the image of its symmetric products $[S^{(n)}]$, its associated Fano variety of lines $F(S)$ and its associated LLSvS variety $Z(S)$ under the Gillet–Soulé motivic realisation and find the possible formulae relating this classes in $\mathrm{K}_0(\mathrm{Chow}_{\mathbb{Q}})$. We require some technical results, namely Theorems 3.3 and 3.7, that allow us to compare beautiful formulae and formulae in the Grothendieck ring of Chow motives $\mathrm{K}_0(\mathrm{Chow}_{\mathbb{Q}})$ via Corollary 3.8. This section is mainly based on [Pop18] and [SP11].

One of the main tools that we will be using to restrict the form of beautiful formulae comes from the theory of Galois representations. By a *Galois representation* of the Galois extension L/K we mean a discrete, finite dimensional representation of the Galois group $\mathrm{Gal}(L/K)$ over $\mathbb{Q}$, where a discrete representation is a group homomorphism $\mathrm{Gal}(L/K) \longrightarrow \mathrm{GL}(V)$ being continuous for $\mathrm{GL}(V)$ equipped with the discrete topology and $\mathrm{Gal}(L/K)$ with the profinite topology. We denote the category of Galois representations of L/K by $\mathrm{Rep}(\mathrm{Gal}(L/K))$. Morphisms are given by morphisms of representations, namely morphisms commuting with the action of the Galois group. We fix the notation $\mathrm{Gal}_K = \mathrm{Gal}(\bar{K}/K)$ for the absolute Galois group.

Denote by $\mathrm{Chow}^0_{\mathbb{Q}} \subset \mathrm{Chow}_{\mathbb{Q}}$ *the subcategory of zero dimensional Chow motives* with objects $\mathcal{M} \otimes \mathbb{L}^n$, where $\mathcal{M} \in \mathrm{Chow}^{\mathrm{Art}}_{\mathbb{Q}}$ is an Artin motive and $n \in \mathbb{Z}$, see Appendix 5.2. Observe that for any Chow motive $(X, \mathrm{id}, \mathrm{n}) \in \mathrm{Chow}^0_{\mathbb{Q}}$, the geometric points $X(\bar{K})$ come with a Gal_K-action. Thus, by assigning a basis vector $b_x \in \mathbb{Q}^{X(\bar{K})}$ to $x \in X(\bar{K})$, we obtain a representation $\mathrm{Gal}_K \longrightarrow \mathrm{GL}(\mathbb{Q}^{X(\bar{K})})$. Denote this representation by $\mathrm{Rep}(X) = \mathrm{Rep}(\mathfrak{h}(X))$. Moreover, $\mathrm{Rep}\colon \mathrm{Chow}^0_{\mathbb{Q}} \longrightarrow \mathrm{Rep}(\mathrm{Gal}_K)$ defines a functor: Let $X, Y \in \mathrm{SmProj}$ be zero dimensional varieties and let $f \in \mathrm{Corr}^0(X, Y)$ be a correspondence. By base changing we get a cycle

$f_{\bar{K}} \in \mathrm{Corr}^0(X_{\bar{K}}, Y_{\bar{K}})$. Hence, we can write uniquely

$$f_{\bar{K}} = \sum_{x \in X(\bar{K}), y \in Y(\bar{K})} \alpha_{x,y}[x \times y],$$

for $\alpha_{x,y} \in \mathbb{Q}$. Additionally, we have $\mathrm{Hom}_{Chow_{\mathbb{Q}}}(\mathfrak{h}(X), \mathfrak{h}(Y)) = \mathrm{Corr}^0(X, Y)$, hence for any morphism $\phi\colon \mathfrak{h}(X) \longrightarrow \mathfrak{h}(Y)$ we define $\mathrm{Rep}(\phi) : \mathrm{Rep}(X) \longrightarrow \mathrm{Rep}(Y)$ to be given by $b_x \longmapsto \sum_y \alpha_{x,y} b_y$. This morphism commutes with the action of Gal_K because $f_{\bar{K}}$ comes from $f \in \mathrm{Corr}^0(X, Y)$ and Gal_K leaves K-points invariant, therefore it leaves f invariant and so does with $f_{\bar{K}}$ because $\mathrm{Corr}^0(X, Y) \longrightarrow \mathrm{Corr}^0(X_{\bar{K}}, Y_{\bar{K}})$ is injective. It can be verified that this definition satisfies $\mathrm{Rep}(\phi \circ \psi) = \mathrm{Rep}(\phi) \circ \mathrm{Rep}(\psi)$.

Lemma 3.1. *Let ρ: $\mathrm{Gal}(L/K) \longrightarrow \mathrm{GL}(V)$ be a Galois representation of an infinite Galois extension L/K. Then there exists a finite Galois subextension K'/K such that the action of* $\mathrm{Gal}(L/K)$ *on V factors through* $\mathrm{Gal}(K'/K)$.

Proof. Let G be the image of $\mathrm{Gal}(L/K)$ under the representation ρ. By definition of a Galois representation, $\mathrm{Gal}(L/K)$ is profinite and $\mathrm{GL}(V)$ is discrete, hence G is compact and discrete, so it is finite. The kernel of the representation $\mathrm{Ker}(\rho)$ is a subgroup of $\mathrm{Gal}(L/K)$, which corresponds to a Galois subextension K'/K. Thus, $\mathrm{Gal}(K'/K) = \mathrm{Gal}(L/K)/\mathrm{Gal}(L/K') = G$ is finite and so is the field extension K'/K. □

Lemma 3.2. *The functor* Rep *induces an equivalence of categories between* $\mathrm{Chow}_{\mathbb{Q}}^{\mathrm{Art}}$ *and* $\mathrm{Rep}(\mathrm{Gal}_K)$.

Proof. The functor Rep defined above induces a functor Rep: $\mathrm{Chow}_{\mathbb{Q}}^{\mathrm{Art}} \longrightarrow \mathrm{Rep}(\mathrm{Gal}_K)$, since $\mathrm{Chow}_{\mathbb{Q}}^{\mathrm{Art}}$ is the pseudo-abelian hull of the category $\mathrm{Chow}_{\mathbb{Q}}$ generated by $\mathfrak{h}(X)$ with $X \in \mathrm{SmProj}$ zero dimensional. Firstly, let us show that

$$\mathrm{Hom}_{\mathrm{Chow}_{\mathbb{Q}}}(\mathfrak{h}(X), \mathfrak{h}(Y)) \longrightarrow \mathrm{Hom}_{\mathrm{Rep}(\mathrm{Gal}_K)}(\mathrm{Rep}(X), \mathrm{Rep}(Y))$$

is a bijection for all zero dimensional varieties X, Y. The injectivity follows from the injectivity of the base change $\mathrm{Corr}^0_K(X, Y) \longrightarrow \mathrm{Corr}^0_{\bar{K}}(X_{\bar{K}}, Y_{\bar{K}})$.
For the surjectivity we note that any representation $\phi \in \mathrm{Rep}(\mathrm{Gal}_K)$ is characterised by a matrix $A = (a_{x,y})_{x \in X_{\bar{K}}, y \in Y_{\bar{K}}}$ with coefficients in $\mathbb{Q}$, where $\phi(b_x) = a_{x,y} b_y$. Then we define $f' = \sum_{x \in X(\bar{K}), y \in Y(\bar{K})} \alpha_{x,y}[x \times y] \in \mathrm{Corr}^0(X_{\bar{K}}, Y_{\bar{K}})$. This cycle descends to $\mathrm{Corr}^0(X, Y)$ since ϕ commutes with the action of Gal_K.
We prove now that the functor Rep is essentially surjective. Let ϕ: $\mathrm{Gal}_K \longrightarrow \mathrm{GL}(V)$ be a Galois representation. By Lemma 3.1, ϕ is a Galois representation of a finite Galois extension K'/K. Since we are working in characteristic zero, Maschke's theorem allows us to assume that ϕ is irreducible, hence ϕ is a direct summand of the regular representation $\mathbb{Q}^{\mathrm{Gal}(K'/K)}$, [FH91].

As Gal_K-sets $\mathrm{Gal}(K'/K)$ and $\mathrm{Spec}(K')(\bar{K}) = \mathrm{Hom}_K(K', \bar{K})$ are isomorphic[3], thus the regular representation is $\mathrm{Rep}(\mathfrak{h}(\mathrm{Spec}(K')))$. Since Rep is fully faithful, $\mathrm{Chow}_{\mathbb{Q}}^{\mathrm{Art}}$ is pseudo-abelian

[3]Note that since K'/K is a Galois extension, for $g \in \mathrm{Gal}_K$ we have $g|_{K'} \in \mathrm{Gal}(K'/K)$, which allows as to consider $\mathrm{Hom}_K(K', \bar{K})$ as a Gal_K-set in the natural form. Then the isomorphism is induced by the inclusion $K' \subset \bar{K}$.

and ϕ is a summand of $\mathbb{Q}^{\mathrm{Gal}(K'/K)}$, hence it can be written as a kernel of a morphism with domain $\mathbb{Q}^{\mathrm{Gal}(K'/K)}$, then ϕ lies in the essential image of Rep. $\square$

For a discrete group G, let $\mathrm{Rep}(G)^g$ be *the ring of graded rational finite dimensional representations of G such that the action of G factors through a finite group.* The Galois group Gal_K acts on the lines contained in $S_{\bar{K}}$, hence it defines a homomorphism to the group of automorphisms of lines $\mathrm{Gal}_K \longrightarrow \mathrm{Aut}(\mathcal{I}) \simeq \mathbb{W}$, where $\mathbb{W}$ denotes the Weyl group of type $\mathbf{E_6}$, see Remark 2.31. Denote the image of the above morphism by $\mathbb{W}_0$, thus we have $\mathrm{Rep}(\mathbb{W}_0)^g \hookrightarrow \mathrm{Rep}(\mathrm{Gal}_k)^g$ induced by the surjection. Additionally, we have a surjection $\mathrm{Rep}(\mathbb{W})^g \twoheadrightarrow \mathrm{Rep}(\mathbb{W}_0)^g$.

Theorem 3.3. *The Grothendieck ring of zero dimensional motives* $\mathrm{K}_0(\mathrm{Chow}^0_{\mathbb{Q}}) \subset \mathrm{K}_0(\mathrm{Chow}_{\mathbb{Q}})$ *is isomorphic to* $\mathrm{Rep}(\mathrm{Gal}_K)^g$*, see Definition 5.12.*

Proof. The equivalence of categories from Proposition 3.2 extends to an equivalence of categories between $\mathrm{Chow}^0_{\mathbb{Q}}$ and the category of graded Galois representations of Gal_K, which is semi-simple. We have that $\mathrm{K}_0(\mathrm{Chow}^0_{\mathbb{Q}}) \subset \mathrm{K}_0(\mathrm{Mot}_{\mathrm{num}})$ [MNP13], where $\mathrm{Mot}_{\mathrm{num}}$ is the semi-simple category of motives modulo numerical equivalence. Hence, $\mathrm{K}_0(\mathrm{Chow}^0_{\mathbb{Q}})$ and $\mathrm{Rep}(\mathrm{Gal}_K)^g$ are isomorphic. $\square$

Remark 3.4. Representation theory of $\mathbb{W}$ over $\mathbb{Q}$ is the same as representation theory over $\mathbb{C}$. Given any irreducible $\mathbb{C}$-representation ϕ, the representation $\phi^{\oplus m_\phi}$, where m_ϕ is the Schur index of ϕ, is defined over $\mathbb{Q}$ since all the characters in Table 1 are rational, [CR66]. Additionally, for representations of the Weyl group $\mathbb{W}$ it was proved in [Ben71] that the Schur index is 1.

Let us now apply 3.3 to obtain obstructions to the form of beautiful formulae for smooth cubic surfaces S by means of representation theory. For this we first need to study the structure of $\mathfrak{h}(S)$.

Lemma 3.5. *Let S be a smooth cubic surface. Then the motive of S is given by*

$$\mathfrak{h}(S) \simeq 1 \oplus (V \otimes \mathbb{L}) \oplus \mathbb{L}^2,$$

where V is the Artin motive corresponding to the Galois representation $\mathrm{NS}(S) \otimes \mathbb{Q}$ *under the equivalence of categories proved in Lemma 3.2.*

Proof. In [KMP07] one has the following decomposition of the motive of a smooth projective surface:

$$\mathfrak{h}(S) = 1 \oplus ((\mathrm{Pic}^0_S)_{\mathrm{red}}(K) \otimes \mathbb{L}) \oplus (\mathrm{NS}(S) \otimes \mathbb{L}) \oplus (T(S) \otimes \mathbb{L}^2) \oplus (\mathrm{Alb}_S(K) \otimes \mathbb{L}^2) \oplus \mathbb{L}^2,$$

where Pic^0_S denotes the Picard variety of S [Gro61], $T(S) = \mathrm{Kern}(Z^2(S)_0 \longrightarrow \mathrm{Alb}_S(K))$ the Albanese–Jacobi kernel, $Z^2(S)_0$ the abelian group generated by of 2-cycles of S being numerically trivial and Alb_S the Albanese variety of S.

From Lemma 4.1.1. in [Huy23] and Proposition 5.10 in [Kle05] follows that $(\mathrm{Pic}^0_S)_{\mathrm{red}}(K) \otimes \mathbb{L}$ vanish. Since the Albanese variety is the conjugate of the Picard variety [Gro61], we also have that $\mathrm{Alb}_S(K) \otimes \mathbb{L}^2$ vanish. Additionally, we have $T(S) = Z^2(S)_0 = 0$, since zero-cycles $\alpha \in Z_0(S) = Z^2(S)$ in smooth projective surfaces are numerically trivial if and only if they are algebraically trivial [ACV17], and any zero-cycle in S is algebraically trivial because S is connected. $\square$

Remark 3.6. Note that we are using the same notation for elements in different rings, namely $\mathbb{L} = [\mathbb{A}^1] \in \mathrm{K}_0(\mathcal{V})$ and $\mathbb{L} = (\mathrm{Spec}(K), \mathrm{id}, -1) \in \mathrm{K}_0(\mathrm{Chow}_{\mathbb{Q}})$. However, under the Gillet–Soulé motivic realisation we have $[\mathbb{P}^1] \in \mathrm{K}_0(\mathcal{V}) \longmapsto \mathfrak{h}(\mathbb{P}^1) \in \mathrm{K}_0(\mathrm{Chow}_{\mathbb{Q}})$. Thus, Example 5.11 implies $\mathbb{L} \in \mathrm{K}_0(\mathcal{V}) \longmapsto \mathbb{L} \in \mathrm{K}_0(\mathrm{Chow}_{\mathbb{Q}})$.

Theorem 3.7. *Let S be a smooth cubic surface. Then the classes of the motives associated to $S^{(n)}, F(S), Z(S)$ lie in the ring of graded representations $\mathrm{Rep}(\mathbb{W}_0)^g$.*

Proof. The geometric points of $F(S)$ and $Z(S)$ correspond to lines and roots in $\mathrm{NS}(S)$ respectively, cf. Remark 2.30 and Theorem 2.39, then $\mathfrak{h}(F(S))$ and $\mathfrak{h}(Z(S))$ are Artin motives. Finally, it was proved in [RN98] that $\mathfrak{h}(\mathrm{Sym}^n X) = \mathrm{Sym}^n \mathfrak{h}(X)$. Thus, by Lemma 3.5 follows that $\mathfrak{h}(S^{(n)})$ is a direct sum of zero dimensional motives. Since the action of the Galois group Gal_K on the associated Galois modules factors through $\mathbb{W}_0$, we obtain the claimed result. □

Corollary 3.8. *Any formula in the Grothendieck ring of varieties for smooth cubic surfaces with their Fano variety $F(S)$ or their LLSvS variety $Z(S)$ descends via the Gillet–Soulé motivic realisation to a formula in the ring $\mathrm{Rep}(\mathbb{W}_0, \mathbb{C})^g$. Moreover, any formula in $\mathrm{Rep}(\mathbb{W}_0, \mathbb{C})^g$ induces a formula in $\mathrm{Rep}(\mathbb{W}, \mathbb{C})^g$.*

Proof. By Theorems 3.3 and 3.7, the images of $S^{(n)}, F(S), Z(S) \in \mathrm{K}_0(\mathcal{V})$ under the Gillet–Soulé motivic realisation lie in $\mathrm{Rep}(\mathbb{W}_0)^g \simeq \mathrm{Rep}(\mathbb{W}_0, \mathbb{C})^g$, hence the image of a beautiful formula along the Gillet–Soulé motivic realisation is a formula in $\mathrm{Rep}(\mathbb{W}_0, \mathbb{C})^g$. Furthermore, we obtain a formula in $\mathrm{Rep}(\mathbb{W}, \mathbb{C})^g$ via the surjection $\mathrm{Rep}(\mathbb{W}, \mathbb{C})^g \twoheadrightarrow \mathrm{Rep}(\mathbb{W}_0, \mathbb{C})^g$. □

Lemma 3.9 ([Pop18])**.** *Let $V \in \mathrm{Chow}_{\mathbb{Q}}^{\mathrm{Art}}$ be the Artin motive corresponding to the Galois representation $\mathrm{NS}(S) \otimes \mathbb{Q}$ under the equivalence of categories proved in Lemma 3.2. Then the class of this Artin motive $[V] \in \mathrm{Rep}(\mathbb{W}, \mathbb{C})^g$ has the following decomposition in irreducible representations of $\mathbb{W}$, see Table 1.*

$$[V] = 1 + \chi_3.$$

Proof. By definition we have $[V] = \mathrm{NS}(S) \otimes \mathbb{C} \in \mathrm{Rep}(\mathbb{W}, \mathbb{C})^g$. The symmetric group S_6 acts on $\mathrm{NS}(S)$ by permutation of its generators $E_1, \ldots, E_6$, leaving the canonical class $K_S = -3E_0 + E_1 + \ldots + E_6$ invariant. Hence, $S_6 \subset \mathbb{W}$. By studying the characters of representations of S_6 we will determine the representation of V. The irreducible representations of $\mathbb{W}$ that could appear in the decomposition of $\mathrm{NS}(S) \otimes \mathbb{C}$ are of dimension $\leqslant 7$ because $\mathrm{NS}(S) \otimes \mathbb{C}$ has dimension 7, hence by Table 1 the possible irreducible representations are $\chi_1, \chi_2, \chi_3, \chi_4$.
Additionally, the canonical class K_S is invariant under the action of $\mathbb{W}$, therefore $[V] = 1 + R$, with R a 6-dimensional representation. In order to understand R, we consider R as a representation of S_6 and analyse how R acts on $K_S^{\perp}$. As proved in Lemma 8.2.6 in [Dol12], the vectors $E_0 - E_1 - E_2 - E_3$ and $E_{i-1} - E_i$ for $i \in \{2, \ldots, 6\}$ form a basis of $K_S^{\perp}$, which implies that R can be decompose into two irreducible representations, one of them corresponding to the vector space $\langle E_0 - E_1 - E_2 - E_3 \rangle$ and the second one corresponding to the vector space generated by $E_{i-1} - E_i$ for $i \in \{2, \ldots, 6\}$. Thus, R can be decomposed into an irreducible representation of dimension 1 and one of dimension 5, in particular R is not a sum of 1-dimensional representations. Since, by Table 1, χ_1, χ_2 are 1-dimensional representations and R

can not be decomposed into 1-dimensional irreducible representations, they don't appear in $[V]$. The representation R is permutational in $\{E_1, \dots, E_6\}$. The trace of an element $g \in S_6$ equals the elements that are left invariant, hence a transposition has trace 4. Since transpositions lie in a conjugacy class of $\mathbb{W}$ and the character is invariant under conjugation, we need to find a conjugacy class of χ_3 or χ_4 with character 4. By Table 1, only χ_3 has an irreducible class with character 4, hence $R = \chi_3$. $\square$

Lemma 3.10. *For any smooth cubic surface S, the following classes of zero dimensional motives have the decomposition in irreducible representations of the Weyl group $\mathbb{W}$ presented below, see Table 1.*

(i) $[S] = 1 + (1 + \chi_3)\mathbb{L} + \mathbb{L}^2$,
(ii) $[S^2] = 1 + (2 + 2\chi_3)\mathbb{L} + (4 + 2\chi_3 + \chi_9 + \chi_{10})\mathbb{L}^2 + (2 + 2\chi_3)\mathbb{L}^3 + \mathbb{L}^4$,
(iii) $[S^{(2)}] = 1 + (1 + \chi_3)\mathbb{L} + (3 + \chi_3 + \chi_{10})\mathbb{L}^2 + (1 + \chi_3)\mathbb{L}^3 + \mathbb{L}^4$,
(iv) $[F(S)] = 1 + \chi_3 + \chi_{10}$.

Proof.
(i) Follows from Lemma 3.5 and Lemma 3.9.
(ii) Follows from the decomposition of $[S]$ and $\chi_3^2 = 1 + \chi_9 + \chi_{10}$, which can be verified by comparing characters, see Table 1.
(iii) For a representation V, we define the n-th symmetric product of the representation as $\mathrm{Sym}^n V$. Given a direct product of vector spaces $A \oplus B$, it holds $\mathrm{Sym}^2(A \oplus B) = \mathrm{Sym}^2(A) \oplus A \otimes B \oplus \mathrm{Sym}^2(B)$. Hence, from the representation of $[S]$ follows:

$$\mathrm{Sym}^2([S]) = 1 + (1 + \chi_3)\mathbb{L} + (2 + \chi_3 + \mathrm{Sym}^2 \chi_3)\mathbb{L}^2 + (1 + \chi_3)\mathbb{L}^3 + \mathbb{L}^4. \tag{3.1}$$

Additionally, for the character of $\chi(g) = \chi_{\mathrm{Sym}^2_{\chi_k}}(g)$ holds $\chi(g) = \frac{1}{2}(\chi_k(g)^2 + \chi_k(g^2))$, [FH91]. In Table 1 can be found in the third, fourth and fifth lines the second, third and fifth powers of the conjugacy classes respectively. Explicit calculations using (3.1) show that $\mathrm{Sym}^2 \chi_3 = 1 + \chi_{10}$, which implies the claimed result.
(iv) Follows from the decomposition of the class of the symmetric power $[S^{(2)}]$ and the X-$F(X)$ relation, see Example 2.17. $\square$

Corollary 3.11. *There is no homogeneous beautiful formula of degree* 2 *for smooth cubic surfaces S.*

Proof. By Corollary 3.8, a homogeneous formula of degree 2 implies a formula in $\mathrm{Rep}(\mathbb{W}, \mathbb{C})^g$ relating the classes. $[S], [S^2], [S^{(2)}]$. However, by Lemma 3.10, the representation of $[S]$ does not have summands χ_9 and χ_{10} but the other two classes have the term χ_{10} in their decomposition, which leaves us with the only possibility $[S^2] - [S^{(2)}] = [S]$. This formula does not hold, since in the left hand side we have the term $\chi_9\mathbb{L}^2$, which does not appear on the right hand side. Since there is no such formula for generic smooth cubic surfaces, we have shown the statement. $\square$

Corollary 3.12 ([Pop18]). *The X-$F(X)$ relation is the unique beautiful formula of degree 2 for smooth cubic surfaces S with the Fano variety $F(S)$.*

Proof. Firstly, by Theorem 2.14 the X-$F(X)$ is a beautiful formula. Additionally, any beautiful formula involving $[F(S)]$ have as coefficient of $[F(S)]$ a polynomial in $\mathbb{L}$ divisible by $\mathbb{L}^2$, since χ_{10} is the coefficient of $\mathbb{L}^2$ in the decomposition of $[S^2]$ and $[S^{(2)}]$.
Assume there is another beautiful formula of degree 2 involving $[F(S)]$. Let $p(\mathbb{L})\mathbb{L}^2$ be the coefficient of $[F(S)]$. Hence, by multiplying the $S - F(S)$ relation by $p(\mathbb{L})$ and subtracting, we obtain an homogeneous beautiful formula of degree 2, which is not possible by Corollary 3.11. $\square$

As proved in 2.39, for smooth cubic surfaces we have that the geometric points of $Z(S)$ correspond to the set of roots $R \subset \mathrm{NS}(S)$, cf. Remark 2.31. Using this description of $Z(S)$ and SageMath, Popov in [Pop18] obtained explicit representatives for each conjugacy class of $\mathbb{W}$ in terms of simple reflections and calculated traces explicitly to determine following decomposition in irreducible representations for the class of $Z(S)$.

Lemma 3.13 ([Pop18]). *The 72-dimensional representation $[Z(S)] \in \mathrm{Rep}(\mathbb{W}, \mathbb{C})^g$ has the following decomposition in irreducible representations, see Table 1.*

$$[Z(S)] = 1 + \chi_3 + \chi_8 + \chi_{10} + \chi_{16}.$$

Lemma 3.14. *For any smooth cubic surface S, the following classes of zero dimensional motives have the decomposition in irreducible representations of the Weyl group $\mathbb{W}$ presented below, see Table 1.*

(i) $[S^{(3)}] = 1 + (1 + \chi_3)\mathbb{L} + (3 + \chi_3 + \chi_{10})\mathbb{L}^2 + (3 + 3\chi_3 + 2\chi_{10} + \chi_{16})\mathbb{L}^3 + (3 + \chi_3 + \chi_{10})\mathbb{L}^4 + (1 + \chi_3)\mathbb{L}^5 + \mathbb{L}^6$,

(ii) $[S \times S^{(2)}] = 1 + (2 + 2\chi_3)\mathbb{L} + (6 + 3\chi_3 + \chi_9 + 2\chi_{10})\mathbb{L}^2 + (6 + 7\chi_3 + \chi_9 + 3\chi_{10} + \chi_{16} + \chi_{20})\mathbb{L}^3 + (6 + 3\chi_3 + \chi_9 + 2\chi_{10})\mathbb{L}^4 + 2(1 + \chi_3)\mathbb{L}^5 + \mathbb{L}^6$,

(iii) $[S^3] = 1 + (3 + 3\chi_3)\mathbb{L} + (9 + 6\chi_3 + 3\chi_9 + 3\chi_{10})\mathbb{L}^2 + (10 + 12\chi_3 + 3\chi_9 + 4\chi_{10} + \chi_{12} + \chi_{16} + 2\chi_{20})\mathbb{L}^3 + (9 + 6\chi_3 + 3\chi_9 + 3\chi_{10})\mathbb{L}^4 + (1 + \chi_3)\mathbb{L}^5 + \mathbb{L}^6$.

Proof. Follows as in Lemma 3.10. $\square$

Corollary 3.15. *There is no beautiful formula of degree 3 for smooth cubic surfaces S with the Fano variety of lines $F(S)$, or the LLSvS variety $Z(S)$.*

Proof. Any beautiful formula for S implies a formula in $\mathrm{Rep}(\mathbb{W}, \mathbb{C})^g$ by Corollary 3.8. The unique class of degree $\leqslant 3$ having the irreducible summand χ_{12} is $[S^3]$ and the unique class having the summand χ_8 is $[Z(S)]$, hence they cannot appear in a formula of degree 3. By excluding these classes, the unique class having the summand χ_{20} is $[S \times S^{(2)}]$, hence it cannot appear in the formula. Now, the only possible class of degree 3 that can appear is $[S^{(3)}]$, however it has the irreducible summand χ_{16}, which does not appear in any class of degree $\leqslant 2$ or $F(S)$. Hence, there is no formula of degree 3 involving $Z(S)$ or $F(S)$. Since there is no such formula for a generic smooth cubic surface, the statement follows. $\square$

Lemma 3.16. *For any smooth cubic surface S, the following classes of zero dimensional motives have the decomposition in irreducible representations of the Weyl group $\mathbb{W}$ presented below, see Table 1.*

(i) $[S^{(4)}] = 1 + (1+\chi_3)\mathbb{L} + (3+\chi_3+\chi_{10})\mathbb{L}^2 + (3+3\chi_3+2\chi_{10}+\chi_{16})\mathbb{L}^3 + (6+4\chi_3+\chi_8+5\chi_{10}+\chi_{16}+\chi_{20})\mathbb{L}^4 + (3+3\chi_3+2\chi_{10}+\chi_{16})\mathbb{L}^5 + (3+\chi_3+\chi_{10})\mathbb{L}^6 + (1+\chi_3)\mathbb{L}^7 + \mathbb{L}^8$,

(ii) $[S \times S^{(3)}] = 1 + (2+2\chi_3)\mathbb{L} + (6+3\chi_3+\chi_9+2\chi_{10})\mathbb{L}^2 + (8+9\chi_3+\chi_9+5\chi_{10}+2\chi_{16}+\chi_{20})\mathbb{L}^3 + (12+10\chi_3+\chi_8+3\chi_9+10\chi_{10}+3\chi_{16}+3\chi_{20}+\chi_{23})\mathbb{L}^4 + (8+9\chi_3+\chi_9+5\chi_{10}+2\chi_{16}+\chi_{20})\mathbb{L}^5 + (6+3\chi_3+\chi_9+2\chi_{10})\mathbb{L}^6 + (2+2\chi_3)\mathbb{L}^7 + \mathbb{L}^8$,

(iii) $[S^4] = 1 + (4\chi_3+4)\mathbb{L} + (12\chi_3+6\chi_9+6\chi_{10}+16)\mathbb{L}^2 + (36\chi_3+12\chi_9+16\chi_{10}+4\chi_{12}+4\chi_{16}+8\chi_{20}+28)\mathbb{L}^3 + (41\chi_3+\chi_7+\chi_8+24\chi_9+29\chi_{10}+4\chi_{12}+2\chi_{13}+7\chi_{16}+2\chi_{17}+12\chi_{20}+3\chi_{23}+3\chi_{25}+40)\mathbb{L}^4 + (36\chi_3+12\chi_9+16\chi_{10}+4\chi_{12}+4\chi_{16}+8\chi_{20}+28)\mathbb{L}^5 + (12\chi_3+6\chi_9+6\chi_{10}+16)\mathbb{L}^6 + (4\chi_3+4)\mathbb{L}^7 + \mathbb{L}^8$,

(iv) $[S^2 \times S^{(2)}] = 1 + (2\chi_3+2)\mathbb{L} + (4\chi_3+\chi_9+3\chi_{10}+8)\mathbb{L}^2 + (12\chi_3+2\chi_9+6\chi_{10}+2\chi_{16}+2\chi_{20}+10)\mathbb{L}^3 + (13\chi_3+\chi_8+4\chi_9+13\chi_{10}+\chi_{13}+3\chi_{16}+\chi_{17}+4\chi_{20}+\chi_{23}+17)\mathbb{L}^4 + (12\chi_3+2\chi_9+6\chi_{10}+2\chi_{16}+2\chi_{20}+10)\mathbb{L}^5 + (4\chi_3+\chi_9+3\chi_{10}+8)\mathbb{L}^6 + (2\chi_3+2)\mathbb{L}^7 + \mathbb{L}^8$,

(v) $[S^2 \times S^{(2)}] = 1+(3\chi_3+3)\mathbb{L}+(7\chi_3+3\chi_9+4\chi_{20}+11)\mathbb{L}^2+(21\chi_3+5\chi_9+10\chi_{10}+\chi_{12}+3\chi_{16}+4\chi_{20}+17)\mathbb{L}^3+(23\chi_3+\chi_8+11\chi_9+19\chi_{10}+\chi_{12}+\chi_{13}+5\chi_{16}+\chi_{17}+7\chi_{20}+2\chi_{23})+\chi_{25}+25)\mathbb{L}^4+(21\chi_3+5\chi_9+10\chi_{10}+\chi_{12}+3\chi_{16}+4\chi_{20}+17)\mathbb{L}^5+(7\chi_3+3\chi_9+4\chi_{10}+11)\mathbb{L}^{16}+(3\chi_3+3)\mathbb{L}^7+\mathbb{L}^8$.

Proof. Follows as in Lemma 3.10. ☐

Corollary 3.17. *There is no homogeneous beautiful formula of degree* 4 *for smooth cubic surfaces S.*

Proof. Similar as in Corollaries 3.11, 3.12 and 3.15 we can show that there is no homogeneous formula of degree 4 by comparing coefficients in the representations of the classes listed in Lemma 3.16. ☐

Corollary 3.18 ([Pop18])**.** *The only possible form up to multiplication of a beautiful formula of degree* 4 *for smooth cubic surfaces S with their LLSvS variety $Z(S)$, is the following relation.*

$$\begin{aligned}\mathbb{L}^4[Z(S)] = [S^{(4)}] - (1-\mathbb{L}+\mathbb{L}^2)[S^{(3)}] - \mathbb{L}[S][S^{(2)}] + (\mathbb{L}+\mathbb{L}^2+\mathbb{L}^3)[S^2] - 2\mathbb{L}^2[S^{(2)}]\\ - (\mathbb{L}-\mathbb{L}^2+\mathbb{L}^3-\mathbb{L}^4+\mathbb{L}^5)[S] + (\mathbb{L}^2+\mathbb{L}^4+\mathbb{L}^6).\end{aligned}\tag{3.2}$$

Proof. A beautiful formula of degree 4 induces a formula in $\mathrm{Rep}(\mathbb{W},\mathbb{C})^g$ by Corollary 3.8. By comparing the irreducible decomposition of the classes involved, we conclude that the formula (3.2) holds in $\mathrm{Rep}(\mathbb{W},\mathbb{C})^g$.

Note that χ_8 appears in classes of symmetric powers of S only as coefficient of $\mathbb{L}^4$. Hence, in any beautiful formula with $[Z(S)]$, the coefficient of $[Z(S)]$ is a polynomial in $\mathbb{L}$ divisible by $\mathbb{L}^4$. Assume that there is a beautiful formula of degree 4 with $[Z(S)]$ different to (3.2). Say that the coefficient of $[Z(S)]$ in such formula is $p(\mathbb{L})\mathbb{L}^4$ with $p(\mathbb{L}) \in \mathbb{C}[\mathbb{L}]$, thus by multiplying (3.2) with $p(\mathbb{L})$ and subtracting the two formulae we obtain an homogeneous formula of degree 4, contradicting Corollary 3.17. ☐

4. Obstructions from stable birational geometry via the stable birational realisation

Our goal in this section is to find obstructions by means of the stable birational realisation to the possible forms that a beautiful formula for smooth cubic surfaces S with their LLSvS variety $Z(S)$ can have. We have proved via the Gillet–Soulé motivic realisation, following [Pop18], that the only possible homogeneous beautiful formula for smooth cubic surfaces S of degree 4 is (3.2), which holds in the Grothendieck ring of Chow motives $\mathrm{K}_0(\mathrm{Chow}_{\mathbb{Q}})$. However, as we will prove in Corollary 4.6, following [Pop18], this formula is not a beautiful formula, which implies that there is no beautiful formula of degree 4 for smooth cubic surfaces with their LLSvS variety.

Let K be an algebraically closed field of characteristic zero. Any smooth cubic surface over K is isomorphic to a blow-up of $\mathbb{P}^2_K$ in 6 distinct points by Lemma 2.28, thus $[S] = 1 + 7\mathbb{L} + \mathbb{L}^2$. Additionally, by Theorem 2.39 we can write the class $[Z(S)]$ as a polynomial in $\mathbb{L}$. Thus, there are various formulae relating these two classes. The structure of smooth cubic surfaces over non-algebraically closed fields is more interesting, therefore we will explore this direction.

Lemma 4.1 (Lang–Nishimura Lemma, [Lan54], [Nis55])**.** *Ler $f\colon X \dashrightarrow Y$ be a rational map of K-schemes with Y proper. If X has a smooth K-point, then Y has a K-point.*

Proof. After blowing-up X, we obtain a morphism $\tilde{X} \dashrightarrow Y$, where $\tilde{X} \dashrightarrow X$ is a finite sequence of blow-ups with smooth centres, [Hir64]. The fibre over the smooth point is rational, hence $\tilde{X}$ has a K-point, which is mapped to a K-point in Y. □

Lemma 4.2. *Let S be a smooth cubic surface over K, then $S^{[3]}(K) \neq \varnothing$.*

Proof. Consider a K-line not contained in S. Then, the intersection with S is defined over K and is a subscheme of length 3, [Har77]. This defines a a K-point in $S^{[3]}$, see Appendix 5.5. □

Lemma 4.3. *Let S be a smooth cubic surface over K and let $n \in \mathbb{N}$ be prime to 3. If $S^{(n)}(K) \neq \varnothing$, then there exists a field extension L/K of degree prime to 3 such that $S(L) \neq \varnothing$.*

Proof. Given a K-point $S^{(n)}$, we can find a K-point in $S^{[n]}$ since the fibres of the Hilbert–Chow morphism are rational, see Appendix 5.5. A K-point of $S^{[n]}$ is a closed subscheme of dimension zero $Z \subset S$ and length n supported at finitely many points $y_1, \ldots, y_d$ with $d \leqslant n$.

Thus, $\mathrm{length}(\mathcal{O}_{Z,y_i}) = [k(y_i)\colon K]\,\mathrm{mult}_{y_i}(Z_{y_i})$ and $n = \mathrm{length}(Z) = \sum_{1\leqslant i\leqslant d} \mathrm{length}(\mathcal{O}_{Z,y_i})$, [EH00]. Assume $[k(y_i)\colon K]$ is divisible by 3 for all $1 \leqslant i \leqslant d$, then n is not prime to 3, which contradicts the hypothesis. In consequence, there exists a y_i such that $[k(y_i) : K]$ is prime to 3. Consider $y_i \subset Z \subset S$ as an $(L = k(y_i))$-point. □

Lemma 4.4. *Let $n \in \mathbb{N}$ prime to 3. There exists a smooth cubic surface S over $K = \mathbb{Q}$ (or $K = \mathbb{Q}_p$) such that $S^{(n)}$ is not stably birationally equivalent to $S^{(3)}$.*

Proof. In [CM04] it was proved that there exists a smooth cubic surface S over $K = \mathbb{Q}$ (or $K = \mathbb{Q}_p$) with no L-points for any field extension $L/\mathbb{Q}$ (or $L/\mathbb{Q}_p$) of degree prime to 3.
Assume that $S^{(n)}$ and $S^{(3)}$ are stably birationally equivalent, thus there exist $n, m \in \mathbb{Z}_{\geqslant 0}$ and a

rational map $S^{[3]} \times \mathbb{P}^n \longrightarrow S^{(3)} \times \mathbb{P}^n \dashrightarrow S^{(n)} \times \mathbb{P}^m$, where the first morphism is induced by the Hibert–Chow morphism. By Lemma 4.2, we have $S^{[3]}(K) \neq \varnothing$, which implies $S^{(n)}(K) \neq \varnothing$ via Lemma 4.1. From Lemma 4.3 follows that there exists a field extension L/K of degree prime to 3 with $S(L) \neq \varnothing$, which contradicts the result proved in [CM04]. □

Theorem 4.5 ([Pop18])**.** *There is no beautiful formula in* $\mathrm{K}_0(\mathcal{V}_{\mathbb{Q}})$ *(or* $K_0(\mathcal{V}_{\mathbb{Q}_p})$*) for smooth cubic surfaces* S *of the form:*

$$[S^{(3)}] = \sum_i [S^{(n_1^i)} \times \cdots \times S^{(n_{k_i}^i)}] \ (\mathit{mod}\ \mathbb{L}), \tag{4.1}$$

if for every i *there is an* n_j^i *prime to 3.*

Proof. If (4.1) is a beautiful formula, then by Corollary 2.24 $S^{(3)}$ is stably birationally equivalent to $S^{(n_1^i)} \times \cdots \times S^{(n_{k_i}^i)}$ for some i, which contradicts Lemma 4.4. □

Corollary 4.6. *The formula* (3.2) *for smooth cubic surfaces* S *is not a beautiful formula in* $\mathrm{K}_0(\mathcal{V})$ *for* $K = \mathbb{Q}$ *(or* $K = \mathbb{Q}_p$*). In particular, it is not a beautiful formula.*

Proof. Assume that the formula (3.2) is a beautiful formula. Then, we have $[S^{(3)}] = [S^{(4)}]$ (mod $\mathbb{L}$), which is a contradiction to Theorem 4.5. □

Remark 4.7. Note that Theorem 4.5 holds for all fields K of characteristic zero such that there exists a smooth cubic surface S without L-points for every field extension L/K of degree n prime to 3. In this sense, Corollary 4.6 could be extended. In particular, Corollary 4.6 holds for any field extension $K/\mathbb{Q}$ (or $K/\mathbb{Q}_p$) of degree prime to 3.

5. Appendix

5.1. Fano variety of lines.

Definition 5.1 (Fano variety of lines)**.** Let $X \subset \mathbb{P}^n_K$ subvariety and let $0 \leqslant m \leqslant n+1$. Define the Fano functor of planes:

$$F(X,m) \colon (Sch/K)^o \longrightarrow (Sets)$$

via

$$F(X,m)(T) := \{L \subset T \times X : L \text{ is } T\text{-flat and } L_t \subset \mathbb{P}^n_{k(t)} \text{ is a } m\text{-dimensional linear subspace}\}.$$

By T-flat we mean that $L \hookrightarrow T \times X \longrightarrow T$ is a flat morphism and the fibres are considered with respect to this morphism. We would be interested in the case $m = 1$, for which we obtain the *Fano functor of lines* denoted by $F(X) = F(X,1)$.

Example 5.2. Some familiar cases of Fano functors are the following.
(i) $F(\mathbb{P}, m) = \mathbb{G}(m, \mathbb{P})$ corresponds to the Grassmann functor. The Grassmann functor is defined as $\mathbb{G}(m, X)$,
(ii) $F(X, 0) = h_X = \mathrm{Hom}_k(-, X)$ corresponds to the functor of points.

We want to give a geometric interpretation to this functor via the representation of functors.

Definition 5.3 (Representable functor)**.** A functor $F\colon \mathcal{C}^{opp} \longrightarrow (Sets)$ is called representable if there exists an object $X \in \mathcal{C}$ and an isomorphism $\eta\colon h_X \longrightarrow F$. The tuple (η, X) is uniquely determined up to unique isomorphism.

Remark 5.4. For our particular case we want to identify the Fano functor of planes with a variety, which we will call a Fano variety of m-planes. The idea behind this is to use the representability of the Grassmannian functor, which comes from the Plücker embedding, and the fact that $F(X, m) \subset G(m, \mathbb{P})$ is a closed subfunctor, [Huy23].

Lemma 5.5 (Proposition 2.1.19, [Huy23])**.** *The Fano variety of lines $F(X)$ of a smooth cubic hypersurface $X \subseteq \mathbb{P}^{n+1}_K$, where $n \geqslant 2$, is a smooth projective variety of dimension $2n-4$.*

5.2. **Grothendieck ring of Chow motives.** This subsection is mostly based on [Sch94] and [MNP13]. Let $X \in \mathrm{SmProj}$ be a smooth projective variety. Define the *cycle group*, $Z^d(X)$, to be the free abelian group generated by irreducible subvarieties of X of codimension $d \geqslant 0$. We will now consider adequate relations on this group, see [MNP13] for a formal definition of adequate relations.

Let $W \subset X \times \mathbb{P}^1$ be a closed irreducible subvariety of dimension $d+1$ and let a, b be distinct points of $\mathbb{P}^1$ such that $X \times a, X \times b$ and W intersect properly, namely $\dim(W \cap X \times a) \leqslant d$ and $\dim(W \cap X \times b) \leqslant d$. The fibre W_a of the morphism $W \longrightarrow \mathbb{P}^1$ is the scheme theoretic intersection $W \cap X \times a$. Identify $X \times a$ as X, then we can think of the fibre W_a as a cycle of X of dimension $\leqslant d$. Based on this discussion, we can define the rational equivalence relation: Two cycles $Z_1, Z_2 \subset X$ are said to be *rational equivalent* if there exists a cycle $W \subset X \times \mathbb{P}^1$ and $a, b \in \mathbb{P}^1$ as described above such that $W_a = Z_1$ and $W_b = Z_2$. Extend this equivalence relation in the natural way over the cycle group.

The intuition behind being rationally equivalent is that one can go from one cycle to the other one through a rational family of cycles, i.e. a family of cycles parameterised by $\mathbb{P}^1$. This definition can be weakened by allowing families of cycles parameterised by smooth projective connected curves. This description gives rise to the so called algebraic equivalence. Two cycles $Z_1, Z_2 \subset X$ are said to be *algebraically equivalent* if there exists a connected curve $C \in \mathrm{SmProj}$ and a cycle $W \subset X \times C$ with $W_a = Z_1$ and $W_b = Z_2$ for two points $a, b \in C$. It follows from the definition that two cycles being rationally equivalent are algebraically equivalent.

Consider the rational equivalence relation $\sim$ and define the *codimension d Chow group* as $\mathrm{CH}^d(X) := Z^d(X) \otimes \mathbb{Q}/\sim$. Given two smooth projective varieties $X, Y \in \mathrm{SmProj}$ such that X is of pure dimension d we define the *group of correspondences of degree r* to be $\mathrm{Corr}^r(X, Y) := \mathrm{CH}^{d+r}(X \times Y)$. More generally, for $X = \bigsqcup_i X_i$ with X_i connected components of X, we define $\mathrm{Corr}^r(X, Y) := \bigoplus_i \mathrm{Corr}^r(X_i \times Y)$.

We want to relate the cycle groups of two different varieties $X, Y \in \mathrm{SmProj}$ related by a proper morphism $f\colon X \longrightarrow Y$ with the objective of defining the composition of correspondences. Firstly, we define the pullback and pushforward of algebraic cycles. Given an irreducible subvariety $W \subset X$ we define the *pushforward* by extending linearly following definition:

$$f^*(W) = \begin{cases} [K(W) : K(f(W))]f(W) & \text{, if } \dim(f(W)) = \dim(W) \\ 0 & \text{, else} \end{cases}.$$

In order to define the pullback, we need the notion of intersection of two smooth subvarieties $V, W \subset X$ of codimension n and m, which intersect in a union of subvarieties of codimension $\leqslant n + m$. If all the intersections have codimension $n + m$, we say that they *intersect properly.* If V, W intersect properly, we define the intersection product, [Har77]:

$$V \cdot W := \sum_Z i(V \cdot W; Z)Z,$$

where Z runs over the irreducible components of $V \cap W$ and $i(V \cdot W; Z)$ is the *intersection number* defined using the Serre's Tor formula as:

$$i(V \cdot W; Z) := \sum_r (-1)^r l_A Tor_r^A(A/I(V), A/I(W)),$$

for $A = \mathcal{O}_{X,Z}$ and $I(V)$ the ideal of V in A.

We define the *pullback* of a subvariety $T \subset Y$ such that Γ_f intersects properly $X \times T$ as:

$$f^*(T) := (pr_X)_*(\Gamma_f \cdot (X \times T)).$$

The correspondences are a generalisations of morphisms of varieties in the following sense: Given a morphism of varieties $f\colon X \longrightarrow Y$, the Graph $\Gamma_f \in \mathrm{Corr}(X, Y) := \oplus_r \mathrm{Corr}^r(X, Y)$. In consequence, we should have an analogous to composition of morphisms. Let $X, Y, Z \in \mathrm{SmProj}$, define

$$\mathrm{Corr}^r(X, Y) \otimes \mathrm{Corr}^s(Y, Z) \longrightarrow \mathrm{Corr}^{r+s}(X, Z)$$

by

$$f \otimes g \longmapsto g \circ f := p_{XZ*}(p_{XY}^* f \cdot p_{YZ}^* g),$$

where p_{XY}, p_{XZ}, p_{YZ} are the projections from $X \times Y \times Z$ to $X \times Y$, $X \times Z$ and $Y \times Z$, respectively.

Definition 5.6 (Category of Chow motives)**.** The category of Chow motives, $\mathrm{Chow}_{\mathbb{Q}}$, is defined as follows: The objects are triples (X, p, n), where $X \in \mathrm{SmProj}$ is a variety, $p = p^2 \in \mathrm{Corr}^0(X, X)$ is an idempotent and n is an integer. If $(X, p, n), (Y, q, m)$ are Chow motives, then

$$\mathrm{Hom}_{\mathrm{Chow}_{\mathbb{Q}}}((X, p, n), (Y, q, m)) := q \,\mathrm{Corr}^{m-n}(X, Y)p,$$

where composition is defined as described above. We call p and q *projectors.*

Definition 5.7 (Category of Artin motives)**.** Let $\mathrm{Chow}_{\mathbb{Q}}^{\mathrm{Art}}$ be the pseudo-abelian subcategory of $\mathrm{Chow}_{\mathbb{Q}}$ generated by the motives $\mathfrak{h}(X) = (X, \mathrm{id}, 0)$ associated to zero dimensional smooth projective varieties $X \in \mathcal{V}$.

Definition 5.8 (Tensor product of motives)**.** We define the tensor product of motives via:

$$(X, p, n) \otimes (Y, q, m) := (X \times Y, p \times q, n + m).$$

Definition 5.9 (Direct sum of motives)**.** The category $\mathrm{Chow}_{\mathbb{Q}}$ is an additive category. If $(X, p, n), (Y, q, m)$ are Chow motives with $n = m$ the direct sum is defined as

$$(X, p, n) \oplus (Y, q, m) := (X \sqcup Y, p \sqcup q, n).$$

Example 5.10. Let $X \in \mathrm{SmProj}$ of dimension d and $e \in X(K)$ a K-point. The cycles $p_0 = e \times X$ and $p_{2d} = X \times e$ define orthogonal projectors, i.e. $p_0 \circ p_{2d} = 0$ and $p_{2d} \circ p_0 = 0$. This implies $\mathfrak{h}(X) = \mathfrak{h}^0(X) \oplus \mathfrak{h}^+(X) \oplus \mathfrak{h}^{2d}(X)$, where $\mathfrak{h}^*(X) = (X, p_*, 0)$ with $p_+ = \mathrm{id} - \mathrm{p}_0 - \mathrm{p}_{2\mathrm{d}}$. It can be shown that $\mathfrak{h}^0(X) = (\mathrm{Spec}(K), \mathrm{id}, 0)$, [MNP13].

Example 5.11. We denote $1 = \mathfrak{h}(\mathrm{Spec}(K)) = (\mathrm{Spec}(K), \mathrm{id}, 0)$, cf. Definition 2.25. By computation of the diagonal $\Delta_{\mathbb{P}^1}$ can be shown $p_+(\mathbb{P}^1) = 0$, which implies $\mathfrak{h}(\mathbb{P}^1) = 1 \oplus \mathbb{L}$, where $\mathbb{L} := \mathfrak{h}^2(\mathbb{P}^1) = (\mathbb{P}^1, \mathbb{P}^1 \times e, 0)$ is called the *Lefschetz motive*, [MNP13]. It can also be proved that $\mathbb{L} = (\mathrm{Spec}(K), \mathrm{id}, -1)$.

Definition 5.12 (Grothendieck ring of Chow motives)**.** Let $\mathcal{A}$ be a a $\mathbb{Q}$-linear tensor category. Denote by $\mathrm{K}_0(\mathcal{A})$ the free abelian group of isomorphism classes $[X]$ of objects of $\mathcal{A}$ modulo the relations $[X \oplus Y] = [X] + [Y]$. The tensor product of $\mathcal{A}$ induces a commutative ring structure on $\mathrm{K}_0(\mathcal{A})$. The Grothendieck ring associated to the category of Chow motives $\mathrm{Chow}_{\mathbb{Q}}$ is called the Grothendieck ring of Chow motives and is denoted by $\mathrm{K}_0(\mathrm{Chow}_{\mathbb{Q}})$.

5.3. **Tools from birational geometry.** Let X be a surface, $P \in X(K)$ be a K-point and $\pi\colon \tilde{X} := \mathrm{Bl}_P(X) \longrightarrow X$ be the blow up of X at the point P.

Lemma 5.13. *The canonical divisor of $\tilde{X}$ is given by $K_{\tilde{X}} = \pi^* K_X + E$.*

Definition 5.14 (Intersection)**.** Let $Z \subset X$ be a proper over K closed subscheme of dimension $d \leqslant n$. The intersection of $\mathcal{L}_1, \ldots, \mathcal{L}_n \in \mathrm{Pic}(X)$ with Z is defined as

$$(\mathcal{L}_1 \ldots \mathcal{L}_2.Z) = \sum_{i_1, \ldots, i_m \subset \{1, \ldots, n\}} (-1)^m \chi(Z, \mathcal{L}_{i_1}^* \otimes \ldots \otimes \mathcal{L}_{i_m}^* |_Z).$$

If $Z = X$ we write $(\mathcal{L}_1 \ldots \mathcal{L}_n)$.

Definition 5.15 (Intersection form)**.** Let X be a smooth projective surface, then for two curves $C, D \subset X$ we define the intersection form as

$$C.D := (\mathcal{O}_X(C).\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X(-D)) + \chi(\mathcal{O}_X(-C - D)).$$

Note that this makes sense since Weil divisors are effective Cartier by smoothness.

Lemma 5.16. *Let X be a smooth projective surface. Then the map $\mathbb{Z} \longrightarrow \mathrm{Pic}(\tilde{X})$ defined by $1 \longmapsto \mathcal{O}_X(E)$, where E is the exceptional divisor of the blow up $\pi\colon \tilde{X} \longrightarrow X$, and the natural map $\pi^*\colon \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\tilde{X})$ determine an isomorphism $\mathrm{Pic}(\tilde{X}) \simeq \mathrm{Pic}(X) \oplus \mathbb{Z}$. The intersection form on $\mathrm{Bl}_P(X)$ is determined by:*

(i) *If $C, D \in \mathrm{Pic}(X)$, then $\pi^* C.\pi^* D = C.D$.*
(ii) *If $C \in \mathrm{Pic}(X)$, then $\pi^* C.E = 0$, where E is the exceptional line.*
(iii) *$E.E = -1$ in $\mathrm{Pic}(\tilde{X})$.*

Theorem 5.17 (Castelnuovo's contraction theorem, [Har77])**.** *Let X be a smooth surface, $C \subset X$ a curve with $C \simeq \mathbb{P}^1$ and $C^2 = -1$. Then there exists a smooth surface Y, a point $y \in Y$ and an isomorphism $X \simeq \mathrm{Bl}_y(Y)$ identifying C with the exceptional divisor.*

Corollary 5.18. *Let X be a smooth surface. Then X is minimal if and only if X contains no (-1)-curves.* □

Theorem 5.19 (Classification of minimal smooth surfaces, [Har77])**.** *Let X be a minimal smooth surface. Then X satisfies exactly one of the following conditions.*

(i) X has Kodaira dimension $\kappa(X) = -\infty$. Hence $X \simeq \mathbb{P}^2$ or X is a $\mathbb{P}^1$-bundle over a curve.

(ii) X has Kodaira dimension $\kappa(X) = 0$. Hence X is an Abelian, $K3$, Enriques or a (quasi)bielliptic surface.

(iii) X has Kodaira dimension $\kappa(X) = 1$. Hence X admits a fibration over a curve $f\colon X \longrightarrow C$ such that all the fibres are smooth elliptic curves.

(iv) X has Kodaira dimension $\kappa(X) = 2$. Hence X is of general type.

Theorem 5.20 (Weak factorisation theorem, [Abr+00])**.** *Let $\phi : X_1 \dashrightarrow X_2$ be a birational map between complete smooth connected varieties, let $U \subset X_1$ be an open set where ϕ is an isomorphism. Then ϕ can be factored into a sequence of blow-ups and blow-downs with smooth centres disjoint from U. There exists a sequence of birational maps*

$$X_1 = V_0 \overset{\phi_1}{\dashrightarrow} V_1 \overset{\phi_2}{\dashrightarrow} \dots \overset{\phi_i}{\dashrightarrow} V_i \overset{\phi_{i+1}}{\dashrightarrow} V_{i+1} \overset{\phi_{i+2}}{\dashrightarrow} \dots \overset{\phi_{l-1}}{\dashrightarrow} V_{l-1} \overset{\phi_l}{\dashrightarrow} V_l = X_2,$$

where $\phi = \phi_l \circ \phi_{l-1} \circ \cdots \phi_2 \circ \phi_1$, such that each factor ϕ_i is an isomorphism over U, and $\phi_i : V_i \dashrightarrow V_{i+1}$ or $\phi_i^{-1} : V_{i+1} \dashrightarrow V_i$ is a morphism obtained by blowing up a smooth centre disjoint from U.

5.4. **Resolutions of rational double point singularities.** The following study of rational double point singularities in surfaces is mostly based on [Dol12]. Throughout this section let X be a normal projective surface and $\pi\colon \tilde{X} \longrightarrow X$ be a minimal resolution of singularities.

Definition 5.21 (Resolution of singularity)**.** A resolution of X is a birational, proper and surjective morphism $\pi : \tilde{X} \longrightarrow X$, where $\tilde{X}$ is a non-singular projective variety. A resolution is called *minimal* if it does not factor non-trivially through another resolution of singularities.

Remark 5.22. It was proved in [Hir64] that resolutions always exist for varieties over fields of characteristic zero.

Lemma 5.23. *For any resolution $\pi : \tilde{X} \longrightarrow X$ we have $\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$.*

Proof. Without loss of generality we restrict to the affine case $X = \mathrm{Spec}(A)$. Since π is proper, $\pi_*\mathcal{O}_{\tilde{X}}$ is coherent. Hence $B = \Gamma(\pi_*\mathcal{O}_{\tilde{X}}, X)$ is a finitely generated A-module. By birationallity of π, A and B have the same quotient field. Additionally, X is normal, hence A is integrally closed, which implies $B = A$. □

Definition 5.24 (Rational singularity)**.** A singularity of X is *rational*, if for the resolution $\pi : \tilde{X} \longrightarrow X$ we have $R^1\pi_*\mathcal{O}_{\tilde{X}} = 0$.

Remark 5.25. The definition of a rational singularity is independent of the chosen resolution. To prove this consider the functor $\pi_* : \mathrm{Sh}(\tilde{X}) \longrightarrow \mathrm{Sh}(X)$, which sends injective objects to Γ-acyclic objects. Hence there exists a spectral sequence

$$E_2^{p,q} = (R^p\Gamma R^q\pi_*)(F) = H^p(X, R^q\pi_*(F)) \Rightarrow R^{p+q}(\Gamma \circ \pi_*)(F) = H^{p+q}(\tilde{X}, F),$$

which is natural in F. It can be verified that this sequence stabilises on the third page, hence

$$H^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) = E_\infty^{2,0} = E_3^{2,0} = E_2^{2,0}/E_2^{0,1} = H^2(X, \mathcal{O}_X)/H^0(X, R^1\pi_*\mathcal{O}_{\tilde{X}}).$$

This implies $p_a(X) - p_a(\tilde{X}) = h^0(X, R^1\pi_*\mathcal{O}_X)$ and the arithmetic genus is a birational invariant for varieties over fields of characteristic zero, [Har77].

Lemma 5.26 ([Dol12]). *The following statements are equivalent:*
(i) x is a rational singularity,
(ii) for every curve C supported in $\pi^{-1}(x)$, one has $H^1(C, \mathcal{O}_C) = 0$,
(iii) for every curve C supported in $\pi^{-1}(x)$ we have $p_a(C) = 1 + \frac{1}{2}C.(C + K_Y) \leqslant 0$.

For $x \in X$, the exceptional curve $E = \pi^{-1}(x)$ is compact and one dimensional, since X is proper and the resolution is a birational morphism. Additionally, it is connected by Zariski's connectedness theorem. Hence, E is the union of finitely many irreducible curves, say $\{E_i\}_{i \leqslant n}$. Let $Z_x = \sum_i n_i E_i$ be a positive cycle minimal (in terms of order on the set of effective divisors) with the property $Z.E_i \leqslant 0$ for all E_i supported in $\pi^{-1}(x)$, we call such a cycle a *fundamental cycle.*

Lemma 5.27. *The components E_i of the exceptional curve are isomorphic to $\mathbb{P}^1$ and are roots in the lattice associated to $\tilde{X}$, i.e. $E_i.K_{\tilde{X}} = 0$ and $E_i^2 = -2$.*

Proof. From Lemma 5.26 we conclude $p_a(E_i) = 0$ for all i, hence $E_i \simeq \mathbb{P}^1$. From corollary 5.18 we have $E_i^2 \leqslant -2$ because the resolution is minimal. By the adjunction formula, $E_i^2 + E_i.K_{\tilde{X}} = -2$ implies $E_i.K_{\tilde{X}} \geqslant 0$. Let Z be a fundamental cycle, then Lemma 5.26 implies $0 = 2 + Z^2 \leqslant -Z.K_{\tilde{X}} = -\sum_i n_i E_i.K_{\tilde{X}}$. This implies $E_i.K_{\tilde{X}} = 0$ for every E_i, hence the adjunction formula yields $E_i^2 = -2$. □

5.5. **Symmetric product.** This section is based on [Mus11]. Let X be a scheme of finite type over K, and let G be a finite group acting from the right on X by automorphism over K. We denote by σ_g the automorphism corresponding to $g \in G$.

Definition 5.28 (Geometric quotient). A geometric quotient of X by G is a pair $(X/G, \pi)$ consisting of a K-scheme X/G and a morphism of K-schemes π with the following properties:
(i) The morphism π is G-invariant, i.e. $\pi \circ \sigma_g = \pi$ for all $g \in G$,
(ii) The morphism π is surjective and the fibres of q over closed points of X/G are exactly the orbits of the closed points of X,
(iii) The scheme X/G carries the quotient topology induced by π,
(iv) The structure sheaf $\mathcal{O}_{X/G} = \pi_*(\mathcal{O}_X^G) \subset \pi_*\mathcal{O}_X$ consists of G-invariant sections.

In the affine case $X = \mathrm{Spec}(A)$, it can verified that $\pi\colon \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(A^G)$ induced by the inclusion $A^G \subset A$ is a geometric quotient, where A^G denote the G-invariant elements in A, see [Mus11]. In the case of X not being affine, we construct the geometric quotient locally. In order to do it, we require the existence of an affine open cover $\{U_i\}_{i\in I}$ of X such that U_i is G-invariant for every $i \in I$. The existence of such cover can be verified in the case of X being quasi-projective. For a detailed construction see [Núñ21].

Theorem 5.29 ([Mus11]). *Let X be a quasi-projective scheme and G a finite group acting on X. Then the geometric quotient $X \longrightarrow X/G$ exists.*

Lemma 5.30 ([Núñ21]). *Under the conditions of Theorem 5.29, we have that X/G is separated and of finite type. Moreover, the properties "reduced" and "projective" descend through the morphism π. In particular, if X is a projective variety, so is X/G.*

Given a smooth surface S, we define the *n-fold symmetric power of S* to be the geometric quotient $S^{(n)} := S^n/S_n$, where S_n is the n-th symmetric group. Additionally, we define the *Hilbert scheme of n points of S* to be $S^{[n]} := \mathcal{H}^n(S)$, see Definition 2.32. Note that a closed subscheme $Z \subset S$ having constant Hilbert polynomial n is zero dimensional and supported at finitely many closed points. Even more, $\dim_K H^0(Z, \mathcal{O}_Z) = n = \sum_i \dim_K(\mathcal{O}_{Z,z_i})$, where z_i runs over the points where Z is supported and we call $\dim_K H^0(Z, \mathcal{O}_Z)$ the *length* of Z.

Corollary 5.31. *The symmetric product of a smooth cubic surface S is a variety.* $\square$

Theorem 5.32 ([Fog68]). *Let S be a smooth projective surface. Then the Hilbert scheme $S^{[n]}$ is an irreducible smooth variety of dimension $2n$ and there is a K-scheme morphism*

$$\begin{aligned} \pi\colon S^{[n]} &\longrightarrow S^{(n)} \\ [Z] &\longmapsto \sum_{x\in S} \dim_K(\mathcal{O}_{Z,x})[x], \end{aligned}$$

called the Hilbert–Chow morphism. *Moreover, the Hilbert–Chow morphism is a resolution of singularities.*

TABLE 1. Character table for the Weyl group of type $\mathbf{E_6}$

Class	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
Order	1	2	2	2	2	3	3	3	4	4	4	4	5	6	6	6	6	6	6	6	8	9	10	12	12
$p = 2$	1	1	1	1	1	6	7	8	3	4	4	4	13	6	7	7	8	8	7	8	9	22	13	19	14
$p = 3$	1	2	3	4	5	1	1	1	9	10	11	12	13	3	3	2	3	2	4	5	21	6	23	10	9
$p = 5$	1	2	3	4	5	6	7	8	9	10	11	12	1	14	15	16	17	18	19	20	21	22	2	24	25
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	1	1	1	1	-1	-1	1	1	1	1	-1	1	-1	1	-1	-1	1	-1	-1	1
χ_3	6	4	-2	2	0	-3	3	0	2	-2	2	0	1	1	1	1	-2	-2	-1	0	0	0	-1	1	-1
χ_4	6	-4	-2	2	0	-3	3	0	2	2	-2	0	1	1	1	-1	-2	2	-1	0	0	0	1	-1	-1
χ_5	10	0	-6	2	0	1	-2	4	2	0	0	-2	0	-3	0	0	0	0	2	0	0	1	0	0	-1
χ_6	15	-5	7	3	-1	-3	0	3	-1	-3	1	1	0	1	-2	-2	1	1	0	-1	1	0	0	0	-1
χ_7	15	-5	-1	-1	3	6	3	0	3	-1	-1	-1	0	2	-1	1	2	-2	-1	0	1	0	0	-1	0
χ_8	15	5	7	3	1	-3	0	3	-1	3	-1	1	0	1	-2	2	1	-1	0	1	-1	0	0	0	-1
χ_9	15	5	-1	-1	-3	6	3	0	3	1	1	-1	0	2	-1	-1	2	2	-1	0	-1	0	0	1	0
χ_{10}	20	10	4	4	2	2	5	-1	0	2	2	0	0	-2	1	1	1	1	1	-1	0	-1	0	-1	0
χ_{11}	20	-10	4	4	-2	2	5	-1	0	-2	-2	0	0	-2	1	-1	1	-1	1	1	0	-1	0	1	0
χ_{12}	20	0	4	-4	0	-7	2	2	4	0	0	0	0	1	-2	0	-2	0	2	0	0	-1	0	0	1
χ_{13}	24	4	8	0	4	6	0	3	0	0	0	0	-1	2	2	-2	-1	1	0	1	0	0	-1	0	0
χ_{14}	24	-4	8	0	-4	6	0	3	0	0	0	0	-1	2	2	2	-1	-1	0	-1	0	0	1	0	0
χ_{15}	30	-10	-10	2	2	3	3	3	-2	4	0	0	0	-1	-1	-1	-1	-1	-1	-1	0	0	0	1	1
χ_{16}	30	10	-10	2	-2	3	3	3	-2	-4	0	0	0	-1	-1	1	-1	1	-1	1	0	0	0	-1	1
χ_{17}	60	10	-4	4	2	6	-3	-3	0	-2	-2	0	0	2	-1	1	-1	1	1	-1	0	0	0	1	0
χ_{18}	60	-10	-4	4	-2	6	-3	-3	0	2	2	0	0	2	-1	-1	-1	-1	1	1	0	0	0	-1	0
χ_{19}	60	0	12	4	0	-3	-6	0	4	0	0	0	0	-3	0	0	0	0	-2	0	0	0	0	0	1
χ_{20}	64	16	0	0	0	-8	4	-2	0	0	0	0	-1	0	0	-2	0	-2	0	0	0	1	1	0	0
χ_{21}	64	-16	0	0	0	-8	4	-2	0	0	0	0	-1	0	0	2	0	2	0	0	0	1	-1	0	0
χ_{22}	80	0	-16	0	0	-10	-4	2	0	0	0	0	0	2	2	0	2	0	0	0	0	-1	0	0	0
χ_{23}	81	9	9	-3	-3	0	0	0	-3	3	-1	-1	1	0	0	0	0	0	0	0	1	0	-1	0	0
χ_{24}	81	-9	9	-3	3	0	0	0	-3	-3	1	-1	1	0	0	0	0	0	0	0	-1	0	1	0	0
χ_{25}	90	0	-6	-6	0	9	0	0	2	0	0	2	0	-3	0	0	0	0	0	0	0	0	0	0	-1

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