Unirationality of hypersurfaces

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Zusammenfassung

Diese Arbeit befasst sich mit der birationalen Geometrie glatter Hyperflächen über einem algebraisch abgeschlossenen Körper der Charakteristik null. Da bereits im Fall von Kubiken die Rationalität in höheren Dimensionen im Allgemeinen sehr schwer zu bestimmen ist, konzentrieren wir uns auf den abgeschwächten Begriff der Unirationalität. Zunächst stellt man fest, dass in den Fällen von Kurven und Flächen die beiden Begriffe äquivalent sind, doch in höheren Dimensionen gilt dies nicht. Erste Beispiele für Letzteres wurden in den 70ern gefunden.

Bereits der italienischen Schule der algebraischen Geometrie war die Rationalität von glatten Quadriken und die Unirationalität von glatten Kubiken bekannt. Morin zeigte im Jahr 1940, dass auch für beliebigen höheren Grad die allgemeine Hyperfläche mit genügend hoher Dimension unirational ist, siehe [M]. Harris, Mazur und Pandharipande verallgemeinerten dieses Resultat 1998 in [HMP]. Sie bewiesen, dass sogar jede glatte Hyperfläche unirational ist, falls ihre Dimension groß ist im Vergleich zu ihrem Grad.

Diese Bachelorarbeit beleuchtet den induktiven Beweis für diese Aussage, ohne ihn vollständig zu reproduzieren. Vielmehr wollen wir den Schwerpunkt auf die Fälle von Kubiken und Quartiken legen, da diese bereits sowohl die grundlegenden Beweisideen als auch die auftretenden Probleme in höheren Graden enthalten. Essentiell für all diese Betrachtungen ist die Existenz von linearen Unterräumen auf Hyperflächen hoher Dimension, auf die wir ebenfalls eingehen werden.

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CONTENTS

INTRODUCTION

This thesis is concerned with the birational geometry of smooth hypersurfaces in projective spaces over algebraically closed fields of characteristic zero. It turns out to be very difficult to determine whether a given variety is rational and hence we will focus on the weaker notion of unirationality. It is easy to see that Lüroth's theorem is just an algebraic restatement of the fact that unirational curves are already rational, and the same can even be proved for surfaces. However, this is wrong for higher dimensions, though the first examples have not been found until the 1970s.

It was already known to the Italian school of algebraic geometry that for example every smooth quadric is rational if its dimension is at least one and every smooth cubic is unirational if its dimension is at least two. The classical proofs crucially involve on one hand the existence of a point on such a quadric and a line on such a cubic on the other hand. Of course the first fact is trivial in our setting. But ensuring the existence of lines on a cubic leads directly to the study of Fano varieties and their dimension.

In 1940 Morin generalized these results by showing that for any degree the general hypersurface is unirational if its dimension is sufficiently high. Again his proof in [M] uses the existence of higher-dimensional planes on such a hypersurface. Morin's result was further generalized later. On one hand Predonzan found that the statements can be extended to the case of complete intersections in 1949 in [P]. The same result was also proved in a more recent paper from Paranjape and Srinivas in 1992, see [PS]. On the other hand, Harris, Mazur, and Pandharipande extended Morin's statement to smooth hypersurfaces in 1998, see [HMP]. The focus of this thesis will be on their work. However, our goal is not to reproduce their already well written proof in a formal way, but rather to give an idea of how the classical results are generalized and what problems have to be dealt with.

The structure of this thesis

In Section 1 the basic notions and examples will be introduced. We will give the classical proof for the rationality of smooth quadrics and see some counter examples to Lüroth's problem, i.e. unirational hypersurfaces that are not rational.

As indicated, we will always need some results on the dimension of Fano varieties of planes on hypersurfaces in the background. In Section 2 the existence of planes on high-dimensional hypersurfaces will be provided.

The second classical result, that is the unirationality of smooth cubics, will be proved in detail in Section 3. In this proof the problem will be reduced to showing that a certain Fano correspondence is rational. In the case of cubics it turns out to be a projective bundle over some Grassmannian. We will also try to extend the proof to the case of smooth quartics, but this attempt will fail, since the Fano correspondence might become reducible and in particular irrational. An explicit example for this phenomenon will be given.

Finally, in Section 4 we will sketch how Harris, Mazur, and Pandharipande prove that for high-dimensional smooth hypersurfaces these Fano correspondences have an irreducible component one can work with and how they deduce an inductive proof for the unirationality of any high-dimensional smooth hypersurface from this. We will also compare their results to the statements in [PS].

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With this bachelor thesis the first part of my studies comes to an end. I welcome this opportunity to thank all those people who have been there for me over the past years. My utmost thanks are due to my family, especially my mother, my brother and my sisters for their constant support. Last but not least I owe thanks to my girlfriend Maren for a great number of things, among them her patience and tolerance during the work on this thesis.

1 RATIONALITY AND UNIRATIONALITY

This first section is dedicated to introducing the notions of rationality and unirationality as well as to giving examples which might be helpful in the context of this thesis. Furthermore Lüroth's problem, which is a classical problem on the relationship between these two notions, will be examined briefly.

In this thesis we will work over some algebraically closed field of characteristic zero, if not stated otherwise, and n will always be a positive integer.

1.1 Rational varieties

Definition 1.1 A (quasi-projective) variety X is said to be *rational*, if it is birationally equivalent to some $\mathbb{P}^n$, i.e. there are mutually inverse dominant rational maps $X \dashrightarrow \mathbb{P}^n$ and $\mathbb{P}^n \dashrightarrow X$. In this case we will also say that the rational maps are "birational".

Remark 1.2 According to the definition of rational maps, rationality of X means that some non-empty open subset of it is isomorphic to an open subset of $\mathbb{P}^n$. Moreover we can understand rational varieties in terms of algebra: dominant rational maps define inclusions of function fields and this yields an arrow-reversing equivalence of the category of varieties with dominant rational maps with the category of finitely generated field extensions of k. Thus X being rational is equivalent to $k(X) \cong k(\mathbb{P}^n) = k(x_1, \ldots, x_n)$ for some n.

At this point some examples of rational and irrational hypersurfaces will be given before we start talking about unirational varieties.

Example 1.3 Quadric hypersurfaces.
Every smooth quadric $X \subseteq \mathbb{P}^{n+1}$ is rational. For $n \in \{1, 2\}$ one can even give a classification up to isomorphism: Every smooth conic is isomorphic to $\mathbb{P}^1$ and every smooth quadric surface is isomorphic to the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, which is birationally equivalent to $\mathbb{P}^2$.
In the general case take the projection π from any point $P \in X$ onto some n-plane H not containing P. By a change of coordinates one may assume that $P = [0 : 0 : \ldots : 0 : 1]$ and $H = \mathrm{Z}(x_{n+1})$. Then $\pi : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^n$ is given by $[x_0 : \ldots : x_{n+1}] \mapsto [x_0 : \ldots : x_n]$ and its restriction to X is rational and generically one-to-one: Assume that X is given as the zero locus of

$$f = \sum_{i,j=0}^{n} a_{ij} x_i x_j + x_{n+1} \sum_{k=0}^{n} a_k x_k + a x_{n+1}^2,$$

where $P \in X$ implies $a = 0$. Then for any $[z_0 : \ldots : z_n] \in \mathbb{P}^n \backslash \mathrm{Z}(\sum_{k=1}^{n} a_k x_k)$ we have

$$\pi|_X^{-1}([z_0 : \ldots : z_n]) = \{[z_0 : \ldots : z_n : \frac{-\sum a_{ij} z_i z_j}{\sum a_k z_k}]\}.$$

Thus $\pi|_X$ is birational. Indeed, the description of the fibres above gives us an set-theoretically inverse map and [Hr, Ex. 7.8] implies that this map is rational (provided that $\mathrm{char}(k) = 0$).
Note that the statement on quadrics is still true if we replace “smooth” by “irreducible”. It also remains true if we allow k to have positive characteristic, provided that there is a rational point on X, see [SR, Prop. 1.3]. This condition is clearly fulfilled over algebraically closed fields.

Example 1.4 Cubic surfaces.
Now let X be a smooth cubic surface in $\mathbb{P}^3$. We show that it is birational to $\mathbb{P}^1 \times \mathbb{P}^1$ and hence rational, cf. [Hu, Prop. 5.17]: Choose a pair of skew lines $l, m \subseteq X$ and note that by a change of coordinates one may assume $l = \mathrm{Z}(x_2, x_3)$ and $m = \mathrm{Z}(x_0, x_1)$. Now consider the rational map:

$$\begin{aligned} \pi : \mathbb{P}^3 &\dashrightarrow l \times m = \mathbb{P}^1 \times \mathbb{P}^1 \\ [x_0 : x_1 : x_2 : x_3] &\longmapsto ([x_0 : x_1], [x_2 : x_3]). \end{aligned}$$

Geometrically, this map sends a point $R \in X$ to the pair (P, Q), where P and Q are the unique points of intersection of l with the linear span $\overline{R, m}$ and of m with $\overline{l, R}$, respectively. Again, its restriction to X is generically one-to-one. This is due to the fact that for a general pair $(P, Q) \in l \times m$ there is a unique third point in $\overline{PQ} \cap X$ (just consider the intersection multiplicities). However, $\pi|_X(R) = (P, Q)$ holds if and only if P, Q, R are collinear.

Example 1.5 We have seen that smooth quadrics and cubics in $\mathbb{P}^3$ are rational, but surfaces of higher degree are not, though this is a bit harder to prove. Roughly, one could argue as follows: Define the geometric genus $p_g(X)$ of a smooth projective variety X to be the dimension of the space of total sections of the canonical sheaf. This is a birational invariant. By using the Euler sequence, one finds that $\omega_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$ and $p_g(\mathbb{P}^2) = 0$. But if X is of degree d then the adjunction formula gives us $\omega_X \cong \mathcal{O}_X(d-3)$ and for $d > 3$ one concludes that $p_g(X) > 0$. For more detail see [Hs, Prop. II.8.20]. The proof does not depend on the characteristic (and we will use this in Example 1.8).

In contrast to the first example, smooth cubics are not rational in arbitrary dimension. The situation is in fact a bit intricate: Elliptic curves are not rational since their geometric genus is one, but cubic curves with a double point are rational. Smooth cubic threefolds are irrational, which is a famous result due to Clemens and Griffiths, see [CG, Thm 13.12]. Moreover, it is unknown whether general cubic hypersurfaces in higher dimensions are rational, cf. [Hr, Ch. 7].

1.2 Unirational varieties and Lüroth's problem

Definition 1.6 A variety X is said to be *unirational* if there is a dominant rational map $\mathbb{P}^n \dashrightarrow X$ for some n.

Remark 1.7 In geometric terms, a variety is unirational if it is covered by a rational variety. Indeed, if $Y \to X$ is a surjective morphism and Y is rational then the composition gives a dominant rational map $\mathbb{P}^n \dashrightarrow X$. Again another equivalent condition using function fields can be given: X is unirational if and only if $k(X)$ can be embedded into some $k(\mathbb{P}^n) = k(x_1, \dots, x_n)$. Since $k \subseteq k(X)$ is a finitely generated field extension of transcendence degree $\dim(X)$, one can always assume that $n = \dim(X)$.

Two natural questions arise: Are there unirational varieties that are not rational? And if so, in what way is this new notion "better" or easier to manage?

Let us first deal with the first question, since the latter (and more imprecise) one will be answered in later chapters. In the case of dimension one, we find that any unirational curve X is in fact rational. Indeed, its function field $k(X)$ is a non-trivial intermediate field of $k \subseteq k(x_1)$ and by Lüroth's theorem any such intermediate field of a simple, purely transcendental field extension is itself a simple, purely transcendental field extension. So $k(X) \cong k(x_1)$ and hence X is rational.

In 1861 this classical observation led to the question if this is true in any dimension, cf. [K]. It became famous as

Lüroth's problem Is every unirational variety rational? Or in other words: Is every non-trivial subfield of some field of rational functions isomorphic to a field of rational functions?

Using another classical theorem (Castelnuovo's rationality criterion) one can also prove that this holds for surfaces over algebraically closed fields of characteristic zero, see [Hs, Thm V.6.2]. However, in higher dimensions the question remained open for nearly hundred years. The answer is negative, as we will see in the following examples. Thus, the notion of unirationality gives us really a new tool to study higher-dimensional varieties.

Example 1.8 Let us first give an example in positive characteristic that was given by Tate already in 1965, see [T]: Let k be algebraically closed with $\mathrm{char}(k) = p > 0$ and consider the Fermat surface of degree d, i.e. $X_d = \mathrm{Z}(w^d + x^d + y^d + z^d) \subseteq \mathbb{P}^3$. The smooth variety X_4 is not rational (because of Example 1.5). But if $p \equiv 3 \pmod 4$ then X_4 is unirational, which we will prove in two steps, following [Sh].
i) Assume $p = 3$. We will find that $X_4 = \mathrm{Z}(w^4 + x^4 + y^4 + z^4)$ is unirational by computing its field of functions. First change coordinates and write the defining equation as $w^4 - x^4 = y^4 - z^4$. Substituting $a = w + x, b = w - x, c = y + z, d = y - z$ and setting $d = 1$ does not change the function field, hence let X_4 be given by

$$ab(a^2 + b^2) = c(c^2 + 1).$$

Now X_4 is birational to the variety Y defined by $a^4(1 + u^2) = v(u^2v^2 + 1)$. Indeed, the rational maps $X_4 \dashrightarrow Y, [a : b : c] \mapsto [a : \frac{b}{a} : \frac{ca}{b}]$, defined on the complement of $\mathrm{Z}(ab)$, and $Y \dashrightarrow X_4, [a : u : v] \mapsto [a : au : uv]$, defined on the complement of $\mathrm{Z}(a, u) \cup \mathrm{Z}(a, v)$, are mutually inverse. Therefore,

$$K := k(X_4) \cong k(Y) \cong \mathrm{Quot}(k[a, u, v]/(a^4(1 + u^2) - v(u^2v^2 + 1))),$$

and it is enough to show that the latter field can be embedded into some field of rational functions. The morphism

$$\begin{aligned} \varphi : k[a,u,v] &\longrightarrow k[t,u,v]/(u^2(t^4-v)^3 - v + t^{12}) \\ a &\longmapsto t^3 \end{aligned}$$

induces an inclusion $K \hookrightarrow K' := \mathrm{Quot}(k[t,u,v]/(u^2(t^4-v)^3 - v + t^{12}))$. Note that the assumption $p = 3$ is being used here to get $(t^4 - u)^3 = t^{12} - u^3$. Now set $s = u(t^4 - v)$ and solve $s^2(t^4 - v) = v - t^{12}$ for v. Hence,

$$K \hookrightarrow K' \cong \mathrm{Quot}(k[t,s,v]/(v - \frac{t^4(s^2+t^8)}{s^2+1})) \cong k(t,s)$$

and X_4 is unirational.
ii) If $p \equiv 3 \pmod 4$, then X_{p+1} is unirational by the same computation. Now X_4 is covered by X_{p+1}, since $[w : x : y : z] \mapsto [w^{\frac{p+1}{4}} : x^{\frac{p+1}{4}} : y^{\frac{p+1}{4}} : z^{\frac{p+1}{4}}]$ defines a surjective regular map. Hence X_4 is unirational.

As we consider mainly fields of characteristic zero, one should note that there are counterexamples for that case too (which are much more difficult to find). We already mentioned that Clemens and Griffiths proved that smooth cubic threefolds are not rational in [CG] in 1972, but in Section 3.1 we will find that they are unirational. In fact we will show that every smooth cubic hypersurface of dimension at least two is unirational. Another example was given by Iskovskih and Manin in 1971: They proved that smooth quartic threefolds are irrational, using the observation that birational equivalence between smooth quartic threefolds is actually a projective equivalence and concluding that their automorphism groups are finite. This implies irrationality, because the automorphism group of $\mathbb{P}^3$ is infinite. On the other hand Segre had already found some unirational smooth quartic threefolds, see [IM, Sect. 1].

As indicated in Section 1.1, it might be very difficult to check whether a given variety is rational or to make statements about the rationality of smooth hypersurfaces of fixed degree and dimension. We have seen that being unirational is a weaker condition on X than being rational. However, it might be still quite difficult to decide whether a given variety X is unirational, but it will be much easier to make general statements about unirationality of hypersurfaces satisfying certain conditions in Section 3 and Section 4.

2 FANO VARIETIES

Our first examples of rational varieties in the previous section have been smooth quadric hypersurfaces. The proof was based on choosing some point on a given quadric and projecting from it to get a rational parametrization. Similarly the proof of the unirationality of a hypersurface of higher degree will begin with choosing some linear subspace on it. In the case of a smooth two-dimensional cubic we know that there are 27 lines on it, but in the more general case of a hypersurface of degree d the existence of k-planes is not obvious. So we will first state some facts about Fano varieties in general and then prove that smooth hypersurfaces of low degree contain linear subspaces.

Let $k < n$ and let $\mathbb{G}(k,n)$ denote the Grassmann variety of k-planes in $\mathbb{P}^n$. The Fano variety of k-planes on a hypersurface $X \subseteq \mathbb{P}^n$ is defined as

$$F_k(X) = \{\Lambda \in \mathbb{G}(k,n) \,|\, \Lambda \subseteq X\},$$

which forms a subvariety of $\mathbb{G}(k,n)$. Our intention is to be able to choose a plane in later proofs, so it would be enough for our purposes to find $\dim(F_k(X)) \geq 0$ for low-degree smooth hypersurfaces. So let us fix the degree d and try to find the dimension of the Fano variety.

The vector space of homogeneous polynomials of degree d in $n+1$ variables is of dimension $\binom{n+d}{d}$. Hence the space of hypersurfaces of degree d in $\mathbb{P}^n$ is parametrized by $\mathbb{P}^N$, where $N = \binom{n+d}{d} - 1$. Now consider the incidence variety

$$\begin{tikzcd} & I := \{(X,\Lambda) \in \mathbb{P}^N \times \mathbb{G}(k,n) \,|\, \Lambda \subseteq X\} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^N & & \mathbb{G}(k,n) \end{tikzcd}$$

and the projections π_1, π_2. Note that $F_k(X)$ is the fibre of π_1 over X and that π_2 is surjective. In order to compute the dimension of the fibres of π_1 we need the dimension of I first.

Lemma 2.1 *The incidence correspondence I is irreducible of dimension $(k+1)(n-k) + N - \binom{k+d}{d}$.*

Proof. This follows from studying the fibres of π_2. Let Λ be a k-plane. Let φ be the canonical epimorphism[1] from the space of polynomials of degree d on $\mathbb{P}^n$ to the space of polynomials of degree d on Λ, more precisely the one induced by projection from $k[x_0, \ldots, x_n]$

[1] Of course this is just the map $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \twoheadrightarrow \Gamma(\Lambda, \mathcal{O}_\Lambda(d))$, cf. [Hs, Ex. II.5.14].

to the homogeneous coordinate ring of Λ. The kernel of this linear map is of dimension $\binom{n+d}{d} - \binom{k+d}{d}$. Now $\mathbb{P}(\ker(\varphi)) \cong \mathbb{P}^{N-\binom{k+d}{d}}$ is the fibre of π_2 over Λ and hence all the fibres are irreducible and of the same dimension. This implies that I is irreducible, see for example [Sf, Thm I.6.8]. Moreover we can compute the dimension of I via the fibres of π_2 and get

$$\dim(I) = \dim(\mathbb{G}(k,n)) + \dim(\pi_2^{-1}(\Lambda)) = (k+1)(n-k) + N - \binom{k+d}{d}. \qquad \square$$

It would be nice if one could apply the same arguments to π_1 to get the dimension of $F_k(X)$. However, π_1 might be not surjective. If we could ensure that π_1 is surjective, the general fibre, that is the Fano variety of a general hypersurface, would have dimension

$$\Phi(n,d,k) := \dim(I) - N = (k+1)(n-k) - \binom{k+d}{d}.$$

This is also called the "expected" dimension of $F_k(X)$. For general X it is actually the dimension, see the following result from Chapter 12 in [Hr]:

Proposition 2.2 *For $d \geq 3$ the Fano variety $F_k(X)$ of a general hypersurface $X \subseteq \mathbb{P}^n$ is empty if $\Phi(n,d,k) < 0$ and of dimension $\Phi(n,d,k)$ otherwise.*

In [HMP] the authors state that for any hypersurface of degree $d \geq 3$ the dimension of $F_k(X)$ is bounded from below by $\Phi(n,d,k)$, if $\Phi(n,d,k)$ is non-negative, and prove equality for high-dimensional smooth hypersurfaces, i.e. that such varieties do not have "too many" k-planes. We will not go into the details here, but rather give an easier proposition ([HMP, Lem. 3.9][2]) that will be sufficient for our purposes in Section 3. For natural numbers d, k let

$$M(d,k) := \binom{d+k-1}{d-1} + k - 1.$$

Proposition 2.3 *If $n > M(d,k+1)$ and $X = \mathrm{Z}(F) \subseteq \mathbb{P}^n$ is any hypersurface of degree d, then $\Lambda \in F_k(X)$ implies that there is some linear subspace $\Gamma \in F_{k+1}(X)$ containing Λ.*

Proof. Choose homogeneous coordinates $Z_0, \ldots, Z_k, W_{k+1}, \ldots, W_n$ on $\mathbb{P}^n$ such that $\Lambda = \mathrm{Z}(W_{k+1}, \ldots, W_n)$ and write

$$F(Z,W) = \sum_{0 \leq |I| \leq d} Z^I \cdot F_I(W),$$

where $F_I \in k[W_{k+1}, \ldots, W_n]$ is homogeneous of degree $d - |I|$. Since for every index $|I| = d$ the polynomial F_I is constant and F vanishes on Λ, we have $\sum_{|I|=d} Z^I F_I = 0$ and

$$F(Z,W) = \sum_{0 \leq |I| < d} Z^I \cdot F_I(W),$$

[2] There is a small inaccuracy in the statement and in the proof in [HMP]: we have to require that n is *strictly* greater than $M(d,k+1)$ in order to get that $n - k > M(d,k+1) - k = \binom{k+d}{k+1}$. E.g. for $k = 0, d = 3, n = M(3,1) = 3$ and X smooth the statement would otherwise imply that every point on X is contained in one of the 27 lines.

where all F_I have positive degree now. These are $\sum_{l=0}^{d-1}\binom{k+l}{l} = \binom{k+d}{k+1}$ polynomials in $n-k$ variables. But the assumption

$$n-k > M(d,k+1)-k = \binom{d+k}{d-1} + k - k = \binom{k+d}{k+1}$$

implies now that $n-k-1 \geq \binom{k+d}{k+1}$ and hence the F_I have a non-trivial common zero $Q = [Q_{k+1} : \ldots : Q_n]$, cf. [Sf, Cor. I.6.2.5]. We claim that the linear span $\Gamma := \overline{\Lambda, Q}$ is contained in X. Note that Γ is parametrized by $\{[Z_0 : \ldots : Z_k : Q_{k+1}Y : \ldots : Q_nY]\}$ and one has homogeneous coordinates $Z_0, \ldots, Z_k, Y$ on Γ. Then for any $(Z,Y) \in \Gamma$ we have

$$\begin{aligned} F(Z,Y) &= \sum_{0 \leq |I| < d} Z^I F_I(Q_{k+1}Y, \ldots, Q_nY) \\ &= \sum_{0 \leq |I| < d} Z^I Y^{d-|I|} F_I(Q) = 0 \end{aligned}$$

and therefore $\Gamma \subseteq X$. $\square$

Since $k \leq k'$ implies $M(d,k) \leq M(d,k')$, we can use this result repeatedly as long as $n > M(d,k)$ holds. In particular we will be able to find at least one k-plane on each smooth hypersurface of sufficiently high dimension.

One of the main theorems from [HMP] says that smooth hypersurfaces are unirational, if their dimension is high enough. Before we turn explicitly towards that theorem we will first discuss the case of cubic and quartic hypersurfaces. We will prove the unirationality of smooth cubics, where the condition “high enough” means at least of dimension two and will enable us to choose a plane on such a cubic according to the preceding section. After that we will try to transfer the proof to the case of quartics and give examples of reducible Fano correspondences in order to see what problems arise in higher degrees. This whole section is based on [HMP, Sect. 2].

3.1 Unirationality of cubic hypersurfaces

Now let us consider the easiest case of the main theorem, that is the case of cubics. The result was already known to Max Noether, cf. [CG, p. 352], yet the proof we present here is from [HMP, Sect. 2.1].

Theorem 3.1 *For $n \geq 3$ any smooth cubic hypersurface in $\mathbb{P}^n$ is unirational.*

Let $X \subseteq \mathbb{P}^n$ be a smooth cubic hypersurface. If $n \geq M(3,1) = 3$ then X contains a line. Indeed, this is a well-known fact for $n = 3$ and for $n > 3$ we can use Proposition 2.3. Though the proof would work with a line, we choose an l-plane $\Gamma \in \mathrm{F}_l(X)$, where $l \geq 1$, having the general case in mind. Consider the set of $(l+1)$-planes that contain Γ, which is just the sub-Grassmannian $\mathbb{G}(0, n-l-1) \subseteq \mathbb{G}(l+1, n)$. So these planes are parametrized by $\mathbb{P}^{n-l-1}$. The next step is to understand what the general $(l+1)$-plane section of X looks like.

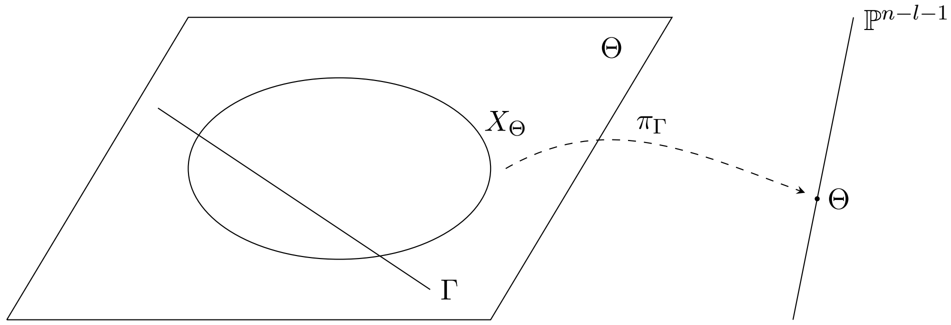
Lemma 3.2 *A generic $(l+1)$-plane Θ containing Γ intersects X in the union of Γ and an irreducible l-dimensional quadric hypersurface X_Θ.*

Proof. To begin with, we have $\deg(\Theta) \cdot \deg(X) = 3$. Therefore, it is enough to show that Γ is the only irreducible component of degree one in $X \cap \Theta$ and that the intersection multiplicity of X and Θ along Γ is one. Then Bézout’s formula implies the lemma, cf. [Hs, Thm I.7.7].

First, one checks that the intersection multiplicity along Γ is one, by showing that Θ and X intersect transversally at a general $p \in \Gamma$, i.e. $\overline{\mathrm{T}_p\Theta, \mathrm{T}_pX} = \mathrm{T}_p\mathbb{P}^n$, which is equivalent to the condition $\dim(\Theta \cap \mathrm{T}_pX) = l$. Since Γ is contained in the intersection $\Theta \cap \mathrm{T}_pX$ the dimension must be at least l. If it were strictly greater than l, one would have $\Theta \subseteq \Theta \cap \mathrm{T}_pX$, so $\Theta \subseteq \mathrm{T}_pX$. However, if the latter were true for a generic Θ, this would imply $\mathrm{T}_pX = \mathbb{P}^n$ and the point p would have to be a singular point of X.

Before we show that Γ is the only component of degree one, let us first note that every irreducible component W of the intersection is l-dimensional. By the projective dimension theorem every component has at least dimension $\dim(\Theta) + \dim(X) - n = l$ and $W \subseteq \Theta \cap X \subsetneq \Theta$ implies $\dim(W) < l + 1$. In particular X and Θ intersect properly, i.e. the intersection has the expected dimension.

Assume now that Γ is not the only component of degree one, i.e. there is another irreducible component W with $\deg(W) = 1$. Then W is also an l-plane and since both W and Γ are contained in the $(l+1)$-plane Θ, there is a point $p \in \Gamma \cap W$. Then W and Γ are both contained in T_pX and moreover $\Theta = \overline{W, \Gamma} \subseteq \mathrm{T}_pX$, which would again imply that p is a singular point of X. □



The projection π_Γ from Γ to $\mathbb{P}^{n-l-1}$ defines a regular map $X \backslash \Gamma \to \mathbb{P}^{n-l-1}$ with generic fibre $\pi_\Gamma^{-1}(\Theta) = X_\Theta \backslash \Gamma$. In order to get a family of quadric hypersurfaces we resolve the map, that is we eliminate indeterminacy of the rational map $\pi_\Gamma : X \dashrightarrow \mathbb{P}^{n-l-1}$ by blowing X up along Γ. In this way one gets a family of quadrics

$$\pi : \tilde{X} = \mathrm{Bl}_\Gamma(X) \to \mathbb{P}^{n-l-1},$$

which is birational to X. We know that the general member of that family is rational, but in order to find a rational parametrization of the total space one would need to choose rational parametrizations of the fibres consistently over some open subset. Giving a rational parametrization of an irreducible quadric is equivalent to giving a point on it, see Example 1.3. Thus one actually needs to find a rational section $\mathbb{P}^{n-l-1} \dashrightarrow \tilde{X}$. To be more precise, we want to use the following

Lemma 3.3 *Let $p : E \to B$ be a family of (generically irreducible) quadric hypersurfaces over a rational base B. If there is a rational section $B \dashrightarrow E$ of p, then E is rational.*

Proof. Let $\sigma : B \dashrightarrow E$ be a rational section and let W be a non-empty open subset $W \subseteq B$ such that σ is regular on W and every fibre $E_b = p^{-1}(b)$ over W is irreducible. For any $b \in W$ the point $\sigma(b) \in E_b$ can be used to project from it and to get a birational isomorphism $\varphi_b : E_b \dashrightarrow \mathbb{P}^{n-1}$. Then we can define a rational map

$$\begin{aligned} f : E &\dashrightarrow B \times \mathbb{P}^{n-1} \\ (x \in E_b) &\longmapsto (b, \varphi_b(x)), \end{aligned}$$

which is regular on $p^{-1}(W)$. Now, f is generically one-to-one, since every φ_b is. Hence f is birational and E is rational. $\square$

As already mentioned there are smooth cubics that are irrational, so Lemma 3.3 implies that one can not find a rational section of π in general. This problem can be eliminated by doing a base change and considering pointed quadrics: Pull the family of quadrics $\tilde{X} \to \mathbb{P}^{n-l-1}$ back to the incidence variety

$$I := \{(\Theta, p) \in \mathbb{P}^{n-l-1} \times \Gamma \,|\, p \in X_\Theta\} \to \mathbb{P}^{n-l-1}$$

to get

$$H := \tilde{X} \times_{\mathbb{P}^{n-l-1}} I = \{(q, \Theta, p) \in \tilde{X} \times \mathbb{P}^{n-l-1} \times \Gamma \,|\, p, q \in X_\Theta\}.$$

Now the projection $\rho : H \to I$ makes H a family of pointed quadrics: The fibre over a generic $(\Theta, p) \in I$ is $\{(q, \Theta, p) \,|\, q \in X_\Theta\} = (X_\Theta, p)$. It is not hard to find a rational section of this family: Just take $\sigma : I \dashrightarrow H, (\Theta, p) \mapsto (p, \Theta, p)$.

$$\begin{tikzcd} H \arrow[r] \arrow[d, "\rho"] & \tilde{X} \arrow[d, "\pi_\Gamma"] \\ I \arrow[r] \arrow[u, dashed, bend left, "\sigma"] & \mathbb{P}^{n-l-1} \end{tikzcd}$$

Unfortunately one can not apply Lemma 3.3 yet, because first the rationality of our base I has to be tackled. Note that $I \to \mathbb{P}^{n-l-1}$ is a family of $(l-1)$-dimensional quadrics and we may try to use the same lemma to get the rationality of I, but here we encounter the same problem as before: A rational section of this family can not be found in general. In fact this approach does not seem to take us much further. However, the other projection $f : I \to \Gamma$ will turn out to be a $\mathbb{P}^{n-l-2}$-bundle by Proposition 3.5 and in particular it has a rational section. Due to local triviality and the fact that $\Gamma \cong \mathbb{P}^l$ we observe that I is birational to $\mathbb{P}^{n-l-2} \times \mathbb{P}^l$ and is thus rational. Therefore, our next step will be to show that I fulfils the requirements of Proposition 3.5:

Lemma 3.4 *The incidence variety I is irreducible and forms a family of $(n-l-2)$-planes in $\mathbb{P}^{n-l-1}$ via f.*

Proof. Each fibre is parametrized by $\mathbb{P}^{n-l-2}$. Indeed, the fibre $f^{-1}(p) = \{(\Theta, p) \,|\, p \in X_\Theta\}$ over $p \in \Gamma$ is just $\{\Theta \,|\, p \in X_\Theta\} \times \{p\}$. Now for a given pair (Θ, p) the condition $p \in X_\Theta$ is equivalent to $\Theta \subseteq \mathrm{T}_pX$ (because $\Theta = \overline{\mathrm{T}_p\Gamma, \mathrm{T}_pX_\Theta}$) and from this we deduce that the fibre over p is again a sub-Grassmannian $\mathbb{G}(0, n-l-2) = \mathbb{P}^{n-l-2}$. Now Γ is irreducible and all the fibres of $f : I \to \Gamma$ are irreducible and of the same dimension. This implies the irreducibility of I. Since I is a closed subvariety of $\mathbb{P}^{n-l-1} \times \Gamma$, the projection $f : I \to \Gamma$ makes it a family of $(n-l-2)$-planes. $\square$

The following proposition states that such families of linear subspaces of constant dimension are locally trivial. Note that it is necessary to require that all the fibres are subspaces of a fixed projective space. Consider for example the family of conics given by projection from a line on a smooth cubic threefold. Each conic is abstractly isomorphic to some $\mathbb{P}^1$, but the cubic is irrational and hence the family of conics can not have a rational section. In particular, it can not form a projective bundle.

Proposition 3.5 *If $p : E \to B$ is a family of k-planes in a fixed $\mathbb{P}^n$, then E is a projective bundle.*

Proof. We have to show local triviality. Let b_0 be a point in B and let $\Lambda \subseteq \mathbb{P}^n$ be an $(n-k-1)$-plane such that $E_{b_0} \cap \Lambda = \emptyset$. We claim that the preimage of the open neighbourhood $U = \{b \in B \,|\, E_b \cap \Lambda = \emptyset\} \subseteq B$ of b_0 is trivial, meaning that there is an isomorphism $E_U := p^{-1}(U) \to U \times \mathbb{P}^k$ making the diagram

$$\begin{tikzcd} E_U := p^{-1}(U) \arrow[rr] \arrow[dr, "p"'] & & U \times \mathbb{P}^k \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

commutative. Projection from Λ defines a regular map $\pi : \mathbb{P}^n \backslash \Lambda \to \mathbb{P}^k$, which restricts to an isomorphism on every E_b for $b \in U$. Then consider $\mathrm{id} \times \pi : U \times (\mathbb{P}^n \backslash \Lambda) \to U \times \mathbb{P}^k$. Restricted to E_U this defines an isomorphism $E_U \to U \times \mathbb{P}^k$ and it makes the diagram above commutative. □

Finally one can apply Lemma 3.3 to conclude the proof: Since $H \to I$ is a family of quadrics over a rational base having a rational section, H is a rational variety. So $\tilde{X}$ is dominated by a rational variety and hence unirational. Of course, the same is true for X.

3.2 The case of quartics and reducible Fano correspondences

Now consider the case of a smooth quartic hypersurface $X \subseteq \mathbb{P}^n$ and let us see how far one gets with the same strategy. Let n be sufficiently high to ensure the existence of an l-plane $\Gamma \in F_l(X)$ with $l \geq 2$. Again the $(l+1)$-planes containing Γ are parametrized by $\mathbb{P}^{n-l-1}$ and a general such plane intersects X in the union of Γ and an l-dimensional cubic hypersurface X_Θ. Projection from Γ defines a rational map $X \dashrightarrow \mathbb{P}^{n-l-1}$ and blowing up along Γ makes this a family of cubics $\tilde{X} \to \mathbb{P}^{n-l-1}$.

The general member of that family is unirational, as was just proved. A unirational parametrization of the total space could be given, if there existed consistent unirational parametrizations of the fibres X_Θ, or – equivalently – if there were a consistent choice of k-planes on the fibres. In the previous section we had $k = 0$, because choosing a point on a quadric was enough to get a rational parametrization of it. Now the fibres are cubics and a unirational parametrization of a cubic was achieved by projecting from a k-plane, where k was at least one.

Again this might not be possible. An interesting example is given in Section 2.2 in [HMP]: If the fibres are cubics of dimension two, one can not choose lines consistently on the X_Θ, since in general the monodromy acts transitively on the 27 lines on X_Θ as Θ varies. Similarly to what was done before, one hence considers the incidence correspondence

$$I = \{(\Theta, \Omega) \in \mathbb{P}^{n-l-1} \times \mathbb{G}(k, \Gamma) \mid \Omega \subseteq X_\Theta\}$$

and introduces the fibre product $H = \tilde{X} \times_{\mathbb{P}^{n-l-1}} I$. Then the fibre of $\rho : H \to I$ over a point $(\Theta, \Omega) \in I$ is just $X_\Theta \times \{(\Theta, \Omega)\}$ and H is a family of cubic hypersurfaces over I, with an obvious choice of a k-plane on $\rho^{-1}(\Theta, \Omega)$, namely $\Omega \subseteq X_\Theta$. In order to prove the unirationality of X it would again be enough to show that H is rational. This would follow from the rationality of I.

Up to this point it was possible to transfer the proof from the previous section with a few modest adjustments and to reduce the problem to the rationality of a certain Fano correspondence. In the case of cubics we noticed that it forms a projective bundle over $\Gamma = \mathbb{G}(0, \Gamma)$ and used this to finish the proof. Now in the quartic case this may be wrong: We will give an example of a Fano correspondence with the property that every fibre is a projective subspace, but not necessarily of the same dimension. If the fibre dimensions of such an I jump, I might be reducible. Since we are interested in the unirationality of X one could take an irreducible component I_0 dominating $\mathbb{P}^{n-l-1}$ in that case. However, such a component might not dominate $\mathbb{G}(k, \Gamma)$.

Example 3.6 Let $\Lambda \subseteq \mathbb{P}^3$ be a 2-plane and $C \subseteq \Lambda$ some smooth cubic curve. The cubic surfaces that contain C form a 10-dimensional linear series: The space V of homogeneous cubic polynomials in $X_0, \ldots, X_3$ has dimension $\binom{6}{3} = 20$. The condition $C \subseteq S = \mathrm{Z}(f)$ is equivalent to f being an element of the ideal generated by a cubic and a linear homogeneous polynomial. This implies that the subspace $U \subseteq V$ of cubic homogeneous polynomials vanishing on C is of dimension $11 = \binom{3}{0} + \binom{5}{2}$. This can also be established by noting that it is the kernel of $\Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) \to \Gamma(C, \mathcal{O}_{\mathbb{P}^3}(3)|_C)$, which is an epimorphism from a 20-dimensional to a 9-dimensional vector space. Thus the cubic surfaces containing C form a 10-dimensional linear series.
Let

$$I = \{(S, L) \in \mathbb{P}^{10} \times \mathbb{G}(1, 3) \mid L \subseteq S\}$$

be the Fano correspondence of lines on the members of this series and let us study the fibres of the dominant projection $\pi : I \to \mathbb{G}(1, 3)$ over three lines: a general one not intersecting C, a general one intersecting C and a special one contained in Λ. We will see that the fibres have different dimensions and that they give rise to irreducible components of I which do not dominate both $\mathbb{P}^{10}$ and $\mathbb{G}(1, 3)$.
i) If $L_1 \in \mathbb{G}(1, 3)$ does not intersect C then the fibre over L_1 is of dimension six: the subspace $W \subseteq V$ of homogeneous cubic polynomials that are contained in the ideal generated by two linear polynomials has codimension $4 = \binom{4}{1}$. Since L_1 is a general line not intersecting C we can see that $\operatorname{codim}(U \cap W \subseteq U) = 4$ and hence $\pi^{-1}(L_1) = \mathbb{P}(U \cap W) = \mathbb{P}^6$. In other words, such a line imposes exactly four independent conditions on the linear series.
Now

$$I_1 = \{(S, L) \in \mathbb{P}^{10} \times \mathbb{G}(1, 3) \mid L \subseteq S, L \cap C = \emptyset\}$$

is an irreducible component of I, because the fibres of the restriction of π are all irreducible and of the same dimension. Moreover, $\dim(I_1) = \dim(\pi(I_1)) + 6 = 10$, because I_1 dominates $\mathbb{G}(1,3)$. Indeed, a general line does not intersect the curve C. But I_1 does not dominate $\mathbb{P}^{10}$, because any cubic surface that contains C and another point in Λ contains already the whole 2-plane Λ.
ii) Now let $L_2 \in \mathbb{G}(1,3)$ intersect C in one point. Such a line imposes three independent conditions and the fibre over L_2 is of dimension seven. This gives rise to an irreducible component

$$I_2 = \{(S,L) \in \mathbb{P}^{10} \times \mathbb{G}(1,3) \,|\, L \subseteq S, |L \cap C| = 1\}$$

of dimension $\dim(I_2) = \dim(\pi(I_2)) + 7 = 10$. Here the dimension of $\pi(I_2)$ is three, since C is a curve and for a fixed point $p \in C$ the lines passing through p not contained in Λ form a dense subset of the $\mathbb{P}^{3-0-1} = \mathbb{P}^2$ of lines through p. In particular this component does not dominate $\mathbb{G}(1,3)$. But it dominates $\mathbb{P}^{10}$. In fact, if we fix a general cubic surface containing C then each line on that cubic has to intersect C. Otherwise it would again contain the 2-plane Λ.
iii) If we start with a line $L_3 \in \mathbb{G}(1,\Lambda) \subseteq \mathbb{G}(1,3)$ then the fibre over L_3 is of dimension 9, since it imposes only one condition on the linear series. We get another irreducible component

$$I_3 = \{(S,L) \in \mathbb{P}^{10} \times \mathbb{G}(1,3) \,|\, L \subseteq S \cap \Lambda\}$$

of dimension $\dim(\mathbb{G}(1,2)) + 9 = 11$, that dominates neither $\mathbb{P}^{10}$ nor $\mathbb{G}(1,3)$.

Before we turn to the general case, let us mention that in the next section we will see that for high-dimensional smooth quartics one irreducible component of this Fano correspondence dominates both factors and specify what "high-dimensional" means: If we set $k = 1$, Proposition 4.5 requires the choice of an l-plane on X, where l is at least 58, that means we need $n > M(4,l) \geq M(4,58) = 36,047$.

4 THE GENERAL CASE

We have seen that one can reduce the proof of the unirationality of a smooth quartic hypersurface to the rationality of a certain Fano correspondence. A priori, the ones that come up in this proof might be reducible and irrational in particular. If this problem can be solved, one might be able to generalize the proof for quartics and hope to get an inductive proof for the unirationality of any smooth hypersurface of high dimension.

4.1 The main theorem

This section gives an overview of the proof for the following result from [HMP]:

Theorem 4.1 *For any $d \geq 3$ there is some $N(d) \in \mathbb{N}$ such that for every $n \geq N(d)$ any smooth hypersurface of degree d in $\mathbb{P}^n$ is unirational.*

The proof of Harris, Mazur, and Pandharipande consists of two steps: First they ensure that under certain conditions there is an irreducible component of a given Fano correspondence such that both projections are dominant, which is Proposition 4.5. On the way, we will need the following lemma and also see that Fano varieties of smooth hypersurfaces of low degree have the expected dimensions. In the second step they use this to give an complete inductive proof of Theorem 4.1.

Lemma 4.2 *Let $\mathcal{D} = \{D_\Lambda \subseteq \mathbb{P}^n\}_{\Lambda \in \mathbb{P}^m}$ be a linear series of hypersurfaces with base-point locus $B \subseteq \mathbb{P}^n$ of dimension b and define $S_k := \{\Lambda \in \mathbb{P}^m \,|\, \dim((D_\Lambda)_{\text{sing}}) \geq k + b\}$. Then for any $k \in \{0, \ldots, n-b-1\}$ one has $\dim(S_k) \leq m - k$.*

Proof. Consider the incidence variety $J = \{(\Lambda, p) \in \mathbb{P}^m \times \mathbb{P}^n \,|\, p \in (D_\Lambda)_{\text{sing}}\}$. The fibre over any $\Lambda \in \mathbb{P}^m$ is the singular locus of D_Λ and hence over any $\Lambda \in S_k \backslash S_{k+1}$ the fibre dimension is $k + b$. Now it is enough to show that $\dim(J) \leq m + b$, as this implies

$$\dim(S_k) \leq \dim(J) - \dim(\pi_1^{-1}(\Lambda)) \leq m + b - k - b = m - k.$$

A point $p \in \mathbb{P}^n \backslash B$ imposes one condition on the linear series and corresponds hence to a hyperplane in $\mathbb{P}^m$. This defines a map $f : \mathbb{P}^n \backslash B \to (\mathbb{P}^m)^*$. The fibre of f over any $p \in W_l := \{p \in \mathbb{P}^n \backslash B \,|\, \text{rank}(df_p) = l\}$ is an $(m-l-1)$-plane. Sard's Theorem implies

$$\dim(W_l) \leq \dim(f^{-1}(f(p))) + l \leq b + 1 + l,$$

because the dimension of the fibres of f is at most $b + 1$. Hence

$$\dim(J) \leq m - l - 1 + l + b + 1 = m + b. \qquad \square$$

As mentioned before, Theorem 4.1 requires very high dimensions already in the case of quartics. This is mainly due to the following result. Recall that we defined $M(d,k)$ to be $\binom{k+d-1}{d-1}+k-1$. Now let $N_0(d,k)$ and $N(d,k)$ be defined recursively by

$$N_0(2,k) = N(2,k) = \binom{k+1}{2} + 3$$

and

$$\begin{aligned} N_0(d,k) &= M(d, N_0(d-1,k) + \binom{k+d-1}{d-1} + 1), \\ N(d,k) &= N_0(d,k) + \binom{k+d}{d} + 2. \end{aligned}$$

Though the proofs for the theorem and its corollary are rather elementary, we will skip them both in order to keep this section short. Recall also that $\Phi(n,d,k)$ denoted the expected dimension of the Fano variety of k-planes on a hypersurface of degree d in $\mathbb{P}^n$ in Section 2.

Theorem 4.3 *If $n \geq N_0(d,k)$ and X is a smooth hypersurface of degree d in $\mathbb{P}^n$, then*

$$\dim(F_k(X)) = \Phi(n,d,k).$$

Corollary 4.4 *If $n \geq N(d,k)$ and $X \subseteq \mathbb{P}^n$ is any hypersurface of degree d, then*

$$\dim(F_k(X)) \leq \max\{\Phi(n,d,k), \Phi(n,d,k) + \dim(X_{\mathrm{sing}}) - 1\}.$$

Now one can deduce that there is an irreducible component of the Fano correspondence such that both projections are dominant, if X contains a $N(d-1,k)$-plane. Note that this enables us to apply Proposition 3.5 to this component.

Proposition 4.5 *Let $l \geq N(d,k)$ and let $\{D_\Lambda \subseteq \mathbb{P}^l\}_{\Lambda\in\mathbb{P}^m}$ be a base-point-free linear series of hypersurfaces of degree d and consider the Fano correspondence I of k-planes on members of this series. Then I has dimension $m + \Phi(l,d,k)$ and every irreducible component of I dominates $\mathbb{P}^m$.*

Proof. The projection $\pi : I \to \mathbb{P}^m$ is surjective, since by Lemma 4.2 the general member of the linear series is smooth and Theorem 4.3 implies that generically the fibre dimension is $\Phi(l,d,k)$. Let S_k be the set of $\Lambda \in \mathbb{P}^m$ such that $\dim((D_\Lambda)_{\mathrm{sing}}) \geq k$. Then by Lemma 4.2 we have $\dim(S_k) \leq m-k-1$. For $\Lambda \in S_k$ Corollary 4.4 implies

$$\dim(\pi^{-1}(\Lambda)) = \dim(F_k(D_\Lambda)) < \Phi(l,d,k) + \operatorname{codim}(S_k \subseteq \mathbb{P}^m).$$

Indeed, if $\dim((D_\Lambda)_{\mathrm{sing}}) \geq 1$ we have

$$\dim(F_k(D_\Lambda)) \leq \Phi(l,d,k) + \dim((D_\Lambda)_{\mathrm{sing}}) - 1 < \Phi(l,d,k) + k + 1$$

and otherwise we see that $k = 0$ and

$$\dim(F_k(D_\Lambda)) \leq \Phi(l,d,k) < \Phi(l,d,k) + k + 1.$$

Hence,

$$\dim(\pi^{-1}(S_k)) < \Phi(l, d, k) + \operatorname{codim}(S_k \subseteq \mathbb{P}^m) + \dim(S_k) = \Phi(l, d, k) + m.$$

Now $\binom{d+k}{k}$ is the maximal number of conditions imposed by a k-plane and any irreducible component I_0 of I has dimension

$$\dim(I_0) \geq m + \dim(\mathbb{G}(k, l)) - \binom{d+k}{k} = m + \Phi(l, d, k),$$

and therefore its image can not be contained in S_k. In particular there is a smooth D_Λ over some $\Lambda \in \pi(I_0)$ and by Theorem 4.3 the minimal fibre dimension over $\pi(I_0)$ is $\Phi(l, d, k)$, cf. Section 2. It follows that π is dominant:

$$\dim(\pi(I_0)) = \dim(I_0) - \Phi(l, d, k) \geq m = \dim(\mathbb{P}^m)$$

Of course we can conclude that one has equality here and hence the dimension estimate for I_0 above is actually an equality too. $\square$

Let us go back to the situation of Section 3.2: Let X be a smooth quartic in $\mathbb{P}^n$ and choose an l-plane on X. Consider the family of cubic surfaces $\mathrm{Bl}_\Gamma(X) \to \mathbb{P}^{n-l-1}$. In order to get unirational parametrizations of the cubics, we need to find k-planes on them, where $k \geq 1$. The intersections of the cubics with Γ are cubic hypersurfaces in $\Gamma = \mathbb{P}^l$ and they form a linear series. If we want to apply Proposition 4.5 to the incidence correspondence of k-planes on the members of that series, we need $l \geq N(3, k)$. More generally we see that we have to require that $l \geq L(d) := N(d-1, L(d-1))$. For $d = 4$ this is $l \geq L(4) = N(3, L(3)) = N(3, 1) = 58$.

Indeed, Proposition 4.5 leads to a proof of Theorem 4.1, but the details of the inductive step require some additional effort. Roughly, the proposition is used to show that a family of l-planed smooth hypersurfaces of degree d with $l \geq L(d)$ is dominated fibre-by-fibre by a rational variety C, which the authors construct by means of Grassmann bundles and so called comb morphisms, see Section 3.2 in [HMP].

4.2 Further remarks

Related results in [HMP]

Let us come back to two of the results we presented: Harris, Mazur, and Pandharipande showed that smooth high-dimensional hypersurfaces are unirational and that the Fano varieties of k-planes on them have the expected dimensions. Actually, they note also that smoothness is not necessary for those statements, but a high codimension of the singular locus is enough. For example for unirationality this reads as follows: If $X \subseteq \mathbb{P}^n$ is a hypersurface of degree d with singular locus X_{sing} and if $\operatorname{codim}(X_{\text{sing}} \subseteq X) \geq \tilde{L}(d)$, then X is unirational. The proof just uses the intersection of X with some m-plane Λ avoiding the singular locus and applying the result for smooth hypersurfaces to $X \cap \Lambda$. The statement for the Fano varieties is similar, see [HMP, Sect. 3].

Comparison with [PS]

As already mentioned, Paranjape and Srinivas showed that Morin's results hold for complete intersection. If we restrict our consideration to hypersurfaces, we might compare their results to the ones in [HMP]. As a lemma they show the existence of linear subspaces first. Indeed, the lemma they prove is a generalization of Proposition 2.3 to complete intersections. In the case of hypersurfaces the two statements are identical and require the same dimension estimate. Their theorem states that a general high-dimensional complete intersection is unirational. Restricted to hypersurfaces, this is just Morin's result and its proof is somewhat shorter than the proof for the corresponding result in [HMP]. This is because in [HMP] the authors have to verify some facts, that are known to be true for general hypersurfaces. Moreover, the required dimensions in the theorem in [PS] are considerably lower than the ones we presented, see the table below.

About the bounds

As one might guess from the case of quartic hypersurfaces, the bounds that are required to prove unirationality in [HMP] are far from being optimal. Though for cubics dimension two was sufficient, for quartics we already needed dimension 36,047. This is due to the very general approach, that does not take the special geometry of cubics and quartics into account, and the fact that we did not make an effort to get optimal bounds in Section 2. For comparison only: Using unirational parametrizations of cubic hypersurfaces that are based on the choice of a point, one can give a proof of the unirationality of smooth quartics that requires only $n \geq 7$, where n is the dimension of the ambient projective space, cf. [HMP, Rem. 2.2].

Moreover, in the following table we compare for low degrees the dimensions that are necessary for the proof of the unirationality of the general hypersurface in [PS] with the bounds that were presented in this thesis. The numbers for [PS] are given by $n(1) := 1$ and

$$n(d) := n(d-1) + \binom{n(d-1)+d-1}{d-1},$$

the ones for [HMP] by $M(d, L(d))$.

Degree	PS	HMP
2	3	2
3	13	3
4	573	36,047
5	$4.571 \cdot 10^{9}$	$8.829 \cdot 10^{76}$
6	$1.662 \cdot 10^{46}$	$9.175 \cdot 10^{4656}$

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