FINITENESS RESULTS AND THE TATE CONJECTURE FOR K3 SURFACES VIA CUBIC FOURFOLDS

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Contents

Introduction

In this thesis we explore the arithmetic of Hassett's association, which assigns K3 surfaces to special cubic fourfolds, cf. Section 2. We demonstrate that this association, which a priori is transcendental, descends to number fields, cf. Section 3. The motivation for this is the Tate Conjecture for K3 surfaces over finite fields, and its relation to finiteness statements, cf. Section 1. More precisely, by a Theorem of Lieblich–Maulik–Snowden, cf. Theorem 1.2.5, the Tate Conjecture for K3 surfaces over finite fields is true if and only if for each fixed finite field there exist only finitely many K3 surfaces over this field. Now, the idea is that we embed the moduli spaces of polarized K3 surfaces into a fixed moduli space (in our case the moduli space of cubic fourfolds) which has only finitely many points over a finite field. While doing so, we try to avoid the Kuga–Satake construction, which plays an important role in several proofs of the Tate conjecture for K3 surfaces in the literature, as well as sophisticated theories like Shimura varieties.

We work towards a proof of the Tate Conjecture in special cases using this approach via a finiteness statement. For this we continue Charles study, cf. [Cha16], of the proof of Lieblich–Maulik–Snowden's theorem and discuss which (weaker) finiteness statements are actually enough to conclude the Tate Conjecture for a given K3 surface, cf. Section 1. In the last section we bring this together with Hassett's association by discussing lifting of K3 surfaces to characteristic 0, and by gathering together our results into a strategy of proof, cf. Section 4. At this point we want to remark, that our ansatz does not provide a complete proof, since we need to assume good reduction modulo p of cubic fourfolds that are associated to K3 surfaces which have good reduction. We leave this question for further study.

Prerequisites. We certainly assume familiarity with scheme theory, and to some extent complex algebraic geometry and Hodge theory. Since we study K3 surfaces and cubic fourfolds in this thesis, we assume some knowledge of their theories. See [HuyK3] for an excellent introduction to K3 surfaces, [Vár17] for a quick introduction to the arithmetic of K3 surfaces, and the notes [HuyC4] for an introduction to cubic hypersurfaces. Most of our approach is driven by various moduli spaces, e.g. moduli spaces of sheaves on K3

Date: September 3, 2018 (revised November 28, 2018).

surfaces, cf. [HL10] and [Lie07]. We require the reader to at least accept the existence and basic properties of these spaces. Occasionally, we use the theory of Fourier–Mukai transformations, cf. [HuyFM] for a general introduction, and [Huy09] for an introduction in the case of twisted sheaves on K3 surfaces.

This being said, we provide ample references so that a reader, who has less acquaintance with some of the mentioned theories, does not get lost. This takes into account that we discuss complex algebraic geometric constructions as well as arithmetic questions. We hope that this thesis serves both the complex geometer, as well as the arithmetic geometer.

Acknowledgment. I would like to thank Daniel Huybrechts very much for proposing the topic for this thesis, discussing various aspects of it, and answering numerous questions. He guided me through all of my mathematical study in the last five years at the University of Bonn, gave many lectures which shaped me mathematically, and served as a role model for how to do mathematics. I sincerely thank him for all this. I am very happy to thank Emma Brakkee for discussions around special cubic fourfolds and Hassett's association, as well as Andrey Soldatenkov for a discussion regarding descent of transcendental constructions. Parts of the work in this thesis were presented on various occasions in a seminar of the Arbeitsgruppe Komplexe Geometrie in Bonn, from which the author profited.

1. The Tate Conjecture for K3 surfaces via finiteness results

1.1. The Tate Conjecture.

Let us briefly recall the Tate Conjecture, focusing on the case of divisors. We refer the reader to [Tat94] and [Mil07] for a general discussion and equivalent formulations.

Conjecture 1.1.1 (Tate Conjecture). *Let X be a geometrically connected, smooth, projective variety over a field k which is finitely generated over its prime field. Let $\ell \neq \operatorname{char}(k)$ be a prime number. Then*

$$\mathrm{CH}^m_{\sim\mathrm{hom}}(X, \mathbb{Q}_\ell) \twoheadrightarrow \mathrm{H}^{2m}_{\text{ét}}(X_{k^{\mathrm{sep}}}, \mathbb{Q}_\ell(m))^{\mathrm{Gal}(k^{\mathrm{sep}}/k)}$$

is surjective for $0 \leq m \leq \dim(X)$.

Remark 1.1.2. The cases $m = 0$ and $m = \dim(X)$ are always satisfied, since both sides become the vector space $\mathbb{Q}_\ell$ (with trivial Galois-action) and the map becomes the identity.

Remark 1.1.3. When k is a finite field, then the Tate Conjecture for cycles of codimension m in conjunction with Standard Conjecture[1] D for cycles of codimension m is independent of the prime ℓ, cf. [Tat94, Theorem 2.9]. Standard Conjecture D is verified for divisors[2], cf. [SGA6, XIII Théorème 4.6], so we will just say 'Tate Conjecture' or 'Tate Conjecture for divisors' in the following, without mentioning some prime ℓ.

The Tate Conjecture is in general wide open, but some instances are known for designated families of varieties. Let us state the results for varieties that are most interesting to us.

Remark 1.1.4. The Tate Conjecture for divisors (i.e. $m = 1$ in the formulation above) is verified for abelian varieties A over k. The case $k = \mathbb{F}_q$ is due to Tate's Isogeny Theorem

$$\mathrm{End}(A) \otimes \mathbb{Z}_\ell \simeq \mathrm{End}_{\mathrm{Gal}(k^{\mathrm{sep}}/k)}(\mathrm{T}_\ell(A)),$$

cf. [Tat66, Main Theorem], and is closely related to the inception of the Tate Conjecture, cf. [Tat65]. The case where k is a function field of positive characteristic greater than 2 was settled by Zarhin in [Zar75; Zar76], and also by Mori in [Mor78]. The case where k is a number field is part of Faltings's work on the Mordell Conjecture, cf. [Fal83], and for a generalization to finitely generated extensions of $\mathbb{Q}$ see [FW92, Chapter VI].

[1] Standard Conjecture D says that numerical equivalence with rational coefficients is the same as ℓ-adic homological equivalence.

[2] We even have that numerical equivalence with rational coefficients coincides with algebraic equivalence.

In this thesis we are concerned with the Tate Conjecture for K3 surfaces. Thus we shall provide an insight into the history of the conjecture in this case. For more details we refer to the surveys [Tot17] and [Ben15].

Remark 1.1.5. Let $k/\mathbb{Q}$ be a finitely generated field, and let S be a K3 surface over k. For the K3 surface S we can consider its associated[3] Kuga–Satake abelian variety $\mathrm{KS}(S)$, which comes with a $\mathrm{Gal}(\overline{k}/k)$-equivariant map

$$\mathrm{H}^2_{\text{ét}}(S_{\overline{k}}, \mathbb{Q}_\ell(1)) \hookrightarrow \mathrm{End}(\mathrm{H}^1_{\text{ét}}(\mathrm{KS}(S)_{\overline{k}}, \mathbb{Q}_\ell)).$$

Now, one can deduce the Tate Conjecture for divisors on S from the case for the abelian variety $\mathrm{KS}(X)$, which is known as remarked above, cf. Remark 1.1.4. For more details we refer to [HuyK3, Section 17.3.2].

Theorem 1.1.6 (Tate Conjecture for K3 surfaces)**.** *Let S be a K3 surface over a field k of positive characteristic. Then S satisfies the Tate Conjecture.*

Reference. The genesis of a complete proof required the work of many mathematicians. Denote by $p = \mathrm{char}(k)$ the characteristic of the field k, let d denote the degree of some polarization of S, and let h denote the height of S. For example $h = \infty$ means that S is supersingular, cf. Definition 4.1.6.

Authors	Year	Reference	Assumptions
Artin–Swinnerton-Dyer	1973	[AS73, Theorem 5.2]	elliptically fibered
Rudakov–Zink–Shafarevich	1983	[RZS83, Theorem 4]	$p \geq 5$, $d = 2$, $h = \infty$
Nygaard	1983	[Nyg83, Corollary 3.4]	$p \geq 5$, $h = 1$
Nygaard–Ogus	1985	[NO85, Theorem 0.2]	$p \geq 5$, $h < \infty$
Charles	2013	[Cha13, Theorem 1]	$p \geq 5$, $h = \infty$
Maulik	2014	[Mau14, Main Theorem]	$p > d + 4$, $h = \infty$
Madapusi Pera	2015	[MP15, Theorem 1]	$p \neq 2$
Charles	2016	[Cha16, Theorem 1.4]	$p \geq 5$ or $\rho(S) \geq 2$
Kim–Madapusi Pera	2016	[KMP16, Theorem A.1]	$p = 2$

The assumption 'elliptically fibered' means that S admits some elliptic fibration. □

1.2. **The Tate Conjecture and finiteness statements.**

Definition 1.2.1. Let X be a quasi-projective variety, then the (cohomological) *Brauer group* of X is $\mathrm{Br}(X) \coloneqq \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}})_{\mathrm{tors}}$.[4]

Remark 1.2.2. In Definition 1.2.1, we make the assumption on X to be quasi-projective so that we do not have to distinguish between the Brauer group and the cohomological Brauer group, cf. [Jon06]. We could also assume that X is a regular surface, cf. [Gro68].

Proposition 1.2.3 (Tate)**.** *Let S be a geometrically connected, smooth, projective surface over $\mathbb{F}_q$. Then the Tate Conjecture holds for S if and only if $\#\,\mathrm{Br}(S) < \infty$.*

Sketch. See [Tat94, Section 4] for details. Let us just sketch the equivalence of the Tate Conjecture with the finiteness of $\mathrm{Br}(S)[\ell^\infty]$, when $\ell \neq \mathrm{char}(\mathbb{F}_q)$ is some prime, and $\mathrm{NS}(S_{\overline{\mathbb{F}}_q})$ has no ℓ-torsion. Applying étale cohomology to the Kummer sequence yields

$$0 \to \mathrm{Pic}(S) \otimes \mathbb{Z}_\ell \to \mathrm{H}^2_{\text{ét}}(S, \mathbb{Z}_\ell(1)) \to \mathrm{T}_\ell \,\mathrm{Br}(S) \to 0.$$

The rightmost term is 0 if and only if $\#\,\mathrm{Br}(S)[\ell^\infty] < \infty$. Note that the middle term is $\mathrm{H}^2_{\text{ét}}(S, \mathbb{Z}_\ell(1)) \simeq \mathrm{H}^2_{\text{ét}}(S_{\overline{\mathbb{F}}_q}, \mathbb{Z}_\ell(1))^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}$ by applying the Hochschild–Serre spectral sequence, cf. [HuyK3, Lemma 2.5]. □

[3]We suppress that we might need to take a finite field extension in order to construct the Kuga–Satake variety.

[4]When X is integral and regular, all elements of $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}})$ are already torsion.

As seen in Remark 1.1.5, the Tate Conjecture for K3 surfaces in characteristic 0 builds upon the case of abelian varieties, which was settled by Faltings. While he does this he proves a few finiteness results, and deduces the Tate Conjecture from the following theorem.

Theorem 1.2.4 (Faltings)**.** *Let $k/\mathbb{Q}$ be a number field and let A be an abelian variety over k. Then there exist only finitely many (up to isomorphism) abelian varieties over k that are isogenous to A.*

Reference. See [Fal83, Satz 1, Satz 2], where (a priori) a slightly weaker statement is proven, which suffices to deduce the Tate Conjecture. See [FW92, Theorem V.3.5] for a more detailed proof of the full statement. □

Now one could wonder whether it is possible to attack the Tate Conjecture for K3 surfaces via some finiteness result for K3 surfaces, avoiding the Kuga–Satake abelian variety. Indeed, this is possible as the next theorem shows.

Theorem 1.2.5 (Lieblich–Maulik–Snowden)**.** *Let $\mathbb{F}_q$ be a finite field with* $\mathrm{char}(\mathbb{F}_q) \geq 5$.

i) *There exist only finitely many K3 surfaces over $\mathbb{F}_q$ that satisfy the Tate Conjecture over all finite extensions of $\mathbb{F}_q$.*
ii) *If there are only finitely many K3 surfaces over $\mathbb{F}_{q^2}$, then every K3 surface over $\mathbb{F}_q$ satisfies the Tate Conjecture over $\mathbb{F}_{q^2}$.*

Reference. See [LMS14, Main Theorem]. □

Remark 1.2.6. The assumption on the characteristic can be weakened. For i) it is sufficient to assume $\mathrm{char}(\mathbb{F}_q) \geq 3$. For ii) one does not need any assumption on the characteristic as observed in Charles modification of the argument[5], cf. [Cha16].

Remark 1.2.7. The K3 surfaces in Theorem 1.2.5.ii) will also satisfy the Tate Conjecture over $\mathbb{F}_q$, since when $k \subset k' \subset k^{\mathrm{sep}}$ is a finite Galois extension, then the validity of the Tate conjecture for divisors over k' implies the validity over k, cf. [HuyK3, Remark 17.3.2].

When trying to prove such a finiteness statement, the first, naive, idea would be to use that the moduli spaces of polarized K3 surfaces of degree d are of finite-type over $\mathbb{F}_q$, so in particular, there exists only finitely many K3 surfaces over the finite field $\mathbb{F}_q$ once we have fixed a degree d. But, since there are infinitely many possible degrees d, we cannot conclude the desired finiteness statement without fixing the degree d. Now, the next hope might be to look at the moduli space of K3 surfaces without a polarization, $\mathrm{K3}_d \hookrightarrow \mathrm{K3}$. But this space is not well-behaved at all, more precisely it is not an algebraic stack, as observed in [Sta06, Proof of Claim 3.5], and cannot be used in the algebraic setting.

The Kuga–Satake construction provides, eventually, a quasi-finite morphism

$$\mathrm{K3}'_d \to \mathrm{AV}_{g,d'},$$

where $g = 2^{19}$.[6] Furthermore, using Zarhin's trick, cf. [Zar75, Section 5.3], we can assemble all the latter moduli spaces in the moduli space of principally polarized abelian varieties

$$\mathrm{AV}_{g,d'} \to \mathrm{PPAV}_{8g}\,.$$

Let us pretend for a moment, that this works over a finite field $\mathbb{F}_q$. Then there are only finitely many $\mathbb{F}_q$-points in PPAV_{8g} and we might try to deduce that there are only finitely many K3 surfaces over $\mathbb{F}_q$. But immediately some technical problem arise, e.g. when infinitely many images of $\mathrm{K3}_d$ intersect at one point in PPAV_{8g}. Also, a priori the

[5]He still assumes $\mathrm{char}(\mathbb{F}_q) \geq 5$ for the case of geometric Picard rank 1, since his finiteness result requires it.

[6]The space $\mathrm{K3}'_d$ is a covering of $\mathrm{K3}_d$, corresponding to additional 'spin structures'.

Kuga–Satake construction is transcendental and it takes quite some work to get it over number fields and eventually over finite fields. This leads to the question if there is a moduli space into which we can embed many $K3_d$, and for which we have some geometric control. The role will be taken by the moduli space of cubic fourfolds.

1.3. **Further finiteness statements for K3 surfaces.** We give some finiteness results for K3 surfaces that are useful for our discussion below, where we refer to them from time to time.

Theorem 1.3.1 (Bridgeland–Maciocia)**.** *Let S be a K3 surface over an algebraically closed field $\overline{k}$ of characteristic $\mathrm{char}(\overline{k}) \neq 2$. Then S has only finitely many Fourier–Mukai partners, i.e. there are only finitely many K3 surfaces S' such that $\mathbf{D}^{\mathrm{b}}(S) \simeq \mathbf{D}^{\mathrm{b}}(S')$.*

Reference. See [HuyK3, Proposition 16.3.10] or the original articles [BM01, Corollary 1.2] for the case $\mathrm{char}(\overline{k}) = 0$, and [LO15, Theorem 1.1.ii)] for the case $\mathrm{char}(\overline{k}) > 0$. □

Theorem 1.3.2 (Huybrechts–Stellari)**.** *Let S be a K3 surface over $\mathbb{C}$. Then S has only finitely many twisted Fourier–Mukai partners, i.e. there are only finitely many pairs (S', α), where S' is a K3 surface over $\mathbb{C}$ and $\alpha \in \mathrm{Br}(S')$, such that $\mathbf{D}^{\mathrm{b}}(S) \simeq \mathbf{D}^{\mathrm{b}}(S', \alpha)$.*

Reference. See [HS05, Corollary 0.5]. □

See Section 1.4 for the definition of the category of α-twisted coherent sheaves, and a fortiori its derived category $\mathbf{D}^{\mathrm{b}}(S, \alpha)$. In Section 2.4 more details regarding Fourier–Mukai partners are provided.

Proposition 1.3.3. *Let S be a K3 surface over $\mathbb{C}$, and let $d \in \mathbb{N}$. Then S admits at most finitely many polarizations of degree d, up to isomorphism. Furthermore, there is no bound on the cardinality of this set.*

Reference. The statement follows from the cone conjecture (which is valid for K3 surfaces), cf. [HuyK3, Corollary 8.4.10], and [Huy18a] for a proof which does not rely on the cone conjecture. □

We discuss the following two finiteness results for motivational purposes. We do not use them in later sections of this thesis. They come from related work of Artin–Swinnerton-Dyer, cf. [AS73], and Orr–Skorobogatov, cf. [OS17].

Proposition 1.3.4. *Let S be a K3 surface. Then S admits at most finitely many elliptic fibrations*[7] *$X \to \mathbb{P}^1$ up to isomorphism.*

Reference. See [HuyK3, Proposition 11.1.3.iii)]. □

This finiteness result can be taken as the last step in Artin–Swinnerton-Dyer's proof of the Tate Conjecture for K3 surfaces with elliptic fibration in positive characteristic. The proof of Lieblich–Maulik–Snowden's theorem, cf. Theorem 1.2.5, is a modern variant of Artin–Swinnerton-Dyer's strategy, which we will sketch now.

Let S be an elliptic K3 surface (with section) over a finite field $\mathbb{F}_q$. By Proposition 1.2.3, we want to show that $\#\,\mathrm{Br}(S) < \infty$. Let us assume the contrary. Then we find infinitely many (suitable) Brauer classes $\alpha_j \in \mathrm{Br}(S)$ that satisfy $d_j := \mathrm{ord}(\alpha_j) \to \infty$ for $j \to \infty$.[8] The Brauer group of S and its Tate-Shafarevich group are isomorphic

$$\mathrm{Br}(S) \simeq Ш(S) = \left\{ (S'/\mathbb{P}^1, \varphi) \;\middle|\; \begin{array}{l} S' \text{ elliptic K3, } \varphi\colon S \xrightarrow{\sim} \mathrm{J}(S') \text{ over } \mathbb{P}^1 \\ \text{and respecting group structures} \end{array} \right\},$$

[7] See [HuyK3, Chapter 11] for an introduction to elliptic K3 surfaces.

[8] Let $\ell \neq \mathrm{char}(\mathbb{F}_q)$ be a prime, and note that we even have $\#\,\mathrm{Br}(S)(\ell) = \infty$. Applying the Kummer sequence, and extracting finiteness for $\mathrm{H}^2_{\text{ét}}(S, \mu_{\ell^n})$ from the Hochschild–Serre spectral sequence over $\mathbb{F}_q$, shows that $\#\,\mathrm{Br}(S)[\ell^n] < \infty$ for each $n \geq 1$.

cf. [HuyK3, Corollary 11.5.5, Proposition 11.5.6], hence each α_j corresponds to an elliptic K3 surface S_j, whose Jacobian fibration is $\mathrm{J}(S_j) \simeq S$. We have $\mathrm{index}(S_j/\mathbb{P}^1) = \mathrm{ord}(\alpha_j) = d_j$, so $\mathrm{J}^{d_j}(S_j) \simeq \mathrm{J}(S_j) \simeq S$, cf. [HuyK3, Remark 11.4.4], and the section of $S/\mathbb{P}^1$ induces a multisection D_j on $S_j/\mathbb{P}^1$ with fibre degree d_j. At this point Artin–Swinnerton-Dyer tweak the multisection D_j so that $(D_j)^2 = d$ is constant, independent of j, and D_j is big and nef. Now, there are only finitely many such K3 surfaces over the fixed finite field $\mathbb{F}_q$, cf. the discussion in Remark 1.4.14. So, infinitely many elliptic fibrations $S_j/\mathbb{P}^1$ are defined on one K3 surface, contradicting Proposition 1.3.4.[9]

Theorem 1.3.5 (Orr–Skorobogatov)**.** *There are only finitely many K3 surfaces of CM type over a number field of given degree, up to isomorphism over* $\overline{\mathbb{Q}}$.

The preceding finiteness result is relevant for us because of its method of proof, which provides a Zarhin trick for K3 surfaces, assembling the moduli spaces of polarized K3 surfaces $\mathrm{K3}_d$ in some common space. This is motivated by Charles's Zarhin trick for K3 surfaces (with suitable polarization), cf. [Cha16, Theorem 2.10, Theorem 3.3], which assembles some moduli spaces of polarized K3 surfaces in a moduli space of irreducible holomorphic symplectic varieties. In contrast, Orr-Skorobogatov's space has no apparent moduli interpretation. Let us sketch their strategy.

Sketch. See [OS17, Theorem 4.1] for details. The construction is on the level of Shimura variety components, cf. [OS17, Section 3] for a quick recollection. Let

$$S_d := \mathrm{Sh}^+_{K_{\Lambda_d}}(\mathrm{SO}(\Lambda_d), \mathrm{D}(\Lambda_d))$$

be the Shimura variety component[10] of orthogonal type associated to the polarized K3 lattice Λ_d, cf. Section 2.2 for the notation around these lattices. We have a period map $\widetilde{\mathrm{K3}}_d \hookrightarrow S_d$ that is is an open immersion, where $\widetilde{\mathrm{K3}}_d \twoheadrightarrow \mathrm{K3}_d$ is the double cover[11] corresponding to trivializing determinants $\det(\mathrm{H}^2_{\text{ét}}(S, \mathbb{Z}_2(1)) \xrightarrow{\sim} \det(\Lambda_d \otimes \mathbb{Z}_2)$.

Now, Orr and Skorobogatov construct a Shimura variety component $S_\#$ of orthogonal type, such that there is a finite morphism $S_d \to S_\#$ for every even natural number d, cf. [OS17, Section 4.3]. This is done via lattice theory by constructing a lattice $\Lambda_\#$ and (chosen) primitive embeddings $\Lambda_d \hookrightarrow \Lambda_\#$. This lattice is unimodular of signature $(2, 26)$, in fact $\Lambda_\# = E_8(-1)^{\oplus 3} \oplus U^{\oplus 2}$. We can now take

$$S_\# := \mathrm{Sh}^+_{K_{\Lambda_\#}}(\mathrm{SO}(\Lambda_\#), \mathrm{D}(\Lambda_\#)).$$

Since the Shimura variety component $S_\#$ is of orthogonal type, it is in particular of abelian type, as witnessed by the Kuga–Satake construction , cf. [OS17, Section 3.3]. This reduces the finiteness of points on $S_\#$, corresponding to K3 surfaces of CM type, to the case of abelian varieties, cf. [OS17, Theorem 2.5, Proposition 3.1].

To finish the proof, one has to handle the situation when infinitely many S_d intersect inside $S_\#$ at a common point. One notes that the transcendental lattices of two K3 surfaces that map to the same point in $S_\#$ are Hodge isometric, cf. [OS17, Lemma 4.3]. This is enough to conclude the finiteness statement (when forgetting the polarization of the K3 surfaces), cf. Section 2.3 for a discussion in a similar situation. □

[9]Technically, the last step in Artin–Swinnerton-Dyer's proof is different, and more in the style of Charles's modification of Lieblich–Maulik–Snowden's argument below. They show that $d_j = \mathrm{index}(S_j/\mathbb{P}^1)$ is bounded, contradicting $d_j \to \infty$, cf. [AS73, Lemma 5.18].

[10]The complex points of this Shimura variety component are $\widetilde{\mathrm{SO}}(\Lambda_d)_+\backslash \mathrm{D}(\Lambda_d)^+$, where $K_{\Lambda_d} := \widetilde{\mathrm{SO}}(\Lambda_d)$ is the discriminant kernel, cf. $\widetilde{\mathrm{O}}(\Lambda_d)$ in Section 2.2, and the plus signs indicate (preservation of) a chosen connected component.

[11]Note that later, when we consider period maps in Theorem 2.2.5, the double cover is not necessary. The reason is that we can use the orthogonal group instead of the special orthogonal group as dictated by the formalism of orthogonal Shimura varieties.

1.4. **A refinement of Lieblich–Maulik–Snowden's theorem.** In this section we sketch the proof of Theorem 1.2.5 and present a refinement due to Charles, cf. [Cha16], which, as we observe, is sufficient to make it compatible with Hassett's numerical conditions introduced in Section 2.

Usually Brauer classes give obstructions, e.g. to the existence of rational points, projective bundles, or universal sheaves, e.g. cf. [HuyK3, Section 10.2.2.ii)]. We have seen in Proposition 1.2.3 that (infinitely many) Brauer classes can also obstruct the Tate Conjecture. We will turn this around and instead use Brauer classes to construct objects, and eventually attack the Tate Conjecture.

Definition 1.4.1. Let $\alpha \in \mathrm{Br}(X)$ and represent it by a Čech-cocycle $(\alpha_{ijk} \in \mathcal{O}^\times(U_{ijk}))_{ijk}$. An $(\alpha_{ijk})_{ijk}$*-twisted sheaf* $\mathcal{E}$ on X consists of sheaves $\mathcal{E}_i$ on U_i together with isomorphism $\varphi_{ij}\colon \mathcal{E}_j|U_{ij} \xrightarrow{\sim} \mathcal{E}_i|U_{ij}$ such that $\varphi_{ii} = \mathrm{id}$, $\varphi_{ij} = \varphi_{ji}^{-1}$, and $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot \mathrm{id}$.

Remark 1.4.2. We can form the category $\mathbf{Coh}(X, \alpha)$ of α-twisted coherent sheaves on X by requiring the $\mathcal{E}_i$ in Definition 1.2.1 to be coherent. Note that different cocycle representations of α yield equivalent categories, cf. [Căl00, Lemma 1.2.8].

For technical reasons it is nevertheless advisable to have a Čech-cocycle or a gerbe representing the Brauer class. We will not do this and refer again the reader to [LMS14] for such details.

Situation 1.4.3. Let S be a K3 surface over $\mathbb{F}_q$, with $p = \mathrm{char}(\mathbb{F}_q)$, and let $\ell \neq p$ be a prime number.

Definition 1.4.4. An ℓ*-adic B-field* is an element $B = \beta/\ell^n \in \mathrm{H}^2_{\text{ét}}(S, \mathbb{Q}_\ell(1))$ where $n \geq 0$ and $\beta \in \mathrm{H}^2_{\text{ét}}(S, \mathbb{Z}_\ell(1))$ is primitive[12]. The Brauer class associated to an ℓ-adic B-field B is defined via

$$\begin{array}{ccccc} \mathrm{H}^2_{\text{ét}}(S, \mathbb{Z}_\ell(1)) & \to & \mathrm{H}^2_{\text{ét}}(S, \mu_{\ell^n}) & \to & \mathrm{Br}(S)[\ell^n] \\ \beta & & \mapsto & & \alpha_n. \end{array}$$

Remark 1.4.5. Let us compare the ℓ-adic approach to associate a Brauer class to a B-field with the complex approach when S is a complex variety. Consider the exponential sequence of sheaves in the complex analytic topology, and push it out along $\mathbb{Z} \to \mu_n$, $1 \mapsto \mathrm{e}^{2\pi i/n}$ to get the Kummer sequence.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\mathbb{Z}} & \xrightarrow{2\pi i} & \mathcal{O}_S & \xrightarrow{\exp} & \mathcal{O}_S^\times & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \exp(2\pi i/n)} & & \big\downarrow{\scriptstyle \exp(\cdot/n)} & & \big\downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mu_n & \xrightarrow{\subset} & \mathcal{O}_S^\times & \xrightarrow{()^n} & \mathcal{O}_S^\times & \longrightarrow & 0 \end{array}$$

Taking (singular) cohomology yields

$$\begin{array}{ccccc} \mathrm{H}^2(S^{\mathrm{an}}, \mathbb{Z}) & \xrightarrow{f} & \mathrm{H}^2(S^{\mathrm{an}}, \mathcal{O}_S) & \xrightarrow{g} & \mathrm{H}^2(S^{\mathrm{an}}, \mathcal{O}_S^\times) \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle g(\cdot/n)} & & \big\downarrow \\ \mathrm{H}^2(S^{\mathrm{an}}, \mu_n) & \xrightarrow{h} & \mathrm{H}^2(S^{\mathrm{an}}, \mathcal{O}_S^\times) & \longrightarrow & \mathrm{H}^2(S^{\mathrm{an}}, \mathcal{O}_S^\times). \end{array}$$

Now take a B-field $B \in \mathrm{H}^2(S^{\mathrm{an}}, \mathbb{Z}[1/\ell])$ and write it as $B = \beta/\ell^k$ where $\beta \in \mathrm{H}^2(S^{\mathrm{an}}, \mathbb{Z})$ is primitive, and specialize the diagrams above to $n = \ell^k$. By definition, the Brauer class associated in the complex sense to the B-field B is $g(f(\beta)/\ell^k)$, while in the ℓ-adic sense we associate $h(p(\beta))$ to it. Now the commutativity of the diagram implies $g(f(\beta)\ell^k) = h(p(\beta))$ as desired. One can further compare singular cohomology with étale cohomology if required.

[12]When we write B in this form, we will always assume that $n \geq 0$ and β is primitive.

Definition 1.4.6. Denote by $\mathrm{T}(S,\mathbb{Z}_\ell) \coloneqq (\mathrm{CH}^1_{\sim\mathrm{hom}}(S,\mathbb{Z}_\ell))^\perp \subset \mathrm{H}^2_{\text{ét}}(S,\mathbb{Z}_\ell(1))$ the ℓ-adic transcendental lattice of S.

Definition 1.4.7. Let $B = \beta/\ell^n$ be an ℓ-adic B-field on S. The B-twisted Chow lattice is

$$\mathrm{CH}^B(S,\mathbb{Z}) \coloneqq \{(a\ell^n, D + a\beta, c) \mid a, c \in \mathbb{Z}, D \in \mathrm{Pic}(S)\},$$

with intersection pairing

$$(a\ell^n, D + a\beta, c) \,.\, (a'\ell^n, D' + a'\beta, c') \coloneqq (D + a\beta) \,.\, (D' + a'\beta) - a\ell^n c' - a'\ell^n c.$$

The twisted Chow lattice is the natural recipient of Mukai-vectors of twisted sheaves.

Definition 1.4.8. Let $\alpha \in \mathrm{Br}(S)$ and $B = \beta/\ell^n$ an ℓ-adic B-field inducing α. Let $\mathcal{E}$ be a locally free (or more generally a perfect complex of) $-\alpha$-twisted[13] sheaf on S of positive rank, then its *twisted Mukai-vector* is

$$\nu^B(\mathcal{E}) \coloneqq \mathrm{e}^B \,\mathrm{ch}(\mathcal{E}^{\otimes \ell^n})^{1/\ell^n} \sqrt{\mathrm{td}(S)}.$$

Proposition 1.4.9. *Let $\mathcal{E}$ be a perfect complex of $-\alpha$-twisted sheaves on S, then we have that $\nu^B(\mathcal{E}) \in \mathrm{CH}^B(S,\mathbb{Z})$.*

Reference. See [LMS14, Lemma 3.3.7]. □

Remark 1.4.10. The Grothendieck–Riemann–Roch Theorem justifies the intersection product on $\mathrm{CH}^{\beta/r}(S,\mathbb{Z})$, cf. [LMS14, Lemma 3.3.7.(2)]. Indeed, calculate

$$\begin{aligned}
-\nu^B(\mathcal{P}) \,.\, \nu^B(\mathcal{Q}) &= -(\mathrm{e}^B \,\mathrm{ch}(\mathcal{P}^{\otimes r})^{1/r} \sqrt{\mathrm{td}(S)}) \,.\, (\mathrm{e}^B \,\mathrm{ch}(\mathcal{Q}^{\otimes r})^{1/r} \sqrt{\mathrm{td}(S)}) \\
&= \deg((\mathrm{e}^B \,\mathrm{ch}(\mathcal{P}^{\otimes r})^{1/r} \sqrt{\mathrm{td}(S)})^\vee (\mathrm{e}^B \,\mathrm{ch}(\mathcal{Q}^{\otimes r})^{1/r} \sqrt{\mathrm{td}(S)})) \\
&= \deg((\mathrm{e}^{-B} \,\mathrm{ch}(\mathcal{P}^{\otimes -r})^{1/r} \sqrt{\mathrm{td}(S)}^\vee)(\mathrm{e}^B \,\mathrm{ch}(\mathcal{Q}^{\otimes r})^{1/r} \sqrt{\mathrm{td}(S)})) \\
&= \deg(\mathrm{ch}((\mathcal{P}^\vee \otimes \mathcal{Q})^{\otimes r})^{1/r} \,\mathrm{td}(S)) \\
&= \deg(\mathrm{ch}(\mathcal{H}om\,(\mathcal{P},\mathcal{Q}))\,\mathrm{td}(S)) \\
&= \chi(\mathcal{P},\mathcal{Q}),
\end{aligned}$$

where the dual $c^\vee$ of a cohomology class c in degree $2k$ means multiplying it by $(-1)^k$. In the equation chain we expand the definitions, use that $\mathrm{td}(S) = \mathrm{td}(S)^\vee$ since $\mathrm{c}_1(S) = 0$ for a K3 surface, and that the twist of $\mathcal{P}^\vee$ and $\mathcal{Q}$ cancels, and in the last step apply the Grothendieck–Riemann-Roch Theorem.

Let $\underline{\mathrm{M}}(\alpha,\nu)$ be the stack of simple $-\alpha$-twisted sheaves on S with twisted Mukai vector ν. The objects over a $\mathbb{F}_q$-scheme T are

$$\underline{\mathrm{M}}(\alpha,\nu)(T) = \left\{ \begin{matrix} \mathcal{F} \in \mathbf{QCoh}(S \times_{\mathbb{F}_q} T, (-\alpha \boxtimes 1)), \\ \varphi\colon \det(\mathcal{F}) \xrightarrow{\sim} \mathcal{O}(D) \end{matrix} \,\middle|\, \begin{matrix} \mathcal{F} \text{ finitely-presented and flat over } T, \\ \nu(\mathcal{F}_{\bar t}) = \nu, \text{ and } \mathrm{End}(\mathcal{F}_{\bar t}) = \mathrm{k}(\bar t) \\ \text{for every geometric point } \bar t \in T \end{matrix} \right\}.$$

Proposition 1.4.11. *Let $B = \beta/\ell^n$ be a B-field with associated Brauer class α, and let $\nu \in \mathrm{CH}^B(S,\mathbb{Z})$ be a twisted Mukai-vector on S. In particular, we have $\mathrm{ord}(\alpha) = \ell^n$.*

i) *If $\mathrm{rk}(\nu) = \ell^n$ and $(\nu)^2 = 0$, then $\underline{\mathrm{M}}(\alpha,\nu)$ is coarsely represented by a K3 surface, which we denote by $\mathrm{M}(\alpha,\nu)$.*

ii) *If, in addition, there exists some twisted Mukai-vector $\upsilon \in \mathrm{CH}^B(S,\mathbb{Z})$ such that $\gcd(\upsilon \,.\, \nu, \ell) = 1$, then $\mathrm{M}(\alpha,\nu)$ is a fine moduli space, and we have*

$$\mathbf{D}^{\mathrm{b}}(\mathrm{M}(\alpha,\nu)) \simeq \mathbf{D}^{\mathrm{b}}(S,\alpha).$$

Reference. See [LMS14, Proposition 3.4.2]. □

[13] We place the minus sign here so that it will not intervene later in the twisted derived equivalences.

Remark 1.4.12. Usually, when discussing moduli spaces of (twisted) sheaves one requires some stability condition on the sheaves. In contrast, we do not require any stability condition on our sheaves. The reason is that our numerical conditions say $\operatorname{rk}(\mathcal{F}) = \ell^n = \operatorname{ord}(\alpha)$, so that there are no α-twisted subsheaves and a fortiori stability conditions become vacuous, cf. [Lie07, Section 2.2.5] where the keyword is "optimal gerbe".

Let us sketch the proof of Lieblich–Maulik–Snowden's theorem, cf. Theorem 1.2.5.ii).

Sketch of Theorem 1.2.5.ii). Fix a polarization D on S of degree, say, d. Let us assume, for the sake of a proof by contradiction, that $\#\operatorname{Br}(S) = \infty$. Then we can find by Proposition 1.4.17 below a transcendental B-field $\beta \in \mathrm{T}(S, \mathbb{Z}_\ell)$ with

$$(\beta)^2 = -d.$$

Note that the B-field β is indeed primitive, cf. [LMS14, Proof of Proposition 3.5.6]. Now, let α_j be the Brauer class associated to the B-field β/ℓ^j. They satisfy $\operatorname{ord}(\alpha_j) = \ell^j$. Also, define the twisted Mukai-vectors

$$\nu_j := (\ell^j, D + \beta, 0) \in \mathrm{CH}^{\beta/\ell^j}(S, \mathbb{Z}).$$

Note that $\operatorname{rk}(\nu_j) = \ell^j$ and $(\nu_j)^2 = (D+\beta)^2 = (\beta)^2 + (D)^2 = -d + d = 0$, since β is transcendental, so $M_j := \mathrm{M}(\alpha_j, \nu_j)$ are K3 surfaces. Furthermore, we have

$$\nu_j \,.\, (\ell^j, \beta, 0) = -d,$$

and choosing an ℓ such that $\ell \nmid -2d$, we see that $\mathrm{M}(\alpha_j, \nu_j)$ is a fine moduli space.

By the finiteness hypothesis, infinitely many $\mathrm{M}(\alpha_j, \nu_j)$ must be isomorphic to, say, M. Now, we see

$$\mathbf{D}^{\mathrm{b}}(S, \alpha_j) \simeq \mathbf{D}^{\mathrm{b}}(\mathrm{M}(\alpha_j, \nu_j)) \simeq \mathbf{D}^{\mathrm{b}}(M),$$

and hence S has infinitely many twisted Fourier–Mukai partners. This is a contradiction (after lifting to characteristic 0) to Theorem 1.3.2. □

The last step of the proof, where we need to lift to characteristic 0, is quite technical. Charles looks instead at the discriminant of the twisted Mukai lattice and can thus avoid these complications, as well as Theorem 1.3.2.

Proposition 1.4.13. *Define λ_j by $(\nu_j \,.\, \mathrm{CH}^{\beta/\ell^j}(S_{\overline{\mathbb{F}}_q}, \mathbb{Z})) = \lambda_j \mathbb{Z}$. Then there exists $t_j \in \mathbb{N}$ such that*

$$\lambda_j^2 p^{t_j} \operatorname{disc} \mathrm{NS}(\mathrm{M}(\alpha_j, \nu_j)_{\overline{\mathbb{F}}_q}) = \ell^{2j} \operatorname{disc} \mathrm{NS}(S_{\overline{\mathbb{F}}_q}).$$

Reference. See [Cha16, Proposition 4.3]. □

In the situation above, we have $\nu_j \,.\, (\ell^j, \beta, 0) = -d$, and hence $\lambda_j^2 \leq d^2$. Looking at the ℓ-adic valuation of $\operatorname{disc} \mathrm{NS}(M_{j,\overline{\mathbb{F}}_q})$, we see that only finitely many can be isomorphic. More precisely, as soon as $j' > \ln(d)/\ln(\ell)$ we have that $M_{j,\overline{\mathbb{F}}_q}$ and $M_{j+j',\overline{\mathbb{F}}_q}$ are not isomorphic.

Remark 1.4.14. Charles considers in [Cha16, Theorem 1.4] the case of geometric Picard rank $\overline{\rho}(S) \geq 2$ separately. He proves finiteness without resorting to cubic fourfolds. Namely, choose a divisor B on S such that $n_0 := (B)^2 > 0$, and[14] $(B \,.\, D) = 0$. From this we get $b_j := (0, B, 0) \in \mathrm{CH}^{\beta/\ell^j}(S, \mathbb{Z})$, satisfying $(b_j \,.\, \nu_j) = 0$ and $(b_j)^2 = b$. We use the injective isometry

$$\nu_j^{\perp}/\mathbb{Z}\nu_j \hookrightarrow \mathrm{NS}(M_j),$$

cf. [Cha16, Theorem 2.4.vi), Equation (4.2)], to get a divisor B_j on M_j for each $j \geq 1$ that satisfies $(B_j)^2 = n_0$, i.e. the degree of B_j is independent of j. But, there exist only finitely many such K3 surfaces, cf. [Cha16, Corollary 3.2].

[14]Here D is not required to be ample.

The idea for the last statement is as follows: There exist natural numbers N and d depending only on n_0, such that M_j is birational to a closed subvariety of $\mathbb{P}^n$ of degree at most d and $n \leq N$, cf. [Cha16, Proposition 3.1]. Since there exist only finitely many closed subvarieties of $\mathbb{P}^n$ of degree at most d, there are only finitely many such varieties up to birationality. But K3 surfaces are minimal, i.e. two birational K3 surfaces are isomorphic, so there are only finitely many such K3 surfaces up to isomorphism.

Proposition 1.4.15. *Continuing with the notation above, in particular S has a polarization D of degree d. Assume that $\rho(S) = 1$, then the K3 surface M_j above admits a polarizations of degree $d\ell^{2j}$.*

Sketch. See [Cha16, Proof of Theorem 1.3] for details. Let us define the twisted Mukai vector $h_j := (\ell^{2j}, \ell^j\beta, -2d) \in \mathrm{CH}^{\beta/\ell^j}(S, \mathbb{Z})$. Then we have $h_j \,.\, \nu_j = 0$ and

$$(h_j)^2 = d\ell^{2j}.$$

We conclude that M_j admits a polarization of degree $d\ell^{jk} = \deg(S, D)\,\mathrm{ord}(\alpha_j)^2$. □

Remark 1.4.16. Doing some lattice yoga, one sees that we can not do better then this degree (having to assume pessimistically that $\rho = 1$ and we only have one choice of β).

When showing the existence of the transcendental B-field β above, there are some requirements on ℓ, but nevertheless there are infinitely many suitable ℓ. We will impose later in Remark 2.1.8 the additional requirement $\ell \equiv 1 \mod 3$. We cannot hope to implement these additional assumptions without modifying Lieblich–Maulik–Snowden's proof, since by Chebotarev's Density Theorem, the density of primes satisfying our extra assumption is $1/\varphi(6) = 1/2$ and the density of primes satisfying Lieblich–Maulik–Snowden's requirements is also less then $1/2$. Conveniently, Charles states the proposition more carefully, and we can apply his version, cf. Proposition 1.4.19.

Proposition 1.4.17. *Assume*[15] *$\#\,\mathrm{Br}(S) = \infty$ and let $d \in \mathbb{Q}$ be some rational number. Then there exist primes $p_1, \ldots, p_r$ such that for every prime $\ell \gg 0$, for which all p_i are squares modulo ℓ, there exists a transcendental B-field $\beta \in \mathrm{T}(S, \mathbb{Z}_\ell)$ with $(\beta)^2 = d$.*

Reference. See [Cha16, Lemma 4.4] and [LMS14, Lemma 3.5.2]. This is proven via a lattice theoretic argument which we do not reproduce here for space reasons. The reader who enjoys lattice theory is invited to look it up. □

Remark 1.4.18. There are indeed infinitely many primes ℓ that satisfy the requirements of Proposition 1.4.17, i.e. given finitely many integers $p_1, \ldots, p_r$, there exist infinitely many primes ℓ such that all p_i are squares modulo ℓ. This is essentially due to Dirichlet's theorem on primes in arithmetic progressions, cf. [Cha16, Lemma 4.6] for details.

Proposition 1.4.19. *There exist integers p_1 and p_2 such that for a prime number $\ell \neq 3$ both p_1 and p_2 are a squares modulo ℓ if and only if $\ell \equiv 1 \mod 3$.*

Proof. Take $p_1 = -1$, i.e. assume -1 is a square modulo ℓ. We claim that $\ell \not\equiv 2 \mod 3$ if and only if 3 is square modulo ℓ, so that we can take $p_2 = 3$. Indeed, let us apply quadratic reciprocity. By our assumption we have $\left(\frac{-1}{\ell}\right) = 1$, which is equivalent to $\ell \equiv 1 \mod 4$. In particular, we have $2 | \frac{\ell-1}{2}$. Now, we have

$$\left(\frac{3}{\ell}\right)\left(\frac{\ell}{3}\right) = (-1)^{\frac{3-1}{2}\frac{\ell-1}{2}} = 1,$$

i.e. we have that 3 is a square modulo ℓ if and only if ℓ is a square modulo 3. But the latter means $\ell \not\equiv 2 \mod 3$, since 2 is not a square modulo 3. □

[15] We implicitly replace $\mathbb{F}_q$ by $\mathbb{F}_{q^2}$ at this point. This is needed because [LM11, Lemma 3.5.1] only provides control over the discriminant of the sublattice fixed by Frobenius *up to a sign*. The quadratic extension counteracts this.

2. Hassett's associated cubic fourfolds

2.1. **Special cubic fourfolds.** The notion of special cubic fourfolds as well as their associated K3 surfaces, which we will use to map moduli spaces of polarized K3 surfaces to the moduli space of cubic fourfolds, was introduced by Hassett in [Has00]. In this section we recall his results. We work in the setting of complex algebraic geometry, i.e. all our varieties are defined over $\mathbb{C}$.

Recall that a cubic fourfold X over a field k is a closed subscheme $X \subset \mathbb{P}^5_k$ of codimension 1 and degree 3. We will be concerned only with smooth cubic fourfolds.

Let us look at the Hodge diamonds of a K3 surface and a cubic fourfold, cf. Figure 1. We see some similarities in the center of the diamonds, that might suggest that the second cohomology of a K3 surface and the primitive fourth cohomology of a cubic fourfold look Hodge isometric. But, this does not work out, e.g. for signature reasons. We see that we need at least a rank 2 modification together with a sign change on the cohomology of a cubic fourfold and should consider primitive cohomology of a polarized K3 surface.

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

(A) Hodge diamond of K3 surface

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 1 & & 21 & & 1 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 1 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

(B) Hodge diamond of cubic fourfold

$$\operatorname{sign} \mathrm{H}^2 = (3, 19)$$
$$\operatorname{sign} \mathrm{H}^2_{\mathrm{prim}} = (2, 19)$$

(C) Signature of K3 surface

$$\operatorname{sign} \mathrm{H}^4 = (21, 2)$$
$$\operatorname{sign} \mathrm{H}^4_{\mathrm{prim}} = (20, 2)$$

(D) Signature of cubic fourfold

Figure 1. Hodge diamonds

Definition 2.1.1.

i) A smooth cubic fourfold X is *special* if $\operatorname{rk} \mathrm{H}^{2,2}(X^{\mathrm{an}}, \mathbb{Z}) \geq 2$.
ii) A *labeling* on it is a positive definite, rank 2, primitive sublattice $K \subset \mathrm{H}^{2,2}(X^{\mathrm{an}}, \mathbb{Z})$ with $h^2 \in K$, where h denotes the cohomology class of a hyperplane section.
iii) The discriminant of a labeled special cubic fourfold (X, K) is $\operatorname{disc} K$.
iv) A *marking*[16] on it is a primitive embedding of a labeling $K \hookrightarrow \mathrm{H}^{2,2}(X^{\mathrm{an}}, \mathbb{Z})$ that preserves h^2.

Remark 2.1.2. For the very general smooth cubic fourfold, we have $\operatorname{rk} \mathrm{H}^{2,2}(X^{\mathrm{an}}, \mathbb{Z}) = 1$, which explains the terminology "special" in Definition 2.1.1, cf. Proposition 2.1.4. The general special cubic fourfold has $\operatorname{rk} \mathrm{H}^{2,2}(X^{\mathrm{an}}, \mathbb{Z}) = 2$, and a fortiori has a unique labeling.

Remark 2.1.3. Since the integral Hodge conjecture is true for smooth cubic fourfolds, cf. [Zuc77, Theorem 3.2] and [Voi13, Theorem 1.4], we see that every element in a labeling K is algebraic, i.e. it is in the image of the cycle class map $\mathrm{CH}^2(X, \mathbb{Z}) \to \mathrm{H}^4(X^{\mathrm{an}}, \mathbb{Z})$.

[16]This should not be confused with a marking in the sense of Hodge theory. When we say 'marked cubic fourfold', we will always refer to the notion defined here.

(⋆)	8	12	14	18	20	24	26	30	32	36	38	42	44	48	50	54	56	60	62
(⋆⋆)			14				26				38	42							62

TABLE 1. Table of numbers (up to 62) satisfying Hassett's conditions

Let us denote by C4 the coarse moduli space of smooth cubic fourfolds[17]. This is an open subscheme of the coarse moduli space of (GIT-stable) cubic fourfolds, which arises as the GIT-quotient $\mathbb{P}(\mathrm{H}^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3)))_{\mathrm{st}} /\!\!/ \mathrm{SL}(6)$, cf. [HuyC4, Section 2.1.3] for details. We denote the locus of special cubic fourfolds admitting a labeling of discriminant d by

$$\mathrm{C4}_d(\mathbb{C}) := \{X \text{ special cubic fourfold} \mid \exists \text{ labeling } K\colon\ \mathrm{disc}(K) = d\} \subset \mathrm{C4}(\mathbb{C}).$$

This set-theoretic definition can be enhanced to a scheme-theoretic one.

Proposition 2.1.4 (Hassett)**.**
i) $\mathrm{C4}_d \subset \mathrm{C4}$ *is an irreducible divisor, for* $d \geq 1$.
ii) $\mathrm{C4}_d \neq \emptyset$ *if and only if*

$$d > 6 \quad \textit{and} \quad d \equiv 0, 2 \mod 6. \tag{$\star$}$$

Reference. See [Has00, Theorem 3.2.3] and [Has00, Theorem 4.3.1, Proposition 3.2.3]. □

Hassett also explains how to associate a polarized K3 surface to a labeled special cubic fourfold with suitable discriminant.

Theorem 2.1.5 (Hassett)**.** *Let* (X, K) *be a labeled special cubic fourfold, say with discriminant d. Then there exists a polarized K3 surface* (S, f) *of degree d together with an isometric isomorphism of Hodge-structures*

$$\mathrm{H}^2(S^{\mathrm{an}}, \mathbb{Z}) \supset \langle f \rangle^{\perp} \simeq K^{\perp} \subset \mathrm{H}^4(X^{\mathrm{an}}, \mathbb{Z})^{-},$$

if and only if

$$\begin{gathered} d > 6,\ 2 \mid d,\ 4 \nmid d,\ 9 \nmid d,\ \textit{and} \\ p \nmid d \textit{ for any odd prime } p \textit{ with } p \equiv 2 \mod 3. \end{gathered} \tag{$\star\star$}$$

Reference. See [Has00, Theorem 5.1.3]. □

Remark 2.1.6. Note that condition (⋆⋆) implies condition (⋆). Indeed d must be even, and then we are asking if $d/2 \equiv 0, 1 \mod 3$. But this is valid, since no prime p with $p \equiv 2 \mod 3$ is allowed to divide $d/2$.

Remark 2.1.7. In the case that d is an even number, and $d > 6$, Hassett's condition (⋆⋆) might be easier to remember as the condition

$$d \mid 2(n^2 + n + 1)$$

for some $n \in \mathbb{N}$, cf. [Add16]. These unintuitive conditions are motivated by the lattice theory behind Hassett's association.

Remark 2.1.8. If d satisfies (⋆⋆), and if ℓ is a prime with $\ell \equiv 1 \mod 3$, then $d' = d\ell^{2k}$ also satisfies (⋆⋆). Indeed, $d' > 6$ and $2 | d'$ is clear, while $4 \nmid d'$ means $\ell \neq 2$ and $9 \nmid d'$ means $\ell \neq 3$. Now, no odd prime p with $p \equiv 2 \mod 3$ is allowed to divide d'. This translates into $\ell \equiv 1 \mod 3$, since we have $\ell \neq 2, 3$ already.

[17]Of course one can analogously consider the moduli spaces C3, C5 etc., while K2, etc. do not make sense. The reader may forgive us this asymmetry.

2.2. **Hassett's association on the level of moduli spaces.** The association comes into life at the level of period domains and is induced by isomorphisms of lattices, which come on one side from K3 surfaces and on the other side from cubic fourfolds. So, let us introduce these lattices, period domains and their arithmetic quotients. We are guided by Huybrechts's notes [Huy18b; Huy18c], and refer for details to Hassett's article [Has00].

First, let us introduce the relevant lattices[18].

Name	Lattice	Signature	Discriminant
Cubic lattice	$\widetilde{\Gamma} \coloneqq \mathrm{H}^4(X^{\mathrm{an}}, \mathbb{Z})^-$	$(2,21)$	1
Primitive cubic lattice	$\Gamma \coloneqq \mathrm{H}^4(X^{\mathrm{an}}, \mathbb{Z})^-_{\mathrm{prim}} \subset \widetilde{\Gamma}$	$(2,20)$	3
K3 lattice	$\Lambda \coloneqq \mathrm{H}^2(S^{\mathrm{an}}, \mathbb{Z})$	$(3,19)$	1
Polarized K3 lattices	$\Lambda_d \coloneqq \mathrm{H}^2(S^{\mathrm{an}}, \mathbb{Z})_{\mathrm{prim}} \subset \Lambda$	$(2,19)$	d
Cubic labeling lattices	$K_d \subset \widetilde{\Gamma}$ (cf. Definition 2.1.1)	$(0,2)$	$d \equiv 0,2 \mod 6$

Remark 2.2.1. The lattice K_d is indeed well-defined, one can write down a specific choice of K_d concretely, cf. [Huy18b, Lecture 1]. Later, Proposition 2.2.3 shows that the choice of such a lattice is not problematic.

Definition 2.2.2. Let V be a vector space over $\mathbb{R}$ with symmetric bilinear form $(.)$ of signature (n_+, n_-) with $n_+ \geq 2$. Define the *period domain* associated to V as

$$\mathrm{D}(V) \coloneqq \{x \in \mathbb{P}(V \otimes_{\mathbb{R}} \mathbb{C}) \mid (x \,.\, x) = 0, (x \,.\, \overline{x}) > 0\}.$$

If V arises from a lattice L via $V = L \otimes_{\mathbb{Z}} \mathbb{R}$, we allow to write $\mathrm{D}(L)$ for $\mathrm{D}(L \otimes_{\mathbb{Z}} \mathbb{R})$.

Let us apply this to our lattices. On the K3 side, we have

$$Q_d \coloneqq \mathrm{D}(\Lambda_d) \quad \subset \quad Q \coloneqq \mathrm{D}(\Lambda)$$

with dimension 19 and 20, respectively. While Q is connected, the space Q_d has two connected components. On the cubic fourfold side, we have

$$D_d \coloneqq \mathrm{D}(K_d^\perp) \quad \subset \quad D \coloneqq \mathrm{D}(\Gamma)$$

which have again dimension 19 and 20, respectively. Both have two connected components.

To get the *arithmetic varieties*[19], and get closer to the moduli spaces, we want to quotient out by suitable isometric group actions. We consider the following groups of isometries.[20]

$$\begin{aligned}
\widetilde{\mathrm{O}}(\Lambda_d) &\coloneqq \{g \in \mathrm{O}(\Lambda) \mid g(\ell) = \ell\} \\
\widetilde{\mathrm{O}}(\Gamma) &\coloneqq \{g \in \mathrm{O}(\widetilde{\Gamma}) \mid g(h^2) = h^2\} \\
\widetilde{\mathrm{O}}^{\mathrm{lab}}(K_d^\perp) &\coloneqq \{g \in \widetilde{\mathrm{O}}(\Gamma) \mid g(K_d) = K_d\} \\
\widetilde{\mathrm{O}}^{\mathrm{mar}}(K_d^\perp) &\coloneqq \{g \in \widetilde{\mathrm{O}}(\Gamma) \mid g|_{K_d} = \mathrm{id}\}
\end{aligned}$$

As promised, the choice of K_d is irrelevant up to the action of $\widetilde{\mathrm{O}}(\Gamma)$.

Proposition 2.2.3 (Hassett)**.** *Let $K \subset \widetilde{\Gamma}$ and $K' \subset \widetilde{\Gamma}$ be two labelings. Then there exists an isometry $\gamma \in \widetilde{\mathrm{O}}(\Gamma)$ such that $\gamma(K) = K'$ if and only if* $\mathrm{disc}(K) = \mathrm{disc}(K')$.

Reference. See [Has00, Proposition 3.2.4]. □

[18]For concreteness, one can write down these lattices using standard lattices, e.g. $\mathbb{Z}(n)$, U, A_2, E_8. This facilitates calculation and pinning down distinguished elements, like the polarization ℓ in Λ, such that $\Lambda_d = \ell^\perp$. We refer to [Huy18b, Lecture 1] for details.

[19]The pedantic reader may forgive us for using the terminology 'arithmetic variety' even when we do not quotient out by a torsion-free group.

[20]Recall that ℓ corresponds to a polarization, so $\Lambda_d = \ell^\perp \subset \Lambda$, and h is the class of a hyperplane section of X.

When L is a lattice with $\mathrm{sign}(L) = (2, n_-)$, the orthogonal group $\mathrm{O}(L)$ acts properly discontinuous on $\mathrm{D}(L)$. This leads to the following arithmetic varieties, and morphisms between them.

$$\mathcal{N}_d := \widetilde{\mathrm{O}}(\Lambda_d)\backslash \mathrm{D}(\Lambda_d)$$
$$\mathcal{C}_d^{\mathrm{mar}} := \widetilde{\mathrm{O}}^{\mathrm{mar}}(K_d^{\perp})\backslash \mathrm{D}(K_d^{\perp}) \to \mathcal{C}_d^{\mathrm{lab}} := \widetilde{\mathrm{O}}^{\mathrm{lab}}(K_d^{\perp})\backslash \mathrm{D}(K_d^{\perp}) \to \mathcal{C} := \widetilde{\mathrm{O}}(\Gamma)\backslash \mathrm{D}(\Gamma)$$

By the Baily–Borel theorem, cf. [BB66], these arithmetic varieties are normal, quasi-projective varieties, and by the Borel extension theorem, cf. [Bor72], the morphisms are algebraic[21]. We denote by $\mathcal{C}_d$ the image of $\mathcal{C}_d^{\mathrm{lab}}$ in $\mathcal{C}$.

Proposition 2.2.4 (Hassett)**.**

i) *The index of* $\widetilde{\mathrm{O}}^{\mathrm{mar}}(K_d^{\perp})$ *in* $\widetilde{\mathrm{O}}^{\mathrm{lab}}(K_d^{\perp})$ *is* 2 *if* $d \equiv 0 \mod 6$*, and is* 1 *if* $d \equiv 2 \mod 6$*. Consequently, the morphism* $\mathcal{C}_d^{\mathrm{mar}} \to \mathcal{C}_d^{\mathrm{lab}}$ *is two-to-one, respectively one-to-one.*

ii) *The subvariety* $\mathcal{C}_d \subset \mathcal{C}$ *is a divisor, which is non-empty if and only if* d *satisfies the condition* $d \equiv 0, 2 \mod 6$*.*

iii) *The morphism* $\mathcal{C}_d^{\mathrm{lab}} \to \mathcal{C}_d$ *is the normalization morphism.*

Reference. See [Has00, Proposition 5.2.1, Theorem 3.2.3, Proposition 3.2.2, Page 7] respectively. □

The period domains parametrize Hodge structures of polarized K3 surfaces and smooth cubic fourfolds, respectively. In the end we are interested in K3 surfaces and cubic fourfolds itself, and, what is more, their moduli spaces. The connection with the period domains, and arithmetic varieties, is established via period maps and Torelli theorems.

Consider the mapping $\tau_d\colon \mathrm{K3}_d \to \mathcal{N}_d$ which maps a polarized K3 surface $(X, f) \in \mathrm{K3}_d(\mathbb{C})$ to its Hodge structure $\mathrm{H}^{2,0}(S^{\mathrm{an}}) \subset \mathrm{H}^2(S^{\mathrm{an}}, \mathbb{Z})_{\mathrm{prim}} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \Lambda_d \otimes_{\mathbb{Z}} \mathbb{C}$. Similarly, consider the mapping $\tau\colon \mathrm{C4} \to \mathcal{C}$ which maps a smooth cubic fourfold $X \in \mathrm{C4}(\mathbb{C})$ to its Hodge structure $\mathrm{H}^{3,1}(X^{\mathrm{an}}) \subset \mathrm{H}^4(X^{\mathrm{an}}, \mathbb{Z})_{\mathrm{prim}} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \Gamma \otimes_{\mathbb{Z}} \mathbb{C}$.

Theorem 2.2.5 (Global Torelli)**.**

i) *The map* $\tau_d\colon \mathrm{K3}_d \to \mathcal{N}_d$ *is algebraic and an open immersion.*

ii) *The map* $\tau\colon \mathrm{C4} \to \mathcal{C}$ *is algebraic and an open immersion.*

Reference. See [HuyK3, Proposition 6.2.8, Theorem 6.3.5] and the original [PSS72] for the case of K3 surfaces. See [Voi86] and [Voi08, Théorème 1] for the case of cubic fourfolds. □

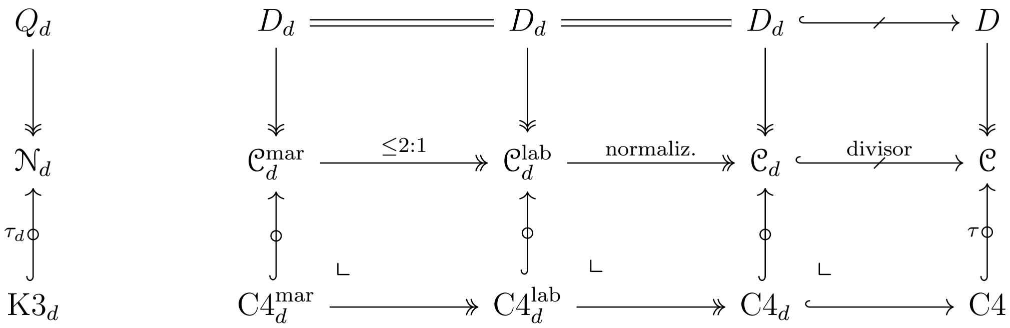
The period map τ is not surjective, which will cause slight difficulties later on. Nevertheless the image can be specified.

Theorem 2.2.6 (Laza, Looijenga)**.** *The image of the period map* $\tau\colon \mathrm{C4} \to \mathcal{C}$ *for smooth cubic fourfolds is* $\mathcal{C} \setminus (\mathcal{C}_2 \cup \mathcal{C}_6)$*.*

Reference. See [Laz10] and [Loo09]. □

Now define $\mathrm{C4}_d^{\mathrm{mar}}$, $\mathrm{C4}_d^{\mathrm{lab}}$, and $\mathrm{C4}_d \subset \mathrm{C4}$ by pull-back via the period map τ. Note that by construction the variety $\mathrm{C4}_d$ parametrizes special cubic fourfolds of discriminant d, the points of $\mathrm{C4}_d^{\mathrm{lab}}$ correspond to special cubic fourfolds together with a labeling K of discriminant d, and the points of $\mathrm{C4}_d^{\mathrm{mar}}$ correspond to special cubic fourfolds together with a marking $K \simeq K_d$ of discriminant d. Let us summarize the situation with the following diagram.

[21]One has to be careful with torsion when applying the Baily–Borel and Borel extension theorem. But, one can find torsion free, normal subgroups of finite index, and then take the quotient in two steps, cf. [Has00, Proof of Proposition 2.2.2] and [HuyK3, Remark 6.4.2].



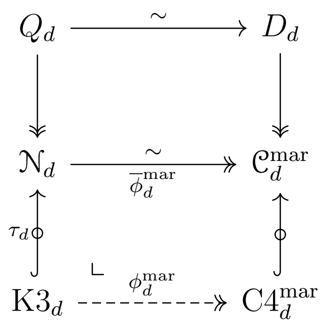
Proposition 2.2.7 (Hassett)**.**

i) *There exists an isomorphism of lattices* $K_d^\perp \simeq \Lambda_d$ *if and only if* d *satisfies the condition* ($\star\star$)*. In particular, we get* $\mathrm{D}(K_d^\perp) \simeq \mathrm{D}(\Lambda_d)$*.*

ii) *We have* $\widetilde{\mathrm{O}}^{\mathrm{mar}}(K_d^\perp) \simeq \widetilde{\mathrm{O}}(\Lambda_d)$ *under such an isomorphism.*

Reference. See [Has00, Proposition 5.1.4] and [Has00, Theorem 5.2.2]. □

In conclusion, for d satisfying ($\star\star$), this yields the diagram



and the maps of moduli spaces $\phi_d \colon \mathrm{K3}_d \dashrightarrow \mathrm{C4}_d^{\mathrm{mar}} \to \mathrm{C4}_d$. Note that by Theorem 2.2.6 the domain of the map ϕ_d is the preimage of $\mathcal{C} \setminus (\mathcal{C}_2 \cup \mathcal{C}_6)$ in $\mathrm{K3}_d$. For every marked cubic fourfold (X, K) of discriminant d there exists a polarized K3 surface (S, f) of degree d, that maps to it, i.e. the map ϕ_d^{mar} is surjective. This comes from the surjectivity of the period map for pseudo-polarized K3 surfaces and the fact that $\mathrm{H}^{2,2}(X^{\mathrm{an}}, \mathbb{Z})_{\mathrm{prim}}$ does not contain any class w with $(w)^2 = +2$, cf. [Voi86, Proposition 4.1], which witnesses that f is a polarization as desired.

For the joy of the reader, we have visualized the situation involving Hassett's maps ϕ_d in Figure 2. The moduli spaces of polarized K3 surfaces float freely, without interaction among each other, above the moduli space of cubic fourfolds, where they are mapped to and assembled in a common space. The gray gaps in the moduli spaces of polarized K3 surfaces signal that ϕ_d is not defined everywhere.

2.3. **Further study of Hassett's association.** Assume, for motivational purposes, that Hassett's association works over finite fields $\mathbb{F}_q$ and gives

$$\#\{\text{K3 surface } S/\mathbb{F}_q \mid S \text{ has a polarization of degree satisfying } (\star\star)\} < \infty.$$

Now, after checking that the K3 surfaces M_j in the proof of Theorem 1.2.5 satisfy Hassett's condition ($\star\star$), we win and prove the Tate Conjecture. Of course we have to be careful, e.g. since the maps ϕ_d are not defined everywhere, might map infinitely many polarized K3 surfaces to the same cubic fourfold (when varying d), and most important arise via a transcendental construction, so that they are, a priori, only available over $\mathbb{C}$. It is the goal of the next sections to work towards solutions or workarounds for these problems.

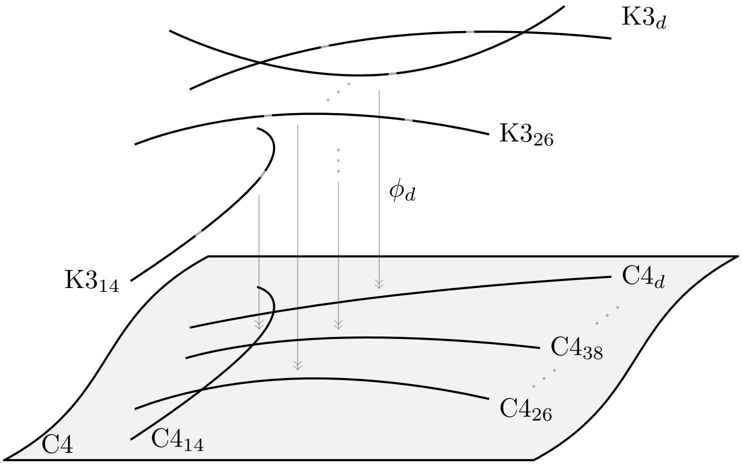


FIGURE 2. Visualization of Hassett's maps from moduli spaces of polarized K3 surfaces to Hassett's divisors.

The next proposition gives control over K3 surfaces that are associated with some cubic fourfold lying on (possibly infinitely) many Hassett divisors $\mathrm{C4}_d$. This is visualized in Figure 3, again for the reader's joy.

Proposition 2.3.1. *Let X be a special cubic fourfold. Then all K3 surfaces associated to X are mutually Fourier–Mukai equivalent*[22]*. In particular, X has only finitely many associated (non-polarized) K3 surfaces.*

Proof. Let (S, f) and (S', f') be two polarized K3 surfaces (of possibly different degree) that are associated with X. Say, the corresponding labelings are K and K'. That means we have Hodge isometries

$$\mathrm{H}^2(S^{\mathrm{an}}, \mathbb{Z})_{\mathrm{prim}} \simeq K^{\perp} \subset \mathrm{H}^4(X^{\mathrm{an}}, \mathbb{Z})^-$$
$$\mathrm{H}^2(S'^{\mathrm{an}}, \mathbb{Z})_{\mathrm{prim}} \simeq K'^{\perp} \subset \mathrm{H}^4(X^{\mathrm{an}}, \mathbb{Z})^-,$$

cf. Theorem 2.1.5. This induces in particular a Hodge isometry of transcendental lattices

$$\mathrm{T}(S) \simeq \mathrm{T}(X)^- \simeq \mathrm{T}(S').$$

By the Derived Torelli Theorem, cf. [Huy09, Theorem 5.13], we conclude that S and S' are Fourier–Mukai partners. Now, recall that a K3 surface has only finitely many Fourier–Mukai partners, cf. Theorem 1.3.1.[23] □

Remark 2.3.2. Examining the proof of Theorem 1.2.5 shows that we only need finiteness of K3 surfaces up to Fourier–Mukai equivalence. So we do not need Theorem 1.3.1 for our applications. Indeed, take Proposition 1.4.13 into account and compare the discriminants of the Néron–Severi lattices under a Fourier–Mukai equivalence, cf. [HuyK3, Corollary 16.2.8].

Now, Proposition 2.3.1 rectifies the problem that the maps φ_d are only generically two-to-one, and also the problem when infinitely many $\mathrm{C4}_d$ intersect at one point. Still, there is the problem that a K3 surface, even one satisfying Hassett's condition ($\star\star$), could have no associated cubic fourfold. Since Hassett's association comes into live on the level of period domains, it is not clear how to compute the associated cubic fourfold for a given K3 surface. Furthermore, criteria for deciding whether a K3 surface has an associated cubic fourfold at all are missing. The only criterion we can give, and will use later, is the following.

[22]This notion is recalled in Definition 2.4.1

[23]The finiteness result in Theorem 1.3.1 is proved via studying the situation on the level of transcendental lattices, so our use of the Derived Torelli Theorem is canceled.

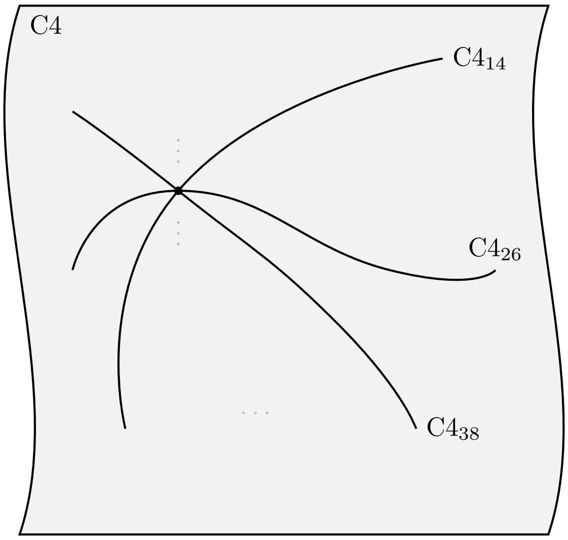


FIGURE 3. Intersection of (possibly infinitely) many Hassett's divisors at a common intersection point.

Proposition 2.3.3. *Let (S, f) be a polarized K3 surface over $\mathbb{C}$. Assume that S has Picard rank $\rho(S) = 1$, and its degree d satisfies Hassett's condition ($\star\star$). Then (S, f) has an associated cubic fourfold, i.e. it lies in the domain of ϕ_d.*

Proof. By Theorem 2.2.6, we need to check that $\bar{\phi}_d^{\mathrm{mar}}$ does not map the period of (S, f) into the Hassett divisors $\mathcal{C}_2 \cup \mathcal{C}_6$. If it would lie on $\mathcal{C}_2$ or $\mathcal{C}_6$, then it would lie on at least two Hassett divisors, since it already lies on $\mathcal{C}_d$. But, then the period point would admit two different labelings K_d and $K_{d'}$, which is impossible, since $\rho(S) = 1$ only leaves space for one labeling. □

Remark 2.3.4. There are different competing notions of what we might mean when we say a K3 surface S is associated to a cubic fourfold X. It could mean

i) $\phi_d(S, f) = X$,
ii) $\mathrm{H}^2(S^{\mathrm{an}}, \mathbb{Z}) \supset \langle f \rangle^\perp \simeq K^\perp \subset \mathrm{H}^4(X^{\mathrm{an}}, \mathbb{Z})^-$, or
iii) $\mathrm{T}(S) \simeq \mathrm{T}(X)^-$.

In the following, we take ii) as our definition of being 'associated'. We have the implications i) ⇒ ii) ⇒ iii). The first one holds by construction, and for the second one see Proposition 2.3.1. Note that the converse implication ii) ⇒ i) does not hold, not even when we trivialize Diagram 2 by requiring $\rho(S) = 1$ and $d \equiv 2 \mod 6$. The problem is that the map ϕ_d is *not* canonical, there are choices involved. But, when we have an association in the sense of ii), then there is a choice of ϕ_d such that $\phi_d^{\mathrm{mar}}(S, f) = (X, K)$ and i) is verified.

See [Add16] for further discussion, which also touches upon the question when a cubic fourfold is rational, i.e. when it is birational to the projective space $\mathbb{P}^4$.

Proposition 2.3.5. *Assume that d satisfies ($\star\star$), and define*[24]

$$G_d := \mathrm{Isom}(\mathrm{Disc}(K_d^\perp), \mathrm{Disc}(-\Lambda_d)).$$

Then the possible choices of Hassett's identification $\bar{\phi}_d^{\mathrm{mar}} \colon \mathcal{N}_d \to \mathcal{C}_d^{\mathrm{mar}}$ are in bijection with $G_d/\{\pm 1\}$. The group $R_{2,d} := \{n \in \mathbb{Z}/d\mathbb{Z} \mid n^2 = 1\}$ acts faithfully and transitively on G_d. In particular

$$\#\{\bar{\phi}_d^{\mathrm{mar}} \colon \mathcal{N}_d \xrightarrow{\sim} \mathcal{C}_d^{\mathrm{mar}}\} = \frac{1}{2}\#R_{2,d}.$$

Reference. See [Has00, Theorem 5.2.3]. □

[24]The notation $\mathrm{Disc}(L)$ denotes the discriminant group of a lattice L.

Proposition 2.3.6.

i) *There is a unique choice of Hassett's association* $\bar{\phi}_d^{\mathrm{mar}}\colon \mathcal{N}_d \to \mathcal{C}_d^{\mathrm{mar}}$ *if and only if we have* $d = 6$ *or* $d = 2p^k$ *where* $k \geq 0$ *and* p *is a prime with* $p \equiv 1 \mod 3$.

ii) *Let* ℓ *be a prime with* $\ell \not\equiv 2 \mod 3$, *and let* $d > 2$ *be coprime to* ℓ. *For* $d' = \ell^k d$ *there are exactly twice as many choices of Hassett's association* $\bar{\phi}_d^{\mathrm{mar}}\colon \mathcal{N}_d \to \mathcal{C}_d^{\mathrm{mar}}$ *as for* d.

Proof. By Proposition 2.3.5 we have to look at the cardinality $f(n,2) := \#R_{2,n}$ of $R_{2,d}$. The function $f(n,2)$ is a multiplicative arithmetic function, cf. [FMS10, Section 2].

ii) Write the equation $a^2 \equiv 1$ as $(a+1)(a-1) \equiv 0$. Since ± 1 are always solutions, let us assume $a \not\equiv \pm 1$. So, $\ell^k \nmid a+1$ and $\ell^k \nmid a-1$, which implies $\ell \mid a+1$ and $\ell \mid a-1$. But this is only possible for $\ell = 2$. So $f(\ell^k, 2) = 2$ as desired.

i) For the case $d = 6$ see [Has00, Theorem 5.2.3], so let us now assume $d \neq 6$. By the multiplicativity of f, and ii), we have $f(2p^k, 2) = f(2,2)f(p^k,2) = 1 \cdot 2$.

Conversely, write $d = \prod_p p^{k_p}$ factored into primes. Since d must satisfy Hassett's conditions ($\star\star$), we have that $k_2 = 1$, $k_3 \leq 1$, and $k_p = 0$ for every odd prime p satisfying $p \equiv 2 \mod 3$. Then $2 = f(d,2) = \prod_p f(p^{k_p}, 2)$, but $f(p^{k_p}, 2) \geq 2$ for $p \geq 3$ and $k_p \geq 1$. So, we must have $d = 2p^k$ with p and k as claimed. □

2.4. **Hassett–Kuznetsov association.**

In this section we discuss an alternative notion for a cubic fourfold to be associated to a K3 surface, due to Kuznetsov, cf. [Kuz10],. The association takes place on the level of derived categories, so let us recall the relevant definitions, cf. [HuyFM] for details.

Definition 2.4.1. Let X and Y be schemes over a field k.

i) Denote by $\mathbf{D}^{\mathrm{b}}(X) := \mathbf{D}^{\mathrm{b}}(\mathbf{Coh}(X))$ the bounded derived category of coherent sheaves on X.

ii) We say that X and Y are *derived equivalent* if there exists a k-linear exact equivalence $\mathbf{D}^{\mathrm{b}}(X) \simeq \mathbf{D}^{\mathrm{b}}(Y)$.

Remark 2.4.2. We will usually say X and Y are Fourier–Mukai equivalent, or Fourier–Mukai partners, when they are derived equivalent. This is justified when X and Y are smooth projective varieties, since then an exact equivalence $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ can be written as a Fourier–Mukai functor, i.e. there exists a *Fourier–Mukai kernel* $\mathcal{P} \in \mathbf{D}^{\mathrm{b}}(X \times_k Y)$ such that

$$\Phi(\mathcal{F}) = \Phi_{\mathcal{P}}(\mathcal{F}) := \mathbf{R}(\mathrm{pr}_Y)_*(\mathrm{pr}_X^* \mathcal{F} \otimes^{\mathbf{L}} \mathcal{P})$$

for $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X)$, cf. [HuyFM, Theorem 5.14] and the survey [CS12].

The first, naive, idea one could have is to say that a K3 surface S and a smooth cubic fourfold X are associated if they are derived equivalent. The problem is that a smooth cubic fourfold is a Fano variety, i.e. $\omega_X^\vee$ is ample, and in particular never isomorphic to a K3 surface, which is a Calabi–Yau variety, i.e. $\omega_S \simeq \mathcal{O}_S$. But, by Bondal–Orlov S and X must be isomorphic as soon as they are derived equivalent, cf. [HuyFM, Proposition 4.11].

Instead, Kuznetsov restricts attention to a part $\mathcal{K}u(X) \subset \mathbf{D}^{\mathrm{b}}(X)$ that looks like the derived category of a K3 surface, if you will, a 'noncommutative K3 surface'. We refer to the articles [Kuz10], [Kuz16], and the notes [MS18] for details.

Definition 2.4.3. For a smooth cubic fourfold X define the *Kuznetsov component* as

$$\begin{aligned}\mathcal{K}u(X) &:= \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\rangle^\perp \\ &= \{\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X) \mid \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(\mathcal{O}_X(i), \mathcal{F}[j]) = 0 \text{ for } i = 0,1,2 \text{ and } j \in \mathbb{Z}\} \\ &= \{\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X) \mid \mathrm{H}^\bullet(X, \mathcal{F}(i)) = 0 \text{ for } i = 0,-1,-2\}.\end{aligned}$$

Remark 2.4.4. The Kuznetsov component $\mathcal{K}u(X)$ looks like a the derived category of a K3 surface. More precisely it is a 2-dimensional Calabi–Yau category, whose Hochschild cohomology look like that of a derived category of a K3 surface.

Definition 2.4.5. We say X is associated to some K3 surface in Kuznetsov's sense when the Kuznetsov component $\mathcal{K}u(X)$ is *geometric*, i.e. there exists a K3 surface S together with a Fourier–Mukai equivalence $\mathcal{K}u(X) \simeq \mathbf{D}^{\mathrm{b}}(S)$.

Remark 2.4.6. The K3 surface S in Definition 2.4.5 does not need to have anything to do with an associated K3 surface in Hassett's sense. But as soon as one knows that $\mathcal{K}u(X)$ is geometric, one can indeed get a Fourier–Mukai equivalence with a Hassett associated K3 surface.

Proposition 2.4.7. *Let (X, K) be a marked cubic fourfold with associated polarized K3 surface (S, f), i.e. we have a Hodge isometry*

$$\varphi\colon\ \mathrm{H}^2(S^{\mathrm{an}}, \mathbb{Z})_{\mathrm{prim}} \simeq K^{\perp} \subset \mathrm{H}^4(X^{\mathrm{an}}, \mathbb{Z})^{-}.$$

If $\mathcal{K}u(X)$ is geometric, then there exists a Fourier–Mukai equivalence $\Phi_{\mathcal{P}}\colon \mathbf{D}^{\mathrm{b}}(S) \to \mathcal{K}u(X)$ that induces φ on cohomology.

Reference. See [AT14, Proposition 5.1]. □

The question remains if X has an associated K3 surface in Hassett's sense if and only if the Kuznetsov component $\mathcal{K}u(X)$ is geometric. This was answered by Addington–Thomas generically, and completed in announced work of Bayer–Lahoz–Macrì–Nuer–Perry–Stellari.

Theorem 2.4.8 (Addington–Thomas, Bayer–Lahoz–Macrì–Nuer–Perry–Stellari)**.** *Let X be a special cubic fourfold of discriminant d that satisfies Hassett's condition ($\star\star$), i.e. it has an associated K3 surface, then $\mathcal{K}u(X)$ is geometric.*

Reference. See [AT14, Theorem 1.1] and [BLM+17, Corollary 4].

Since this theorem is central to us, let us give a few words about its proof. First, one uses concrete geometric knowledge of cubic fourfolds in $\mathrm{C4}_8$ to see that their Kuznetsov component is Fourier–Mukai equivalent to the derived category of a twisted K3 surface. Then one looks for, and finds, special cubic fourfolds X that lie on both $\mathrm{C4}_8$ and $\mathrm{C4}_d$, where d satisfies Hassett's condition ($\star\star$). One goes on to show that the twist above must be trivial for such X, and a fortiori $\mathcal{K}u(X)$ is geometric.

Now, Proposition 2.4.7 shows that one can find the 'right' K3 surface S and Fourier–Mukai equivalence. This Fourier–Mukai equivalence is then deformed along with X and S, which themselves deform in 'synchronization' according to a chosen Hassett map ϕ_d. This is visualized in Figure 4. Now, to complete the result, one has to take 'limits' of the Fourier–Mukai equivalences, as made possible by a suitable moduli space of stable objects, cf. [BLM+17]. □

3. Arithmetic of Hassett's association

In the end we would like to have Hassett's association available over a finite field $\mathbb{F}_q$. We will not go as far and instead employ lifting to characteristic 0 in Section 4 and then apply Hassett's association. Nevertheless, we study the descent of Hassett's association to subfields of $\mathbb{C}$, like $\overline{\mathbb{Q}}$, or (local)[25] number fields.

[25] After choosing an embedding $\overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$.

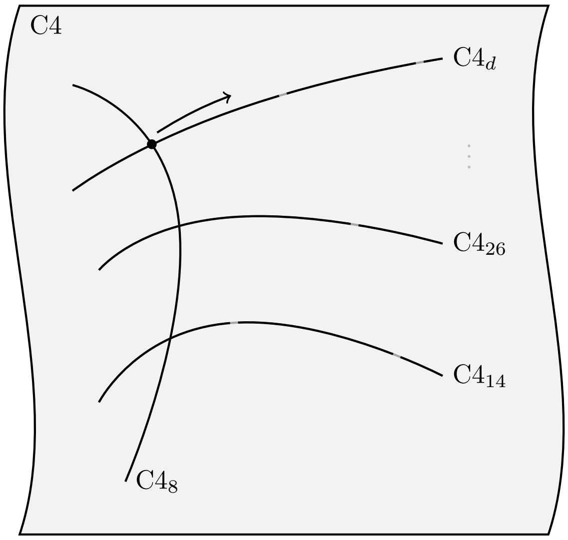


FIGURE 4. Hassett's divisors in the situation of the proof of Theorem 3.1.5.

3.1. **Descend of Hassett's associated cubic fourfold.** So, as a first step, let us discuss the descent of associated special cubic fourfolds. In this regard we want to answer the following question.

Question 3.1.1. Let $k \subset \mathbb{C}$ be a field, and S a K3 surface over k with a cubic fourfold $X_{\mathbb{C}}$ associated to $S_{\mathbb{C}}$. Is $X_{\mathbb{C}}$ defined over the field k?

Theorem 3.1.2 (Hulek–Kloosterman). *Let k be a number field and assume that $\mathrm{Hilb}^2(S)$ has a k-rational point, that*[26] $\mathrm{F}(X_{\mathbb{C}}) \simeq \mathrm{Hilb}^2(S_{\mathbb{C}})$, *and that the associated line bundle*[27] *descends to k. Then $X_{\mathbb{C}}$ is defineable over k.*

Reference. See [HK07, Theorem 1.1]. □

Remark 3.1.3. If we assume that S has a k-rational point, then there is a universal bound on the degree of the field extension k'/k that is needed to descend the associated line bundle, cf. Remark 4.1.21. So after replacing k by a finite extension of controlled degree we can remove the line bundle descent assumption in the theorem, trading it for a k-rational point.

Remark 3.1.4. The theorem does not help in our situation, since in order to ensure that there is an isomorphism $\mathrm{F}(X_{\mathbb{C}}) \simeq \mathrm{Hilb}^2(S_{\mathbb{C}})$, we want $d = 2(n^2+n+1)$ for some $n \in \mathbb{N}$ and $X_{\mathbb{C}}$ is generic, cf. [Has00, Theorem 6.1.4], or, alternatively, $da^2 = 2(n^2+n+1)$ for some $a, n \in \mathbb{N}$, $d \equiv 0 \mod 6$, and $\rho(S_{\mathbb{C}}) = 1$, cf. [Bra18, Proposition 4.3, Proposition 4.5]. But we cannot assume that $\ell^{2k} d_0$ will be of this form for $k \gg 1$.

We will give ourselves a bit of freedom and ask in Question 3.1.1 whether $X_{\mathbb{C}}$ is defined over some finite extension of k, or equivalently by spreading out (when we do not control the degree of this extension), whether it is defined over $\overline{k}$.

Theorem 3.1.5 (Addington–Thomas). *Let (S, f) be a polarized K3 surface over $\mathbb{C}$ and let (X, K) be some labeled cubic fourfold, which is associated to (S, f). Then the Hodge isometry $\mathrm{H}^2(S^{\mathrm{an}}, \mathbb{Z}) \supset \langle f \rangle^{\perp} \simeq K^{\perp} \subset \mathrm{H}^4(X^{\mathrm{an}}, \mathbb{Z})^{-}$ from Theorem 2.1.5 is induced by an algebraic cycle in $\mathrm{CH}^3(S \times_{\mathbb{C}} X, \mathbb{Q})$.*

[26] Here $\mathrm{F}(X_{\mathbb{C}})$ denotes the Fano variety of lines on $X_{\mathbb{C}}$, cf. [HuyC4, Chapter 3].

[27] The associated line bundle is a line bundle on $S_{\mathbb{C}}$ defined as follows. Pull-back the Plücker polarization to $\mathrm{Hilb}^2(S_{\mathbb{C}})$ via $\mathrm{Hilb}^2(S_{\mathbb{C}}) \simeq \mathrm{F}(X_{\mathbb{C}})$. Now, write this line bundle as $\mathcal{O}(f + aE)$ using the canonical decomposition $\mathrm{Pic}(\mathrm{Hilb}^2(S_{\mathbb{C}})) \simeq \mathrm{Pic}(S_{\mathbb{C}}) \times \frac{1}{2}\mathbb{Z} \cdot E$, where E is the exceptional divisor on $\mathrm{Hilb}^2(S_{\mathbb{C}})$ (corresponding to the locus of non-reduced subschemes), cf. [Fog73, Corollary 6.3]. Finally, call $\mathcal{O}(f)$ the associated line bundle.

Reference. See [AT14, Theorem 1.2]. This is a corollary of Theorem 2.4.8, by taking Mukai vectors of the Fourier–Mukai equivalences there. By taking limits of algebraic cycles, the generic version of the theorem by Addington–Thomas is sufficient. ☐

Let $\sigma \in \mathrm{Aut}(\mathbb{C})$ and let X be a scheme over $\mathbb{C}$. Recall that the conjugation X^σ of X by the automorphism σ is defined as the pull-back

$$\begin{array}{ccc} X^\sigma & \xrightarrow{\sigma^{-1}} & X \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \xrightarrow{\mathrm{Spec}(\sigma)} & \mathrm{Spec}(\mathbb{C}). \end{array}$$

Via pull-back this also explains f^σ for a divisor f on X, or e.g. K^σ for a marking K on a special cubic fourfold X, cf. Remark 2.1.3.

Corollary 3.1.6. *Let (S, f) be a polarized K3 surface, associated to the marked cubic fourfold (X, K). Let $\sigma \in \mathrm{Aut}(\mathbb{C})$ be an automorphism of the field $\mathbb{C}$. Then the polarized K3 surface (S^σ, f^σ) is associated to the marked cubic fourfold (X^σ, K^σ).*

Proof. By Theorem 3.1.5, we know that the association

$$\mathrm{H}^2(S^{\mathrm{an}}, \mathbb{Z}) \supset \langle f \rangle^\perp \simeq K^\perp \subset \mathrm{H}^4(X^{\mathrm{an}}, \mathbb{Z})^- \tag{3.1.1}$$

is induced by an algebraic correspondence $\gamma \in \mathrm{CH}^3(S \times_{\mathbb{C}} X, \mathbb{Q})$. Conjugating γ by σ gives us the algebraic correspondence γ^σ, which still induces a Hodge isometry, now for (S^σ, f^σ) and (X^σ, K^σ), cf. [CS11, Section 2] for a discussion about conjugating Hodge classes.

Since the cycle γ has rational coefficients, it is not clear if γ^σ induces an isometry of *integral* Hodge structures. The reason behind this is that γ arises from a Fourier–Mukai equivalence via taking Mukai vectors, cf. Theorem 2.4.8 and Theorem 3.1.5, and this Fourier–Mukai equivalence preserves not the usual integral structure on cohomology, but instead the integral structure coming from topological K-theory, cf. [AT14, Section 2].

So, let us take a Fourier–Mukai kernel $\mathcal{P}$ inducing[28] the Hodge isometry in Equation 3.1.1, cf. Theorem 2.4.8 and Proposition 2.4.7. We can now consider $\mathcal{P}^\sigma$, which induces the Hodge isometry of integral Hodge structures

$$\mathrm{H}^2((S^\sigma)^{\mathrm{an}}, \mathbb{Z}) \supset \langle f^\sigma \rangle^\perp \simeq (K^\sigma)^\perp \subset \mathrm{H}^4((X^\sigma)^{\mathrm{an}}, \mathbb{Z})^-,$$

as desired, cf. [AT14, Section 5.1]. ☐

Remark 3.1.7. As observed in the proof of Corollary 3.1.6, it is not just a corollary of Theorem 3.1.5, but uses the complete version of Theorem 2.4.8.

We might try to conclude, that when S is defined over a subfield $k \subset \mathbb{C}$, with associated cubic fourfold X over $\mathbb{C}$, then also X is definable over k. The problem is that, even when there is a canonical choice of ϕ_d, we only know that $X \simeq X^\sigma$ for some isomorphism over $\mathbb{C}$, and when varying σ they will in general not satisfy the cocycle condition needed by Galois descent, cf. [MilAG, Section 16.f].

Proposition 3.1.8 (Galois descent for subvarieties and morphisms)**.** *Let $k \subset \mathbb{C}$ be a subfield*[29]*, and let X be a variety over k. Define the action of $\mathrm{Aut}(\mathbb{C}/k)$ on the $\mathbb{C}$-rational points $X(\mathbb{C}) = \mathrm{Mor}_{/k}(\mathrm{Spec}(\mathbb{C}), X)$ of X via its action on $\mathrm{Spec}(\mathbb{C})$, or when thinking in coordinates it is just the coordinate-wise action of $\mathrm{Aut}(\mathbb{C}/k)$. Then we have the following:*

[28] Take the Mukai vector $\nu(\mathcal{P}) = \mathrm{ch}(\mathcal{P})\sqrt{\mathrm{td}(S \times_{\mathbb{C}} X)}$ and view it as a cohomological correspondence from S to X.

[29] Or more generally, let Ω be a separably closed field and $k \subset \Omega$ a subfield such that $k = \Omega^{\mathrm{Aut}(\Omega/k)}$.

i) *If $Y_{\mathbb{C}} \hookrightarrow X_{\mathbb{C}}$ is a closed subvariety such that $Y_{\mathbb{C}}(\mathbb{C}) \subset X(\mathbb{C})$ is stable under the action of* $\mathrm{Aut}(\mathbb{C}/k)$*, then there exists a subvariety $Y \hookrightarrow X$ over k whose base change to $\mathbb{C}$ is $Y_{\mathbb{C}}$.*

ii) *Let X and Y be varieties over k, and let $f_{\mathbb{C}} \colon X_{\mathbb{C}} \to Y_{\mathbb{C}}$ be a morphism over $\mathbb{C}$. If $f_{\mathbb{C}}$ commutes with the action of* $\mathrm{Aut}(\mathbb{C}/k)$*, i.e. we have $f(x) = \sigma.\, f(\sigma^{-1}.\, x)$ for every automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}/k)$ and $x \in X(\mathbb{C})$, then there exists a unique morphism $f \colon X \to Y$ over k whose base change to $\mathbb{C}$ is $f_{\mathbb{C}}$.*

Reference. See [MilAG, Proposition 16.8, Proposition 16.9]. □

Remark 3.1.9. Note that when the subvariety is just a point, then one does not need the machinery of Galois descent, but can argue just using Galois theory.

Proposition 3.1.10. *Let $\overline{k} \subset \mathbb{C}$ be an algebraically closed subfield. If $G \subset \mathrm{Aut}(\mathbb{C}/\overline{k})$ is a subgroup of finite index, then the fixed field is $\mathbb{C}^G = \overline{k}$.*

Proof. This proof is based on [MilFT, Theorem 9.29]. Let us assume that there is an element $\alpha \in \mathbb{C}^G$ that is transcendental over $\overline{k}$. Choose automorphisms $\varphi_i \in \mathrm{Aut}(\mathbb{C}/\overline{k})$ extending the mapping $\alpha \mapsto \alpha + i$ for $i \in \mathbb{N}$. Fix a system of representatives $\{h_1, \ldots, h_n\}$ of $\mathrm{Aut}(\mathbb{C}/\overline{k})/G$ and write $\varphi_i = h_{j_i} \circ g_i$ with some suitable $g_i \in G$. We arrive at a contradiction by choosing $i \in \mathbb{N}$ large enough, since we have $\alpha + i = \varphi_i(\alpha) = h_{j_i}(g_i(\alpha)) = h_{j_i}(\alpha)$. □

Proposition 3.1.11. *Let (S, f) be a polarized K3 surface over $\mathbb{C}$ with associated cubic fourfold (X, K) of discriminant d. Denote by $x \in \mathrm{C4}(\mathbb{C})$ the point corresponding to X in the moduli space of cubic fourfolds. If (S, f) is definable over $k \subset \mathbb{C}$, then the point x descends to a k'-rational point of* C4*, where k'/k is a Galois extension with bounded degree $[k' \colon k] \leq C_1(d) \coloneqq (\#R_{2,d})!$.*

Proof. For every $\sigma \in \mathrm{Aut}(\mathbb{C}/k)$ we know that $(S, f) = (S, f)^\sigma$ is associated to $(X, K)^\sigma$, cf. Corollary 3.1.6. Hence, the cardinality of the set $\{(X, K)^\sigma \mid \sigma \in \mathrm{Aut}(\mathbb{C}/k)\}/{\simeq_{\mathbb{C}}}$ is bounded by $\frac{1}{2}\#R_{2,d}$, cf. Proposition 2.3.5. The subgroup $G \subset \mathrm{Aut}(\mathbb{C}/k)$ of automorphism that fix every element of this set is a normal subgroup of finite index, which is bounded by $C_1(d) = (\frac{1}{2}\#R_{2,d})!$. Now, Galois theory provides us with a finite Galois extension[30] k'/k with $G = \mathrm{Aut}(\mathbb{C}/k')$, and whose degree is bounded by $C_1(d)$. Finally, Galois descent, cf. Proposition 3.1.8, shows that x is a k'-rational point. □

Remark 3.1.12. Proposition 3.1.11 is also valid with the roles of (X, k) and (S, f) exchanged.

Remark 3.1.13. If one does not want a normal field extension in Proposition 3.1.11, one can get rid of the factorial in the bound.

3.2. **Level structures on cubic fourfolds.** Let X be a special cubic fourfold of discriminant d with associated polarized K3 surface (S, f). Assume that (S, f) is definable over some subfield $k \subset \mathbb{C}$. In the last section we have seen that the point corresponding to X in the moduli space C4 of cubic fourfolds is k'-rational, where k'/k is a finite extension whose degree is bounded by $C_1(d)$. We would like to conclude that X is definable over k'. The problem is that the moduli space C4 is only a coarse moduli space, and hence we only know over the algebraically closed field $\overline{k'}$ that $\mathrm{C4}(\overline{k'})$ is the set of smooth cubic fourfolds over $\overline{k'}$. We will use level structures on cubic fourfolds to remedy this problem and get a uniform bound on the degree of a field extension k''/k' such that X is definable over k''.

Situation 3.2.1. Fix a perfect field k_0 and let $\overline{k}$ be its algebraic closure, e.g. $k_0 = \mathbb{Q}$ and $\overline{k} = \overline{\mathbb{Q}}$. We work (for simplicity) relative to this setup, so in the following k denotes an intermediate field $k_0 \subset k \subset \overline{k}$.

[30]Note that by Proposition 3.1.10 we can first replace $\mathbb{C}$ by $\overline{k}$ and then G is a closed subgroup $G \subset \mathrm{Gal}(\overline{k}/k)$, since it is defined by fixing points in the moduli space of marked cubic fourfolds that correspond to the $(X, K)^\sigma$.

Definition 3.2.2. Let X be a scheme over $\overline{k}$.

i) We say k is a *field of definition* of X if X is definable over k, i.e. there exists a scheme X_k over k such that $X_k \times_k \overline{k} \simeq X$.
ii) We say k is a *field of moduli* of X if we have an isomorphism $X^\sigma \simeq X$ over $\overline{k}$ for every automorphism $\sigma \in \mathrm{Aut}(\overline{k}/k)$.[31]

Remark 3.2.3. Analogous to Definition 3.2.2, we can talk about the field of definition and field of moduli of a scheme X over $\overline{k}$ endowed with some extra structure and some required properties. We shall assume that these extra structures can be base changed along field extensions, and that the properties are stable under such base changes, so that an extension k'/k of a field of definition k of X is also a field of definition.

Remark 3.2.4. If k is a field of definition of X then it is also a field of moduli of X.

Remark 3.2.5. Assume X (together with extra structures and properties) is parametrised by a coarse moduli space M over k_0. If k is a field of moduli of X, then X corresponds to a k-rational point of M. Indeed, by the definition of the field of moduli we see that the point corresponding to X over $\overline{k}$ is invariant under every automorphism $\sigma \in \mathrm{Aut}(\overline{k}/k)$. Now, since k is a perfect field, we have $k = \bar{k}^{\mathrm{Aut}(\bar{k}/k)}$, cf. [MilAG, Proposition 16.1], and we are done.

Proposition 3.2.6. *Assume that X (together with extra structures and properties) is parametrised by a* fine *moduli space M over k_0. If X corresponds to some k-rational point p of M (in particular k is a field of moduli of X), then k is a field of definition of X.*

Proof. Since M is a fine moduli space, there exists a universal family $\mathfrak{X}$ over it. Now the fibre of $\mathfrak{X}$ over the k-rational point p is a model of X over k as desired. □

The moduli space of smooth cubic fourfolds is coarse but not fine. The problem is that some smooth cubic fourfolds have nontrivial automorphisms, cf. [HuyC4, Section 2.1.5]. So let us get a feeling for the automorphism group of a smooth cubic fourfold. Some of the following statements are specializations of more general results about hypersurfaces or complete intersections to the case of cubic fourfolds.

Proposition 3.2.7. *The general cubic fourfold X has trivial automorphism group. More precisely, there exists a non-empty open subset $U \subset \mathrm{C4}_{k_0}$, such that we have $\overline{X} \in U(\overline{k})$ if and only if $\mathrm{Aut}(\overline{X}) = \{\mathrm{id}\}$.*

Reference. More generally, this is true for hypersurfaces of degree $d \geq 3$ and dimension $n \geq 1$, except when $(n,d) = (1,3)$ or $(n,d) = (2,4)$. See [Poo05] and [HuyC4, Theorem 1.3.12] for an overview. In [KS99, Lemma 11.8.5] it is proven, that U is open. See [MM64a, Theorem 5] and [KS99, p. 10.6.18] for the non-emptiness of U. □

Remark 3.2.8. More generally, let X be a smooth cubic fourfold, then $\mathrm{Aut}(X)$ is finite, cf. [HuyC4, Corollary 1.3.6]

Remark 3.2.9. The open set U in Proposition 3.2.7 is the fine moduli space of smooth cubic fourfolds with trivial geometric[32] automorphism group, i.e. there exists a universal family over U, cf. [HuyC4, Remark 2.1.14.ii)].

Corollary 3.2.10. *Let X be a smooth cubic fourfold over $\overline{k}$ with trivial automorphism group. Say, it corresponds to some geometric point $x\colon \mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(\mathrm{k}(x)) \to \mathrm{C4}_{k_0}$, in particular $\mathrm{k}(x)$ is the*[33] *field of moduli of X. Then $\mathrm{k}(x)$ is the field of definition of X.*

[31] See also the note [HH03] for a discussion of the field of moduli, where the focus is on curves.

[32] Since the automorphism group is finite, in particular discrete, it cannot grow under a base-change of algebraically closed fields.

[33] In the sense of 'the smallest' relative to k_0.

Proof. Let $U \subset \mathrm{C4}$ be the open subset of cubic fourfolds with trivial automorphism group, cf. Proposition 3.2.7. By Remark 3.2.9, we know that there is a universal family $\mathcal{X}$ over U. Now, the fibre $\mathcal{X}_x$ of this family over the point $x\colon \operatorname{Spec}(\mathrm{k}(x)) \to \mathrm{C4}_{k_0}$ provides a model of X over $\mathrm{k}(x)$ as desired. □

Proposition 3.2.11. *Let X be a generic special cubic fourfold over $\mathbb{C}$, i.e. we have the equality $\operatorname{rk}(\mathrm{H}^{2,2}(X^{\mathrm{an}}, \mathbb{Z})) = 2$, with discriminant d satisfying ($\star\star$). Then we have that $\operatorname{Aut}(X) \subset \mathbb{Z}/2\mathbb{Z}$. If in addition $d \equiv 2 \mod 6$, then we have that $\operatorname{Aut}(X) = \{\mathrm{id}\}$.*

Proof. Let $f \in \operatorname{Aut}(X)$ be some isomorphism. Denote by K the[34] labeling on X, and let S be an associated K3 surface, cf. Theorem 2.1.5. We have $f^*(K) \subset K$, since f^* preserves algebraic classes. It follows that $f^*(K) = K$, by calculating with the lattice or, simpler in our situation, by looking at f^{-1}.

Assume now that f is an isomorphism of marked cubic fourfolds, i.e. we have $f^*|_K = \mathrm{id}$. We see that $f^* \in \widetilde{\mathrm{O}}^{\mathrm{mar}}(K_d^{\perp})$, and a fortiori $f^* \in \widetilde{\mathrm{O}}(\Lambda_d)$ by Proposition 2.2.7.ii). The Global Torelli Theorem implies that f^* corresponds to some automorphism of S, cf. [HuyK3, Theorem 7.5.3]. Since $\rho(S) = 1$, this automorphism must be trivial and a fortiori we have $f^* = \mathrm{id}$, cf. [HuyK3, Corollary 15.2.12]. But f acts faithfully on the cohomology $\mathrm{H}^4(X, \mathbb{Z})$, cf. Proposition 3.2.13, and we see $f = \mathrm{id}$.

We conclude that in general $\operatorname{Aut}(X) \hookrightarrow \mathrm{O}(K, h^2) = \{g \in \mathrm{O}(K) \mid g(h^2) = h^2\}$. By [Has00, Proposition 5.2.1] we have $\mathrm{O}(K, h^2) = \{\mathrm{id}\}$ when $d \equiv 2 \mod 6$, and we have $\mathrm{O}(K, h^2) = \{\mathrm{id}, \iota\}$ when $d \equiv 0 \mod 6$ □

Remark 3.2.12. The involution ι on K_d in the case $d \equiv 0 \mod 6$ comes from an involution on the cubic lattice $\widetilde{\Gamma}$, cf. [Has00, Proposition 5.2.1]. In [Bra18, Theorem 1], the action of the involution on $\mathrm{K3}_d$ is determined.

Proposition 3.2.13. *Let X be a smooth cubic fourfold over $\overline{k}$. Let $N \geq 3$ and let ℓ be a prime, both coprime to $\operatorname{char}(\overline{k})$. Then the following two maps are injective:*

$$\operatorname{Aut}(X) \hookrightarrow \operatorname{Aut}(\mathrm{H}^4_{\text{ét}}(X, \mathbb{Q}_\ell))$$
$$\operatorname{Aut}(X) \hookrightarrow \operatorname{Aut}(\mathrm{H}^4_{\text{ét}}(X, \mathbb{Z}/N\mathbb{Z}))$$

Reference. See [CPZ15, Theorem 1.6] and [JL17, Corollary 2.2]. □

The proposition shows that we can eliminate automorphism of our cubic fourfolds X by equipping them with an isomorphism

$$\mathrm{H}^4_{\text{ét}}(X, \mathbb{Z}/N\mathbb{Z}) \xrightarrow{\sim} \widetilde{\Gamma} \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z}.$$

Such an isomorphism is a called[35] a *level N structure* on X. One can formulate the moduli problem of smooth cubic fourfolds with level N structure and arrives at the stack $\underline{\mathrm{C4}}_{[N]}$ over $\mathbb{Z}[1/N]$ parametrizing these objects, cf. [JL17, Section 3]. Eventually we will consider the stack $\underline{\mathrm{C4}}_{[N]}$ over $\mathbb{Q}$ only.

Theorem 3.2.14. *Let $N \geq 3$ be a natural number.*

i) *The stack $\underline{\mathrm{C4}}_{[N]}$ is represented by a smooth, affine scheme $\mathrm{C4}_{[N]}$ over $\mathbb{Z}[1/N]$.*

ii) *The stack $\underline{\mathrm{C4}}_{[N]}$ is a $\mathrm{GL}_{23}(\mathbb{Z}/N\mathbb{Z})$-torsor over the stack $\underline{\mathrm{C4}}$ of cubic fourfolds*[36].

In particular, the induced morphism $\mathrm{C4}_{[N]} \to \mathrm{C4}$ of (coarse) moduli space is finite.

Reference. See [JL17, Theorem 3.2]. □

[34] Note that it is unique, since $\operatorname{rk}(\mathrm{H}^{2,2}(X^{\mathrm{an}}, \mathbb{Z})) = 2$.

[35] To be compatible with our discussion in Section 3.3 we should require that $h^2 \in \mathrm{H}^4_{\text{ét}}(X, \mathbb{Z}/N\mathbb{Z})$, where h is the class of a hyperplane section, is mapped to the abstract element $h^2 \in \widetilde{\Gamma} \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z}$.

[36] The stack of cubic fourfolds is the quotient stack $[\mathrm{PGL}_6 \backslash \mathrm{Hilb}_{3,4}]$ where $\mathrm{Hilb}_{3,4}$ is the Hilbert scheme of degree 4 hypersurfaces in $\mathbb{P}^5$.

Corollary 3.2.15. *Let $x \in \mathrm{C4}(k)$ be some k-rational point, and let X be the smooth cubic fourfold over $\overline{k}$ corresponding to x, so X has k as a field of moduli. Then there exists a field extension k'/k, whose degree is bounded by the constant $\#\,\mathrm{GL}(23, \mathbb{F}_3)$, such that X descends to k', i.e. k' is a field of definition for X.*

Proof. As soon as we have a k'-rational point of $\mathrm{C4}_{[3]}$ we get a cubic fourfold over k', since the moduli space is fine[37]. Now we lift a k-rational point to a k'-rational point through the finite morphism $\mathrm{C4}_{[3]} \to \mathrm{C4}$. ☐

3.3. **Descent of Hassett's association.** In this section we want to discuss the descent of Hassett's association $\phi_d\colon \mathrm{K3}_d \dashrightarrow \mathrm{C4}_d$. The fact that ϕ_d is not canonical in general introduces some difficulties. As a start, let us discuss the case when ϕ_d is canonical.

Proposition 3.3.1. *Let $d = 2p^k$, where p is some odd prime with $p \equiv 1 \mod 3$ and $k \geq 1$. Let (S, f) be in the domain of ϕ_d, and set $(X, K) = \phi_d^{\mathrm{mar}}(S, f)$ Then we have $\phi_d^{\mathrm{mar}}(S^\sigma, f^\sigma) = (X^\sigma, K^\sigma)$ for every $\sigma \in \mathrm{Aut}(\mathbb{C})$. In particular, Hassett's association ϕ_d descends to $\mathbb{Q}$.*

Proof. By Corollary 3.1.6, we know that (S^σ, f^σ) is associated to the marked cubic fourfold (X^σ, K^σ). So, by Remark 2.3.4, we see that there is some choice of ϕ_d^{mar} such that we have $\phi_d^{\mathrm{mar}}(S^\sigma, f^\sigma) = (X^\sigma, K^\sigma)$. But, for $d = 2p^k$ as in the hypothesis, there is only one choice of ϕ_d^{mar}, cf. Proposition 2.3.6.

Now, by Corollary 3.3.1 the map ϕ_d commutes with conjugation by $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. Hence by Galois descent, cf. Proposition 3.1.8, the map (as well as the domain where it is defined) is definable over $\mathbb{Q}$. ☐

Proposition 3.3.2. *Let $d \in \mathbb{N}$ be a natural number that satisfies Hassett's condition ($\star\star$), and let $\phi_d\colon \mathrm{K3}_d \dashrightarrow \mathrm{C4}$ be a Hassett map. Then ϕ_d descends to a number field $k/\mathbb{Q}$, whose degree is bounded by $[k : \mathbb{Q}] \leq \#R_{2,d}/2$.*

Proof. If the subgroup $G \subset \mathrm{Aut}(\mathbb{C})$ which fixes ϕ_d under conjugation[38], i.e. $\phi_d^\sigma = \phi_d$ for each $\sigma \in G$, has finite index, which is bounded by $\#R_{2,d}/2$, then the fixed field $k := \mathbb{C}^G$ is a number field as desired[39] We get this control over G as soon as we verify that for every choice of Hassett map ϕ'_d, also ${\phi'_d}^\sigma$ is a Hassett map. Indeed, then $\mathrm{Aut}(\mathbb{C})$ acts on the set $A := \{\phi_d^\sigma \mid \sigma \in \mathrm{Aut}(\mathbb{C})\}$ by $\tau\,.\,\phi := \phi^\tau$, and $\#A \leq \#R_{2,d}/2$, cf. Proposition 2.3.5, shows that $G = \mathrm{Stab}(\phi_d^{\mathrm{id}})$ is as desired.

Now, take $\sigma \in \mathrm{Aut}(\mathbb{C})$ and fix some Hassett map[40] ϕ_d. We want to show that ϕ_d^σ is again one of Hassett's maps. Recall that Hassett's maps are only rational maps, so we shrink the domain to an open U, where ϕ_d^σ and all choices of Hassett's map ϕ'_d are defined[41]. If ϕ_d^σ would not be a Hassett map, then the locus where it coincides with some is a closed subscheme E of codimension at least 1. Now, we consider an irreducible divisor D in the domain U, and not lying in E, and consider all the images $\phi'_d(D)$. By shrinking U we can assume that the images are pairwise disjoint. Now, Proposition 3.3.2 shows that

$$\phi_d^\sigma(D) \subset \bigcup_{\phi'_d} \phi'_d(D),$$

[37]Actually it is enough to use that cubic fourfolds with level 3 structure have trivial automorphism group, cf. Proposition 3.2.13

[38]Recall that $\sigma \in \mathrm{Aut}(\mathbb{C})$ acts (locally) coordinatewise, so we have pointwise $\phi^\sigma(s) = \sigma.\,\phi(\sigma^{-1}.\,s)$.

[39]As above, we can apply Proposition 3.1.10 in order assume $G \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is of finite index. Note that it is a closed subgroup, since it is defined as the fixed group of the map ϕ_d, and the involved spaces are of finite type. Now we can apply Galois theory to see that k satisfies $[k : \mathbb{Q}] \leq \#R_{2,d}/2$.

[40]From this point on, we consider Hassett's maps $\phi_d^{\mathrm{mar}}\colon \mathrm{K3}_d \dashrightarrow \mathrm{C4}_d^{\mathrm{mar}}$, but still refer to them by ϕ_d to keep the indices and superscript in the notation under control.

[41]Recall that there are only finitely many choices. This fact is also important for the remaining steps of this proof.

and since D is irreducible, there exists a choice ϕ'_d such that $\phi^\sigma_d(D) \subset \phi'_d(D)$. We conclude that $\phi^\sigma_d|_D = \phi'_d|_D$, again using Proposition 3.3.2. Indeed, for every point $s \in D$ the image point $\phi^\sigma_d(s)$ lies in $\phi'_d(D)$ and is 'associated' to s. The only possibility is that $\phi^\sigma_d(s) = \phi'_d(s)$. This reaches a contradiction as desired.

Last but not least, use that $\mathrm{K3}_d$ is irreducible, and that $\mathrm{C4}^{\mathrm{mar}}_d$ is separated to extend the equality $\phi^\sigma_d = \phi'_d$ beyond the open set U. □

Remark 3.3.3. For the proof of Proposition 3.3.2, we only need the generic version of Theorem 2.4.8. Indeed, in the proof, we only work on some open subset U, and this can take the open subset from the generic version of Theorem 2.4.8 into account.

Note that a postiori, this removes the dependence of Proposition 3.1.11 on the full version of Theorem 2.4.8, by establishing it as a Corollary of Proposition 3.3.2.

Let us also look at the situation on the level of arithmetic varieties. We will demonstrate descent to $\overline{\mathbb{Q}}$, when using suitable level structures, without depending on Theorem 2.4.8 or Theorem 3.1.5.

Proposition 3.3.4 (Rigid descent)**.** *Let* $\mathbf{P}$ *be a property of morphisms of schemes over* $\mathbb{Q}$ *that is stable under base change and compatible with limits of schemes. Let* X *be a variety over a field* $\Omega/\overline{\mathbb{Q}}$ *that satisfies* $\mathbf{P}$*. If* X *is* $\mathbf{P}$*-rigid, i.e. every deformation* $\mathcal{X}$ *of* X *over a connected variety* T *over* Ω *which satisfies* $\mathbf{P}$*, is the trivial deformation, then* X *is definable over* $\overline{\mathbb{Q}}$*.*

Proof. Spread out X to $\mathcal{X}$ over $T = \mathrm{Spec}(A)$, where $\mathbb{Q} \subset A \subset \Omega$ is of finite-type over $\mathbb{Q}$, non-empty, and connected. Since X/Ω satisfies $\mathbf{P}$ and $\mathbf{P}$ is compatible with limits of schemes, we can assume that $\mathcal{X}/T$ satisfies $\mathbf{P}$. The inclusion $A \subset \Omega$ gives an Ω-point of T whose image is the generic point η of T. Now base-changing back to Ω we get a deformation of X which satisfies $\mathbf{P}$, since the latter is stable under base change. By our hypothesis, it must be the trivial deformation. So for every closed point $t \in T$, say with associated Ω-rational point $\bar{t} \in T(\Omega)$, we have $\mathcal{X}_{\bar{t}}$ is isomorphic to X. But now, $\mathcal{X}_t$ is definable over $\overline{\mathbb{Q}}$, since T is of finite-type over $\mathbb{Q}$. □

A more principled approach and exposition of rigidity and rigid descent can be found in [Pet17]. Our version which does not mention deformation theory is sufficient for our applications.

Proposition 3.3.5 (Piatetski-Shapiro–Shafarevich)**.** *Let* X *and* Y *be varieties over* $\overline{\mathbb{Q}}$*, and let* $\phi\colon X_{\mathbb{C}} \to Y_{\mathbb{C}}$ *be a morphism over* $\mathbb{C}$*. Assume that* X *is irreducible, and that* Y *is quasi-projective. If there exists an everywhere dense, in the Zariski-topology, subset* $\mathcal{S} \subset X(\mathbb{C})$ *such that every* $x \in \mathcal{S}$ *is a* $\overline{\mathbb{Q}}$*-rational point*[42]*, and every* $\phi(x) \in Y(\mathbb{C})$ *is a* $\overline{\mathbb{Q}}$*-rational point, then* ϕ *descends to a morphism* $\phi\colon X \to Y$ *over* $\overline{\mathbb{Q}}$*.*

Sketch. See [PSS75, Lemma 9] for details. By pulling-back projective coordinates $Y_1, \ldots, Y_N$ on the codomain Y, we can reduce to the case where ϕ is a rational function. Now write

$$\phi = \frac{p(X_1, \ldots, X_n)}{q(X_1, \ldots, X_n)},$$

where $p, q \in \mathbb{C}[X_1, \ldots, X_n]$, and $X_1, \ldots, X_n$ are rational coordinates on X. The infinite linear system

$$q(x_1, \ldots, x_n)\phi(x) = p(x_1, \ldots, x_n), \quad x \in \mathcal{S}$$

solving for the coefficients of p and q has by definition a solution over $\mathbb{C}$. Since $\mathbb{C}$ is flat over $\overline{\mathbb{Q}}$, the system also has a solution over $\overline{\mathbb{Q}}$, say $\widetilde{p}$, $\widetilde{q}$. We have $\phi(x) = \frac{\widetilde{p}(x)}{\widetilde{q}(x)}$ for every

[42]Fix some embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ in order to fix the notion of $\overline{\mathbb{Q}}$-rational point.

point $x \in \mathcal{S}$, so the density of $\mathcal{S} \subset X(\mathbb{C})$ and the irreducibility of X show that ϕ is a rational function over $\overline{\mathbb{Q}}$. □

We can descend morphisms using Proposition 3.3.5, but it is unhelpful for descending the domain of definition of a rational map. To remedy this, we look at the level of arithmetic varieties instead of moduli spaces. Furthermore, we use level structures to guarantee that the involved arithmetic varieties are definable over $\overline{\mathbb{Q}}$. See [AT14, Proposition 5.2] for a similar discussion.

Recall the period domains from Section 2.2, Q_d for polarized K3 surfaces of degree d, D for cubic fourfolds, and D_d for special cubic fourfolds of discriminant d. Instead of taking quotients by $\widetilde{\mathrm{O}}(\Lambda_d)$, $\widetilde{\mathrm{O}}(\Gamma)$, and $\widetilde{\mathrm{O}}^{\mathrm{mar}}(K_d^{\perp})$ respectively, to arrive at the arithmetic varieties $\mathcal{N}_d$, $\mathcal{C}$, and $\mathcal{C}_d^{\mathrm{mar}}$ respectively, we take quotients by the finite-index subgroups

$$\begin{aligned} \widetilde{\mathrm{O}}(\Lambda_d)_{[N]} &\coloneqq \{g \in \widetilde{\mathrm{O}}(\Lambda_d) \mid g \equiv \mathrm{id} \mod N\}, \\ \widetilde{\mathrm{O}}(\Gamma)_{[N]} &\coloneqq \{g \in \widetilde{\mathrm{O}}(\Gamma) \mid g \equiv \mathrm{id} \mod N\}, \\ \widetilde{\mathrm{O}}^{\mathrm{mar}}(K_d^{\perp})_{[N]} &\coloneqq \{g \in \widetilde{\mathrm{O}}^{\mathrm{mar}}(K_d^{\perp}) \mid g \equiv \mathrm{id} \mod N\}. \end{aligned}$$

We arrive at the arithmetic varieties

$$\mathcal{N}_{d,[N]} \coloneqq \widetilde{\mathrm{O}}(\Lambda_d)_{[N]} \backslash Q_d, \qquad \mathcal{C}^{\mathrm{mar}}_{d,[N]} \coloneqq \widetilde{\mathrm{O}}^{\mathrm{mar}}(K_d^{\perp})_{[N]} \backslash D_d, \qquad \mathcal{C}_{[N]} \coloneqq \widetilde{\mathrm{O}}(\Gamma)_{[N]} \backslash D,$$

corresponding to polarized K3 surfaces of degree d with level structure, marked cubic fourfolds of discriminant d with level structure, and cubic fourfolds with level structure respectively.

Now, Proposition 2.2.7 establishes a Hassett map $\bar{\phi}_{d,[N]} \colon \mathcal{N}_{d,[N]} \to \mathcal{C}^{\mathrm{mar}}_{d,[N]} \to \mathcal{C}_{d,[N]}$, which becomes a Hassett map $\bar{\phi}_d$ from before when forgetting level structures, i.e. when $N = 1$.

We want to consider the arithmetic varieties over $\overline{\mathbb{Q}}$. For this let us recall the notion of neat group, cf. [Bor69, Section 17.1].

Definition 3.3.6. A matrix $g \in \mathrm{GL}_n(k)$ is called *neat* if the subgroup of $\overline{k}^{\times}$ generated by its eigenvalues is torsion-free. A subgroup $G \subset \mathrm{GL}_n(k)$ is called neat if all its elements are neat.

Proposition 3.3.7. *The arithmetic varieties $\mathcal{N}_{d,[N]}$, and $\mathcal{C}_{[N]}$ respectively, are definable over $\overline{\mathbb{Q}}$ if the groups $\widetilde{\mathrm{O}}(\Lambda_d)_{[N]}$, and $\widetilde{\mathrm{O}}(\Gamma)_{[N]}$ respectively, are neat.*

Reference. This is a special case of a Theorem by Faltings, cf. [Fal84, Theorem 1], or [Pet17, Proposition 3.1] for another exposition of his result. □

Remark 3.3.8. Let us examine for which $N \in \mathbb{N}$ the groups $\widetilde{\mathrm{O}}(\Lambda_d)_{[N]}$, and $\widetilde{\mathrm{O}}(\Gamma)_{[N]}$ respectively, are neat. As a first approximation, we have that for $N \geq 3$ the group $\mathrm{GL}_n(\mathbb{Z})_{[N]} \coloneqq \{g \in \mathrm{GL}_n(\mathbb{Z}) \mid g = \mathrm{id} \mod N\}$ is torsion-free, cf. [Min87], and a fortiori the groups $\widetilde{\mathrm{O}}(\Lambda_d)_{[N]}$ and $\widetilde{\mathrm{O}}(\Gamma)_{[N]}$ are also torsion-free. But, the condition $N \geq 3$ is also sufficient to guarantee neatness, cf. [Sch10, Section 4.3].

Proposition 3.3.9. *The morphism $\bar{\phi}_{d,[N]} \colon \mathcal{N}_{d,[N]} \to \mathcal{C}_{[N]}$ is definable over $\overline{\mathbb{Q}}$, when we have $N \geq 3$.*

Proof. Consider the set $\mathcal{S} \subset \mathcal{N}_{d,[N]}$ of periods of singular K3 surfaces $(S, f, \varphi) \in \mathrm{K3}_{d,[N]}(\mathbb{C})$, i.e. we have $\rho(S) = 20$. By rigid descent, cf. Proposition 3.3.4, singular K3 surfaces are definable over $\overline{\mathbb{Q}}$.[43]

[43] A fortiori the points of $\mathcal{S}$ are definable over $\overline{\mathbb{Q}}$, since the period map $\mathrm{K3}_{d,[N]} \to \mathcal{N}_{d,[N]}$ is definable over $\overline{\mathbb{Q}}$, cf. [PSS75, Lemma 7]. In fact their argument also considers singular K3 surfaces, and shows that $\mathcal{S}$ is definable over $\overline{\mathbb{Q}}$ by drawing on the Kuga–Sata construction, cf. [PSS75, Page 50] and [PSS75, Corollary to Lemma 4].

Indeed, denote by **P** the property to be a family of polarized K3 surfaces of degree d over a $\mathbb{Q}$-scheme, having fibres of Picard rank 20. Note that **P** satisfies the hypotheses of Proposition 3.3.4, use, among others, that the Picard rank can only increase under specialization, but Picard rank 20 is already maximal in characteristic 0. Also, every singular K3 surface is **P**-rigid, since (in characteristic 0) a smooth, proper family of K3 surfaces over a connected, quasi-projective base is isotrivial, cf. [HuyK3, Remark 6.2.10].

In the same way, elements[44] of the set $\bar{\phi}_{d,[N]}(\mathcal{S}) \subset \mathcal{C}_{[N]}$, which is zero dimensional as just noticed, are definable over $\overline{\mathbb{Q}}$.

We can now conclude using Proposition 3.3.5, since $\mathcal{S} \subset \mathcal{N}_{d,[N]}$ is indeed everywhere dense, cf. [HuyK3, Proposition 6.2.9] applied to $\mathrm{K3}_{d,[N]}$ and using induction. ☐

Remark 3.3.10. When S is a singular K3 surface over $\mathbb{C}$, i.e. $\rho(S) = 20$, we just saw that S is definable over $\overline{\mathbb{Q}}$. This was originally observed in [SI77, Theorem 6]. One can determine the number field over which S is defined concretely, cf. [Sch07, Proposition 4.1].

This definability over $\overline{\mathbb{Q}}$ is true in greater generality for K3 surfaces of CM type, cf. [PSS75, Theorem 4].

Remark 3.3.11. Going back to the level of moduli spaces, this shows again descent of Hassett's maps $\phi_d\colon \mathrm{K3}_d \dashrightarrow \mathrm{C4}$ to $\overline{\mathbb{Q}}$.

Remark 3.3.12. If one wants to descend the morphisms $\bar{\phi}_{d,[N]}$ to number fields of controlled degree, one first needs to find models of the arithmetic varieties we considered, that are defined over number fields of controlled degree. Hence, one should replace the arithmetic varieties by Shimura varieties. We do not carry on in this thesis with the discussion in this direction.

3.4. **Reduction of Kuznetsov association.** We want to reduce Hassett's association modulo p. We consider the association on the level of derived categories as discussed in Section 2.4. The following ideas should be considered as a rough sketch or even just a hope that rely on announced results by Bayer–Lahoz–Macrì–Nuer–Perry–Stellari, cf. [BLM+17]. Before we consider Fourier–Mukai equivalences, let us recall a classical theorem of Matsusaka–Mumford which allows for of K3 surfaces to reduce the property to be isomorphic modulo p.

Theorem 3.4.1 (Matsusaka–Mumford)**.** *Let X and Y be families of varieties over a discrete valuation ring A, say with generic point η and special point s. Assume that X is proper over A, and that the special fibre Y_s is geometrically non-ruled.*

i) *Assume that (X, f) and (Y, g) are smooth, projective, polarized schemes over A. Then any isomorphism of generic fibres $(X, f)_\eta$ and $(Y, g)_\eta$ extends to an isomorphism of (X, f) with (Y, g) over A.*
ii) *If the generic fibres X_η and Y_η are birational, then also the special fibres X_s and Y_s are birational.*

Proof. See [MM64b, Theorem 2] and [MM64b, Theorem 1]. ☐

Corollary 3.4.2. *Let X and Y be K3 surfaces over a discrete valuation ring A, say with generic point η and special point s. If the generic fibres X_η and Y_η are isomorphic, then also the special fibres X_s and Y_s are isomorphic.*

They also demonstrate that a singular K3 surface is definable over $\overline{\mathbb{Q}}$, cf. [PSS75, Lemma 8]. Namely, given a singular, polarized K3 surface (S, f), we have to show that $\{(S, f)^\sigma \mid \sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})\}$ is finite. We have $\mathrm{NS}(S^\sigma) \simeq \mathrm{NS}(S)$, and see $\mathrm{disc}(\mathrm{T}(S^\sigma)) = \mathrm{disc}(\mathrm{T}(S))$, which shows by lattice-theory that the lattices $\mathrm{T}(S^\sigma)$ fall into finitely many isomorphism classes. Note that the transcendental lattice $\mathrm{T}(S)$ of a singular K3 surface determines the latter up to isomorphism, cf. [PSS72, Theorem 8]. To get finiteness when also considering polarizations use Proposition 1.3.3.

[44] Shrink $\mathcal{S}$ such that $\bar{\phi}_{d,[N]}(\mathcal{S}) \cap (\mathcal{C}_{2,[N]} \cup \mathcal{C}_{6,[N]}) = \emptyset$, so that we can talk about smooth cubic fourfolds instead of their periods.

Proof. By Theorem 3.4.1.ii), the K3 surfaces X_s and Y_s are birational. Since both X_s and Y_s are minimal surfaces (meaning that their canonical bundles are nef), this birational map extends to an isomorphism. □

Remark 3.4.3. Note that Corollary 3.4.2 says that for K3 surfaces the property to be isomorphic reduces modulo p, not that a particular isomorphism reduces modulo p.

The following desideratum is inspired by announced results in [BLM+17], see also the lecture notes [MS18, Section 5].

Desideratum 3.4.4. *Let A be a discrete valuation ring with generic point η and special point s. Let $\mathcal{S}$ be a relative K3 surface over A and let $\mathcal{X}$ be a relative smooth cubic fourfolds over A. If there exists a Fourier–Mukai equivalence $\mathbf{D}^{\mathrm{b}}(\mathcal{S}_\eta) \simeq \mathcal{K}u(\mathcal{X}_\eta)$, then there also exists a Fourier–Mukai equivalence $\mathbf{D}^{\mathrm{b}}(\mathcal{S}_s) \simeq \mathcal{K}u(\mathcal{X}_s)$.*

Sketch. Write S as a moduli space of stable objects in $\mathcal{K}u(X)$, say $S \simeq \mathrm{M}(\mathcal{K}u(X), \nu, \sigma)$ and formulate the induced relative moduli problem

$$\mathcal{M} := \mathrm{M}(\mathcal{K}u(\mathcal{X}), \nu, \sigma).$$

Now, the special fibre $\mathcal{M}_s$ is a K3 surface by the numerics of ν, and since it arises as a moduli space of sheaves we get a Fourier–Mukai equivalence $\mathbf{D}^{\mathrm{b}}(\mathcal{M}_s) \simeq \mathcal{K}u(\mathcal{X}_s)$. Using Corollary 3.4.2, we see that $\mathcal{S}_s \simeq \mathcal{M}_s$, since the generic fibres of $\mathcal{S}$ and $\mathcal{M}$ are isomorphic. □

Remark 3.4.5. Analogous to Remark 3.4.3, in Desideratum 3.4.4 it is the *property* to be Fourier–Mukai equivalent, that specialises, not a particular Fourier–Mukai equivalence.

Remark 3.4.6. The Desideratum can also be used with valuation rings like $\mathcal{O}_{\overline{\mathbb{Q}}_p}$. These rings are not noetherian, hence not discrete valuation rings, but one can apply spreading out to the situation in order to reduce the problem to one over the *discrete* valuation ring $\mathcal{O}_K$ of a finite field extension $K/\mathbb{Q}_p$.

Question 3.4.7. In Desideratum 3.4.4, we assumed that $\mathcal{X}$ is a smooth hypersurface over a discrete valuation ring. To get into this situation, the question arises if a smooth cubic fourfold, that is associated to a K3 surface which has good reduction modulo p, also has good reduction as a hypersurface.

Eventually, we want to apply this desideratum over an algebraically closed subfield $\overline{k} \subset \mathbb{C}$, like $\overline{\mathbb{Q}}$ or $\overline{\mathbb{Q}}_p$. So we want that if S and X are varieties over $\overline{k}$ that are[45] Fourier–Mukai equivalent over $\mathbb{C}$, then they are already Fourier–Mukai equivalent over $\overline{k}$.

Proposition 3.4.8. *Let S be a K3 surface over K, and $\Phi_{\mathcal{P}}$ a Fourier–Mukai equivalence between S and some variety[46] X. Then the Fourier–Mukai kernel $\mathcal{P} \in \mathbf{D}^{\mathrm{b}}(S \times_K X)$ is rigid, i.e. has no nontrivial deformation. In particular, if $\overline{k} \subset K$ is an algebraically closed subfield, then $\Phi_{\mathcal{E}}$ descends to a Fourier–Mukai equivalence over $\overline{k}$.*

Proof. The set of deformations of $\mathcal{P}$ is a torsor under $\mathrm{H}^1(S \times_K X, \mathcal{H}om(\mathcal{P}, \mathcal{P}))$. We show that this cohomology group is trivial. The local to global Ext spectral sequence $E_2^{pq} = \mathrm{H}^p(\mathcal{E}xt^q(\mathcal{P}, \mathcal{P})) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{P}, \mathcal{P})$ shows that $\mathrm{H}^1(\mathcal{H}om(\mathcal{P}, \mathcal{P})) = 0$ if $\mathrm{Ext}^1(\mathcal{P}, \mathcal{P}) = 0$. Now, use that

$$\mathrm{Ext}^\bullet(\mathcal{P}, \mathcal{P}) \simeq \mathrm{Ext}^\bullet(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}),$$

cf. [AT14, Section 6.1], and the latter is by definition Hochschild cohomology $\mathrm{HH}^\bullet(S)$. The modified Hochschild–Kostant–Rosenberg isomorphism identifies

$$\mathrm{HH}^\bullet(S) \simeq \bigoplus_{i+j=\bullet} \mathrm{H}^i(\wedge^j \mathcal{T}_S),$$

[45] Or S is Fourier–Mukai equivalent to $\mathcal{K}u(X)$ when X is a cubic fourfold.

[46] Or Kuznetsov component.

cf. [AT14, Section 6.3]. But, we have $\mathrm{HH}^1(S) = 0$, since S is a K3 surface.

Now apply rigid descent for sheaves, cf. Proposition 3.3.4[47], to see that $\mathcal{P}$ is already definable over $\overline{k}$. It remains to note that $\mathcal{P}$ over $\overline{k}$ induces a Fourier–Mukai equivalence. Indeed, the base change from $\overline{k}$ to K is faithfully flat, and thus we can check the criterion for being a Fourier–Mukai equivalence over K, cf. [HuyFM, Lemma 1.50]. □

Remark 3.4.9. In the proposition, we do not need the full strength of the assumption that S is a K3 surface. We could instead assume that $\mathrm{H}^0(S, \mathcal{T}_S) = 0$ and $\mathrm{H}^1(S, \mathcal{O}_S) = 0$.

4. Towards the Tate Conjecture for K3 surfaces

In order to apply Lieblich–Maulik–Snowden's theorem, cf. Theorem 1.2.5, we want to prove a finiteness statement for K3 surfaces. We aim to do this using special cubic fourfolds. So far we have considered associated cubic fourfolds over (local) number fields. To get into this realm, we need to lift our K3 surfaces to characteristic 0.

4.1. **Lifting K3 surfaces.**

In this section we will discuss results about lifting K3 surfaces to characteristic 0. Let us remark that most of this section is a digression, as our main application will only use Theorem 4.1.2.

The motivation for this section is the quest to lift possibly infinitely many K3 surfaces from a finite field to characteristic 0 simultaneously. This would help in lifting all the moduli spaces of sheaves appearing in Lieblich–Maulik–Snowden's argument.

Situation 4.1.1. Let k be a perfect field of characteristic $\mathrm{char}(k) = p > 0$. Let S be a K3 surface over k, and let $\mathcal{L} \in \mathrm{Pic}(S)$ be a non-trivial line bundle on S.

Theorem 4.1.2 (Deligne)**.** *Let S be a K3 surface over a perfect field k, and let $\mathcal{L}$ be an ample line bundle on S. Then there exists a smooth, proper lift of $(S, \mathcal{L})$ over some complete discrete valuation ring W that is finite over* $\mathrm{W}(k)$*.*

Reference. See [Del81, Corollaire 1.7]. □

Remark 4.1.3. Lifting a K3 surface formally over $\mathrm{W}(k)$ is comparatively easier, cf. [HuyK3, Proposition 9.5.2]. The extra difficulty lies in lifting an ample line bundle $\mathcal{L}$ along with S. Then one can apply Grothendieck's algebraization theorem, cf. [FGAex, Theorem 8.4.10], to see that the lift is given by a scheme.

Remark 4.1.4. Note that a proper, smooth lift of a K3 surface is again a K3 surface.

Since the Tate Conjecture for K3 surfaces over finite fields is verified, we know by Theorem 1.2.5 that there are only finitely many K3 surfaces over a fixed finite field $\mathbb{F}_q$ of characteristic at least 3.

Corollary 4.1.5. *Let $\mathbb{F}_q$ be a finite field of characteristic at least* 3*. Then there exists a complete discrete valuation ring W that is finite over* $\mathrm{W}(\mathbb{F}_q)$ *such that every K3 surface S over $\mathbb{F}_q$ lifts to a proper, smooth scheme over W.*

For our purposes we would certainly like to deduce this corollary without using the Tate Conjecture. In the rest of this section we work towards this end, but will not quite achieve it. Another question one can ask is: What is the degree of the extension $W/\mathrm{W}(\mathbb{F}_q)$? We will give a uniform bound (independent of q) for this degree, cf. Corollary 4.1.23.

We will now discuss a class of K3 surfaces that are special to positive characteristic and provide most difficulties in lifting, more precisely this is the class of supersingular K3 surfaces and in particular superspecial K3 surfaces.

[47]There we discuss rigid descent for varieties.

Definition 4.1.6. Let S be a K3 surface over an algebraically closed field $k = \overline{k}$ of characteristic $\mathrm{char}(k) = p > 0$.

i) We call S *(Artin-)supersingular* if its formal Brauer group[48] $\widehat{\mathrm{Br}}_S$ has height ∞, i.e. we have $\widehat{\mathrm{Br}}_S \simeq \widehat{\mathbb{G}}_{\mathrm{a}}$.

ii) We call S *Shioda-supersingular* if $\rho(S) = 22$.

Remark 4.1.7. In characteristic 0 there cannot be any Shioda-supersingular K3 surfaces, since Hodge theory implies $\rho \leq 20$, cf. [HuyK3, Section 1.3.3]. Also the formal Brauer group becomes uninteresting, since in characteristic 0 all one-dimensional formal group laws are isomorphic, cf. [Haz12, Theorem 1.6.2].

Remark 4.1.8. As remarked in Section 1, Nygaard and Nygaard–Ogus, cf. [Nyg83] and [NO85], prove the Tate Conjecture for non-supersingular K3 surfaces in positive characteristic. They find lifts of these K3 surfaces to characteristic 0 with nice properties, so called '(quasi) canonical lifts', and can then conclude the Tate Conjecture from the characteristic 0 case.

Conjecture 4.1.9 (Artin)**.** *The notion to be Artin-supersingular is equivalent to the notion to be Shioda-supersingular.*

Remark 4.1.10. In view of Remark 4.1.8 the remaining cases of the Tate Conjecture for K3 surfaces over finite fields become equivalent to the above conjecture of Artin. So nowadays, one can work with the simpler definition, namely Shioda-supersingular[49].

Recall that the deformation functor for our K3 surface S with non-trivial line bundle $\mathcal{L}$ is given by

$$\begin{aligned} \underline{\mathrm{Def}}(S,\mathcal{L})\colon\ \mathbf{Art}/\mathrm{W}(k) &\to \mathbf{Set} \\ A &\mapsto \{\text{flat, proper lift of } S \text{ over } A \text{ together with a lift of } \mathcal{L}\} \\ f &\mapsto f^*, \end{aligned}$$

where $\mathbf{Art}\,/\,\mathrm{W}(k)$ is the category of Artinian local algebras over $\mathrm{W}(k)$ with residue field k.

Theorem 4.1.11. *The functor* $\underline{\mathrm{Def}}(S,\mathcal{L})$ *is pro-representable by a formal scheme* $\mathrm{Def}(S,\mathcal{L})$ *which is flat and of relative dimension* 19 *over* $\mathrm{Spf}\,\mathrm{W}(k)$.

Reference. See [HuyK3, Theorem 9.5.4] and the original [Del81, Théorème 1.6]. □

Remark 4.1.12. Now, if $\mathrm{Def}(S,\mathcal{L})$ is (formally) smooth over $\mathrm{W}(k)$, then there exists a formal lift of $(S,\mathcal{L})$. Using Remark 4.1.3, this lift is algebraic.

Let us recall the Hodge filtration and the conjugate filtration. The Hodge filtration is a descending filtration $\mathrm{F}^i\,\mathrm{H}^2_{\mathrm{dR}}$ on the algebraic de Rham cohomology $\mathrm{H}^2_{\mathrm{dR}}(S/k)$. It is induced by the filtration

$$\mathrm{F}^i\,\Omega^\bullet_S \coloneqq (\cdots \to 0 \to \Omega^i_S \to \Omega^{i+1}_S \to \dots)$$

on the algebraic de Rham complex, which also induces the Hodge–de Rham spectral sequence $E_1^{pq} = \mathrm{H}^q(X,\Omega^p_S) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{dR}}(S/k)$. This spectral sequence degenerates at the E_1-page in our situation. In our case we get

$$0 = \mathrm{F}^3\,\mathrm{H}^2_{\mathrm{dR}} \subset \mathrm{F}^2\,\mathrm{H}^2_{\mathrm{dR}} \subset \mathrm{F}^1\,\mathrm{H}^2_{\mathrm{dR}} \subset \mathrm{F}^0\,\mathrm{H}^2_{\mathrm{dR}} = \mathrm{H}^2_{\mathrm{dR}}(S/k)$$

with subquotients $\mathrm{F}^i\,\mathrm{H}^2_{\mathrm{dR}}/\mathrm{F}^{i+1}\,\mathrm{H}^2_{\mathrm{dR}} \simeq \mathrm{H}^{2-i}(S,\Omega^i_S)$.

[48] See [HuyK3, Section 18.1.3, Section 18.3.1] for a definition of the formal Brauer group and its height.

[49] Of course we have to be more careful here, since we do not want to presuppose the Tate Conjecture.

Now let us come to the conjugate filtration $\mathrm{F}^i_{\mathrm{con}}\,\mathrm{H}^2_{\mathrm{dR}}$, which is also a descending filtration on $\mathrm{H}^2_{\mathrm{dR}}(S/k)$. Denote by $\mathrm{Fr}_{S/k}\colon S \to S^{(p)}$ the relative Frobenius of S over k. Consider the spectral sequence for hypercohomology

$$E_2^{ij} = \mathrm{H}^i(S^{(p)}, \mathcal{H}^j((\mathrm{Fr}_{S/k})_*\,\Omega^\bullet_{S/k})) \Rightarrow \mathrm{H}^{i+j}_{\mathrm{dR}}(S/k),$$

which also degenerates at its first page[50]. The Cartier isomorphism, cf. [DI87, Théorème 1.2],

$$\mathcal{H}^j((\mathrm{Fr}_{S/k})_*\,\Omega^\bullet_{S/k}) \xrightarrow{\sim} \Omega^j_{S^{(p)}/k}$$

identifies the E_2-page with $\mathrm{H}^i(S^{(p)}, \Omega^j_{S^{(p)}/k})$, and in conclusion we get the filtration

$$0 = \mathrm{F}^3_{\mathrm{con}}\,\mathrm{H}^2_{\mathrm{dR}} \subset \mathrm{F}^2_{\mathrm{con}}\,\mathrm{H}^2_{\mathrm{dR}} \subset \mathrm{F}^1_{\mathrm{con}}\,\mathrm{H}^2_{\mathrm{dR}} \subset \mathrm{F}^0_{\mathrm{con}}\,\mathrm{H}^2_{\mathrm{dR}} = \mathrm{H}^2_{\mathrm{dR}}(S/k)$$

with subquotients $\mathrm{F}^i_{\mathrm{con}}\,\mathrm{H}^2_{\mathrm{dR}}/\mathrm{F}^{i+1}_{\mathrm{con}}\,\mathrm{H}^2_{\mathrm{dR}} \simeq (\mathrm{Fr}_{S/k})^*\,\mathrm{H}^i(S, \Omega^{2-i}_S)$.

Definition 4.1.13. We call a K3 surface S over k *superspecial* if the de Rham filtration and conjugate filtration coincide, i.e. $\mathrm{F}^i\,\mathrm{H}^2_{\mathrm{dR}}(S/k) = \mathrm{F}^i_{\mathrm{con}}\,\mathrm{H}^2_{\mathrm{dR}}(S/k)$ for $i = 0, 1, 2, 3$.

Proposition 4.1.14 (Ogus)**.**

i) *Assume that* $k = \overline{k}$, *and* $\rho(S) = 22$. *Then* S *is superspecial if and only if we have* $\operatorname{disc}\mathrm{NS}(S) = -p^2$.

ii) *There exists a* unique *superspecial K3 surface* S *over* $\overline{k}$ *with* $\rho(S) = 22$.

Reference. See [Ogu79, Proposition 7.1] and [Ogu79, Corollary 7.14, Remark 2.4]. □

Remark 4.1.15. If S is a superspecial K3 surface, then it is also Artin-supersingular, cf. [GK00, Proposition 7.1, Lemma 9.6][51]. In particular, embracing Artin's conjecture, cf. Conjecture 4.1.9), there is a unique superspecial K3 surface over $\overline{k}$.

Remark 4.1.16. The unique superspecial K3 surface $\overline{S}$ over $\overline{k}$ admits a model S over $\mathbb{F}_p$, cf. [Sch12, Theorem 1]. It has Picard rank $\rho(S) = 21$, which is optimal by Remark 4.1.22 below. Let us reiterate that $\overline{S}$ can have (finitely many) different models over a given finite field, cf. [LMS14, Proposition 2.4.1].

The first Chern class has image in $\mathrm{F}^1\,\mathrm{H}^2_{\mathrm{dR}}$, cf. [HuyK3, Proposition 2.1] or [Ogu79, Corollary 1.4], and we define $\mathrm{c}_1^{\mathrm{Hodge}}$ as the composition

$$\begin{array}{ccc} \mathrm{c}_1\colon \mathrm{NS}(S) \otimes_{\mathbb{Z}} \mathbb{F}_p & \longrightarrow & \mathrm{F}^1\,\mathrm{H}^2_{\mathrm{dR}} \\ & \searrow^{\mathrm{c}_1^{\mathrm{Hodge}}} & \downarrow \\ & & \mathrm{F}^1\,\mathrm{H}^2_{\mathrm{dR}}/\mathrm{F}^2\,\mathrm{H}^2_{\mathrm{dR}} \simeq \mathrm{H}^1(S, \Omega^1_S). \end{array}$$

Theorem 4.1.17 (Ogus)**.** *Assume that* $k = \overline{k}$. *If* $\mathrm{c}_1^{\mathrm{Hodge}}(\mathcal{L}) \neq 0$, *then the deformation space* $\mathrm{Def}(S, \mathcal{L})$ *is smooth over* $\mathrm{W}(k)$.

Reference. See [Ogu79, Proposition 2.2]. □

[50] The degeneration of the spectral sequences is linked with the Cartier isomorphism, cf. [DI87]. We can formulate the isomorphism as $\mathcal{H}^i((\mathrm{Fr}_{S/k})_*\,\Omega^\bullet_{S/k}) \xrightarrow{\sim} \mathcal{H}^i(\bigoplus_l \Omega^l_{S^{(p)}/k}[-l])$ and ask if it comes from a morphism $(\mathrm{Fr}_{S/k})_*\,\Omega^\bullet_{S/k} \xrightarrow{\sim} \bigoplus_i \Omega^i_{S^{(p)}/k}[-i]$ in $\mathbf{D}^{\mathrm{b}}_{\mathrm{qcoh}}(S^{(p)})$. If this is the case, then we have that $\mathrm{H}^n_{\mathrm{dR}}(S/k) = \mathbb{H}^n(S, \Omega^\bullet_{S/k}) \simeq \mathbb{H}^n(S^{(p)}, (\mathrm{Fr}_{S/k})_*\,\Omega^\bullet_{S/k}) \simeq \bigoplus_i \mathrm{H}^{n-i}(S^{(p)}, \Omega^i_{S^{(p)}/k})$ and the spectral sequences must degenerate by looking at dimensions, already at its E_1-page. This goes through in our situation, since S lifts to a smooth, proper scheme over $\mathrm{W}_2(k)$.

[51] And use that $\dim \mathrm{F}^1\,\mathrm{H}^2_{\mathrm{dR}} = 21$.

Remark 4.1.18. If on the contrary $c_1^{\mathrm{Hodge}}(\mathcal{L}) = 0$, then studying the de Rham and conjugate filtration one concludes that S must be superspecial, cf. [Ogu79, Section 1]. So in view of Remark 4.1.10, and embracing Artin's Conjecture, cf. Conjecture 4.1.9, we know that there is only one K3 surface over $\overline{k}$ for which Theorem 4.1.17 does not apply immediately.

Proposition 4.1.19 (Esnault-Oguiso)**.** *Assume that $k = \overline{k}$, and embrace Artin's conjecture. Then there exists a primitive, ample line bundle $\mathcal{L}$ on S such that $c_1^{\mathrm{Hodge}}(\mathcal{L}) \neq 0$.*

Sketch. See [EO15, Proposition 4.2] for details. Consider the case when S is not Artin-supersingular. Then the map $c_1^{\mathrm{Hodge}} \colon \mathrm{NS}(S) \otimes_{\mathbb{Z}} \mathbb{F}_p \hookrightarrow \mathrm{H}^1(S, \Omega^1_{S/k})$ is already injective and we just have to pick some primitive, ample line bundle.

In the case that X is Shioda-supersingular, van der Geer–Katsura show that c_1^{Hodge} is not the zero map, cf. [GK00, Proposition 11.9], so we find some line bundle $\mathcal{M}$ which satisfies $c_1^{\mathrm{Hodge}}(\mathcal{M}) \neq 0$. From this we get an ample line bundle by considering $\mathcal{M} + mpH$, where H is some ample line bundle on S and $m \gg 1$.

The case discrimination is exhaustive, since we embraced Artin's conjecture. □

In conclusion, embracing Artin's conjecture, we find for every K3 surface $\overline{S}$ over an algebraically closed field $\overline{k}$ of positive characteristic, a primitive ample line bundle $\overline{\mathcal{L}}$ on $\overline{S}$ such that $\mathrm{Def}(\overline{S}, \overline{\mathcal{L}})$ is smooth over $\mathrm{W}(\overline{k})$. In order to lift a K3 surface S over k, which need not be algebraically closed, we want to descend the line bundle $\overline{\mathcal{L}}$ on $\overline{S}$ to one on S.

Proposition 4.1.20. *Assume that $k = \mathbb{F}_q$ is a finite field. Define $\widetilde{k}$ to be the degree* 6 983 776 800 *extension of k. Then $\mathrm{NS}(S_{\widetilde{k}}) \xrightarrow{\sim} \mathrm{NS}(S_{\overline{k}})$.*

Sketch. See [LMS14, Section 2.3] for details. Let us just say that we want to take a field extension that kills the Galois-action on $\mathrm{NS}(S_{\overline{k}})$. This eventually becomes linear algebra, and eventually we want to eliminate roots-of-unity that can appear as a zero of a polynomial of degree 21. Now, note that

$$\begin{aligned}
&\mathrm{lcm}\{m \in \mathbb{N} \mid \varphi(m) \leq 21\} \\
&= \mathrm{lcm}\{1, \dots, 22, 24, 25, 26, 27, 28, 30, 32, 33, 34, 36, 38, 40, 42, 44, 48, 50, 54, 60, 66\} \\
&= 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \\
&= 6\,983\,776\,800.
\end{aligned}$$

□

Remark 4.1.21. In [HuyK3, Lemma 17.2.6], besides the case of finite fields, the case when the field k has characteristic 0 is considered and a universal bound is obtained when the K3 surface in question has a k-rational point. The bound is given by

$$\#\,\mathrm{GL}(\overline{\rho}, \mathbb{F}_3) = \prod_{k=0}^{\overline{\rho}-1} (3^{\overline{\rho}} - 3^k),$$

which is already for $\overline{\rho} = 5$ larger then the bound in Proposition 4.1.20. For the worst case of $\overline{\rho} = 22$ the decimal representation of $\#\,\mathrm{GL}(\overline{\rho}, \mathbb{F}_3)$ has 231 digits.

Remark 4.1.22. A universal bound as in Proposition 4.1.20 must be an even number. Indeed in [Art74, (6.8)] the following is shown. Assume S is supersingular and $k = \mathbb{F}_{p^r}$, where r is an odd number, then the Galois action of $\mathrm{Gal}(\overline{k}/k)$ on $\mathrm{NS}(S_{\overline{k}})$ is non-trivial. In particular $\rho(S) < \rho(S_{\overline{k}})$.

Corollary 4.1.23. *All K3 surfaces over $k = \mathbb{F}_q$ can be lifted to a smooth, proper scheme over $\mathrm{W}(\widetilde{k})$, where $\widetilde{k}$ is the field extension from Proposition 4.1.20.* □

Remark 4.1.24. Note that in Corollary 4.1.23 we only have to take the field extension $\tilde{k}/k$ if $S_{\overline{k}}$ is superspecial, cf. Remark 4.1.18. So for most K3 surfaces S over a finite field we do not need to take any extension, cf. Proposition 4.1.14.

Proposition 4.1.25 (Lieblich–Maulik, Lieblich–Olsson)**.** *Let S be a K3 surface over a field $k = \overline{k}$ of characteristic $\mathrm{char}(k) \neq 2$. Let $E \subset \mathrm{Pic}(S)$ be a saturated subgroup that contains an ample line bundle. If S is not supersingular, then there exists a complete discrete valuation ring W which is finite and flat over $\mathrm{W}(k)$, together with a relative K3 surface $\mathcal{S}$ over W lifting S, such that $\mathrm{Pic}(\mathcal{S}_{\overline{\eta}}) \simeq E$.*

Reference. See [LM11, Corollary 4.2] and [LO15, Proposition A.1]. See also [Cha16, Proposition 1.5]. □

Remark 4.1.26. If we only need that E embeds into $\mathrm{Pic}(S_{\overline{\eta}})$, then we can take $W = \mathrm{W}(k)$, where k is a perfect field that does not need to be algebraically closed, cf. [LM11, Corollary 4.2].

Lieblich–Olsson have a version of Proposition 4.1.25 for supersingular K3 surfaces S, where they assume $\mathrm{rk}(E) \leq 10$. In the argument a further deformation (from a supersingular K3 surface to a non-supersingular one) is needed, which has as a consequence that the residue field κ of W is just some field extension $\kappa/\overline{k}$.

Once we have lifted a K3 surface, a further lifting problem is whether ℓ-adic B-fields can be lifted. We will need this in Section 4.2.

Proposition 4.1.27. *Let W be an complete*[52] *discrete valuation ring with residue field k, say $i\colon \mathrm{Spec}(k) \to \mathrm{Spec}(W)$. Let $\mathcal{F}$ be an étale sheaf on Spec(W). Then we have*

$$\mathrm{H}^r_{\text{ét}}(\mathrm{Spec}(W), \mathcal{F}) \xrightarrow{\sim} \mathrm{H}^r_{\text{ét}}(\mathrm{Spec}(k), i^*\mathcal{F}).$$

Reference. See [MilADT, Proposition II.1.1], or for more details [Maz73]. □

Proposition 4.1.28. *Let W be a complete discrete valuation ring with perfect residue field k of cohomological dimension $\mathrm{cd}(k) \leq 1$, e.g. $k = \mathbb{F}_q$.*[53] *Let $f\colon \mathcal{X} \to \mathrm{Spec}(W)$ be a smooth, proper scheme over W, and let $\mathcal{F}$ be an étale torsion sheaf on $\mathcal{X}$ with torsion order prime to $\mathrm{char}(k)$. Then we have*

$$\mathrm{H}^r_{\text{ét}}(X, \mathcal{F}|_X) \simeq \mathrm{H}^r_{\text{ét}}(\mathcal{X}, \mathcal{F})$$

whenever $\mathrm{H}^{r-1}_{\text{ét}}(X_{\overline{k}}, \mathcal{F}|_{X_{\overline{k}}}) = 0$.

Proof. We compute the Leray spectral sequence $E_2^{r,s} = \mathrm{H}^r_{\text{ét}}(W, \mathbf{R}^s_{\text{ét}} f_*\mathcal{F}) \Rightarrow \mathrm{H}^{r+s}_{\text{ét}}(\mathcal{X}, \mathcal{F})$, which is depicted below.

$$\begin{array}{c|ccc}
s=2 & \mathrm{H}^0_{\text{ét}}(W, \mathbf{R}^2_{\text{ét}} f_*\mathcal{F}) & \mathrm{H}^1_{\text{ét}}(W, \mathbf{R}^2_{\text{ét}} f_*\mathcal{F}) & \mathrm{H}^2_{\text{ét}}(W, \mathbf{R}^2_{\text{ét}} f_*\mathcal{F}) \\
s=1 & \mathrm{H}^0_{\text{ét}}(W, \mathbf{R}^1_{\text{ét}} f_*\mathcal{F}) & \mathrm{H}^1_{\text{ét}}(W, \mathbf{R}^1_{\text{ét}} f_*\mathcal{F}) & \mathrm{H}^2_{\text{ét}}(W, \mathbf{R}^1_{\text{ét}} f_*\mathcal{F}) \\
s=0 & \mathrm{H}^0_{\text{ét}}(W, \mathbf{R}^0_{\text{ét}} f_*\mathcal{F}) & \mathrm{H}^1_{\text{ét}}(W, \mathbf{R}^0_{\text{ét}} f_*\mathcal{F}) & \mathrm{H}^2_{\text{ét}}(W, \mathbf{R}^0_{\text{ét}} f_*\mathcal{F}) \\
\hline
 & r=0 & r=1 & r=2
\end{array}$$

Since we have $\mathrm{cd}(k) \leq 1$, we also have $\mathrm{cd}(W) \leq 1$, cf. Proposition 4.1.27. So starting from the third column $(r \geq 2)$, the spectral sequence is zero. Hence, the spectral sequence degenerates at the E_2-page, and we extract the short exact sequences

$$0 \to \mathrm{H}^1_{\text{ét}}(W, \mathbf{R}^{r-1}_{\text{ét}} f_*\mathcal{F}) \to \mathrm{H}^r_{\text{ét}}(\mathcal{X}, \mathcal{F}) \to \mathrm{H}^0_{\text{ét}}(W, \mathbf{R}^r_{\text{ét}} f_*\mathcal{F}) \to 0.$$

[52] Or, more generally, an excellent henselian discrete valuation ring.

[53] See [Fu11, Theorem 4.5.5].

When $\mathrm{H}^{r-1}_{\text{ét}}(X_{\overline{k}}, \mathcal{F}|_{X_{\overline{k}}}) = 0$, then by proper base change, cf. [Fu11, Theorem 7.3.1], we have that $(\mathbf{R}^{r-1}_{\text{ét}} f_* \mathcal{F})_{\overline{k}} = 0$. Using smooth base change, cf. [Fu11, Theorem 7.7.2], we see $(\mathbf{R}^{r-1}_{\text{ét}} f_* \mathcal{F})_{\overline{\eta}} = 0$, and hence we conclude $\mathbf{R}^{r-1}_{\text{ét}} f_* \mathcal{F} = 0$.

In conclusion, we get $\mathrm{H}^r_{\text{ét}}(\mathcal{X}, \mathcal{F}) \simeq \mathrm{H}^0_{\text{ét}}(W, \mathbf{R}^r_{\text{ét}} f_* \mathcal{F})$, and analogously, replacing W by k, we get $\mathrm{H}^r_{\text{ét}}(X, \mathcal{F}|_X) \simeq \mathrm{H}^0_{\text{ét}}(k, \mathbf{R}^r_{\text{ét}} f_* \mathcal{F}|_X)$. Together, using proper base change once more and applying Proposition 4.1.27, they provide the desired morphism. □

Remark 4.1.29. Let $\mathcal{S}$ be a relative K3 surface over W, and let $\mathcal{F} = \underline{\mu_{\ell^n}}$, where ℓ is prime, which is prime to $\operatorname{char}(k)$. We know that $\mathrm{H}^1_{\text{ét}}(S_{\overline{k}}, \underline{\mu_{\ell^n}}) = 0$, cf. [HuyK3, Remark 1.3.7][54], so Proposition 4.1.28 can be applied and yields $\mathrm{H}^2_{\text{ét}}(S, \underline{\mu_{\ell^n}}) \simeq \mathrm{H}^2_{\text{ét}}(\mathcal{S}, \underline{\mu_{\ell^n}})$.

4.2. Putting it all together.

We now come to to the proof of the main theorem, which is conditional on the following assumption whose assertions are left for further study.

Assumption 4.2.1. Let us make the following two assumptions.

- i) Desideratum 3.4.4 concerning the reduction modulo p of Fourier–Mukai equivalences is true, and
- ii) cubic fourfolds associated to K3 surfaces with good reduction have good reduction, cf. Question 3.4.7.

Theorem 4.2.2 (Main Theorem). *Let S be a K3 surface over $\mathbb{F}_q$ that admits a polarization, whose degree d satisfies Hassett's condition $(\star\star)$. Assume the validity of Assumption 4.2.1. Then S satisfies the Tate Conjecture over $\overline{\mathbb{F}}_q$, i.e. over all finite extensions of $\mathbb{F}_q$.*

Proof. First of all, we can, and will, assume[55] that $\overline{\rho}(S) = 1$, by handling the case $\overline{\rho}(S) \geq 2$ using finiteness results that do not depend on cubic fourfolds, cf. Remark 1.4.14. Also, if S is not supersingular, then Proposition 4.1.25 can be used to lift to a K3 surface $\widetilde{S}$ with geometric Picard rank $\overline{\rho}(\widetilde{S}) = 1$ which is enough for the argument below.[56]

We collect the results proven so far and argue in several steps, which are depicted in Figure 5 and Figure 6.

Step 1) We will apply Lieblich–Maulik–Snowden's strategy and continue with the setup and notation of the proof of Theorem 2.2.6 on page 9. In particular, we have Brauer classes $\alpha_j \in \operatorname{Br}(S)$ coming from a B-field β, and Mukai vectors $\nu_j \in \mathrm{CH}^{\beta/\ell^j}(S, \mathbb{Z})$ such that the moduli spaces $M_j := \mathrm{M}_{\alpha_j}(\nu_j)$ of $-\alpha_j$-twisted sheaves on S with Mukai vector ν_j are again K3 surfaces. By Proposition 1.4.19 we can assume that $\ell \equiv 1 \mod 3$, and using Proposition 1.4.15 we find polarizations of degree $d_j := d\ell^{2j}$ on M_j.

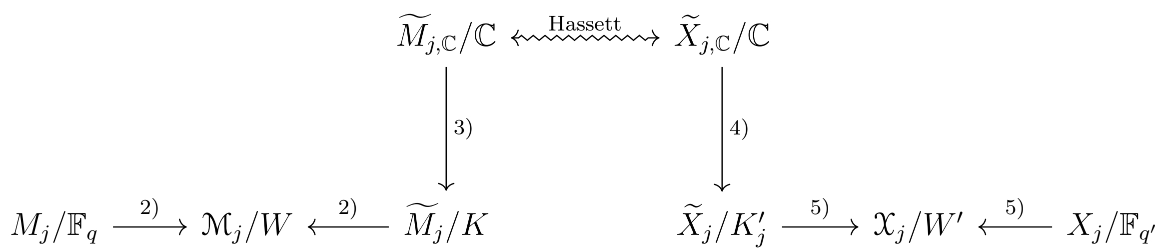


FIGURE 5. Step 2) to 5) of the proof of Theorem 4.2.2

[54] Or use the Kummer sequence together with the fact that $\operatorname{Pic}(S_{\overline{k}}) \simeq \mathrm{H}^1_{\text{ét}}(S_{\overline{k}}, \mathbb{G}_{\mathrm{m}})$ is torsion free.

[55] We need the assumption $\overline{\rho}(S) = 1$ only to ensure that there is an associated cubic fourfold. If other criteria to check this are available, one could (try to) use those.

[56] In case S is supersingular one could try to first deform S to a non supersingular K3 surface, as in [LO15, Proposition A.1], and continue from there carrying a second deformation/lifting and degeneration/reduction through the argument.

Step 2) We lift the (polarized) K3 surface S to characteristic 0 using Theorem 4.1.2, say the lift is realized over the complete discrete valuation ring W which is finite over $\mathrm{W}(\mathbb{F}_q)$. Now, we lift the Brauer classes α_j, or more precisely the B-field β along with S, cf. Proposition 4.1.28 and Remark 4.1.29, build corresponding lifts $\widetilde{\nu_j}$ of the Mukai vectors ν_j, and take the relative moduli space $\mathcal{M}_j := \mathrm{M}_{\widetilde{\alpha_j}}(\widetilde{\nu_j})$ over W. So, the moduli spaces M_j also lift to K3 surfaces $\widetilde{M_j}$ over $K := \mathrm{Quot}(\mathrm{W}(\mathbb{F}_q))$. Note that $\overline{\rho}(\widetilde{M_j}) = 1$, since $\overline{\rho}(M_j) = \overline{\rho}(S) = 1$, cf. [Cha16, Equation 4.2] or [LMS14, Proposition 3.5.6.(2)].

Step 3) We choose some embedding $K \hookrightarrow \mathbb{C}$ and base change $\widetilde{M_j}$ to $\widetilde{M}_{j,\mathbb{C}}$ over $\mathbb{C}$. Since d_j satisfies ($\star\star$), cf. Remark 2.1.8, and $\rho(\widetilde{M}_{j,\mathbb{C}}) = 1$, we can associate[57] a cubic fourfold $\widetilde{X}_{j,\mathbb{C}}$ to it, cf. Theorem 2.1.5.

Step 4) We descend the cubic fourfold $\widetilde{X}_{j,\mathbb{C}}$ to $\widetilde{X}_j$ over a finite field extension K'_j/K, whose degree is bounded by a constant C_d that is independent of j, cf. Proposition 3.1.11, Corollary 3.2.15 and Proposition 2.3.6.ii).

Step 5) Reducing modulo p we get a cubic fourfold X_j over $\mathbb{F}_{q'}$ where $[\mathbb{F}_{q'} : \mathbb{F}_q] \leq C_d$.

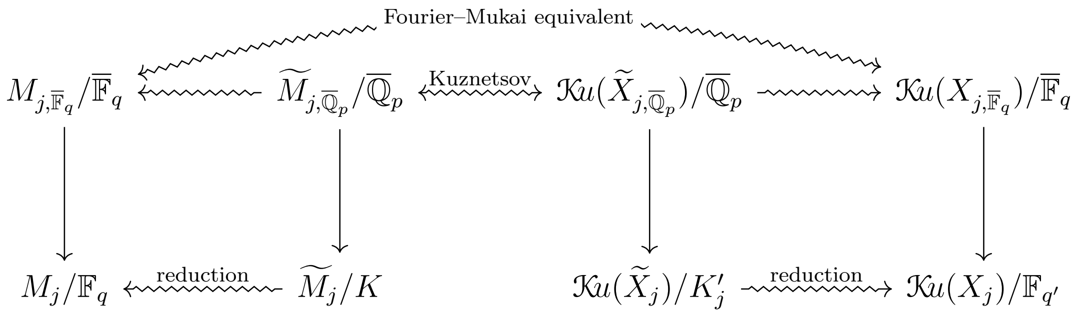


FIGURE 6. Step 6) of the proof of Theorem 4.2.2

Step 6) We know that $\mathbf{D}^{\mathrm{b}}(\widetilde{M}_{j,\overline{\mathbb{Q}}_p}) \simeq \mathcal{K}u(\widetilde{X}_{j,\overline{\mathbb{Q}}_p})$, cf. Theorem 2.4.8, Proposition 2.4.7 and Proposition 3.4.8, and applying Assumption 4.2.1, Dessiderate 3.4.4 and Remark 3.4.6, we conclude that $\mathbf{D}^{\mathrm{b}}(M_{j,\overline{\mathbb{F}}_q}) \simeq \mathcal{K}u(X_{j,\overline{\mathbb{F}}_q})$. If X_j and $X_{j'}$ are isomorphic over $\mathbb{F}_{q'}$, then they are in particular isomorphic over $\overline{\mathbb{F}}_q$. So, we get $M_{j,\overline{\mathbb{F}}_q}$ and $M_{j',\overline{\mathbb{F}}_q}$ are Fourier–Mukai equivalent.

Step 7) Since there are only finitely many cubic fourfolds over $\mathbb{F}_{q'}$, we conclude that there appear only finitely many $M_{j,\overline{\mathbb{F}}_q}$ up to Fourier–Mukai equivalence. Then, by Theorem 1.3.1, there are only finitely many $M_{j,\overline{\mathbb{F}}_q}$ up to isomorphism[58], which is the desired contradiction, since $\mathrm{val}_\ell(\mathrm{disc}(\mathrm{NS}(M_{j,\overline{\mathbb{F}}_q}))) \to \infty$ for $j \to \infty$, cf. Proposition 1.4.13. □

Let us recap. We constructed (in a very ad-hoc way) a map Ξ from the set of K3 surfaces over some p-adic field K, that have geometric Picard rank 1, to the set of (possibly singular) cubic fourfolds over a finite field $\mathbb{F}_{q'}$. Actually we would rather like to have as a domain of the map Ξ the set of K3 surfaces over $\mathbb{F}_q$, but we do not have enough control over the lifting to characteristic 0 to implement this, cf. the discussion in Section 4.1. Instead we just lift the K3 surfaces M_j that appear in the argument. Now, the codomain of Ξ, the set of cubic fourfolds over $\mathbb{F}_{q'}$, is already finite, so we would like Ξ to be injective. Again, we do not have enough control over this, but we only need injectivity up to Fourier–Mukai equivalence over an algebraically closed field. So, we are in the situation of Section 2.4, where we wanted to reduce a Fourier–Mukai equivalence modulo p. This is also the place where Assumption 4.2.1 becomes relevant.

[57]Just choose some cubic fourfold that is associated at the level of Hodge structures, or, if you want, choose and fix Hassett maps ϕ_{d_j}.

[58]We could also apply Remark 2.3.2 instead.

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