Zero divisors in the Grothendieck ring of varieties

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1 Introduction

The Grothendieck ring of varieties is an important part of current research in algebraic geometry. For an algebraically closed field k it is defined to be the ring $K_0(\mathrm{Var}_k)$ that is generated over $\mathbb{Z}$ by isomorphism classes of algebraic k-varieties subject to the relations $[X] = [X \setminus Y] + [Y]$ for closed subvarieties $Y \subseteq X$, together with the product over k.

The Grothendieck ring of varieties appeared for the first time as the "K-group" in Grothendieck's letter [3] to Serre dated 16 August 1964. Since then, many facts about this ring have been brought to light. For instance, several invariants of $K_0(\mathrm{Var}_k)$ (which formally are ring homomorphisms $\lambda\colon K_0(\mathrm{Var}_k) \to A$) are well-known, such as Euler characteristics [12], Hodge and Poincaré polynomials [12], and stable birationality classes [10]. Indeed, Larsen and Lunts [10] proved that varieties are stably birational if and only if their classes in $K_0(\mathrm{Var}_k)$ are equal modulo the class $\mathbb{L}$ of the affine line.

For many theories involving the Grothendieck ring $\mathbb{L}$ plays an important role. Especially the *cancellation problem* that asks whether $\mathbb{L}$ is a zero divisor in $K_0(\mathrm{Var}_k)$ is of particular interest: several statements have been proved under the assumption that $\mathbb{L}$ is not a zero divisor, for instance propositions on the rationality of smooth cubic fourfolds or the study of certain Fano varieties [6]. Although Poonen [14] showed in 2002 that the Grothendieck ring is not an integral domain, the cancellation problem remained unresolved. Finally, in 2014 Borisov [1] succeeded in proving that (contrary to widespread belief) $\mathbb{L}$ is a zero divisor in $K_0(\mathrm{Var}_k)$ for any algebraically closed field k of characteristic zero.

Another problem that has been open just as long as the cancellation question is the *cut-and-paste problem*: given two k-varieties X and Y with the same class in $K_0(\mathrm{Var}_k)$, is it possible to cut them into finitely many pairwise disjoint locally closed subvarieties $\{X_i\}$ resp. $\{Y_i\}$ such that $X_i \cong Y_i$ for all i?

For some special cases this is known to be true, cf. [12]. Larsen and Lunts [11, Thm. 3.3] examined the rationality of motivic zeta functions based on the conjecture that the cut-and-paste problem always has a positive solution. Nevertheless, Borisov's proof of $\mathbb{L}$ being a zero divisor also entails the negative solution of the general cut-and-paste problem. The correspondence of both problems is reasonable especially in view of [12, Rem. 16], which states that the cut-and-paste problem would have a positive answer for varieties of dimension two (which still is unrefuted) if $\mathbb{L}$ would not be a zero divisor.

This bachelor thesis is dedicated to Borisov's proof of $\mathbb{L}$ being a zero divisor and the resulting negative solution to the cut-and-paste problem.

To begin with, we will see a couple of useful examples and properties of the Grothendieck ring. Moreover, we introduce an important technical tool for constructions in $K_0(\mathrm{Var}_k)$, namely Zariski locally trivial fibrations.

Section 3 reproduces Borisov's proof of $\mathbb{L}$ being a zero divisor. Using results from Section 2, we first construct an element in $K_0(\mathrm{Var}_k)$ that annihilates $\mathbb{L}$. In order to show that this element is nonzero, we then introduce the concept of maximal rationally connected fibrations (MRC-fibrations for short).

In the final Section 4 we deduce from previous constructions that the cut-and-paste conjecture fails.

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German summary. Diese Bachelorarbeit beschäftigt sich mit dem Grothendieck-Ring der Varietäten $K_0(\mathrm{Var}_k)$. Für einen algebraisch abgeschlossenen Körper k ist er definiert als der Quotient der freien abelschen Gruppe erzeugt durch Isomorphieklassen von k-Varietäten ausgeteilt nach Relationen $[X] = [X \setminus Y] + [Y]$ für abgeschlossene Untervarietäten $Y \subseteq X$; die Multiplikation ist durch das Faserprodukt über k gegeben.

Die Klasse der affinen Gerade $\mathbb{L} = [\mathbb{A}^1]$ spielt bei vielen Problemen, die mit dem Grothendieck-Ring in Beziehung stehen, eine wichtige Rolle. Obwohl Poonen [14] 2002 zeigte, dass der Grothendieck-Ring kein Integritätsbereich ist, war lange ungeklärt, ob $\mathbb{L}$ ein Nullteiler ist. Borisov [1] gelang es 2014 schließlich zu beweisen, dass $\mathbb{L}$ entgegen einiger Vermutungen (vgl. [6, Conj. 2.7]) tatsächlich ein Nullteiler ist. Sein Beweis erbrachte außerdem mit geringem zusätzlichen Aufwand die Lösung des *cut-and-paste-Problems*: Gegeben seien zwei Varietäten X und Y, deren Klassen in $K_0(\mathrm{Var}_k)$ übereinstimmen. Ist es möglich, sie in paarweise disjunkte, lokal abgeschlossene Untervarietäten $\{X_i\}_{i=1}^n$ bzw. $\{Y_i\}_{i=1}^n$ zu zerlegen, sodass jedes X_i isomorph zu Y_i ist? Borisov [1] bewies, dass dies nicht immer möglich ist.

Zu Beginn dieser Arbeit werden wichtige Beispiele, Aussagen und Arbeitsstrategien zur Konstruktion von Gleichungen in $K_0(\mathrm{Var}_k)$ diskutiert. Der Hauptteil beschäftigt sich dann mit Borisovs Beweis der Aussage, dass $\mathbb{L} \in K_0(\mathrm{Var}_k)$ ein Nullteiler ist. Es folgt außerdem ein Beweis der negativen Lösung des cut-and-paste-Problems.

2 The Grothendieck ring of varieties

The main aim of this section is to introduce the Grothendieck ring of varieties and several interesting working methods, particularly the concept of Zariski locally trivial fibrations. Moreover, we will discuss presentations of classes in the Grothendieck ring for a couple of examples, such as the Grassmannian or the general linear group. Most statements will be given with regard to their application in later sections, whereby we always try to convey an idea of how to think in the Grothendieck ring.

2.1 Definition and first properties

In this entire Section 2 we take k to be an algebraically closed field.

Definition 2.1. *A variety over k (or k-variety) is a reduced and separated scheme of finite type over k. We denote by Var_k the category of k-varieties.*

Note that any variety is by definition of finite type over k, hence noetherian and especially quasi-compact.

Definition 2.2. *The Grothendieck ring of varieties $K_0(Var_k)$ is the quotient of the free abelian group generated by isomorphism classes of k-varieties by relations*

$$[X] = [X \setminus Y] + [Y] \tag{1}$$

for any closed subvariety $Y \subset X$; the multiplication is given by the product of varieties (that is the fibre product).

We observe that the multiplication is well-defined. In particular, the fibre product $X \times_k Y$ of any two varieties X and Y is again a reduced, separated scheme of finite type over k. Note that this is in general not true over a field that is not algebraically closed, since the product of reduced schemes over a non-perfect field k is not necessarily again reduced. Nevertheless, in this case the multiplication can be defined by the reduction of the product over k.

It follows immediately from Definition 2.2 that the zero in $K_0(Var_k)$ is given by the class of the empty set, and we have $1 = [\mathrm{Spec}(k)]$.

Definition 2.3. *We denote by $\mathbb{L}$ the class of the affine line $\mathbb{A}^1$ in the Grothendieck ring.*

For many theories involving the Grothendieck ring (as well as for this bachelor thesis), $\mathbb{L}$ plays an important role. Note that one has $\mathbb{L}^n = [\mathbb{A}^n]$ in $K_0(\mathrm{Var}_k)$.

As a first easy example we consider the projective space $\mathbb{P}^n$ and its equivalence class in the Grothendieck ring. Considering a standard open set $\mathbb{A}^n \subset \mathbb{P}^n$ and its complement $Y \cong \mathbb{P}^{n-1}$ we obtain by relation (1) the description

$$[\mathbb{P}^n] = [\mathbb{A}^n] + [\mathbb{P}^{n-1}] = \sum_{i=0}^{n} \mathbb{L}^i. \tag{2}$$

The relation (1) is often called *scissor relation* as we think of cutting a variety into disjoint subvarieties. In fact, this works not only for closed subvarieties and their open complement, but more generally for locally closed decompositions:

Lemma 2.4. *Let X be a k-variety with a finite decomposition $X = \bigsqcup X_i$ into pairwise disjoint, locally closed subvarieties $X_i \subset X$. Then in the Grothendieck ring of varieties one has*

$$[X] = \sum [X_i].$$

Proof. We prove the statement by induction on the dimension of X and the number of irreducible components in X with maximal dimension. The case $\dim(X) = 0$ is trivial.

Let $X = \bigcup_{j=1}^{m} Z_j$ be the (up to permutation unique) decomposition of X into irreducible components where Z_1 is of maximal dimension. Let $\eta \in Z_1$ be its generic point and i such that $\eta \in X_i$. Then we have $Z_1 \subseteq \overline{X}_i$ (and trivially $X_i \subseteq \overline{X}_i$ is open), hence $X_i \cap Z_1$ is open in Z_1.

We define $U \coloneqq X_i \cap (X \setminus \bigcup_{j\neq 1} Z_j)$. Then U is non-empty as $\eta \in U$. Moreover, $X \setminus \bigcup_{j \neq 1} Z_j$ is open in Z_1. Hence, U is open in $X \setminus \bigcup_{j\neq 1} Z_j$ (using that $X_i \cap Z_1 \subseteq Z_1$ is open), which in turn is open in X. Therefore, U is a non-empty open set in X with $U \subseteq X_i \cap Z_1$. Using relation (1), we find that

$$[X_i] = [U] + [X_i \setminus U], \; [X] = [U] + [X \setminus U]. \tag{3}$$

Additionally, it holds $X \setminus U = (Z_1 \setminus U) \cup \bigcup_{j\neq 1} Z_j = (X_i \setminus U) \sqcup \bigsqcup_{j\neq i} X_j$ and $\dim(Z_1 \setminus U) < \dim(Z_1)$. If Z_1 is the only irreducible component of maximal dimension, then $\dim(X \setminus U) < \dim(X)$. Otherwise, $X \setminus U$ has fewer irreducible components of maximal dimension than X. In both cases we obtain by induction that $[X \setminus U] = [X_i \setminus U] + \sum_{j\neq i}[X_j]$. With (3) the claim follows immediately. $\square$

A note on irreducibility. Let $X \in Var_k$ be a variety with irreducible components $X = \bigcup_{i=1}^{n} X_i$. Then in the Grothendieck ring of varieties one has

$$\big[X\big] = \big[\bigcup_{i=0}^{n-1} X_i\big] + \big[X_n \setminus \big(X_n \cap \bigcup_{i=0}^{n-1} X_i\big)\big],$$

where $V_n := X_n \setminus \left(X_n \cap \bigcup_{i=0}^{n-1} X_i\right)$ is open in X_n and hence irreducible. Inductively we can write

$$[X] = \sum_{i=0}^{n} [V_i]$$

for certain irreducible varieties V_i. Hence, the isomorphism classes of irreducible varieties already generate $K_0(\mathit{Var}_k)$.

In the literature k-varieties often are defined to be irreducible. After all, this does not effect the definition and properties of the Grothendieck ring.

Definition 2.5. *Varieties $X, Y \in K_0(\mathrm{Var}_k)$ are called stably birational if there exist $n, m \geq 0$ such that $X \times \mathbb{P}^n$ and $Y \times \mathbb{P}^m$ are birational. We denote by SB the multiplicative monoid of stable birationality classes of smooth, projective, irreducible varieties, where the multiplication is again given by the product of varieties.*

Note that the multiplication is well-defined: let $[X] = [X']$ in SB, i.e. $X \times \mathbb{P}^n$ and $X' \times \mathbb{P}^m$ are birational for some $n, m \geq 0$. Then for any variety Y also $Y \times X \times \mathbb{P}^n$ and $Y \times X' \times \mathbb{P}^m$ are birational, and hence one has $[Y \times X] = [Y \times X']$ in SB if Y is smooth, projective and irreducible.

Proposition 2.6. *Let k be algebraically closed with $\mathrm{char}(k) = 0$. There is a natural isomorphism of rings $K_0(\mathrm{Var}_k)/(\mathbb{L}) \cong \mathbb{Z}[SB]$ where $\mathbb{Z}[SB]$ is the monoid ring. Under this isomorphism the class of a smooth, projective, irreducible variety X in the Grothendieck ring is identified with its stable birationality class.*

This Proposition has been proved by Larsen and Lunts [10, Sec. 2] for $k = \mathbb{C}$; indeed their proof is more generally valid over any algebraically closed field of characteristic zero.

The statement is very interesting in view of equalities in the Grothendieck ring: the classes of two smooth, projective, irreducible varieties are equal in $K_0(\mathrm{Var}_k)$ modulo $\mathbb{L}$ if and only if the varieties are stably birational. This will be used in Section 3 in order to prove that $\mathbb{L}$ is a zero divisor in $K_0(\mathrm{Var}_k)$.

2.2 Zariski locally trivial fibrations

The main tool for calculations in the Grothendieck ring of varieties are Zariski locally trivial fibrations. In this subsection we will see important definitions, properties and examples that will be frequently used for computations in Section 3.

Recall that the ground field k is algebraically closed in this section.

Definition 2.7. *A morphism of varieties $\phi\colon X \to Y$ is called a Zariski locally trivial fibration with fibre $F \in Var_k$ if it fulfills the following property: for every closed point $y \in Y$ there exist a (Zariski) open set $y \in U \subseteq Y$ and an isomorphism $\phi^{-1}(U) \cong U \times F$ such that ϕ corresponds to the projection onto U.*

In the Grothendieck ring Zariski locally trivial fibrations behave like products. More specifically, the following holds.

Proposition 2.8. *For a Zariski locally trivial fibration $\phi\colon X \to Y$ with fibre F it holds in the Grothendieck ring that $[X] = [Y][F]$.*

Proof. Considering for any $y \in Y$ an open neighborhood U like in Definition 2.7 yields a covering of Y. Being a variety, Y is quasi-compact. Hence, we find a finite covering $Y = \bigcup_{i=1}^{n} Y_i$ of such open subvarieties, i.e. $X_i := \phi^{-1}(Y_i) \cong Y_i \times F$ and $\phi\colon X_i \to Y_i$ corresponds to the projection onto Y_i for all i. We obtain $X = \bigcup_{i=1}^{n} X_i$ and $[X_i] = [Y_i][F]$ for all i.

Let $X^r := \bigcup_{i=1}^{r} X_i$ and $Y^r := \bigcup_{i=1}^{r} Y_i$. Since $X = X^n$ and $Y = Y^n$, it is enough to show that

$$\left[X^r\right] = \left[Y^r\right]\left[F\right].$$

for all $r \in \{1, \dots, n\}$. By assumption this holds for $r = 1$. Let it be true for some r. Since $\phi\colon X_i \cong Y_i \times F \to Y_i$ is the projection for all i, one has $X_{r+1} \cap X^r = \phi^{-1}(Y_{r+1} \cap Y^r) \cong (Y_{r+1} \cap Y^r) \times F$. It follows that

$$\begin{aligned}
\left[X^{r+1}\right] &= \left[X^r\right] + \left[X_{r+1}\right] - \left[X_{r+1} \cap X^r\right] \\
&= \left[Y^r\right]\left[F\right] + \left[Y_{r+1}\right]\left[F\right] - \left[Y_{r+1} \cap Y^r\right]\left[F\right] \\
&= \left[Y^{r+1}\right]\left[F\right].
\end{aligned}$$

Inductively this proves the proposition. □

Corollary 2.9. *Let $f \in k[Z_0, \dots, Z_n]$ be a homogeneous polynomial, $Y := V(f) \subseteq \mathbb{P}^n$ the corresponding projective variety, and $X \subseteq \mathbb{A}^{n+1}$ the affine cone over Y. Then in $K_0(Var_k)$ it holds that*

$$[X] = (\mathbb{L} - 1)[Y] + 1.$$

Note that this is exactly what one would assume: The affine cone over Y consists, figuratively spoken, of the origin plus an affine line without the zero over any point of Y.

Proof. Consider the morphism $\pi\colon X \setminus \{0\} \to Y$ mapping a closed point $0 \neq (a_0, \dots, a_n)$ to the line $[a_0 : \dots : a_n] \in Y$ containing this point. Then all fibres are isomorphic to $\mathbb{A}^1 \setminus \{0\}$.

For simplicity we may assume that $y \in Y$ is contained in the open subset $U := Y \cap \{Z_0 \neq 0\}$. Observe that

$$\begin{aligned}\pi^{-1}(U) &= \big\{a \in X \setminus \{0\} \;\big|\; a_0 \neq 0,\ f(a) = 0\big\}\\ &\cong \big(\mathbb{A}^1 \setminus \{0\}\big) \times \big\{[a] = [1 : a_1 : \ldots : a_n] \;\big|\; f(a) = 0\big\}\\ &= \big(\mathbb{A}^1 \setminus \{0\}\big) \times U,\end{aligned}$$

where $\pi_{|\pi^{-1}(U)}$ corresponds to the projection onto U. Hence, π is a Zariski locally trivial fibration. By Proposition 2.8 it follows that $[X \setminus \{0\}] = (\mathbb{L} - 1)[Y]$ and therefore $[X] = (\mathbb{L} - 1)[Y] + 1$. □

Taking $f = 0$ in Corollary 2.9 it follows $\big[\mathbb{A}^{n+1} \setminus \{0\}\big] = \big[\mathbb{A}^1 \setminus \{0\}\big]\big[\mathbb{P}^n\big]$ even though $\mathbb{A}^{n+1} \setminus \{0\}$ and $\big(\mathbb{A}^1 \setminus \{0\}\big) \times \mathbb{P}^n$ are not isomorphic. This explains why Zariski locally trivial fibrations are very useful for calculations in $K_0(\mathit{Var}_k)$: they often make it possible to represent the class of a variety as a product although the variety itself is not a product.

Example 2.10. The general linear group. We claim that in $K_0(\mathit{Var}_k)$ one has

$$\big[\mathrm{Gl}_k(n)\big] = \prod_{i=0}^{n-1} \big(\mathbb{L}^n - \mathbb{L}^i\big).$$

Here, the description in the Grothendieck ring is again very figurative: we understand an invertible matrix as the collection of n linearly independent vectors in k^n. Then the factor $(\mathbb{L}^n - 1)$ corresponds to the first (nonzero) vector, and every other factor $(\mathbb{L}^n - \mathbb{L}^i)$ corresponds to a vector that is not contained in the span of the previous $i - 1$ vectors. Despite the fact that this does not give a global product of varieties, we can prove the equality in the Grothendieck ring as follows.

Let $V := k^n$ with the standard basis $e_1, \ldots, e_n$. We define for $r = 1, \ldots, n$ the variety $V_r \subset \mathbb{A}^{n\times r}$ to be the set of r-tuples of nonzero linearly independent vectors in V. We find $V_1 = \mathbb{A}^n \setminus \{0\}$ and $V_n = \mathrm{Gl}_k(n)$. We show inductively that $[V_r] = \prod_{i=0}^{r-1}(\mathbb{L}^n - \mathbb{L}^i)$, which is trivial for $r = 1$ and proves the claim for $r = n$.

Consider the projection $\pi_r \colon V_r \to V_{r-1}$ that forgets the last vector. It is enough to show that this is a Zariski locally trivial fibration with fibre $\mathbb{A}^n \setminus \mathbb{A}^{r-1}$ since then $[V_r] = [V_{r-1}](\mathbb{L}^n - \mathbb{L}^{r-1})$.

We define rational functions β_i to be the solutions of the $r-1$ equations $v_{rj} = \sum_{i=1}^{r-1} \beta_i v_{ij}$ for formal variables $v_{rj}, v_{ij},\ i, j \leq r - 1$. Then only the variables v_{ii} appear in the denominators of the β_i. Hence, the β_i are regular on $\pi_r^{-1}(U)$ for the open set $U := \{(v_1, \ldots, v_{r-1}) \in V_{r-1} \mid \forall i\colon v_{ii} \neq 0\} \subset V_{r-1}$ where we write $v_i = \sum_{j=1}^{n} v_{ij} e_j$.

For any $(v_1, \ldots, v_{r-1}) \in U, v_r \in V$ we define a vector $v'_r \in V$ with components

$$v'_{rj} \coloneqq \begin{cases} v_{rj} & \text{if } j \leq r-1 \\ v_{rj} - \sum_{i=1}^{r-1} \beta_i v_{ij} & \text{if } r \leq j. \end{cases}$$

Then v'_r is contained in $\langle e_1, \ldots, e_{r-1} \rangle$ if and only if the β_i are solutions of all n equations $v_{rj} = \sum_{i=1}^{r-1} \beta_i v_{ij}$, which is the case if and only if $v_r \in \langle v_1, \ldots, v_{r-1} \rangle$. Therefore, we obtain an isomorphism

$$\begin{array}{rcl} \pi_r^{-1}(U) & \xrightarrow{\sim} & U \times (\mathbb{A}^n \setminus \mathbb{A}^{r-1}) \\ (v_1, \ldots, v_r) & \mapsto & (v_1, \ldots, v_{r-1}, v'_r). \end{array}$$

Analogously, the fibre over any $U_\sigma \coloneqq \{(v_1, \ldots, v_{r-1}) \in V_{r-1} \mid \forall i\colon v_{i\sigma(i)} \neq 0\}$, $\sigma \in S_n$ is isomorphic to $U_\sigma \times (\mathbb{A}^n \setminus \mathbb{A}^{r-1})$. These sets form an open cover of V_{r-1} and hence π_r is a Zariski locally trivial fibration with fibre $\mathbb{A}^n \setminus \mathbb{A}^{r-1}$, which was to be proven.

Remark 2.11. Note that $\mathrm{Gl}(n)$ is an irreducible variety: It is the complement in $\mathbb{A}^{n \times n}$ of the zero locus of the determinant and hence irreducible as an open subvariety of the (irreducible) affine space.

Example 2.12. The Grassmannian. We take a look at the Grassmannian $G(r, n)$ for $r \leq n$, that is the set of all r-dimensional linear subspaces of k^n. The Grassmannian becomes a projective variety by the Plücker embedding $G(r, n) \hookrightarrow \mathbb{P}(\Lambda^r k^n)$, where an r-space $W = \langle w_1, \ldots, w_r \rangle \subset k^n$ is mapped to $[w_1 \wedge \ldots \wedge w_r] \in \mathbb{P}(\Lambda^r k^n)$, cf. [7, Lect. 6].

For a vector space V over k of dimension n, we also write $G(r, V) \coloneqq G(r, n)$. The r-planes in $\mathbb{P}(V)$ correspond to the $(r+1)$-dimensional linear subspaces of V. Hence, the projective Grassmannian variety $G\big(r, \mathbb{P}(V)\big) \coloneqq G(r+1, V)$ parameterizes the set of r-planes in $\mathbb{P}(V)$.

In the Grothendieck ring the class of the Grassmannian fulfills for $2 \leq r < n$ the equation

$$[G(r, n)] = [G(r, n-1)] + \mathbb{L}^{n-r}[G(r-1, n-1)]. \tag{4}$$

This can be seen by taking a fixed hyperplane $H \coloneqq k^{n-1} \subset k^n$. Then $W \in G(r, n)$ either is contained in $G(r, H) = G(r, n-1)$, or $H \cap W \in G(r-1, n-1)$. To examine the latter case, observe that the morphism

$$\begin{array}{rcl} \big\{W \in G(r, n) \bigm| W \not\subset H\big\} & \to & G(r-1, n-1) \\ W & \mapsto & W \cap H \end{array}$$

is a Zariski locally trivial fibration with fibre $\mathbb{A}^{n-r}$ (see [13, Prop. 2.1] for a detailed proof). Applying Proposition 2.8, we obtain equation (4).

In Section 3 we will repeatedly use the following open covering of the Grassmannian. For a set of (pairwise distinct) indices $I = \{i_1, \ldots, i_r\}$ we consider the open subset

$$U_I \coloneqq \left\{ W = \left\langle e_{i_1} + \sum_{j \notin I} \alpha_{1j} e_j, \ldots, e_{i_r} + \sum_{j \notin I} \alpha_{rj} e_j \right\rangle \;\middle|\; \alpha_{ij} \in k \right\} \subset G(r, n),$$

where $\{e_j\}$ is the standard basis of k^n. In other words, U_I is the set of r-spaces $W = \langle w_1, \ldots, w_r \rangle$ such that $\check{e}_{i_1} \wedge \ldots \wedge \check{e}_{i_r}(w_1 \wedge \ldots \wedge w_r) \neq 0$. Then $U_I \cong \mathbb{A}^{r(n-r)}$. Moreover, these open sets form a finite open cover of $G(r, n)$. Since the U_I are irreducible and their pairwise intersection is non-empty, the following lemma immediately implies that the Grassmannian is irreducible with $\dim\big(G(r, n)\big) = \dim(U_I) = r(n - r)$.

Lemma 2.13. *Let X be a topological space with a finite decomposition $X = \bigcup U_i$ into irreducible, open subsets such that $U_i \cap U_j \neq \emptyset$ for all i, j. Then X is irreducible.*

Proof. Let $X_1, X_2 \subseteq X$ be closed such that $X = X_1 \cup X_2$. Then $U_i = (U_i \cap X_1) \cup (U_i \cap X_2)$, and $U_i \cap X_j$ is closed in U_i for all i, j. Since the U_i are irreducible, we obtain $U_i \subseteq X_1$ or $U_i \subseteq X_2$ for each i.

Assume that $U_1 \subseteq X_1$. Then clearly $U_1 \cap U_j \subseteq X_1$ for all j. Moreover, $U_1 \cap U_j$ is non-empty by assumption and hence dense in U_j. Therefore, one has $U_j \subseteq X_1$ for all j, which immediately implies $X = X_1$. Thus, X is irreducible, which was to be proven. □

3 The cancellation problem

With the theoretical foundations from Section 2, we will now examine the cancellation problem which asks whether $\mathbb{L} = [\mathbb{A}^1]$ is a zero divisor in the Grothendieck ring. Contrary to expectations of Galkin and Shinder [6, Conj. 2.7], Borisov [1] proved that this is true over any algebraically closed field k of characteristic zero.

In this section we elaborate on his construction of an element in $K_0(\mathit{Var}_k)$ that annihilates $\mathbb{L}$. The proof of this element being nonzero is based on the concept of maximal rational connected fibrations and will be discussed in Subsection 3.3.

3.1 The construction

In this subsection we take k to be algebraically closed with $\mathrm{char}(k) = 0$. We compute an equation in $K_0(\mathrm{Var}_k)$ from which results that $\mathbb{L}$ is a zero divisor.

Notations 3.1. Consider the vector space $V = k^7$. One has a canonical linear map $\Lambda^2 V^\vee \hookrightarrow \mathrm{Hom}_k(V, V^\vee)$, which is given as follows. For a skew form $\omega \in \Lambda^2 V^\vee$ we look at the linear map $k \to \Lambda^2 V^\vee$, $a \mapsto a\omega$. Tensoring with V then yields

$$V \;\to\; (\Lambda^2 V^\vee) \otimes V \;\hookrightarrow\; V^\vee \otimes V^\vee \otimes V \;\to V^\vee.$$

Whenever we speak of the rank resp. the kernel of a skew form ω, we refer to the rank resp. the kernel of the corresponding linear map $V \to V^\vee$.

Let now $\{e_1, \ldots, e_7\}$ be the standard basis of V, and let $\{\check{e}_1, \ldots, \check{e}_7\}$ be the corresponding dual basis of $V^\vee$. For $v \in V$, we denote by $\check{v}$ the image of v in $V^\vee$ under the isomorphism $e_i \mapsto \check{e}_i$. This notation will be used throughout this section.

With this fixed basis, we can describe the linear map corresponding to a totally decomposable form $\omega = \check{v}_1 \wedge \check{v}_2$ as

$$\omega\colon V \to V^\vee, \quad v \mapsto \check{v}_1(v) \cdot \check{v}_2 - \check{v}_2(v) \cdot \check{v}_1.$$

Note that the 7×7-matrix describing the linear map corresponding to an arbitrary ω is skew symmetric.

It will always be clear from the context whether we refer to ω as a linear map $V \to V^\vee$ or as a bilinear map $\Lambda^2 V \to k$.

Lemma 3.2. *A nonzero skew form $\omega \in \Lambda^2 V^\vee$ is totally decomposable if and only if its rank is two.*

Proof. Let $0 \neq \omega$ be totally decomposable, i.e. $\omega = \check{v}_1 \wedge \check{v}_2$ for linearly independent vectors $v_1, v_2 \in V$, where we denote by $\check{v}_j$ the image of v_j under the isomorphism $V \cong V^\vee$, $e_i \mapsto \check{e}_i$ like before. We can complete v_1, v_2 to a basis $v_1, \ldots, v_7$ of V such that $\check{v}_1(v_j) = \check{v}_2(v_j) = 0$ for all $j \geq 3$. Then $\mathrm{Ker}(\omega) = \langle v_3, \ldots, v_7 \rangle$ and hence $\mathrm{rk}(\omega) = 2$.

Conversely, let $\omega \in \Lambda^2 V^\vee$ with $\mathrm{rk}(\omega) = 2$. Choose a basis $v_1, \ldots, v_7$ of V such that $\mathrm{Ker}(\omega) = \langle v_3, \ldots, v_7 \rangle$ and $\check{v}_i(v_j) = \delta_{ij}$ for all i, j. We write $\omega = \sum_{i<j} \omega_{ij} \check{v}_i \wedge \check{v}_j$. Then we have

$$0 = \omega(v_k) = \sum_{i<j} \omega_{ij} \big(\check{v}_i(v_k) \cdot \check{v}_j - \check{v}_j(v_k) \cdot \check{v}_i \big) = \sum_{k<j} \omega_{kj} \check{v}_j - \sum_{i<k} \omega_{ik} \check{v}_i$$

for all $k \geq 3$. Since $\check{v}_1, \ldots, \check{v}_7$ are linearly independent, we obtain $\omega_{ik} = \omega_{kj} = 0$ for all $k \geq 3$, $i < k < j$. Hence, we find $\omega = \omega_{12} \check{v}_1 \wedge \check{v}_2$, i.e. ω is totally decomposable. $\square$

Essential elements of Borisov's construction are two particular varieties X_W and Y_W that will prove themselves to be Calabi–Yau threefolds. Let

$W \subset \Lambda^2 V^\vee$ be a seven-dimensional linear subspace. We define X_W to be the subvariety of the Grassmannian $G(2,V)$ given by

$$X_W := \big\{T \in G(2,V) \;\big|\; \forall \omega \in W\colon\; \omega_{|T} = 0\big\}.$$

Here, we interpret a plane $T = \langle t_1, t_2\rangle \in G(2,V)$ via the Plücker embedding as the element $[t_1 \wedge t_2] \in \mathbb{P}(\Lambda^2 V)$ like in Example 2.12. Then $\omega_{|T} = 0$ translates into $\omega(t_1 \wedge t_2) = 0$.

Let $Pf(V)$ denote the Pfaffian variety of V, that is

$$Pf(V) := \big\{[\omega] \in \mathbb{P}(\Lambda^2 V^\vee) \;\big|\; \mathrm{rk}(\omega) < 6\big\}\,.$$

We define Y_W to be the subvariety

$$Y_W := Pf(V) \cap \mathbb{P}(W) \;\subset \mathbb{P}(\Lambda^2 V^\vee).$$

Remark 3.3. A matrix has rank r if and only if there exists a non-zero $r \times r$ minor while all larger minors are zero. Moreover, the rank of any skew symmetric matrix over a field k with $\mathrm{char}(k) \neq 2$ is even. Therefore, any $[\omega] \in Y_W$ is of rank two or four. Furthermore, the Pfaffian is the zero locus of all 6×6 minors and as such indeed a projective variety. Note that by construction also X_W and Y_W are projective varieties.

Lemma 3.4. *For a general choice of W and $\omega \in W$ one has*

$$\mathrm{rk}(\omega) = \begin{cases} 4 & \textit{if } [\omega] \in Y_W \\ 6 & \textit{if } [\omega] \in \mathbb{P}(W) \setminus Y_W. \end{cases}$$

Proof. Consider the subset $\mathcal{W}$ of the Grassmannian defined by

$$\mathcal{W} := \big\{W \in G(7, \Lambda^2 V^\vee) \;\big|\; \forall \omega \in W,\; \omega \neq 0\colon \mathrm{rk}(w) > 2\big\}.$$

Since the Grassmannian is irreducible (see Example 2.12), it is enough to show that $\mathcal{W}$ is a non-empty open subvariety of $G(7, \Lambda^2 V^\vee)$, as the assertions immediately follow for any $W \in \mathcal{W}$.

We denote by Z the closed subvariety in $\Lambda^2 V^\vee \cong k^{21}$ of skew symmetric two-forms of rank less or equal two. Then $\mathcal{W}$ is the locus of 7-planes that intersect trivial with Z.

Consider the incidence variety

$$\Sigma := \Big\{\big(\mathbb{P}(W), [\omega]\big) \in G\big(6, \mathbb{P}(\Lambda^2 V^\vee)\big) \times \mathbb{P}(\Lambda^2 V^\vee) \;\Big|\; \omega \in W\Big\}$$

with the natural projections $\pi_1\colon \Sigma \to G\big(6, \mathbb{P}(\Lambda^2 V^\vee)\big)$ and $\pi_2\colon \Sigma \to \mathbb{P}(\Lambda^2 V^\vee)$. For $W = \langle \omega_1, \ldots, \omega_7\rangle$ the condition $\omega \in W$ is equivalent to $\omega \wedge \omega_1 \wedge \ldots \wedge \omega_7 = 0$, which is a closed condition on $G\big(6, \mathbb{P}(\Lambda^2 V^\vee)\big) \times \mathbb{P}(\Lambda^2 V^\vee)$. Hence, Σ is a projective variety.

Note that the projection $G\big(6, \mathbb{P}(\Lambda^2 V^\vee)\big) \times \mathbb{P}(\Lambda^2 V^\vee) \to G\big(6, \mathbb{P}(\Lambda^2 V^\vee)\big)$ is proper by base change. Since Σ is a closed subvariety of this product, π_1 is proper as well. Therefore,

$$\pi_1\big(\pi_2^{-1}(\mathbb{P}(Z))\big) = \big\{\mathbb{P}(W) \;\big|\; \mathbb{P}(W) \cap \mathbb{P}(Z) \neq \emptyset\big\}$$

is a closed subvariety in $G\big(6, \mathbb{P}(\Lambda^2 V^\vee)\big)$. We obtain that

$$\mathcal{W} = G\big(6, \mathbb{P}(\Lambda^2 V^\vee)\big) \setminus \pi_1\big(\pi_2^{-1}(\mathbb{P}(Z))\big)$$

is indeed an open subvariety, where we use the identification $G(7, \Lambda^2 V^\vee) \cong G\big(6, \mathbb{P}(\Lambda^2 V^\vee)\big)$.

It remains to show that $\mathcal{W}$ is non-empty. For this, we observe that Z is the locus of totally decomposable forms in $\Lambda^2 V^\vee$ by Lemma 3.2. Hence, the projectivization $\mathbb{P}(Z)$ is exactly the image of the Plücker embedding $G(2, V^\vee) \hookrightarrow \mathbb{P}(\Lambda^2 V^\vee)$ introduced in Example 2.12. Thus, $\dim(\mathbb{P}(Z)) = \dim\big(G(2, V^\vee)\big) = 10$.

We know $\dim\big(\mathbb{P}(\Lambda^2 V^\vee)\big) = 20$. Since the intersection of all hyperplanes $H \subset \mathbb{P}(\Lambda^2 V^\vee)$ is empty, there is a hyperplane H not containing $\mathbb{P}(Z)$ and hence fulfilling $\dim\big(\mathbb{P}(Z) \cap H\big) < \dim\big(\mathbb{P}(Z)\big) = 10$. Inductively, we find a nine-dimensional plane $H \subset \mathbb{P}(\Lambda^2 V^\vee)$ such that $\mathbb{P}(Z) \cap H = \emptyset$. Therefore, one has $\emptyset \neq G(6, H) \subset \mathcal{W}$, which was to be proven. $\square$

Proposition 3.5. (cf. [2, Thm. 0.3, Rem. 0.5]) *For a general choice of W the following assertions hold:*

(i) *Both X_W and Y_W are irreducible, smooth Calabi–Yau threefolds (i.e. the canonical bundles are trivial).*

(ii) *The varieties X_W and Y_W are not birational to each other.*

From now on we take W to be a general space fulfilling both Lemma 3.4 and Proposition 3.5.

Proposition 3.6. *We define the Cayley hypersurface of X_W by*

$$H \coloneqq \big\{(T, [\omega]) \;\big|\; \omega \in W,\ \omega_{|T} = 0\big\} \;\subset G(2, V) \times \mathbb{P}(W).$$

Then in the Grothendieck ring of varieties one has

$$[H] = [G(2, V)][\mathbb{P}^5] + [X_W]\mathbb{L}^6.$$

Remark 3.7. Consider $T = \langle t_1, t_2\rangle \in G(2, V)$ with $t_i = \sum_{j=1}^7 t_{ij} e_j,\ i = 1, 2$, and $\omega \in W \subset \Lambda^2 V^\vee = \langle \check{e}_i \wedge \check{e}_j \mid i < j\rangle$ with $\omega = \sum_{i<j} \lambda_{ij} \check{e}_i \wedge \check{e}_j,\ \lambda_{ij} = -\lambda_{ji}$. Then one has

$$\omega_{|T} = 0 \;\Leftrightarrow\; \omega(t_1 \wedge t_2) = 0 \;\Leftrightarrow\; \sum_{i \neq j} t_{1i} t_{2j} \lambda_{ij} = 0.$$

Hence, $\omega_{|T} = 0$ is a linear condition in ω. Moreover, H is indeed a hypersurface in $G(2,V) \times \mathbb{P}(W)$ as it is described by one equation.

Proof of Proposition 3.6. We consider the projection onto the first factor $\pi_1 \colon H \to G(2,V)$. By Remark 3.7 we know that $\omega_{|T} = 0$ is a linear condition in ω. We obtain for any $T \in G(2,V)$ and the corresponding hyperplane $X_T \coloneqq \{[\omega] \in \mathbb{P}(\Lambda^2 V^\vee) \mid \omega_{|T} = 0\}$ that the fibres of π_1 are given by

$$\pi_1^{-1}(T) \cong \big(\mathbb{P}(W) \cap X_T\big) \cong \begin{cases} \mathbb{P}(W) = \mathbb{P}^6 & \text{if } T \in X_W, \text{ i.e. } \mathbb{P}(W) \subset X_T \\ \mathbb{P}^5 & \text{if } T \notin X_W. \end{cases}$$

Since $\pi_1^{-1}(X_W) = X_W \times \mathbb{P}(W) \subset H$ is closed, we obtain

$$\big[H\big] = \big[H \setminus \big(X_W \times \mathbb{P}(W)\big)\big] + \big[X_W\big]\big[\mathbb{P}^6\big].$$

It remains to show that $\pi_1 \colon H \setminus \big(X_W \times \mathbb{P}(W)\big) \to G(2,V) \setminus X_W$ is a Zariski locally trivial fibration with fibre $\mathbb{P}^5$. It then follows by Proposition 2.8 that

$$\begin{aligned} [H] &= \big([G(2,V)] - [X_W]\big)[\mathbb{P}^5] + [X_W][\mathbb{P}^6] \\ &= [G(2,V)][\mathbb{P}^5] + [X_W]\big([\mathbb{P}^6] - [\mathbb{P}^5]\big) \\ &= [G(2,V)][\mathbb{P}^5] + [X_W]\mathbb{L}^6. \end{aligned}$$

Let $\tilde{T} \in G(2,V) \setminus X_W$. We construct an open neighborhood of $\tilde{T}$ fulfilling the condition in Definition 2.7. We may assume for simplicity that $\tilde{T} = \langle e_1, e_2 \rangle$ and $W = \langle \omega_k \coloneqq \check{e}_{i_k} \wedge \check{e}_{j_k} \mid k = 1, \ldots, 7 \rangle$ for certain indices $i_k \neq j_k$ where $\omega_1 = \check{e}_1 \wedge \check{e}_2$. Consider for $\tilde{T}$ the open neighborhood

$$\begin{aligned} U \coloneqq &\big\{T \in G(2,V) \setminus X_W \;\big|\; \omega_{1|T} \neq 0\big\} \\ = &\big\{T \in G(2,V) \setminus X_W \;\big|\; T = \langle t_1, t_2 \rangle,\ t_{11}t_{22} - t_{12}t_{21} \neq 0\big\}. \end{aligned}$$

For $T = \langle t_1, t_2 \rangle \in U$ we always assume that $t_{11}t_{22} - t_{12}t_{21} = 1$ by rescaling t_1 if necessary.

Note that for arbitrary $\omega = \sum_{k=1}^{7} \lambda_k \omega_k \in W$ and $T = \langle t_1, t_2 \rangle$ the condition $\omega_{|T} = 0$ is equivalent to

$$\begin{aligned} \omega(t_1 \wedge t_2) &= \sum_{k=1}^{7} \lambda_k (\check{e}_{i_k} \wedge \check{e}_{j_k})(t_1 \wedge t_2) \\ &= \sum_{k=1}^{7} \lambda_k (t_{1i_k} t_{2j_k} - t_{1j_k} t_{2i_k}) = 0. \end{aligned}$$

Therefore, the morphism

$$\begin{aligned} \varphi \colon \pi_1^{-1}(U) &\longrightarrow U \times \mathbb{P}^5 \\ (T, [\omega]) &\longmapsto (T, [\lambda_2 : \ldots : \lambda_7]) \end{aligned}$$

is well-defined and bijective: for any $(T, [\lambda_2 : \ldots : \lambda_7]) \in U \times \mathbb{P}^5$ we define $\lambda_1 := -\sum_{k=2}^{7} \lambda_k (t_{1i_k} t_{2j_k} - t_{1j_k} t_{2i_k})$. Then $\omega := \sum_{k=1}^{7} \lambda_k \omega_k \in W$ fulfills $\omega_{|T} = 0$, and $[\omega]$ is the unique completion of $[\lambda_2 : \ldots : \lambda_7]$ with this property. In fact, φ is an isomorphism that respects the projection π_1. Hence,

$$\pi_1 \colon \pi_1^{-1}\big(G(2,V) \setminus X_W\big) \to G(2,V) \setminus X_W$$

is a Zariski locally trivial fibration with fibre $\mathbb{P}^5$, which was to be proven. $\square$

Proposition 3.8. *Let $\tilde{H} \subset \big(V \times V \times \mathbb{P}(W)\big)$ be the frame bundle of H, that is the set of triples $(v_1, v_2, [\omega])$ such that v_1, v_2 are linearly independent and fulfill $\omega(v_1, v_2) = 0$. Then in the Grothendieck ring one has*

$$\big[\tilde{H}\big] = \big[H\big]\big[\mathrm{Gl}(2)\big] = \big[H\big]\big(\mathbb{L}^2 - 1\big)\big(\mathbb{L}^2 - \mathbb{L}\big).$$

Proof. It is enough to show that the projection $\varphi \colon \tilde{H} \to H$ that maps a triple $(v_1, v_2, [\omega])$ to the pair $\big(\langle v_1, v_2 \rangle, [\omega]\big) \in H$ is a Zariski locally trivial fibration with fibre $\mathrm{Gl}(2)$.

Considering a point $(T = \langle t_1, t_2 \rangle, [\omega]) \in H$ we find its fibre to be

$$\begin{aligned} \varphi^{-1}(T, [\omega]) &= \big\{(v_1, v_2, [\omega]) \;\big|\; T = \langle v_1, v_2 \rangle\big\} \\ &= \big\{(v_1, v_2, [\omega]) \;\big|\; \exists A = (a_{ij}) \in \mathrm{Gl}(2) \colon v_i = a_{i1} t_1 + a_{i2} t_2, i = 1, 2\big\} \\ &\cong \mathrm{Gl}(2). \end{aligned}$$

Note that for this identification with $\mathrm{Gl}(2)$ we need the fixed basis $t_1, t_2 \in T$.

Consider the standard open covering of $G(2,V)$ from Example 2.12 consisting of sets that are up to permutation of the basis vectors of the form $U := \big\{\langle e_1 + \sum_{i=3}^{7} \alpha_{1i} e_i,\ e_2 + \sum_{i=3}^{7} \alpha_{2i} e_i \rangle \;\big|\; \alpha_{ji} \in k\big\} \cong \mathbb{A}^{10}$. Then the sets of the form $U' := H \cap (U \times \mathbb{P}(W))$ cover H and we find

$$\varphi^{-1}(U') = \big\{(v_1, v_2, [\omega]) \;\big|\; \langle v_1, v_2 \rangle \in U, \omega(v_1, v_2) = 0\big\} \cong U' \times \mathrm{Gl}(2). \quad (5)$$

Here, $\big((T, [\omega]), A = (a_{ij})\big) \in U' \times \mathrm{Gl}(2)$ is identified with $(v_1, v_2, [\omega])$, $v_i = a_{i1} t_1 + a_{i2} t_2$ where the basis t_1, t_2 of T is uniquely determined by the description of U. Moreover, $\varphi_{|\varphi^{-1}(U')}$ corresponds to the projection onto U' and hence is indeed Zariski locally trivial with fibre $\mathrm{Gl}(2)$. $\square$

Remark 3.9. Note that the same argument shows that

$$[G(r,n)][\mathrm{Gl}(r)] = \prod_{i=0}^{r-1} (\mathbb{L}^n - \mathbb{L}^i) = [V_r],$$

where $V_r \subset \mathbb{A}^{n \times r}$ is the variety of r-tuples of linearly independent vectors in k^n, see Example 2.10. The projection $\pi \colon V_r \to G(r,n)$ mapping vectors $(v_1, \ldots, v_r)$ to $\langle v_1, \ldots, v_r \rangle \in G(r,n)$ is a Zariski locally trivial fibration with fibre $\mathrm{Gl}(r)$: on the standard open sets U of the Grassmannian we have $\pi^{-1}(U) \cong U \times \mathrm{Gl}(r)$ analogously to (5).

Here again, the Grothendieck ring reflects our figurative understanding: for fixed linearly independent vectors $v_1, \ldots, v_r \in V_r$ every basis of $\langle v_1, \ldots, v_r \rangle \in G(r, n)$ is obtained by applying elements of the general linear group $\mathrm{Gl}(r)$.

For a further description of $[\tilde{H}]$ in the Grothendieck ring we look at the projection $\tilde{H} \to \mathbb{P}(W)$ mapping $(v_1, v_2, [\omega])$ to $[\omega]$. Then $\tilde{H}$ is the disjoint union of the closed preimage of $Y_W \subset \mathbb{P}(W)$, denoted by $\tilde{H}_1$, and its complement $\tilde{H}_2$, that is the preimage of $\mathbb{P}(W) \setminus Y_M$. Hence, in $K_0(\mathit{Var}_k)$ one has $[\tilde{H}] = [\tilde{H}_1] + [\tilde{H}_2]$.

Proposition 3.10. *In the Grothendieck ring the following relation holds:*

$$[\tilde{H}_1] = [Y_W]\big((\mathbb{L}^7 - \mathbb{L})(\mathbb{L}^3 - 1) + (\mathbb{L}^6 - \mathbb{L})(\mathbb{L}^7 - \mathbb{L}^3)\big).$$

Proof. Let $\tilde{H}_{1,1}$ be the closed set in $\tilde{H}_1$ of all triples $(v_1, v_2, [\omega])$ such that $v_1 \in \mathrm{Ker}(\omega)$. We denote by $\tilde{H}_{1,2}$ the complement of $\tilde{H}_{1,1}$ in $\tilde{H}_1$. Then $[\tilde{H}_1] = [\tilde{H}_{1,1}] + [\tilde{H}_{1,2}]$, hence it is enough to show that the following two equations hold:

$$[\tilde{H}_{1,1}] = [Y_W](\mathbb{L}^7 - \mathbb{L})(\mathbb{L}^3 - 1) \tag{6}$$

$$[\tilde{H}_{1,2}] = [Y_W](\mathbb{L}^6 - \mathbb{L})(\mathbb{L}^7 - \mathbb{L}^3). \tag{7}$$

For (6) we consider the projection

$$\pi_{1,1} \colon \tilde{H}_{1,1} \ \to X_{1,1} := \big\{(v_1, [\omega]) \ \big|\ 0 \neq v_1 \in V, [\omega] \in Y_W, v_1 \in \mathrm{Ker}(\omega)\big\}$$

that maps $(v_1, v_2, [\omega])$ to $(v_1, [\omega])$. We claim that $\pi_{1,1}$ is a Zariski locally trivial fibration with fibre $\mathbb{A}^7 \setminus \mathbb{A}^1$.

Let $(v_1, [\omega]) \in X_{1,1}$. The sets $U_i := \{(v_1, [\omega]) \in X_{1,1} \mid v_{1i} \neq 0\} \subset X_{1,1}$ form an open cover of $X_{1,1}$. We may assume for simplicity that $(v_1, [\omega]) \in U_1$, then a second basis of V is given by $v_1, e_2, \ldots, e_7$. Moreover, $(v_1, v_2, [\omega])$ is contained in $\pi_{1,1}^{-1}(v_1, [\omega])$ if and only if $v_2 \in V \setminus \langle v_1 \rangle$. Hence, for such a $v_2 = a_1 v_1 + \sum_{i=2}^{7} a_i e_i$ we have at least one nonzero coefficient $a_i, i \geq 2$. Therefore, the vector $v_2' := \sum_{i=1}^{7} a_i e_i$ is contained in $V \setminus \langle e_1 \rangle$. We find $\pi_{1,1}^{-1}(U_1) \cong U_1 \times (V \setminus \langle e_1 \rangle)$, where $(v_1, v_2, [\omega]) \in \pi_{1,1}^{-1}(U_1)$ is identified with $((v_1, [\omega]), v_2')$.

This proves the claim that $\pi_{1,1}$ is Zariski locally trivial with fibre $\mathbb{A}^7 \setminus \mathbb{A}^1$ and we obtain by Proposition 2.8 that $[\tilde{H}_{1,1}] = [X_{1,1}](\mathbb{L}^7 - \mathbb{L})$.

We now look at the projection $X_{1,1} \to Y_W$. It can be shown that this in turn is Zariski locally trivial with fibre $\mathbb{A}^3 \setminus \{0\}$. Indeed, the fibre over $[\omega] \in Y_W$ can be identified with the kernel of ω, which is three-dimensional by Proposition 3.4. Then equation (6) immediately follows.

In order to prove equation (7) we claim similarly that the morphism

$$\pi_{1,2}\colon \tilde{H}_{1,2} \ \to X_{1,2} := \big\{(v_1,[\omega]) \ \big|\ 0 \neq v_1 \in V, \omega \in Y_W, v_1 \notin \operatorname{Ker}(\omega)\big\}$$

which again forgets v_2 is a Zariski locally trivial fibration with fibre $\mathbb{A}^6 \setminus \mathbb{A}^1$.

We write a skew form $\omega \in \Lambda^2 V^\vee$ as $\omega = \sum_{i<j} \lambda_{ij} \check{e}_i \wedge \check{e}_j,\ \lambda_{ij} = -\lambda_{ji}$ as in Remark 3.7. Let $U := \{(v_1,[\omega]) \in X_{1,2} \mid \sum_{i\neq 1} \lambda_{i1} v_{1i} \neq 0, v_{12} \neq 0\}$ which is open in $X_{1,2}$. Then by Remark 3.7 it follows that for $(v_1,[\omega]) \in U, v_2 \in V$ one has

$$\begin{aligned} \omega(v_1,v_2) = 0 \ &\Leftrightarrow \sum_{i\neq j\neq 1} v_{1i} v_{2j} \lambda_{ij} \ + \sum_{i\neq 1} v_{1i} v_{21} \lambda_{i1} = 0 \\ &\Leftrightarrow \ v_{21} = -\frac{1}{\sum_{i\neq 1} \lambda_{i1} v_{1i}} \sum_{i\neq j\neq 1} v_{1i} v_{2j} \lambda_{ij}. \end{aligned} \tag{8}$$

We define an injective morphism $\varphi\colon \pi_{1,2}^{-1}(U) \to U \times \mathbb{A}^6$ by mapping $(v_1,v_2,[\omega])$ to $\big((v_1,[\omega]),(v_{22},\ldots,v_{27})\big)$. Then for an element in $U \times \mathbb{A}^6$ defining v_{21} by equation (8) is the unique completion of $(v_{22},\ldots,v_{27})$ to a vector $v_2 \in V$ such that $\omega(v_1,v_2) = 0$. Moreover, $(v_1,v_2,[\omega])$ is then contained in $\pi_{1,2}^{-1}(U)$ if and only if $v_2 \notin \langle v_1 \rangle$, and it is easy to check that this in turn holds if and only if $(v_{22},\ldots,v_{27}) \notin \langle (v_{12},\ldots,v_{17}) \rangle$. Hence, the modified morphism $\psi\colon \pi_{1,2}^{-1}(U) \to U \times (\mathbb{A}^6 \setminus \langle e_1 \rangle)$ that maps $(v_1,v_2,[\omega])$ to $\big((v_1,[\omega]),(v_{22}, v_{2j}v_{12} - v_{22}v_{1j})_{j>2}\big)$ is an isomorphism (note that this is the same trick as in the construction of v'_{rj} for $r = 2$ in Example 2.10).

Analogously, we find $\pi_{1,2}^{-1}(U_{lm}) \cong U_{lm} \times (\mathbb{A}^6 \setminus \mathbb{A}^1)$ for all open sets $U_{lm} := \{(v_1,[\omega]) \mid \sum_{i\neq l} \lambda_{il} v_{1i} \neq 0, v_{1m} \neq 0\} \subset X_{1,2}$. These sets form an open cover of $X_{1,2}$: By definition one has $v_1 \notin \operatorname{Ker}(\omega)$ for any $(v_1,[\omega]) \in X_{1,2}$, and hence there is an l such that $\sum_{i\neq l} \lambda_{il} v_{1i} \neq 0$. Therefore, $\pi_{1,2}$ is Zariski locally trivial with fibre $\mathbb{A}^6 \setminus \mathbb{A}^1$. We find by Proposition 2.8 that $[\tilde{H}_{1,2}] = [X_{1,2}](\mathbb{L}^6 - \mathbb{L})$.

Now consider the projection $X_{1,2} \to Y_W$. The fibre over any closed point is isomorphic to $V \setminus \operatorname{Ker}(\omega) \cong \mathbb{A}^7 \setminus \mathbb{A}^3$. It can be shown that this is Zariski locally trivial and hence equation (7) holds, which was to be proven. □

Proposition 3.11. *In the Grothendieck ring the following relation holds:*

$$[\tilde{H}_2] = ([\mathbb{P}^6] - [Y_W])(\mathbb{L}^7 - \mathbb{L})(\mathbb{L}^6 - 1).$$

Proof. The proof is completely analogous to the one of Proposition 3.10. Splitting up $\tilde{H}_2$ into the two subsets $\tilde{H}_{2,1}$, defined by $v_1 \in \operatorname{Ker}(\omega)$, and $\tilde{H}_{2,2} := \tilde{H}_2 \setminus \tilde{H}_{2,1}$ one finds that

$$\begin{aligned} [\tilde{H}_{2,1}] &= \big([\mathbb{P}(W)] - [Y_W]\big)(\mathbb{L}^7 - \mathbb{L})(\mathbb{L} - 1) \\ [\tilde{H}_{2,2}] &= \big([\mathbb{P}(W)] - [Y_W]\big)(\mathbb{L}^6 - \mathbb{L})(\mathbb{L}^7 - \mathbb{L}). \end{aligned}$$

Here, the only difference from the case in Proposition 3.10 is that a skew form $[\omega] \in \mathbb{P}(W) \setminus Y_W$ has rank six and hence its kernel is of dimension one. $\square$

Proposition 3.12. *In the Grothendieck ring one has the following equality:*

$$[\tilde{H}] = [\mathbb{P}^6](\mathbb{L}^7 - \mathbb{L})(\mathbb{L}^6 - 1) + [Y_W](\mathbb{L}^2 - 1)(\mathbb{L} - 1)\mathbb{L}^7.$$

Proof. This follows directly from $[\tilde{H}] = [\tilde{H}_1] + [\tilde{H}_2]$ using Proposition 3.10 and 3.11. $\square$

Proposition 3.13. *The following relation holds in* $K_0(\mathrm{Var}_k)$*:*

$$\big([X_W] - [Y_W]\big)(\mathbb{L}^2 - 1)(\mathbb{L} - 1)\mathbb{L}^7 = 0.$$

Proof. First note that $[\mathbb{P}^6](\mathbb{L}^6 - 1) = [\mathbb{P}^5](\mathbb{L}^7 - 1)$ by using equation (2) and expanding both sides. Moreover, from Remark 3.9 combined with the description of $[\mathrm{Gl}(2)]$ in Example 2.10 one deduces

$$[G(2,7)](\mathbb{L}^2 - 1)(\mathbb{L}^2 - \mathbb{L}) = (\mathbb{L}^7 - 1)(\mathbb{L}^7 - \mathbb{L}).$$

Using these two equations and Proposition 3.6 and 3.8, we obtain

$$\begin{aligned}
[\tilde{H}] = [H](\mathbb{L}^2 - 1)(\mathbb{L}^2 - \mathbb{L}) &= \big([G(2,7)][\mathbb{P}^5] + \mathbb{L}^6[X_W]\big)(\mathbb{L}^2 - 1)(\mathbb{L}^2 - \mathbb{L}) \\
&= [\mathbb{P}^5](\mathbb{L}^7 - 1)(\mathbb{L}^7 - \mathbb{L}) + [X_W](\mathbb{L}^2 - 1)(\mathbb{L} - 1)\mathbb{L}^7 \\
&= [\mathbb{P}^6](\mathbb{L}^6 - 1)(\mathbb{L}^7 - \mathbb{L}) + [X_W](\mathbb{L}^2 - 1)(\mathbb{L} - 1)\mathbb{L}^7.
\end{aligned}$$

In view of Proposition 3.12 the claim immediately follows. $\square$

Remark 3.14. In fact, the stronger formula $([X_W] - [Y_W])\mathbb{L}^6 = 0$ has been proved by Martin [13], which a few months later was improved to $([X_W] - [Y_W])\mathbb{L} = 0$, see [8]. From each one of these formulas we will obtain in the following subsection that $\mathbb{L}$ is a zero divisor in the Grothendieck ring.

3.2 Maximal rationally connected fibrations

In order to deduce from Proposition 3.13 that $\mathbb{L}$ is a zero divisor in the Grothendieck ring of varieties, we need the concept of maximal rationally connected fibrations (MRC-fibrations for short). In this subsection we introduce key definitions and give an overview of the required statements. Most propositions are elaborated by Kollár [9, Chap. IV] or Debarre [4] and will not be proven here.

We initially take the ground field k to be arbitrary.

Definition 3.15. *A variety X is called rationally chain connected if there exist varieties U and Y with morphisms $g\colon U \to Y$ and $u\colon U \to X$ such that*

(i) *g is a flat and proper morphism whose fibres are connected curves and whose geometric fibres have only rational components,*

(ii) *the induced morphism $u^{(2)} \colon U \times_Y U \to X \times X$ is dominant.*

A rationally chain connected variety X is called rationally connected if the geometric fibres of g are rational curves.

Note that a pair of points (x_1, x_2) is contained in the image of $u^{(2)}$ if and only if there exists a $y \in Y$ such that $x_1, x_2 \in u(U_y)$. Figuratively speaking, there is a chain of rational curves (resp. a rational curve) on X connecting the points x_1 and x_2. Indeed, the next proposition confirms this notion for very general points $x_1, x_2 \in X$.

Definition 3.16. *Let X be a variety over an uncountable field k. We say that a very general point of X has a property P if there exist open dense subvarieties $U_i \subseteq X, i \in \mathbb{N}$ such that every point $x \in \bigcap_{i\in\mathbb{N}} U_i$ has the property P. Analogously, we say that a collection of very general points in X has a property Q if every collection of points in a countable intersection of open dense subvarieties has Q.*

Remark 3.17. It is necessary that k is uncountable.

A variety X over k is locally given by $\operatorname{Spec}(k[z_1, ..., z_n]/\mathfrak{a})$ for some ideal $\mathfrak{a} \subset k[z_1, ..., z_n]$. If k is countable, then $k[z_1, ..., z_n]$ is countable as well, and hence X has only countable many points. Let $X = \{x_i,\ i \in \mathbb{N}\}$, then the open sets $U_i \coloneqq X \setminus \{x_i\}$ fulfill $\bigcap_{i\in\mathbb{N}} U_i = \emptyset$. Hence, the notion of a very general point of a variety defined over a countable field makes no sense.

Proposition 3.18. [9, Prop. IV.3.6] *Let k be algebraically closed and uncountable. Then a k-variety X is*

(i) *rationally chain connected if and only if for very general closed points $x_1, x_2 \in X$ there exists a connected curve $C \subset X$ containing x_1 and x_2 such that every irreducible component of C is rational,*

(ii) *rationally connected if and only if for very general closed points $x_1, x_2 \in X$ there exists a rational curve $C \subset X$ containing x_1 and x_2.*

Example 3.19. [9, Ex. IV.3.2.6.1] The projective space $\mathbb{P}^n$ is rationally connected.

Over an uncountable and algebraically closed field k this immediately follows from Proposition 3.18 (ii) since any two points in $\mathbb{P}^n$ are contained in a line.

More specifically, we consider the incidence variety

$$U \coloneqq \big\{(p, l) \in \mathbb{P}^n \times G(1, \mathbb{P}^n) \;\big|\; p \in l\big\}$$

of lines in $\mathbb{P}^n$ over an algebraically closed field k, together with the projections $g\colon U \to G(1, \mathbb{P}^n)$ and $u\colon U \to \mathbb{P}^n$. The fibres of g are isomorphic to $\mathbb{P}^1_{\kappa(l)}$, hence g fulfills condition (i) in Definition 3.15. Moreover, any two points in $\mathbb{P}^n$ are contained in a line $l \subset \mathbb{P}^n$, hence u fulfills condition 3.15(ii). Therefore, $\mathbb{P}^n$ is rationally connected over any algebraically closed field.

Actually, one can define the Grassmannian variety over an arbitrary field k (analogously to the construction over $\mathbb{Z}$, cf. [5, 5.1.6]). Similar to the above argument it then follows that $\mathbb{P}^n$ is rationally connected.

Definition 3.20. *Let X be a normal and proper variety over k.*

(i) *A rational map $\pi\colon X \dashrightarrow Z$ is called a fibration if it restricts on a dense open subset $X^0 \subset X$ to a proper surjective morphism $\pi^0\colon X^0 \to Z$.*

(ii) *A fibration $\pi\colon X \dashrightarrow Z$ is called rationally chain connected if the fibres of π^0 are rationally chain connected, and $\pi^0_*\mathcal{O}_{X^0} \cong \mathcal{O}_Z$.*

(iii) *A rationally chain connected fibration $\pi\colon X \dashrightarrow Z$ is called maximal (MRCC-fibration) if it fulfills the following universal property. For all rationally chain connected fibrations $\pi'\colon X \dashrightarrow Z'$ there exists a unique rational map $\tau\colon Z' \dashrightarrow Z$ such that $\pi = \tau \circ \pi'$.*

Remark 3.21. The condition $\pi^0_*\mathcal{O}_{X^0} \cong \mathcal{O}_Z$ in (ii) can be dropped in characteristic zero, cf. [4, Def. 5.12 and Rem. 1.13].

Theorem 3.22. [9, Thm. IV.5.2] *Let X be a normal proper variety. Then the MRCC-fibration $\pi\colon X \dashrightarrow Z$ exists and is unique up to birational equivalence.*

Idea of proof. Consider the equivalence relation on X where two points $x_1, x_2 \in X$ are said to be equivalent if they can be connected by a chain of rational curves in X. We want Z to be some kind of a quotient of X by this relation. Set theoretically this exists but there is in general no such morphism $p\colon X \to Z$ with all fibres being the equivalence classes of a relation, see [9, Chap. IV]. Nevertheless, in our case the "quotient" exists on an open $X^0 \subset X$ and yields the requested rationally chain connected fibres.

Proposition 3.23. [9, Thm. IV.3.10.3] *Let X be a smooth variety over a field of characteristic zero. Then X is rationally chain connected if and only if X is rationally connected.*

With this given, the main tool for the proof of $\mathbb{L}$ being a zero divisor can now be introduced. Subsequently, we list some facts that will be needed later.

Remark and Definition 3.24. Let X be a smooth proper variety over a field k of characteristic zero, and let $\pi\colon X \dashrightarrow Z$ be its MRCC-fibration with π^0 like in Definition 3.20. We can assume that π^0 is smooth using generic smoothness in characteristic zero (cf. [15, Thm. 25.3.3]). By Proposition 3.23 every fibre of π^0 is rationally connected. Therefore, we call π the maximal rationally connected fibration (MRC-fibration) of X. The base Z is often denoted by $R(X)$ and is called the MRC-quotient.

Corollary 3.25. *Let k be a field of characteristic zero, and let $X \in Var_k$ be smooth and proper. Then the MRC-fibration $\pi\colon X \dashrightarrow R(X)$ exists and is unique up to birational equivalence.* $\square$

Proposition 3.26. (cf. [9, Compl. IV.5.2.1]) *Let $X \in Var_k$ be a smooth proper variety over an uncountable field of characteristic zero. Then a rationally chain connected fibration $\pi\colon X \dashrightarrow Z$ is an MRC-fibration if and only if it has the following property: If $z \in Z$ is a very general point and $C \subset X$ a rational curve with $C \cap \pi^{-1}(z) \neq \emptyset$, then C is already contained in the preimage of z.*

Proposition 3.27. [9, Thm. IV.5.5] *Let k be a field with* $\mathrm{char}(k) = 0$. *Let X_1, X_2 be smooth proper varieties over k with a dominant rational map $f_X\colon X_1 \dashrightarrow X_2$. Denote by $\pi_i\colon X_i \dashrightarrow Z_i$ their MRC-fibrations. Then there exists a rational map $f_Z\colon Z_1 \dashrightarrow Z_2$ such that $\pi_2 \circ f_X = f_Z \circ \pi_1$. In particular, birationality of X_1 and X_2 implies birationality of Z_1 and Z_2.*

In order to apply MRC-fibrations and their properties to the varieties X_W and Y_W that have been constructed in Subsection 3.1, we need the notion of uniruled varieties. To relate both concepts, we will use the scheme of morphisms from $\mathbb{P}^1$ to a projective variety. Then we will finally be able to prove that $\mathbb{L}$ is indeed a zero divisor in the Grothendieck ring of varieties.

Definition 3.28. *Let X be an integral k-scheme of dimension n. Then X is called uniruled if there exists a k-scheme Y of dimension $n-1$ and a dominant map $\mathbb{P}^1 \times Y \dashrightarrow X$.*

Lemma 3.29. [4, Rem. 4.2.2] *A variety X is uniruled if and only if there exist a variety Y, an open subvariety $U \subset \mathbb{P}^1 \times Y$ with a dominant morphism $e\colon U \to X$, and a point $y \in Y$ such that $U \cap (\mathbb{P}^1 \times \{y\}) \neq \emptyset$ and e is not the contraction to a point on this set.*

Proposition 3.30. *Let $X \in Var_k$ be smooth, proper and uniruled in characteristic zero. We denote by ω_X its canonical bundle. Then one has $H^0(X, \omega_X^m) = 0$ for all $m > 0$.*

Proof. This follows from [9, Cor. IV.1.11] in view of [9, Ex. IV.1.1.6.1]. $\square$

Remark 3.31. (cf. [4, Sec. 2.1]) Let k be algebraically closed with $\mathrm{char}(k) = 0$. A morphism $f\colon \mathbb{P}^1 \to \mathbb{P}^n$ of degree d corresponds to polynomials $F_0, \ldots, F_n \in k[u,v]_d$ that have no nontrivial common zero (here we denote by $k[u,v]_d \subset k[u,v]$ the homogeneous degree d polynomials in two variables). By Hilbert's Nullstellensatz the F_i have no nontrivial common zero if and only if there is an $m \in \mathbb{N}$ such that $(u,v)^m \subset (F_0, \ldots, F_n)$. This is the case if and only if the linear map of k-vector spaces

$$\begin{aligned} \varphi_m\colon \big(k[u,v]_{m-d}\big)^{n+1} &\longrightarrow k[u,v]_m \\ (G_0, \ldots, G_n) &\longmapsto \sum F_i G_i \end{aligned}$$

is surjective, which in turn holds if and only if there is a nonzero $(m+1)$ minor of the matrix M_{φ_m} determining φ_m. Note that the entries of M_{φ_m} are given by linear combinations of the coefficients of the F_i, hence φ_m being surjective is an open condition on these coefficients.

Altogether, identifying a morphism f given by polynomials F_i with the coefficients $[F_{00} : \ldots : F_{0d} : F_{10} : \ldots : F_{nd}] \in \mathbb{P}^{n(d+1)}$, we obtain that

$$\mathrm{Mor}_d(\mathbb{P}^1, \mathbb{P}^n) \coloneqq \bigcup_{m \in \mathbb{N}} \big\{[F_{00} : \ldots : F_{nd}] \in \mathbb{P}^{n(d+1)} \bigm| \varphi_m \text{ is surjective}\big\}$$

parameterizes all degree d morphisms $f\colon \mathbb{P}^1 \to \mathbb{P}^n$ and is a quasi-projective variety since $\mathrm{Mor}_d(\mathbb{P}^1, \mathbb{P}^n) \subset \mathbb{P}^{n(d+1)}$ is open. We denote by

$$\mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^n) \coloneqq \bigsqcup_{d \geq 0} \mathrm{Mor}_d(\mathbb{P}^1, \mathbb{P}^n)$$

the scheme parameterizing all morphisms from $\mathbb{P}^1$ to $\mathbb{P}^n$.

Consider a projective variety $X \subset \mathbb{P}^n$ given by the zero locus of homogeneous polynomials G_j. In characteristic zero a polynomial in $k[z_0, \ldots, z_n]$ is zero if and only if it is zero evaluated at all points $z \in k^{n+1}$. Hence, a morphism $f\colon \mathbb{P}^1 \to \mathbb{P}^n$ maps to X if and only if $G_j(F_0(u,v), \ldots, F_n(u,v)) = 0$ is the zero polynomial. This translates into polynomial conditions on the coefficients of the F_i, hence all morphisms $f\colon \mathbb{P}^1 \to X$ are given by the scheme

$$\mathrm{Mor}(\mathbb{P}^1, X) \coloneqq \big\{f \in \mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^n) \bigm| G_j(F_0, \ldots, F_n) = 0\big\},$$

which is the disjoint union of countably many quasi-projective varieties $\mathrm{Mor}_d(\mathbb{P}^1, X) \subset \mathrm{Mor}_d(\mathbb{P}^1, \mathbb{P}^n)$.

Proposition 3.32. *Let X be an irreducible smooth projective variety over an uncountable algebraically closed field k of characteristic zero. If X is not uniruled, then its MRC-fibration is given by the identity* $\mathrm{id}_X\colon X \to X$.

Proof. We claim that a very general point of X is not contained in a rational curve $C \subset X$. Then by the criterion for MRC-fibrations 3.26 it follows immediately that the identity is the MRC-fibration of X.

Note that the case $\dim(X) = 0$ is trivial. Therefore, we may assume $\dim(X) \geq 1$.

Assume that a very general point of X is contained in a rational curve $C \subset X$. Consider the evaluation map

$$\varphi\colon \mathbb{P}^1 \times \mathrm{Mor}(\mathbb{P}^1, X) \to X,$$

whose image contains every rational curve on X. By assumption, a very general point of X is contained in

$$\bigcup_{d>0} \varphi\big(\mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X)\big) \subset \varphi\big(\mathbb{P}^1 \times \mathrm{Mor}(\mathbb{P}^1, X)\big).$$

We denote by V_d the closure of $\varphi\big(\mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X)\big)$ in X. Suppose that all V_d are proper closed subsets of X for $d > 0$. Then $X \setminus \big(\bigcup_{d>0} V_d\big)$ contains a very general point x of X. Since x is contained in a rational curve in X, one has $x \in \bigcup_{d>0} V_d$, which is a contradiction. Therefore, we obtain $V_d = X$ for some $d > 0$, and hence the restriction

$$\varphi_d\colon \mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X) \to X$$

is dominant. Note that φ_d is a morphism of varieties, whereas φ is only a morphism of schemes.

Since $d > 0$, the evaluation morphism does not contract sets of the form $\mathbb{P}^1 \times \{f\} \subset \mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X)$. Hence, Lemma 3.29 implies that X is uniruled by choosing $U \coloneqq \mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, X)$ and $e = \varphi_d$. This is a contradiction to the assumption and hence proves the claim. $\square$

3.3 The main theorem

The goal of this subsection is to prove that $\mathbb{L}$ is a zero divisor in the Grothendieck ring of varieties. With the notion of MRC-fibrations from the previous subsection, we can now deduce this result from Proposition 3.13.

Theorem 3.33. *Let k be an algebraically closed field of characteristic zero. Then $\mathbb{L}$ is a zero divisor in the Grothendieck ring of k-varieties.*

Proof. Recall that we know from Proposition 3.5 that X_W and Y_W are projective (and hence proper), irreducible, smooth Calabi–Yau threefolds. By Proposition 3.13 it is enough to show that in $K_0(\mathrm{Var}_k)$ one has

$$\alpha \coloneqq \big([X_W] - [Y_W]\big)(\mathbb{L}^2 - 1)(\mathbb{L} - 1) \neq 0.$$

As a first step, we prove the theorem under the additional assumption that k is uncountable.

Assume that α is zero in the Grothendieck ring. Then it is also zero modulo $\mathbb{L}$ and hence $[X_W] = [Y_W]$ modulo $\mathbb{L}$. By Proposition 2.6 it follows that X_W and Y_W are stably birational. Since they are of the same dimension, there exists an $m \geq 0$ such that $X_W \times \mathbb{P}^m$ and $Y_W \times \mathbb{P}^m$ are birationally equivalent.

We denote by ω_{X_W} the canonical bundle of X_W like before. As X_W is a projective Calabi–Yau variety, one has

$$H^0(X_W, \omega_{X_W}^m) = H^0(X_W, \mathcal{O}_{X_W}) = k$$

for all $m > 0$. We obtain by Proposition 3.30 that X_W is not uniruled. Therefore, its MRC-fibration is the identity morphism, and a very general point of X_W is not contained in a rational curve, see Proposition 3.32.

We claim that the MRC-fibration of $X_W \times \mathbb{P}^m$ is given by the projection $\pi\colon X_W \times \mathbb{P}^m \to X_W$. Since π is proper and all fibres are isomorphic to $\mathbb{P}^m$, π is a rationally chain connected fibration. Let now $x \in X_W$ be a very general point, which is then not contained in a rational curve. Then any rational curve $C \subset X_W \times \mathbb{P}^m$ with $C \cap \pi^{-1}(x) \neq \emptyset$ is contracted by π. By Proposition 3.26 we obtain that π is the MRC-fibration of $X_W \times \mathbb{P}^m$ as claimed.

Analogously, we obtain that the MRC-fibration of Y_W is given by the projection $Y_W \times \mathbb{P}^m \to Y_W$. From Proposition 3.27 we know that the MRC-fibration is a birational invariant. Since $X_W \times \mathbb{P}^m$ and $Y_W \times \mathbb{P}^m$ are birationally equivalent, their MRC-quotients X_W and Y_W are birational as well. This is a contradiction to Proposition 3.5 (ii), which proves the theorem in the case of k being uncountable.

Let now k be a countable, algebraically closed field of characteristic zero. We assume $\alpha = 0$ and obtain that $X_W \times \mathbb{P}^m$ and $Y_W \times \mathbb{P}^m$ are birationally equivalent as before.

Consider a field extension $k \subset K$ such that K is algebraically closed and uncountable, then $X_W \times_k \mathbb{P}^m_K$ and $Y_W \times_k \mathbb{P}^m_K$ are birational as well. We denote by X'_W resp. Y'_W the varieties constructed in Subsection 3.1 over the groud field K. Since these are defined by equations with integral coefficients, we find $X'_W = X_W \times_k \operatorname{Spec}(K)$ resp. $Y'_W = Y_W \times_k \operatorname{Spec}(K)$. Therefore, $X'_W \times_K \mathbb{P}^m_K = X_W \times_k \mathbb{P}^m_K$ and $Y'_W \times_K \mathbb{P}^m_K = Y_W \times_k \mathbb{P}^m_K$ are birationally equivalent. Since K is uncountable, this yields a contradiction by the first step and hence proves the theorem. $\square$

Remark 3.34. Note that this result equally follows from the stronger formulas stated in Remark 3.14: Assume that $[X_W] - [Y_W] = 0$ in the Grothendieck ring, then Proposition 2.6 again yields that X_W and Y_W are stably birational. The contradiction follows just like before.

4 The cut-and-paste problem

Borisov [1] did not only prove that $\mathbb{L} \in K_0(\mathrm{Var}_k)$ is a zero divisor; his proof of this theorem also shows with little additional effort that the cut-and-paste problem stated by Larsen and Lunts [10, Quest. 1.2] has a negative solution. In this Section we introduce the concept of piecewise isomorphism, present and discuss the problem and finally reproduce Borisov's proof of the negative answer [1, Thm. 2.13] to the cut-and-paste question.

Definition 4.1. *Let $X, Y \in Var_k$. Then X and Y are piecewise isomorphic if there exist decompositions $X = \bigsqcup_{i=1}^n X_i$ and $Y = \bigsqcup_{i=1}^n Y_i$ into locally closed subvarieties such that $X_i \cong Y_i$ for all i.*

Remark 4.2. For two piecewise isomorphic k-varieties X and Y it follows from Lemma 2.4 that $[X] = [Y]$. This leads to the following question.

Cut-and-paste problem 4.3. *Let X and Y be k-varieties such that their classes in $K_0(\mathrm{Var}_k)$ are equal. Are X and Y piecewise isomorphic?*

At first sight a positive answer to this question seems to be reasonable as the only relation (1) on the Grothendieck ring is about decomposing a variety into subvarieties. On the other hand, embedding varieties X and Y into a variety Z such that their complements in Z are piecewise isomorphic implies $[X] = [Y]$ but does not give any indication whether X and Y are piecewise isomorphic.

Liu and Sebag [12] showed that the Cut-and-paste problem (for k algebraically closed and of characteristic zero) has a positive answer in the case of dimension at most one, in the case of smooth connected projective varieties of pure dimension two, and in the case of varieties that contain only finitely many rational curves. The general cut-and-paste problem is of particular interest in view of motivic zeta functions: Larsen and Lunts [11, Thm. 3.3] showed (for $k = \mathbb{C}$) the irrationality of the zeta function after inverting $\mathbb{L}$ under the assumption that the cut-and-paste problem has a positive solution. However, contrary to their conjecture Borisov [1] proved the following.

Theorem 4.4. *For an algebraically closed field k of characteristic zero the cut-and-paste problem has a negative solution.*

Proof. This statement follows with little additional effort from Borisov's construction [1] that we discussed in Section 3. Using the notation from Subsection 3.1, we define $X := X_W \times \mathrm{Gl}(2) \times \mathbb{A}^6$ and $Y := Y_W \times \mathrm{Gl}(2) \times \mathbb{A}^6$. From Proposition 3.13 we obtain the equality $[X] = [Y]$ in the Grothendieck ring in view of $[\mathrm{Gl}(2)] = (\mathbb{L}^2 - 1)(\mathbb{L}^2 - \mathbb{L})$. Moreover, as the product of irreducible varieties, X and Y are irreducible.

We assume that the cut-and-paste problem has a positive solution. Then X and Y are piecewise isomorphic, hence there are locally closed decompositions $X = \bigsqcup_{i=1}^{n} X_i$ and $Y = \bigsqcup_{i=1}^{n} Y_i$ such that $\emptyset \neq X_i \cong Y_i$ for all i. We claim that X and Y are birational. It is enough to show that $X_j \subseteq X$ and $Y_j \subseteq Y$ are open for some j.

Since X is irreducible and $X = \bigcup_{i=1}^{n} \overline{X_i}$, one has $X = \overline{X_j}$ for some j. As X_j is locally closed, it is the intersection of an open set $U_j \subseteq X$ and a closed set $V_j \subseteq X$. Then we already have $V_j = X$, as $X = \overline{X_j} \subseteq V_j$. Hence, $X_j \subseteq X$ is open. It remains to show that also $Y_j \subseteq Y$ is open. This follows from a look at the dimensions: one has

$$\dim(Y_i) = \dim(X_i) = \dim(X) = \dim(Y),$$

hence $Y_i \subseteq Y$ is dense and thus open, which proves that X and Y are birational.

Considering an affine, open and hence dense $U \subset \mathrm{Gl}(2) \times \mathbb{A}^6$, we obtain that also $X_W \times U$ and $Y_W \times U$ are birational. This in turn implies stable birationality of X_W and Y_W, which is in contradiction to X_W and Y_W not being birational (analogous to the proof of Theorem 3.33). Hence, the cut-and-paste problem has indeed a negative solution. $\square$

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