Special subvarieties in the locus of intermediate Jacobians of cubic threefolds

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Contents

1. Introduction

In this thesis, we study special subvarieties, i.e., subvarieties containing a dense subset of CM points, of the moduli space A_5 of principally polarized abelian varieties of dimension five, generically contained in the locus of intermediate Jacobians of cubic threefolds. The analogous question for Jacobians of curves is related to a conjecture of Coleman–Oort and has been studied by Shimura, Mostow, De Jong–Noot, Rohde, Moonen, Oort, Frediani, Ghigi and others.

Adapting methods of Frediani, Ghigi and Penegini [FGP15], we give a sufficient condition ensuring that the closure of the image of a family of smooth cubic threefolds with prescribed automorphisms via the period map is a special subvariety of A_5 and classify the positive-dimensional families of cubic threefolds satisfying our condition. In particular, we discover two examples of positive-dimensional special subvarieties in the intermediate Jacobian locus that contain the intermediate Jacobian of the Klein cubic threefold.

1.1. Before giving a more precise overview of the results of this thesis, let us discuss the analogous question regarding the existence of special subvarieties generically contained in the locus of Jacobians of curves. For details, see the expository article by Moonen and Oort [MO13]. Let A_g denote the coarse moduli space of principally polarized abelian varieties of dimension g.

In [Col87, Conj. 6], Coleman conjectured that for sufficiently large genus, there are only finitely many curves admitting complex multiplication on their Jacobians. Using the André–Oort Conjecture, recently proven by Tsimerman [Tsi18], one can reformulate the conjecture as follows:

Conjecture 1.1 (Coleman–Oort)**.** *For $g \geqslant 8$, there are no positive-dimensional special subvarieties $Z \subseteq A_g$ generically contained in the Torelli locus.*

Without the assumption on the genus, the conjecture fails. In [FGP15], Frediani, Ghigi and Penegini establish a sufficient condition ensuring that the image of a family of Galois coverings of the projective line via the period map is a special subvariety of A_g, generalizing earlier work by de Jong, Noot, Rohde, Moonen, Oort and others, see [JN91], [Roh09], [Moo10] and [MO13].

Using this criterion, they find exactly 30 positive-dimensional special subvarieties of A_g for $g \leqslant 7$ that are generically contained in the Torelli locus, see [FGP15, Thm. 1.9].

As it serves as the main inspiration for large parts of this note, we briefly describe their criterion. A Galois covering $C \longrightarrow \mathbb{P}^1$ is determined by the ramification data $m := (m_1, \dots, m_r)$, the Galois group G, the branching points $t_1, \dots, t_r \in \mathbb{P}^1$ and an epimorphism

$$\theta \colon \pi_1(\mathbb{P}^1 \backslash \{t_1, \dots, t_r\}, t_0) \twoheadrightarrow G.$$

Fixing the datum (m, G, θ) and varying the points $t_1, \dots, t_r \in \mathbb{P}^1$, one obtains a family of curves. Let $Z(m, G, \theta)$ denote the closure of the set of Jacobians of these curves in A_g. By the Torelli theorem for curves, this an $(r-3)$-dimensional subvariety of A_g.

Theorem 1.2 ([FGP15, Thm. 1.4]). *Let (m, G, θ) be a datum as above. Assume that*

$$(\star) \qquad \dim Z(m, G, \theta) = \dim(S^2 H^0(C, K_C))^G.$$

Then, $Z(m, G, \theta)$ is a special subvariety of PEL-type of A_g, generically contained in the Torelli locus.

Building on ideas of de Jong and Noot [JN91], Moonen [Moo10] proved that for cyclic G the condition $(\star)$ is also necessary. Moreover, he gives a complete classification of those data $(m, \mathbb{Z}/n\mathbb{Z}, \theta)$ for which $Z(m, \mathbb{Z}/n\mathbb{Z}, \theta)$ is a special subvariety.

1.2. The aim of this note is to study the analogous question for the locus of intermediate Jacobians of cubic threefolds. For a general reference on cubic threefolds, we refer to [Huy23, Ch. 5]. Denote by $M := H^0(\mathbb{P}^4, \mathcal{O}(3))_{\mathrm{sm}} / \mathrm{GL}(5, \mathbb{C})$ the coarse moduli space of smooth cubic threefolds. Similar to the case of curves, the Torelli theorem for cubic threefolds asserts that the map

$$J \colon M \longrightarrow A_5,$$

sending a cubic threefold to its intermediate Jacobian is a locally closed embedding. A monodromy computation due to Beauville [Bea85, Thm. 4] shows that the closure of $J(M)$ in A_5 is not a special subvariety.[1]

For a finite group $G \subseteq \mathrm{GL}(5, \mathbb{C})$, we let M_G denote the image of $H^0(\mathbb{P}^4, \mathcal{O}(3))^G_{\mathrm{sm}}$ in the moduli space of smooth cubic threefolds. In analogy to Theorem 1.2, we establish the following criterion:

Theorem 1.3 (See Thm. 4.3). *Assume that*

$$(\star\star) \qquad \dim M_G = \dim(S^2 H^{2,1}(Y))^G$$

holds for some smooth cubic threefold $Y \in M_G$. Then, the closure of $J(M_G)$ in A_5 is a special subvariety of PEL-type.

[1] In [ACT11], Allcock, Carlson and Toledo give a description of the moduli space M of smooth cubic threefolds as an open subset of a ten-dimensional complex ball quotient via an "occult" period map. In particular, there is a notion of special subvarieties of M with respect to this period map. However, note that if the closure of $Z \subseteq M$ in the ten-dimensional ball quotient is a special subvariety, then, in general, the closure of $J(Z) \subseteq A_5$ will not be a special subvariety of A_5. For example, (the closure of) M is a special subvariety in the "occult" sense, but the closure of $J(M)$ in A_5 is not a special subvariety.

By the work of Allcock, Carlson and Toledo [ACT02], it is known that the image of the locus of cyclic cubic threefolds $M^{\mathrm{cyc}} = M_{\langle \mathrm{diag}(\zeta_3,1,1,1,1)\rangle} \subseteq M$ is a special subvariety, see also [Ach13]. Using the classification of groups acting faithfully on smooth cubic threefolds by Wei and Yu [WY20], we show the following:

Theorem 1.4 (See Thm. 8.1)**.** *Let $G \subseteq \mathrm{GL}(5,\mathbb{C})$ be a finite subgroup such that there is no $G \subsetneq G' \subseteq \mathrm{GL}(5,\mathbb{C})$ with $M_{G'} = M_G$. Then ($\star\star$) is satisfied if and only if one of the following holds:*

(i) $\dim M_G = 0$;

(ii) $M_G \subseteq M^{\mathrm{cyc}}$; *or*

(iii) *M_G is the unique family of cubic threefolds admitting a faithful action by $\mathrm{Alt}(4)$ (resp. $\mathrm{Alt}(5)$) such that M_G contains the Klein cubic threefold.*

In particular, in these cases, the closure of $J(M_G)$ in A_5 is a special subvariety.

The locus of cyclic cubic threefolds is four-dimensional, while the loci mentioned in (iii) are two-(resp. one-)dimensional. One can visualize the situation as follows, see Sec. 8 for more explanation:

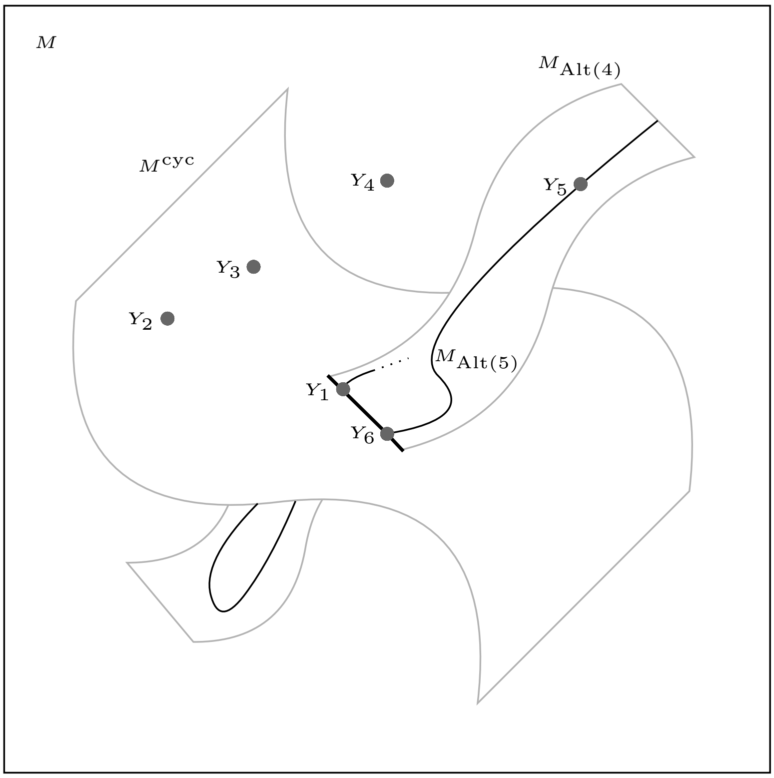

FIGURE 1. Families of cubic threefolds that give rise to special subvarieties in the intermediate Jacobian locus (see Sec. 8)

We conclude this introduction by remarking that ($\star\star$) is sufficient but not necessary for the closure of $J(M_G)$ to be a special subvariety. In Remark 4.4, we give examples of groups $G \subsetneq H$, where G does not satify ($\star\star$), but $M_G = M_H$ and H satisfies ($\star\star$). In particular, the closure of $J(M_G) = J(M_H)$ is a special subvariety.

It would be interesting to know whether there are examples of groups $G \subseteq \mathrm{GL}(5,\mathbb{C})$ for which $\overline{J(M_G)}$ is a special subvariety but there is no subgroup $H \subseteq \mathrm{GL}(5,\mathbb{C})$ with $M_G = M_H$ satisfying ($\star\star$). On a more general note, it would be interesting to know whether there are special subvarieties that are generically contained in the locus of intermediate Jacobians of cubic threefolds but not contained in the examples that we found.

1.3. The plan of this thesis is as follows:

In Section 2, we recall the notion special subvarieties and a particular kind of special subvarieties arising from the existence of extra automorphisms.

In Section 3, we briefly recall the classification of automorphisms groups of smooth cubic threefolds and liftings of such groups to $\mathrm{GL}(5,\mathbb{C})$.

Section 4 is devoted to the proof of Theorem 1.3.

In Section 5, we discuss the known special subvariety arising from the family of cyclic cubic threefolds.

Section 6 is devoted to the discussion of new examples arising from certain subgroups of the automorphism group of the Klein cubic threefold.

In Section 7, we show that there are no further families satisfying ($\star\star$).

In Section 8, we conclude Theorem 1.4.

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2. Special subvarieties

Let A_g denote the coarse moduli space of g-dimensional principally polarized abelian varieties. Recall, e.g., from [Kem12, Sec. 7], that the Siegel upper half-space

$$\mathbb{H}_g = \{J \in \mathrm{GL}(2g,\mathbb{R}) \mid J^2 = -I, J^*E = E, E(x,Jx) > 0, \forall x \neq 0\}.$$

parametrizes complex structures on $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$, compatible with the principal polarization given by the standard alternating form $E\colon \mathbb{Z}^{2g} \times \mathbb{Z}^{2g} \longrightarrow \mathbb{Z}$ of type $(1,\ldots,1)$. The group $\mathrm{Sp}(2g,\mathbb{Z})$ acts properly discontinuous on $\mathbb{H}_g$ by conjugation and the quotient $\mathrm{Sp}(2g,\mathbb{Z})\backslash\mathbb{H}_g$ identifies with A_g, see [Mil05, Sec. 6]. Denote the principally polarized abelian variety corresponding to a complex structure $J \in \mathbb{H}_g$ by (A_J,Θ_J). Observe that $\mathrm{Aut}(A_J,\Theta_J)$ is the stabilizer of the action of $\mathrm{Sp}(2g,\mathbb{Z})$ at the point $J \in \mathbb{H}$.

On $\mathbb{H}_g$, there is a natural variation of rational Hodge structures, with local system $\mathbb{H}_g \times \mathbb{Q}^{2g}$ and corresponding to the Hodge decomposition of $\mathbb{C}^{2g}$ in $\pm i$ eigenspaces for J. By definition, a subvariety $Z \subseteq A_g$ is a *special subvariety* if it is the image of a Hodge locus of this variation of Hodge structures, see [MO13].

Recall that an abelian variety A of dimension g admits *complex multiplication* if $\mathrm{End}(A) \otimes \mathbb{Q}$ contains a commutative $\mathbb{Q}$-algebra of dimension $2g$ over $\mathbb{Q}$. It turns out that zero-dimensional special subvarieties of A_g precisely correspond to abelian varieties admitting complex multiplication:

Proposition 2.1. *A point in $[A] \in A_g$ forms a special subvariety if and only if the corresponding abelian variety A admits complex multiplication. In this case, we call $[A] \in A_g$ a special point.*

The André–Oort conjecture for A_g, recently proven by Tsimerman [Tsi18], building on the work of Pila and many others, yields an alternative description for special subvarieties in terms of special points.

Theorem 2.2 (André–Oort, [And89], [Tsi18])**.** *A subvariety $Z \subseteq A_g$ is a special subvariety if and only if the set of special points in Z is a Zariski-dense subset.*

Let G be a finite subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$. One can show that the set of points of $\mathbb{H}_g$ fixed by G forms a smooth connected submanifold $\mathbb{H}_g^G \subseteq \mathbb{H}_g$, see [FGP15, Lem. 3.3]. Let $Z_G \subseteq A_g$ denote the image of $\mathbb{H}_g^G$ in A_g.

Proposition 2.3 ([FGP15, Prop. 3.7])**.** *The subvariety $Z_G \subseteq A_g$ is a special subvariety (of PEL-type).*

Alternatively, the subvariety $Z_G \subseteq A_g$ can be described as the set of complex structures $J \in \mathbb{H}_g$ for which $G \subseteq \mathrm{Aut}(A_J, \Theta_J) \subseteq \mathrm{Sp}(2g, \mathbb{Z})$.

One can compute the dimension of Z_G as follows:

Proposition 2.4 ([FGP15, Lem. 3.8])**.** *Let (A, Θ) be a principally polarized abelian variety corresponding to a point in $Z_G \subseteq A_g$. Then, we have*

$$\dim Z_G = \dim(S^2 H^{0,1}(A))^G.$$

Remark 2.5. Note that $\dim(S^2 H^{0,1}(A))^G = \dim(S^2 H^{1,0}(A))^G$.

As zero-dimensional special subvarieties are nothing but CM points, we obtain the following consequence:

Corollary 2.6 ([FGP15, Cor. 3.10])**.** *Let (A, Θ) be a principally polarized abelian variety and let $G \subseteq \mathrm{Aut}(A, \Theta)$ be a group of automorphisms. If*

$$\dim(S^2 H^{0,1}(A))^G = 0,$$

then A admits complex multiplication.

3. Cubic threefolds with extra automorphisms

First, recall the description of the coarse moduli space M of smooth cubic threefolds as an affine quotient, see [Huy23, Ch. 3.2]: Let $U := H^0(\mathbb{P}^4, \mathcal{O}(3))_{\mathrm{sm}} \subseteq H^0(\mathbb{P}^4, \mathcal{O}(3))$ denote the open set of homogeneous cubic polynomials defining smooth cubic threefolds. Note that U is the affine variety $\mathrm{Spec}(A)$, where the ring A is the homogeneous localization of the polynomial ring $\mathbb{C}[H^0(\mathbb{P}^4, \mathcal{O}(3))^\vee]$ with respect to the discriminant $\Delta \in \mathbb{C}[H^0(\mathbb{P}^4, \mathcal{O}(3))^\vee]_{80}$. Then, M is the affine quotient

$$U = \mathrm{Spec}(A) \longrightarrow M = \mathrm{Spec}(A^{\mathrm{GL}(5,\mathbb{C})}) = U/\mathrm{GL}(5, \mathbb{C}).$$

3.1. Recall that every automorphism of a smooth cubic threefold extends to an automorphism of the ambient projective space, see [Huy23, Sec. 1.3]. The groups acting faithfully on smooth cubic threefolds have been classified by Wei and Yu in [WY20], see also [GL11].

Theorem 3.1 ([WY20]). *A group G has a faithful action on some smooth cubic threefold if and only if G is isomorphic to a subgroup of one of the following six groups:*

$$(\mathbb{Z}/3\mathbb{Z})^4 \rtimes \mathrm{Sym}(5),\ (((\mathbb{Z}/3\mathbb{Z})^2 \rtimes \mathbb{Z}/3\mathbb{Z}) \rtimes \mathbb{Z}/4\mathbb{Z}) \times \mathrm{Sym}(3),$$
$$\mathbb{Z}/24\mathbb{Z},\ \mathbb{Z}/16\mathbb{Z},\ \mathrm{PSL}(2, 11),\ \mathbb{Z}/3\mathbb{Z} \times \mathrm{Sym}(5).$$

Example 3.2. The following six smooth cubic threefolds realize the maximal automorphism groups, see [WY20, Ex. 3.1]:

	F	$\mathrm{Aut}(V(F))$
Y_1	$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3$	$(\mathbb{Z}/3\mathbb{Z})^4 \rtimes \mathrm{Sym}(5)$
Y_2	$x_0^3 + x_1^3 + x_2^3 + 3(\sqrt{3}-1)x_0x_1x_2 + x_3^3 + x_4^3$	$(((\mathbb{Z}/3\mathbb{Z})^2 \rtimes \mathbb{Z}/3\mathbb{Z}) \rtimes \mathbb{Z}/4\mathbb{Z}) \times \mathrm{Sym}(3)$
Y_3	$x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^3 + x_4^3$	$\mathbb{Z}/24\mathbb{Z}$
Y_4	$x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^3$	$\mathbb{Z}/16\mathbb{Z}$
Y_5	$x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_0$	$\mathrm{PSL}(2, 11)$
Y_6	$x_0^3 + x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_1$	$\mathrm{Sym}(5) \times \mathbb{Z}/3\mathbb{Z}$

Let us recall some definitions concerning the liftability of group actions on $\mathbb{P}^n$, following [OY19]:

Definition 3.3. Let $F \in \mathbb{C}[x_0, \ldots, x_n]_d$ be a homogeneous polynomial of degree d and H a finite subgroup of $\mathrm{PGL}(n+1, \mathbb{C})$. A subgroup $G \subseteq \mathrm{GL}(n+1, \mathbb{C})$ is an *F-lifting* of H if G and H are isomorphic via the natural projection $\mathrm{GL}(n+1, \mathbb{C}) \longrightarrow \mathrm{PGL}(n+1, \mathbb{C})$ and $A.F = F$ for all $A \in G$.

Automorphism groups of cubic threefolds always admit F-liftings:

Theorem 3.4 ([WY20, Thm. 4.11]). *If $Y = V(F) \subseteq \mathbb{P}^4$ is a smooth cubic threefold and $H \subseteq \mathrm{Aut}(Y) \subseteq \mathrm{PGL}(5, \mathbb{C})$ is a group of automorphisms of Y, then there is an F-lifting $G \subseteq \mathrm{GL}(5, \mathbb{C})$ of H.*

Remark 3.5. If $3 \mid |H|$, then the F-lifting may not be unique. For example, if $A \in \mathrm{GL}(5, \mathbb{C})$ is an element of order three then we have $(\zeta_3 A).F = A.F$. However, if $G_1, G_2 \subseteq \mathrm{GL}(5, \mathbb{C})$ are two F-liftings of a finite group $H \subseteq \mathrm{PGL}(5, \mathbb{C})$, then $\langle G_1, \zeta_3 \mathrm{id}_5 \rangle = \langle G_2, \zeta_3 \mathrm{id}_5 \rangle \subseteq \mathrm{GL}(5, \mathbb{C})$ and hence a cubic polynomial is G_1-invariant if and only if it is G_2-invariant, see [WY20, App. B].

3.2. From now on, we assume that $G \subseteq \mathrm{GL}(5,\mathbb{C})$ is a finite subgroup for which the projection $G \longrightarrow \mathrm{PGL}(5,\mathbb{C})$ is injective. As in the introduction, let $M_G \subseteq M$ denote the image of U^G in $M = U/GL(5,\mathbb{C})$.

Lemma 3.6. *The subset $M_G \subseteq M$ is irreducible.*

Proof. Immediately follows from the fact the U^G is irreducible. ☐

Note that M_G depends only on the conjugacy class of the subgroup $G \subseteq \mathrm{GL}(5,\mathbb{C})$. For an abstract group H, let

$$\widetilde{M}_H := \{[Y] \in M \mid H \subseteq \mathrm{Aut}(Y)\} \subseteq M$$

denote the subset of M consisting of all smooth cubic threefolds admitting a faithful action by H. By Theorem 3.4, we have

$$\widetilde{M}_H = \bigcup_{H \simeq G \subseteq \mathrm{GL}(5,\mathbb{C})} M_G.$$

As there are only finitely many subgroups of $\mathrm{GL}(5,\mathbb{C})$ isomorphic to H up to conjugation, the subset $\widetilde{M}_H \subseteq M$ has finitely many irreducible components.

We conclude this section by computing the dimension of M_G in terms of U^G and the centralizer $C_{\mathrm{GL}(5,\mathbb{C})}(G)$ of G in $\mathrm{GL}(5,\mathbb{C})$.

Lemma 3.7. *If $M_G \neq \varnothing$, then*

$$\dim M_G = \dim U^G - \dim C_{\mathrm{GL}(5,\mathbb{C})}(G).$$

Proof. Note that the normalizer $N_{\mathrm{GL}(5,\mathbb{C})}(G)$ naturally acts on U^G by conjugation. The induced morphism

$$U^G/N_{\mathrm{GL}(5,\mathbb{C})}(G) \longrightarrow U/\mathrm{GL}(5,\mathbb{C}) = M$$

is finite by [Lun75, Main Thm.]. As the stabilizers of the action of the normalizer on U^G are finite, this implies that

$$\dim M_G = \dim U^G - \dim N_{\mathrm{GL}(5,\mathbb{C})}(G).$$

Since the quotient $N_{\mathrm{GL}(5,\mathbb{C})}(G)/C_{\mathrm{GL}(5,\mathbb{C})}(G)$ is isomorphic to a subgroup of the finite group $\mathrm{Aut}(G)$, the claim follows. ☐

4. Special subvarieties in the locus of intermediate Jacobians

In this section, we consider special subvarieties generically contained in the locus of intermediate Jacobians of cubic threefolds. Let $Y \subseteq \mathbb{P}^4$ be a cubic threefold. Recall that the intermediate Jacobian

$$J(Y) := \frac{H^{1,2}(Y)}{H^3(Y,\mathbb{Z})} \simeq \frac{H^{2,1}(Y)^\vee}{H_3(Y,\mathbb{Z})}$$

is a principally polarized abelian variety of dimension five. Denote the distinguished theta divisor by Ξ. Analogous to the case of curves, there is a Torelli theorem:

Theorem 4.1 (Clemens–Griffiths, Tyurin)**.** *Let $Y, Y' \subseteq \mathbb{P}^4$ be smooth cubic threefolds. Then the following assertions are equivalent:*

(i) *There is an isomorphism $Y \simeq Y'$.*

(ii) *There is an isomorphism of principally polarized abelian varieties* $(J(Y), \Xi) \simeq (J(Y'), \Xi')$. *Furthermore, there is a natural isomorphism*

$$\mathrm{Aut}(J(Y), \Xi) \simeq \mathrm{Aut}(Y) \times \langle -1 \rangle.$$

Proof. See [CG72]. For the claim about automorphism groups, see [Zhe21, Prop. 1.6]. □

As in the case of curves, the image of the morphism

$$J\colon M \longrightarrow A_5,$$

sending a smooth cubic hypersurface to the isomorphism class of its intermediate Jacobian, is locally closed. Note that M is ten-dimensional and A_5 is fifteen-dimensional.

The aim of this section is to prove Theorem 1.3. As a crucial input, let us first recall an explicit description of the action of $\mathrm{Aut}(Y)$ on $H^{2,1}(Y)$.

Lemma 4.2. *Let* $G \subseteq \mathrm{GL}(5, \mathbb{C})$ *be a finite subgroup and denote the character of the standard representation of* G *on* $\mathbb{C}^5$ *by* χ. *Let* $F \in U^G$ *be a* G*-invariant cubic polynomial defining a smooth cubic threefold* $Y \subseteq \mathbb{P}^4$. *Then, the character of the natural action of* G *on* $H^{2,1}(Y)^\vee$ *is given by* $\det(\chi) \otimes \chi$.

Proof. This is an application of Griffiths' Residue calculus, cf. [Gri69] and [Bea09, Sec. 3]. Let $V := \mathbb{P}^4 \backslash Y$ denote the complement of Y. The Gysin exact sequence yields a natural isomorphism

$$\mathrm{Res}\colon H^4(V, \mathbb{C}) \xrightarrow{\sim} H^3(Y, \mathbb{C}).$$

Let $\Omega := \sum_{i=0}^4 (-1)^i x_i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_4$. The results of [Gri69] imply that the map

$$\begin{aligned} H^0(\mathbb{P}^4, \mathcal{O}(1)) &\longrightarrow H^3(Y, \mathbb{C}) \\ L &\longmapsto \mathrm{Res}\, \frac{L\Omega}{F^2} \end{aligned}$$

induces an isomorphism $H^0(\mathbb{P}^4, \mathcal{O}(1)) \xrightarrow{\sim} H^{2,1}(Y)$. Now use that F is G-invariant, that the action of G on $H^0(\mathbb{P}^4, \mathcal{O}(1))$ has character χ and that G acts on Ω via determinants to conclude the proof. □

In particular, we have

$$\dim(S^2 H^{2,1}(Y))^G = \langle S^2(\det(\chi) \otimes \chi), \chi_{\mathrm{triv}} \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of characters.

Theorem 4.3 (Thm. 1.3)**.** *Let* $G \subseteq \mathrm{GL}(5, \mathbb{C})$ *be a finite group and let* χ *denote the character of the standard representation of* G *on* $\mathbb{C}^5$. *If* $M_G \neq \varnothing$ *and*

$$(\star\star) \qquad \dim M_G = \langle S^2(\det(\chi) \otimes \chi), \chi_{\mathrm{triv}} \rangle,$$

then the closure of $J(M_G)$ *in* A_5 *is a special subvariety (of PEL-type).*

Proof. Take a smooth cubic threefold $Y = V(F) \subseteq \mathbb{P}^4$ such that F is G-invariant. By the Torelli theorem, G acts faithfully on $H^3(Y,\mathbb{Z})$. Fixing an isomorphism $(H^3(Y,\mathbb{Z}),\cup) \simeq (\mathbb{Z}^{10}, E)$, we may thus identify G with a subgroup of $\mathrm{Sp}(10,\mathbb{Z})$. Note that the special subvariety $Z_G \subseteq A_5$ does not depend on this choice of isomorphism.

We claim that $J(M_G) \subseteq Z_G$. Let $F' \in U^G$ be another G-invariant cubic polynomial defining a smooth cubic threefold $Y' \subseteq \mathbb{P}^4$. Then, there is a path in U^G connecting F and F'. Parallel transport along this path gives rise to an isomorphism $(H^3(Y,\mathbb{Z}),\cup) \simeq (H^3(Y',\mathbb{Z}),\cup)$ such that the diagram

$$\begin{tikzcd} G \arrow[r, hook] \arrow[dr, hook] & \mathrm{Sp}(H^3(Y,\mathbb{Z}),\cup) \arrow[d, "\simeq"] \\ & \mathrm{Sp}(H^3(Y',\mathbb{Z}),\cup) \end{tikzcd}$$

commutes. The claim follows.

Since Z_G is irreducible, it suffices to show that $\dim J(M_G) = \dim Z_G$. By the Torelli theorem, we have $\dim M_G = \dim J(M_G)$, and by [FGP15, Lem. 3.8], see Proposition 2.4, we have

$$\dim Z_G = \dim(S^2 H^{2,1}(Y))^G.$$

As a consequence of Lemma 4.2, we have $\dim(S^2H^{2,1}(Y))^G = \langle S^2(\det(\chi)\otimes\chi), \chi_{\mathrm{triv}}\rangle$. Hence, condition $(\star\star)$ implies $\dim J(M_G) = \dim Z_G$ and thus $\overline{J(M_G)} = Z_G$ is a special subvariety. □

Remark 4.4. The condition $(\star\star)$ is only sufficient but not necessary for $\overline{J(M_G)} \subseteq A_5$ to be a special subvariety. As an example, consider the cyclic group $\mathbb{Z}/3\mathbb{Z} \simeq G := \langle \mathrm{diag}(\zeta_3,\zeta_3,1,1,1)\rangle \subseteq \mathrm{GL}(5,\mathbb{C})$. Then, $\dim M_G = 1$ but $\dim Z_G = 3$.

However, one can show that, up to coordinate change, every $Y \in M_G$ can be written as

$$V(x_0^3 + x_1^3 + F(x_2,x_3,x_4)) \subseteq \mathbb{P}^4,$$

where F is a homogeneous cubic polynomial. In particular, the group

$$\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z} \simeq H := \langle \mathrm{diag}(\zeta_3,1,1,1), \mathrm{diag}(1,\zeta_3,1,1,1)\rangle \subseteq \mathrm{GL}(5,\mathbb{C})$$

acts on Y and we have $M_G = M_H$. A simple computation shows that H satisfies $(\star\star)$. Thus, $\overline{J(M_G)} = \overline{J(M_H)}$ is a special subvariety.

It would be interesting to known whether there are examples of groups $G \subseteq \mathrm{GL}(5,\mathbb{C})$ for which $\overline{J(M_G)}$ is a special subvariety but $\overline{J(M_G)} \neq Z_H$ for any $H \subseteq \mathrm{Sp}(10,\mathbb{Z})$.

Corollary 4.5. *The intermediate Jacobians of the smooth cubic threefolds $Y_1,\dots,Y_6$ with maximal automorphism group, see Example 3.2, admit complex multiplication.*

Proof. For an explicit description of the groups acting on $Y_1,\dots,Y_6$ see [WY20, Ex. 3.1]. The case of the Klein cubic threefold Y_5 will be recalled in Section 6. Using [GAP22], one verifies that in these cases, we have

$$\dim M_G = 0 = \langle S^2(\det(\chi)\otimes\chi), \chi_{\mathrm{triv}}\rangle$$

in the notation of Theorem 4.3. Hence, the intermediate Jacobians $J(Y_k)$ admit CM. □

Remark 4.6. At least for Y_1, Y_5 and Y_6, the above is well-known. For Y_1 and Y_6, we have $J(Y_k) \sim E^5$, where E is the Fermat elliptic curve with complex multiplication by $\mathbb{Q}(\zeta_3)$, see [Rou09, Sec. 4] and [GY16, Prop. 1.7]. For the Klein cubic threefold Y_5, Adler has shown in [Adl81] that $J(Y_5) \sim E_{11}^5$, where E_{11} is an elliptic curve with complex multiplication by $\mathbb{Q}(\zeta_{11})$. See also [GMZ05].

Let us conclude this section by a small refinement of the arguments in the proof of Theorem 4.3, leading to the observation that $(\star\star)$ is satisfied if the special subvariety Z_G is generically contained in the intermediate Jacobian locus.

Proposition 4.7. *Let $G \subseteq \mathrm{Sp}(10, \mathbb{Z})$ be a finite group. If $Z_G \subseteq \overline{J(M)}$ and $Z_G \cap J(M) \neq \varnothing$, then there is a finite subgroup $H \subseteq \mathrm{GL}(5, \mathbb{C})$ isomorphic to $\langle G, -1\rangle/\langle -1\rangle$ such that $\overline{J(M_H)} = Z_G$. In particular, H satisfies $(\star\star)$* of Theorem 4.3.

Proof. Let $Y = V(F) \subseteq \mathbb{P}^4$ be a smooth cubic threefold with $J(Y) \in Z_G$. Then, we have $G \subseteq \mathrm{Aut}(J(Y), \Xi)$. Let H denote the image of G under the epimorphism

$$\mathrm{Aut}(J(Y), \Xi) \twoheadrightarrow \mathrm{Aut}(Y)$$

described in [Zhe21, Prop. 1.6]. In particular, we then have $H \simeq \langle G, -1\rangle/\langle -1\rangle$. By Theorem 3.4, we can identify H with a subgroup of $\mathrm{GL}(5, \mathbb{C})$ such that F is H-invariant. Thus, $Y \in M_H$. By the proof of Theorem 4.3, we have $M_H \subseteq Z_G$.

As Z_G is irreducible and, up to conjugation in $\mathrm{GL}(5, \mathbb{C})$, there are only finitely many choices for H as above, the claim follows. □

5. Cyclic cubic threefolds

In this section, we briefly recall the well-known special subvariety generically contained in the locus of intermediate Jacobians that arises as a family of cyclic cubic threefolds, cf. [ACT02] and [CT13].

Definition 5.1. A smooth cubic threefold $Y \subseteq \mathbb{P}^4$ is called a *cyclic cubic threefold* if there is a cyclic triple cover $Y \longrightarrow \mathbb{P}^3$ ramified along a smooth cubic surface.

Observe that the above is equivalent to the existence of an automorphism $\varphi \in \mathrm{Aut}(Y)$ conjugate to $[\mathrm{diag}(\zeta_3, 1, 1, 1, 1)] \in \mathrm{PGL}(5, \mathbb{C})$. Let

$$M^{\mathrm{cyc}} := M_{\langle \mathrm{diag}(\zeta_3,1,1,1,1)\rangle} \subseteq M$$

denote the locus of cyclic cubic threefolds. Let M^{surf} denote the coarse moduli space of cubic surfaces. The morphism

$$M^{\mathrm{surf}} \longrightarrow M^{\mathrm{cyc}} \subseteq M,$$

mapping a cubic surface $S \subseteq \mathbb{P}^3$ to a cyclic triple cover $Y \longrightarrow \mathbb{P}^3$ ramified along S is generically injective, see [Huy23, Rem. 5.22]. Refining this fact, Allcock, Carlson and Toledo have shown that one can embed M^{surf} in $\overline{J(M^{\mathrm{cyc}})} \subseteq A_5$, see [ACT02].

Proposition 5.2. *The closure of $J(M^{\mathrm{cyc}})$ in A_5 is a special subvariety.*

Proof. As discussed in the proof of [ACT02, Lem. 9.2], the closure of $J(M^{\text{cyc}}) \subseteq A_5$ is a totally geodesic subvariety,[2] isomorphic to a four-dimensional complex ball quotient. By [Moo98, Thm. 4.3], a subvariety $Z \subseteq A_g$ is a special subvariety if and only if it is totally geodesic and contains a CM point. Since the Fermat cubic threefold Y_1 is a cyclic cubic threefold and $J(Y_1) \sim E^5$ admits complex multiplication, the claim follows.

Alternatively, apply Theorem 4.3 to $G = \langle \operatorname{diag}(\zeta_3, 1, 1, 1, 1) \rangle \subseteq \mathrm{GL}(5, \mathbb{C})$. □

We have the following consequence of Proposition 5.2:

Corollary 5.3. *Let $G \subseteq \mathrm{GL}(5, \mathbb{C})$ be a finite subgroup containing an element φ conjugate to $\operatorname{diag}(\zeta_3, 1, 1, 1, 1) \in \mathrm{GL}(5, \mathbb{C})$. If $M_G \neq \varnothing$, then the closure of $J(M_G)$ in A_5 is a special subvariety.*

Proof. In this case, we have $Z_G \subseteq Z_{\langle \varphi \rangle} = \overline{J(M^{\text{cyc}})} \subseteq A_5$. By the arguments given in the proof of Proposition 4.7, it follows that the closure of $J(M_G)$ is equal to Z_G and thus a special subvariety. □

We conclude this section by noting that the family M^{cyc} is special among the families M_G as it is the only one containing cubic threefolds with simple, i.e., irreducible up to isogeny, intermediate Jacobian:

Proposition 5.4. *Let Y be a smooth cubic threefold with simple intermediate Jacobian. Then,*

$$\operatorname{Aut}(Y) \subseteq \mathbb{Z}/3\mathbb{Z}.$$

If $\operatorname{Aut}(Y) \simeq \mathbb{Z}/3\mathbb{Z}$ and $J(Y)$ is simple, then Y is a cyclic cubic threefold. Conversely, there are cyclic cubic threefolds with simple intermediate Jacobian.

Proof. Suppose there is a prime number $p \neq 3$ and an automorphism $\varphi \in \operatorname{Aut}(Y)$ of order p. By [GL11], we have $p \in \{2, 5, 11\}$. For $p = 2, 5$, one uses the explicit description in Table 6 to check that $0 < \dim H^{1,2}(Y)^{\varphi} < 5$, which implies that $\operatorname{im}(1 - \varphi^*) \subseteq J(Y)$ is a non-trivial proper abelian subvariety of $J(Y)$, contradicting simplicity of $J(Y)$. For $p = 11$, X. Roulleau has shown in [Rou09] that Y is the Klein cubic threefold Y_5. However, in [Adl81], A. Adler proves that $J(Y_5)$ is isogeneous to E^5, where E is an elliptic curve admitting complex multiplication by $\mathbb{Q}(\zeta_{11})$.

Hence, we either have $\operatorname{Aut}(Y) = \{1\}$ or $|\operatorname{Aut}(Y)| = 3^k$ for some $k \geqslant 1$. In the latter case, Y admits an automorphism φ of order three. As above, one uses the explicit description in Table 6 to verify that $0 < \dim H^{1,2}(Y)^{\varphi} < 5$ if φ is not conjugate to $\operatorname{diag}(\zeta_3, 1, \ldots, 1)$. Hence, simplicity of $J(Y)$ implies that φ is (up to conjugation) given by $\operatorname{diag}(\zeta_3, 1, \ldots, 1)$. In particular, Y is a cyclic cubic threefold. By [Ach13, Prop. 3.6], $\operatorname{End}(J(Y)) \otimes \mathbb{Q}$ is the composition of $\mathbb{Q}(\zeta_3)$ and a totally really field. Therefore, $k = 1$ and $\operatorname{Aut}(Y) \simeq \mathbb{Z}/3\mathbb{Z}$.

On the other hand, it follows from [Ach13, Prop. 3.6] that there are cyclic cubic threefolds with simple intermediate Jacobian. □

[2] See [Moo98] for a discussion of totally geodesic subvarieties of A_g and their relation to special subvarieties.

6. New examples

In this section, we discuss new examples of special subvarieties generically contained in the intermediate Jacobian locus that arise from families of cubic threefolds admitting an action by the alternating groups $\mathrm{Alt}(4)$ and $\mathrm{Alt}(5)$.

Table 1. Irreducible characters of $\mathrm{PSL}(2,11)$ of degree $\leqslant 5$

	$1a$	$2a$	$3a$	$5a$	$5b$	$6a$	$11a$	$11b$
χ_1	1	1	1	1	1	1	1	1
χ_2	5	1	-1	0	0	1	$\frac{-1+\sqrt{-11}}{2}$	$\frac{-1-\sqrt{11}}{2}$
χ_3	5	1	-1	0	0	1	$\frac{-1-\sqrt{-11}}{2}$	$\frac{-1+\sqrt{11}}{2}$

For later use, let us first recall an explicit description of an F-lifting of the action of $\mathrm{PSL}(2,11)$ on the Klein cubic threefold $Y_5 = V(F) \subseteq \mathbb{P}^4$, where

$$F := x_0x_1^2 + x_1x_2^2 + x_2x_3^2 + x_3x_4^2 + x_4x_0^2,$$

see [Adl78]. The group $\mathrm{PSL}(2,11)$ has exactly eight conjugacy classes with representatives of order $1,2,3,5,5,6,11$ and 11. We denote these classes by $1a, 2a, 3a, 5a, 5b, 6a, 11a$ and $11b$. The characters of $\mathrm{PSL}(2,11)$ of degree at most five are given in Table 1. In [Adl78], Adler shows that there is a faithful representation $\rho\colon\ \mathrm{PSL}(2,11) \longrightarrow \mathrm{GL}(5,\mathbb{C})$ with character[3] χ_2 such that the image $G := \mathrm{im}(\rho) \subseteq \mathrm{GL}(5,\mathbb{C})$ is an F-lifting of $\mathrm{Aut}(Y_5)$. Using [GAP22], one computes

$$\dim M_G = \langle S^3\chi_2, \chi_1\rangle - 1 = 0 \quad \text{and} \quad \dim Z_G = \langle S^2(\det(\chi_2)\otimes\chi_2), \chi_1\rangle = 0,$$

confirming the well-known fact that $J(Y_5) \in A_5$ is a special point.

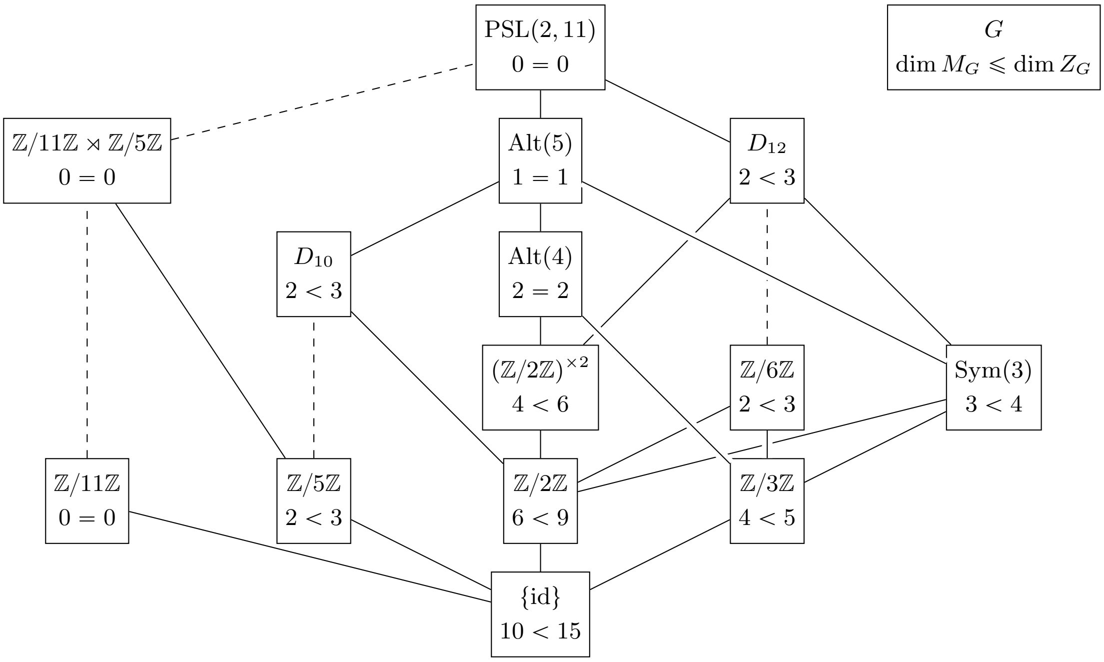
6.1. From now on, we identify $\mathrm{PSL}(2,11)$ with its image in $\mathrm{GL}(5,\mathbb{C})$ via the representation ρ as discussed in the previous section. Up to isomorphism, the subgroups of $\mathrm{PSL}(2,11)$ are $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})^2$, $\mathbb{Z}/6\mathbb{Z}$, $\mathrm{Sym}(3)$, D_{10}, $\mathbb{Z}/11\mathbb{Z}$, $\mathrm{Alt}(4)$, D_{12}, $\mathbb{Z}/11\mathbb{Z} \rtimes \mathbb{Z}/5\mathbb{Z}$, $\mathrm{Alt}(5)$ and $\mathrm{PSL}(2,11)$. Using the character table of $\mathrm{PSL}(2,11)$, one easily verifies the following lemma:

Lemma 6.1. *If G_1 and G_2 are subgroups of $\mathrm{PSL}(2,11) \subseteq \mathrm{GL}(5,\mathbb{C})$ that are isomorphic as abstract groups, then G_1 and G_2 are conjugate in $\mathrm{GL}(5,\mathbb{C})$. In particular, $M_{G_1} = M_{G_2}$.*

With the help of [GAP22], we compute $\dim M_G$ and $\dim Z_G$ for all $G \subseteq \mathrm{PSL}(2,11)$. The results are listed in Figure 2, where vertical lines express the subgroup relation. Pairs of subgroups $G \subsetneq G' \subseteq \mathrm{PSL}(2,11)$ satisfying $M_G = M_{G'}$ are indicated by dashed lines.

[3]There is an automorphism $\alpha \in \mathrm{Aut}(\mathrm{PSL}(2,11))$, explicitly given as conjugation by $\mathrm{diag}(-1,1) \in \mathrm{PGL}(2,11)$, that interchanges the conjugacy classes $11a$ and $11b$ and leaves the remaining classes invariant. Hence, if $\rho_2, \rho_3\colon\ \mathrm{PSL}(2,11) \longrightarrow \mathrm{GL}(5,\mathbb{C})$ are five-dimensional representations with characters $\chi(\rho_i) = \chi_i$ for $i = 2,3$, then their images $\mathrm{im}(\rho_2), \mathrm{im}(\rho_3) \subseteq \mathrm{GL}(5,\mathbb{C})$ are conjugate subgroups. Hence, for our purposes, one could also take a representation with character χ_3.

FIGURE 2. Subgroups of PSL(2, 11)



Up to isomorphism, there are exactly two subgroups which satisfy $\dim M_G > 0$ and ($\star\star$). These are the alternating groups Alt(4) and Alt(5). In the remaining part of this section, we explain the computations and discuss the two resulting special subvarieties.

6.2. Let us now explain the computations going into Figure 2 for the subgroup $\mathrm{Alt}(4) \subseteq \mathrm{PSL}(2, 11)$.

Proposition 6.2. *Let $G \subseteq PSL(2, 11) \subseteq \mathrm{GL}(5, \mathbb{C})$ be a subgroup isomorphic to* Alt(4)*. Then, the closure of $J(M_G)$ in A_5 is a two-dimensional special subvariety.*

Proof. The group Alt(4) has exactly four conjugacy classes $1a, 2a, 3a$ and $3b$ with representatives of order $1, 2, 3$ and 3. The character table is given in Table 2. The restriction of

$$\rho\colon \mathrm{PSL}(2, 11) \longrightarrow \mathrm{GL}(5, \mathbb{C})$$

to Alt(4) has character $\chi := \chi_2 + \chi_3 + \chi_4$. Using [GAP22], we compute

$$\dim U^G = \langle S^3\chi, \chi_{\mathrm{triv}}\rangle = 5 \quad \text{and} \quad \langle S^2(\det(\chi) \otimes \chi), \chi_{\mathrm{triv}}\rangle = 2. \tag{6.1}$$

TABLE 2. Character table of Alt(4)

	$1a$	$2a$	$3a$	$3b$
χ_1	1	1	1	1
χ_2	1	1	$\frac{-1+\sqrt{-3}}{2}$	$\frac{-1-\sqrt{-3}}{2}$
χ_3	1	1	$\frac{-1-\sqrt{-3}}{2}$	$\frac{-1+\sqrt{-3}}{2}$
χ_4	3	-1	0	0

In view of Lemma 3.7, it remains to determine the dimension of the centralizer of the image of $\mathrm{Alt}(4)$ in $\mathrm{GL}(5,\mathbb{C})$. As a subgroup of the symmetric group $\mathrm{Sym}(4)$, $\mathrm{Alt}(4)$ is generated by the permutations (243) and $(12)(34)$. It is easy to verify that

$$(243) \longmapsto \begin{pmatrix} \zeta_3 & 0 & 0 & 0 & 0 \\ 0 & \zeta_3^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (12)(34) \longmapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

gives a representation $\mathrm{Alt}(4) \longrightarrow \mathrm{GL}(5,\mathbb{C})$ with character χ. Let $G \subseteq \mathrm{GL}(5,\mathbb{C})$ denote its image. Using the explicit description, one easily checks that the centralizer $C_{\mathrm{GL}(5,\mathbb{C})}(G)$ is given by

$$C_{\mathrm{GL}(5,\mathbb{C})}(G) = \left\{ \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_3, \lambda_3) \in \mathrm{GL}(5,\mathbb{C}) \mid \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}^\times \right\},$$

and that a general member of M_G is isomorphic to

$$V(x_0^3 + x_1^3 + x_2x_3x_4 + ax_0(x_2^2 + \zeta_3^2 x_3^2 + \zeta_3 x_4^2) + bx_1(x_2^2 + \zeta_3 x_3^2 + \zeta_3^2 x_4^2)) \subseteq \mathbb{P}^4,$$

for some $a, b \in \mathbb{C}$. In particular, we have $\dim C_{\mathrm{GL}(5,\mathbb{C}}(G) = 3$. Combined with (6.1), this yields

$$\dim M_G = \langle S^3\chi, \chi_{\mathrm{triv}} \rangle - \dim C_{\mathrm{GL}(5,\mathbb{C})}(G) = 2 = \langle S^2(\det(\chi) \otimes \chi), \chi_{\mathrm{triv}} \rangle = \dim Z_G.$$

We conclude that $Z_G \subseteq A_5$ is a special subvariety. $\square$

Remark 6.3. By construction, the Klein cubic threefold is contained in the family considered above. Therefore, the family is not contained in the locus of cyclic cubic threefolds. Looking at the explicit equations, we observe that the intersection with the locus of cyclic cubic threefolds is one-dimensional. In fact, it follows from the classification of automorphism of smooth cubic surfaces in [Dol12, Thm. 9.5.8] that the intersection of M^{cyc} and M_G as in the previous proposition is precisely the locus of cubic threefolds that arise as triple cyclic covers of $\mathbb{P}^3$ ramified along a cubic surface admitting a faithful action by $\mathrm{Sym}(4)$.

Proposition 6.4. *The locus of smooth cubic threefolds admitting a faithful action by* $\mathrm{Alt}(4)$ *decomposes into two two-dimensional irreducible components*

$$\widetilde{M}_{\mathrm{Alt}(4)} = M_{G_1} \cup M_{G_2},$$

where we may take $\mathrm{Alt}(4) \simeq G_1 = G$ *as in Proposition 6.2 and* $\mathrm{Alt}(4) \simeq G_2 \subseteq \mathrm{GL}(5,\mathbb{C})$ *to be the image of* $\mathrm{Alt}(4)$ *under the representation*

$$\mathrm{Alt}(4) \subseteq \mathrm{Sym}(4) \longrightarrow \mathrm{GL}(5,\mathbb{C}),$$

given by permutation of the first four coordinates. Both the Fermat cubic threefold and the smooth cubic threefold Y_6 *described in Example 3.2 are contained in the intersection* $M_{G_1} \cap M_{G_2}$*.*

Proof. See Table 2 for the character table of $\mathrm{Alt}(4)$. The characters of faithful actions on $\mathbb{C}^5$ are given by $\chi_{i,j} := \chi_i + \chi_j + \chi_4$ with $1 \leqslant i, j \leqslant 3$. As a subgroup of the symmetric group

Sym(4), Alt(4) is generated by the permutations (243) and (12)(34). It is easy to verify that

$$(243) \longmapsto \begin{pmatrix} \zeta_3^{i-1} & 0 & 0 & 0 & 0 \\ 0 & \zeta_3^{j-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (12)(34) \longmapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

gives a representation $\mathrm{Alt}(4) \longrightarrow \mathrm{GL}(5, \mathbb{C})$ with character $\chi_{i,j}$. Let $G_{i,j} \subseteq \mathrm{GL}(5, \mathbb{C})$ denote its image. We distinguish two cases:

- If $i = j$, then we have

$$H^0(\mathbb{P}^4, \mathcal{O}(3))^{G_{i,i}} = \langle x_0^3,\, x_0^2x_1,\, x_0x_1^2,\, x_1^3,\, x_2x_3x_4 \rangle.$$

  We obtain that $M_{G_{i,i}} \subseteq M$ that $M_{G_{i,i}}$ is independent of i and $M_{G_2} = M_{G_{i,i}}$.
- If $i \neq j$, then we have

$$\begin{aligned} H^0(\mathbb{P}^4, \mathcal{O}(3))^{G_{i,j}} = \langle & x_0^3, x_1^3, x_2x_3x_4, \\ & x_0(x_2^2 + \zeta_3^{i-1}x_3^2 + \zeta_3^{2(i-1)}x_4^2), \\ & x_1(x_2^2 + \zeta_3^{j-1}x_3^2 + \zeta_3^{2(j-1)}x_4^2) \rangle. \end{aligned}$$

  By conjugation with $\mathrm{diag}(1, 1, \zeta_3^a, \zeta_3^b, \zeta_3^c)$, we observe that $M_{G_{i,j}} \subseteq M$ is independent of the choice of $i \neq j$. In particular, we have $M_{G_1} = M_{G_{i,j}}$.

We conclude that $\widetilde{M}_{\mathrm{Alt}(4)}$ decomposes into the two irreducible components M_{G_1} and M_{G_2} described above. □

6.3. At last, we consider the family of smooth cubic threefolds admitting an action by Alt(5). The group Alt(5) has exactly five conjugacy classes $1a, 2a, 3a, 5a$ and $5b$ with representatives of order $1, 2, 3, 5$ and 5. The character table is given in Table 3.

TABLE 3. Character table of Alt(5)

	$1a$	$2a$	$3a$	$5a$	$5b$
χ_1	1	1	1	1	1
χ_2	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
χ_3	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

We have the following description of $\widetilde{M}_{\mathrm{Alt}(5)}$, cf. Proposition 6.4:

Proposition 6.5. *The locus of of smooth cubic threefolds admitting an action by* Alt(5) *has two irreducible components*

$$\widetilde{M}_{\mathrm{Alt}(5)} = M_{H_1} \cup M_{H_2},$$

where $\mathrm{Alt}(5) \simeq H_1 \subseteq \mathrm{PSL}(2,11)$ *and* H_2 *is the image of* $\mathrm{Alt}(5)$ *under the representation*

$$\mathrm{Alt}(5) \subseteq \mathrm{Sym}(5) \longrightarrow \mathrm{GL}(5,\mathbb{C}),$$

given by permutation of coordinates. Moreover, we have $M_{H_1} \cap M_{H_2} = \{Y_1, Y_6\}$.

Proof. We may assume $G_i \subseteq H_i$ for $i = 1, 2$.

Recall that, by the classification in [GL11], we have $\widetilde{M}_{\mathbb{Z}/5\mathbb{Z}} = M_{\langle \mathrm{diag}(1,\zeta_5,\zeta_5^2,\zeta_5^3,\zeta_5^4)\rangle}$. In particular, if $\varphi \in \mathrm{GL}(5,\mathbb{C})$ is the F-lifting of an automorphism of order five of a smooth cubic threefold $Y = V(F) \subseteq \mathbb{P}^4$, then $\mathrm{Tr}(\varphi) = 0$. Hence, if $H \subseteq \mathrm{GL}(5,\mathbb{C})$ is the F-lifting of a group of automorphisms isomorphic to $\mathrm{Alt}(5)$, then the character of the corresponding representation is either given by χ_5 or $\chi_1 + \chi_4$. The character χ_5 corresponds to H_1 and $\chi_1 + \chi_4$ corresponds to H_2.

If $\mathrm{Aut}(Y)$ contains subgroups isomorphic to $\mathrm{Alt}(5)$ that are not conjugate to each other, then $Y \in \{Y_1, Y_6\}$ by the classification of automorphism groups of cubic threefolds in [WY20]. □

Furthermore, we can show that M_{H_1} is exactly the intersection of M_{G_1} and the locus of smooth cubic threefolds admitting an automorphism of order five.

Lemma 6.6. *For* $i = 1, 2$*, we have*

$$M_{G_i} \cap \widetilde{M}_{\mathbb{Z}/5\mathbb{Z}} = M_{H_i}.$$

Proof. By considering the groups occuring in the classification of automorphism groups of cubic threefolds given in [WY20], we observe that a cubic threefold Y admits an action by $\mathrm{Alt}(5)$ if and only if it admits an action by $\mathrm{Alt}(4)$ and $\mathbb{Z}/5\mathbb{Z}$, i.e., we have

$$\widetilde{M}_{\mathrm{Alt}(5)} = \widetilde{M}_{\mathrm{Alt}(4)} \cap \widetilde{M}_{\mathbb{Z}/5\mathbb{Z}}.$$

By construction, we have $M_{H_i} \subseteq M_{G_i} \cap \widetilde{M}_{\mathbb{Z}/5\mathbb{Z}}$. Towards a contradiction, suppose that Y is a smooth cubic threefold contained in $M_{G_i} \cap \widetilde{M}_{\mathbb{Z}/5\mathbb{Z}}$ but not in M_{H_i}. Then, $\mathrm{Aut}(Y)$ contains a subgroup isomorphic to $\mathrm{Alt}(5)$ and two subgroups isomorphic to $\mathrm{Alt}(4)$ that are not conjugate to each other. By going through the list of automorphism groups of smooth cubic threefolds given in [WY20], this already implies $Y \in \{Y_1, Y_6\}$. But both Y_1 and Y_6 are contained in $M_{H_i} \cap M_{H_j}$. As desired, this contradicts our assumption on Y. □

Proposition 6.7. *The closure of* $J(M_{H_1})$ *in* A_5 *is a one-dimensional special subvariety. Moreover,* M_{H_1} *contains the Klein cubic threefold and is thus not contained in the locus of cyclic cubic threefolds.*

Proof. Since $Z_{H_1} \subseteq Z_{G_1} = \overline{J(M_{G_1})}$, Proposition 4.7 implies that $\overline{J(M_{H_1})}$ is equal to Z_{H_1}. In particular, $H_1 \subseteq \mathrm{GL}(5,\mathbb{C})$ satifies ($\star\star$) and $\overline{J(M_{H_1})}$ is a special subvariety of A_5.

It remains to show that M_{H_1} is one-dimensional. The character of the action of H_1 on $\mathbb{C}^5$ equals χ_5. Using [GAP22], we compute

$$\dim U^{H_1} = \langle S^3\chi_5, \chi_{\mathrm{triv}}\rangle = 2 \quad \text{and} \quad \langle S^2(\det(\chi_5) \otimes \chi_5), \chi_{\mathrm{triv}}\rangle = 1.$$

As the character χ_5 is irreducible, the centralizer of $H_1 \subseteq \mathrm{GL}(5,\mathbb{C})$ is one-dimensional. By Lemma 3.7, we conclude that M_{H_1} is one-dimensional. □

Remark 6.8. The intermediate Jacobians of members of the family M_{H_1} are all isogeneous to the self-product of an elliptic curve:[4] Let W_5 denote the five-dimensional irreducible $\mathbb{Q}$-valued representation of $\mathrm{Alt}(5)$ with character χ_5. The endomorphism algebra of W_5 is isomorphic to $\mathbb{Q}$. By [BL04, Thm. 13.6.2], this implies that for each cubic threefold $Y \in M_{H_1}$, there is an elliptic curve E such that the self-product E^5 is isogeneous to $J(Y)$. Note that this yields an alternative proof of Proposition 6.7. See Remark 4.6 for a description of E for $Y_1, Y_5, Y_6 \in M_{H_1}$.

In particular, the set of intermediate Jacobians with maximal Picard number in the family $J(M_{H_1})$ is analytically dense, cf. [Bea14, Prop. 3 and Prop. 4].

7. Excluding further examples

It turns out that the examples discussed in the two previous sections are the only examples of positive-dimensional special subvarieties generically contained in the intermediate Jacobian locus that arise from the criterion given in Theorem 1.3.

Theorem 7.1. *Let $G \subseteq \mathrm{GL}(5, \mathbb{C})$ be a finite subgroup. If $\dim M_G > 0$ and $Z_G \subseteq \overline{J(M)}$,[5] then either $M_G \subseteq M^{\mathrm{cyc}}$, or G is isomorphic to $\mathrm{Aut}(4)$ or $\mathrm{Aut}(5)$ and M_G contains the Klein cubic threefold. If $\dim M_G = 0 = \dim Z_G$, then $M_G = \{Y_i\}$ for some Y_i as in Example 3.2.*

Remark 7.2. In the latter case, M_G is one of the families described in the previous section, see Proposition 6.4 and Proposition 6.5.

Remark 7.3. The cubic threefold Y_4 with $\mathrm{Aut}(Y_4) \simeq \mathbb{Z}/16\mathbb{Z}$ and the Klein cubic threefold Y_5 are not cyclic cubic threefolds. On the other hand, the cubic threefolds Y_1, Y_2, Y_3 and Y_6 in Example 3.2 are cyclic cubic threefolds.

Proof of Theorem 7.1. Recall from the proof of Theorem 4.3 that $\overline{J(M_G)} \subseteq Z_G$ with equality if and only ($\star\star$) holds. By Proposition 4.7, the condition that Z_G is contained in $\overline{J(M)}$ already implies that G satisfies ($\star\star$).

By [WY20], the groups admitting a faithful action on a smooth cubic threefold are precisely the subgroups of the six groups listed in Theorem 3.1. The strategy of this proof is to use [GAP22] to list all five-dimensional representations of these groups and to check when these satisfy ($\star\star$).

First, let us fix some notation. Let H be a subgroup of one of the six groups described in Theorem 3.1. Using [GAP22], we obtain a complete list of characters that correpond to representations $H \longrightarrow \mathrm{GL}(5, \mathbb{C})$ for which the composition $H \longrightarrow \mathrm{GL}(5, \mathbb{C}) \longrightarrow \mathrm{PGL}(5, \mathbb{C})$ is injective. In the following, let χ be such a character, $\rho\colon H \longrightarrow \mathrm{GL}(5, \mathbb{C})$ a representation with character χ, and let $G \subseteq \mathrm{GL}(5, \mathbb{C})$ denote its image. Since, up to conjugation in $\mathrm{GL}(5, \mathbb{C})$, the representation ρ is determined by χ, the subset M_G only depends on the character χ.

[4]Thanks to B. van Geemen for pointing this out. In [GY16], van Geemen and Yamauchi show that if Y admits an automorphism of order five, i.e., $Y \in \widetilde{M}_{\mathbb{Z}/5\mathbb{Z}}$, then $J(Y)$ is isogenous to $E \times B^2$, where E is an elliptic curve and B is an abelian surface, and give an explicit description of B for general $Y \in \widetilde{M}_{\mathbb{Z}/5\mathbb{Z}}$.

[5]Here, we identify $G \subseteq \mathrm{GL}(5, \mathbb{C})$ with a subgroup of $\mathrm{Sp}(10, \mathbb{Z})$ via the inclusion $G \subseteq \mathrm{Aut}(Y) \xrightarrow{\sim} \mathrm{Aut}(J(Y)) \subseteq \mathrm{Sp}(H^3(Y, \mathbb{Z})) \simeq \mathrm{Sp}(10, \mathbb{Z})$ for some $Y \in M_G$. As in the proof of Theorem 4.3, we note that $Z_G \subseteq A_5$ only depends on $G \subseteq \mathrm{GL}(5, \mathbb{C})$ and not on the choice of Y or the isomorphism $H^3(Y, \mathbb{Z}) \simeq \mathbb{Z}^{10}$.

One can compute the dimension of the centralizer of G in $\mathrm{GL}(5,\mathbb{C})$ using the following elementary lemma:

Lemma 7.4. *If $\chi = \sum n_i \chi_i$ is a decomposition of χ into irreducible characters, then*

$$\dim C_{\mathrm{GL}(5,\mathbb{C})}(G) = \sum_i n_i^2.$$

If the group G is abelian, then every irreducible character is of degree one. Moreover, in [WY20, App. B], Wei and Yu give a list of possible F-liftings of abelian groups acting faithfully on smooth cubic threefolds. Hence, we can use [GAP22] to compute $\dim M_G$ and $\dim Z_G$ for all abelian groups acting on smooth cubic threefolds. The results are listed in Table 6.

In the case that we have $\dim M_G = \dim Z_G$ in Table 6, the subgroup $G \subseteq \mathrm{GL}(5,\mathbb{C})$ contains an element conjugate to $\mathrm{diag}(\zeta_3, 1, 1, 1, 1)$ except for the families No. 43, 48, 54, 58, 59, 61, 63, 66 and 67. By determining the corresponding sets U^G of G-invariant homogeneous cubic polynomials, one can show that $M_G \subseteq M^{\mathrm{cyc}}$ holds also in these cases:

Lemma 7.5. *The families No.* $43, 48, 54, 58, 59, 61, 63, 66, 67$ *in Table 6 are contained in M^{cyc}.*

Proof. It is easy to verify that in these cases, every G-invariant homogeneous cubic polynomial is conjugate to a polynomial of the form $x_0^3 + F(x_1, \ldots, x_4)$.

For example, consider Family No. 43. Then, we have

$$G := \langle \mathrm{diag}(1, \zeta_3^1, 1, 1, \zeta_3^1), \mathrm{diag}(1, 1, \zeta_3^1, \zeta_3^2, \zeta_3^2) \rangle \subseteq \mathrm{GL}(5,\mathbb{C}).$$

One easily checks that we have

$$H^0(\mathbb{P}^4, \mathcal{O}(3))^G = \langle x_0^3, x_1^3, x_2^3, x_3^3, x_4^3, x_0 x_2 x_3 \rangle \subseteq H^0(\mathbb{P}^4, \mathcal{O}(3))$$

and thus $M_G \subseteq M^{\mathrm{cyc}}$. The remaining cases are left to the reader. $\square$

This finishes the proof in the case of abelian groups.

In order to carry out the computations in the non-abelian case, let us recall a few criteria for determining whether a subgroup $G \subseteq \mathrm{GL}(5,\mathbb{C})$ satisfies $\dim M_G \neq \varnothing$.

Lemma 7.6. *Let $G \subseteq \mathrm{GL}(5,\mathbb{C})$ be a subgroup for which the projection $G \longrightarrow \mathrm{PGL}(5,\mathbb{C})$ is injective and $M_G \neq \varnothing$. Let $\varphi \in G$ be an element of order n.*

- *If $n = 2$, then φ is conjugate to* $\mathrm{diag}(-1, 1, 1, 1, 1)$ *or* $\mathrm{diag}(-1, -1, 1, 1, 1)$.
- *If $n = 4$, then φ is not conjugate to* $\mathrm{diag}(\zeta_4, 1, 1, 1, 1)$ *and* $\mathrm{diag}(-\zeta_4, 1, 1, 1, 1)$.
- *If $n = 5$, then φ is conjugate to* $\mathrm{diag}(1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4)$.

Proof. Follows from [WY20, Tab. 2]. $\square$

For every non-abelian group H occuring in the classification of automorphism groups of smooth cubic threefolds, we use [GAP22] to create a list of all characters of H of degree five. Then, we apply Lemma 7.4, Lemma 7.6 and the dimension formulas given in Proposition 2.4 and Lemma 3.7 to exclude groups H for which ($\star\star$) can not be satisfied. Furthermore, we remove the characters χ for which the images of the corresponding representations contain an element conjugate to a scalar multiple of $\mathrm{diag}(\zeta_3, 1, 1, 1, 1)$ from our list, because the associated families of cubic threefolds are contained in the locus of cyclic cubic threefolds. The groups that survive this process are $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z} \times \mathrm{Sym}(3)$, $(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \rtimes \mathbb{Z}/3\mathbb{Z}$, $((\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \rtimes \mathbb{Z}/3\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$, $\mathrm{Alt}(4)$ and $\mathrm{Alt}(5)$.

In the remaining part of this proof, we show that only the alternating groups $\mathrm{Alt}(4)$ and $\mathrm{Alt}(5)$ admit representations for which $\dim M_G > 0$, $M_G \not\subseteq M^{\mathrm{cyc}}$ and $\dim M_G = \dim Z_G$.

Lemma 7.7. *If $G \subseteq \mathrm{GL}(5, \mathbb{C})$ is isomorphic to $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$, then $\dim M_G \neq \dim Z_G$.*

Proof. The group $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$ has exactly six conjugacy classes $1a, 4a, 2a, 3a, 4b$ and $6a$ with representatives of order $1, 4, 2, 3, 4$ and 6. The character table is given in Table 4.

Table 4. Character table of $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$

	$1a$	$4a$	$2a$	$3a$	$4b$	$6a$
χ_1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	1
χ_3	1	$-i$	-1	1	$-i$	-1
χ_4	1	i	-1	1	i	-1
χ_5	2	0	-2	-1	0	1
χ_6	2	0	2	-1	0	-1

Going through all characters of degree five and applying the process described above, the characters $2\chi_1 + \chi_3 + \chi_6$ and $2\chi_1 + \chi_4 + \chi_6$ are the only possible candidates for which $(\star\star)$ could hold. Let us consider the case $\chi := 2\chi_1 + \chi_4 + \chi_6$. One checks that

$$(0,1) \longmapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \zeta_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (1,0) \longmapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \zeta_3 & 0 \\ 0 & 0 & 0 & 0 & \zeta_3^2 \end{pmatrix}$$

gives rise to a representation $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathrm{GL}(5, \mathbb{C})$ with character χ. Let $G \subseteq \mathrm{GL}(5, \mathbb{C})$ denote its image. The space of G-invariant homogeneous cubic polynomials is of dimension

$$\dim H^0(\mathbb{P}^4, \mathcal{O}(3))^G = \langle S^3\chi, \chi_{\mathrm{triv}} \rangle = 7$$

and spanned by

$$H^0(\mathbb{P}^4, \mathcal{O}(3))^G = \langle x_0^3, x_0^2x_1, x_0x_1^2, x_1^3, x_0x_3x_4, x_1x_3x_4, x_3^3 + x_4^3 \rangle \subseteq H^0(\mathbb{P}^4, \mathcal{O}(3)).$$

As the variable x_2 does not occur among these polynomials, we conclude that $M_G = \varnothing$.

The character $2\chi_1 + \chi_3 + \chi_6$ is excluded using similar arguments. $\square$

Lemma 7.8. *If $G \subseteq \mathrm{GL}(5, \mathbb{C})$ is isomorphic to one of*

$$\mathbb{Z}/3\mathbb{Z} \times \mathrm{Sym}(3), (\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \rtimes \mathbb{Z}/3\mathbb{Z}, ((\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \rtimes \mathbb{Z}/3\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z},$$

and $\dim M_G = \dim Z_G$, then $M_G \subseteq M^{\mathrm{cyc}}$.

Proof. Let us consider the group $G \simeq \mathbb{Z}/3\mathbb{Z} \times \mathrm{Sym}(3)$. The remaining cases are left to the reader. The group $\mathbb{Z}/3\mathbb{Z} \times \mathrm{Sym}(3)$ has exactly nine conjugacy classes with representatives $1a, 2a, 3a, 3b, 6a, 3c, 3d, 6b$ and $3e$ of order $1, 2, 3, 3, 6, 3, 3, 6$ and 3. The character table is given in Table 5.

Table 5. Character table of $\mathbb{Z}/3\mathbb{Z} \times \mathrm{Sym}(3)$

	$1a$	$2a$	$3a$	$3b$	$6a$	$3c$	$3d$	$6b$	$3e$
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	1	1	-1	1
χ_3	1	-1	ζ_3^2	1	$-\zeta_3^2$	ζ_3	ζ_3^2	$-\zeta_3$	ζ_3
χ_4	1	-1	ζ_3	1	$-\zeta_3$	ζ_3^2	ζ_3	$-\zeta_3^2$	ζ_3^2
χ_5	1	1	ζ_3^2	1	ζ_3^2	ζ_3	ζ_3^2	ζ_3	ζ_3
χ_6	1	1	ζ_3	1	ζ_3	ζ_3^2	ζ_3	ζ_3^2	ζ_3^2
χ_7	2	0	2	-1	0	2	-1	0	-1
χ_8	2	0	$2\zeta_3$	-1	0	$2\zeta_3^2$	$-\zeta_3$	0	$-\zeta_3^2$
χ_9	2	0	$2\zeta_3^2$	-1	0	$2\zeta_3$	$-\zeta_3^2$	0	$-\zeta_3$

Going through the list of characters of degree five and applying the process described above, we see that the only characters we have to check are those of the form $\chi_i + \chi_j + \chi_k$, where

$$(i, j, k) \in \{(1, 7, 9), (5, 7, 9), (1, 7, 8), (6, 7, 8), (5, 8, 9), (6, 8, 9)\}.$$

A straightforward computation as in Lemma 7.5 shows that for these characters, we have $M_G \subseteq M^{\mathrm{cyc}}$. In fact, M_G agrees with the one-dimensional family No. 32 in Table 6. □

We conclude that if $\dim M_G > 0$ and ($\star\star$) are satified, then either $G \simeq \mathrm{Alt}(4), \mathrm{Alt}(5)$ and M_G contains the Klein cubic threefold, or M_G is contained in the locus of cyclic cubic threefolds.

Using the classification of automorphism groups in [WY20], in particular the unicity results in [WY20, Tab. 2], a straightforward computation using [GAP22] shows that $\dim M_G = 0 = \dim Z_G$ already implies that $M_G = \{Y_i\}$ for one of the maximally symmetric cubic threefolds listed in Example 3.2.

This finishes the proof of Theorem 7.1. □

8. Summary of the results

One can summarize the results of the previous sections as follows:

Theorem 8.1. *Let $G \subseteq \mathrm{GL}(5,\mathbb{C})$ be a finite subgroup such that there is no $G \subsetneq G' \subseteq \mathrm{GL}(5,\mathbb{C})$ satisfying $M_G = M_{G'}$. Then ($\star\star$) is satisfied if and only if one of the following holds:*

(i) $\dim M_G = 0$ *and* $M_G = \{Y_i\}$ *for some* $i \in \{1,\ldots,6\}$ *and* Y_i *as in Example 3.2.*

(ii) $M_G \subseteq M^{\mathrm{cyc}}$*; or*

(iii) G *is a subgroup of* $\mathrm{GL}(5,\mathbb{C})$ *isomorphic to* $\mathrm{Alt}(4)$ *(resp.* $\mathrm{Alt}(5)$*) and* M_G *is the unique irreducible component of* $\widetilde{M}_G$ *that contains the Klein cubic threefold* Y_5*.*

In particular, in these cases, the closure of $J(M_G)$ *in* A_5 *is a special subvariety.*

We can also determine the intersection of M^{cyc} with the loci described in (iii):

Proposition 8.2. *The intersection of* M^{cyc} *and the irreducible component of* $\widetilde{M}_{\mathrm{Alt}(4)}$ *containing the Klein cubic threefold is the one-dimensional locus of cyclic cubic threefolds that arise as cyclic triple covers of* $\mathbb{P}^3$ *ramified along cubic surfaces admitting a faithful action by* $\mathrm{Sym}(4)$*.*

The intersection of M^{cyc} *and the irreducible component of* $\widetilde{M}_{\mathrm{Alt}(5)}$ *containing the Klein cubic threefold is given by* $\{Y_1, Y_6\}$*.*

Proof. The first claim was proven in Remark 6.3. For the second claim, see Proposition 6.5. □

The situation is depicted in Figure 3. Note that M^{cyc} is four-dimensional, the family of cubic threefolds with an $\mathrm{Alt}(4)$-action that contains the Klein cubic threefold is two-dimensional and contains a one-dimensional family of cubic threefolds admitting an $\mathrm{Alt}(5)$-action. See Example 3.2 for an explicit description of the cubic threefolds $Y_1,\ldots,Y_6$.

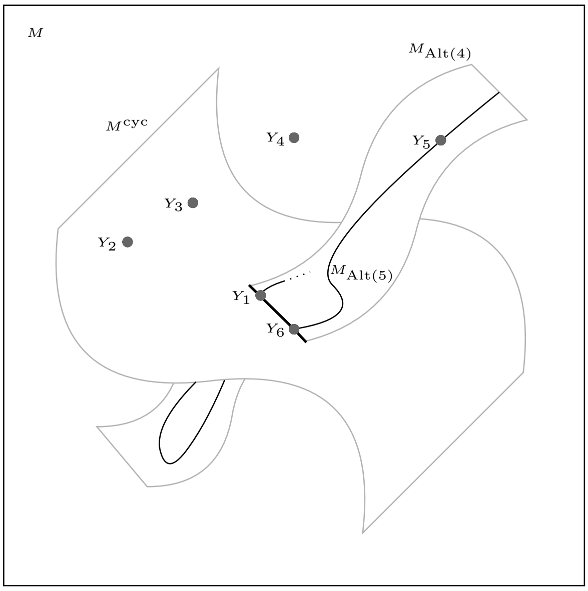


Figure 3. Families of cubic threefolds that give rise to special subvarieties in the intermediate Jacobian locus

9. Appendix

The representations of faithful abelian group actions on smooth cubic threefolds were listed by Wei and Yu in [WY20, Tab. 2]). In Table 6, we complement the list by the results of the dimension computations that were used in the proof of Theorem 8.1.

Table 6: Abelian groups acting on smooth cubic threefolds (cf. [WY20])

No.	$H \subseteq \mathrm{Aut}(X)$	generator(s) of an F-lifting G of H	$\dim M_G$	$\dim Z_G$	$=$
1	$\mathbb{Z}/2\mathbb{Z}$	$\mathrm{diag}(-1,1,1,1,1)$	7	11	No
2	$\mathbb{Z}/2\mathbb{Z}$	$\mathrm{diag}(-1,-1,1,1,1)$	6	9	No
3	$\mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(1,1,1,\zeta_3^1,1)$	4	4	Yes
4	$\mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(1,1,1,\zeta_3^1,\zeta_3^1)$	1	3	No
5	$\mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(1,1,1,\zeta_3^1,\zeta_3^2)$	4	7	No
6	$\mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(1,1,\zeta_3^1,\zeta_3^1,\zeta_3^2)$	4	5	No
7	$\mathbb{Z}/4\mathbb{Z}$	$\mathrm{diag}(\zeta_4^1,-1,1,1,1)$	3	4	No
8	$\mathbb{Z}/4\mathbb{Z}$	$\mathrm{diag}(\zeta_4^1,-1,1,-1,1)$	3	5	No
9	$\mathbb{Z}/4\mathbb{Z}$	$\mathrm{diag}(\zeta_4^1,-1,1,\zeta_4^3,1)$	3	5	No
10	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathrm{diag}(-1,1,1,1,1)$, $\mathrm{diag}(1,1,-1,1,1)$	5	8	No
11	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathrm{diag}(-1,-1,1,1,1)$, $\mathrm{diag}(1,-1,-1,1,1)$	4	6	No
12	$\mathbb{Z}/5\mathbb{Z}$	$\mathrm{diag}(1,\zeta_5^1,\zeta_5^2,\zeta_5^3,\zeta_5^4)$	2	3	No
13	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(-1,1,-1,1,1)$, $\mathrm{diag}(1,1,1,\zeta_3^1,1)$	2	2	Yes
14	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(-1,1,-1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,\zeta_3^1,1)$	1	2	No
15	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(-1,1,-1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1)$	1	2	No
16	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(-1,1,-1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,\zeta_3^1,\zeta_3^2)$	2	3	No
17	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(-1,1,1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,1,1)$	3	3	Yes
18	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(-1,1,1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,1,\zeta_3^1)$	1	3	No
19	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(-1,1,1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,1,\zeta_3^2)$	3	5	No

20	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(-1,1,1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1)$	1	2	No
21	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(-1,1,1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,\zeta_3^1,\zeta_3^2)$	2	3	No
22	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(-1,1,1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,\zeta_3^2,\zeta_3^2)$	2	3	No
23	$\mathbb{Z}/8\mathbb{Z}$	$\mathrm{diag}(\zeta_8^1,\zeta_8^6,\zeta_8^4,1,1)$	1	2	No
24	$\mathbb{Z}/8\mathbb{Z}$	$\mathrm{diag}(\zeta_8^1,\zeta_8^6,\zeta_8^4,1,\zeta_8^2)$	1	2	No
25	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathrm{diag}(\zeta_4^1,-1,1,1,1)$, $\mathrm{diag}(1,1,1,-1,1)$	2	3	No
26	$\mathbb{Z}/9\mathbb{Z}$	$\mathrm{diag}(\zeta_9^1,\zeta_9^7,\zeta_9^4,1,1)$	0	0	Yes
27	$\mathbb{Z}/9\mathbb{Z}$	$\mathrm{diag}(\zeta_9^1,\zeta_9^7,\zeta_9^4,1,\zeta_9^3)$	0	1	No
28	$\mathbb{Z}/9\mathbb{Z}$	$\mathrm{diag}(\zeta_9^1,\zeta_9^7,\zeta_9^4,1,\zeta_9^6)$	0	1	No
29	$\mathbb{Z}/9\mathbb{Z}$	$\mathrm{diag}(\zeta_9^1,\zeta_9^7,\zeta_9^4,\zeta_9^3,\zeta_9^3)$	0	0	Yes
30	$\mathbb{Z}/9\mathbb{Z}$	$\mathrm{diag}(\zeta_9^1,\zeta_9^7,\zeta_9^4,\zeta_9^3,\zeta_9^6)$	0	1	No
31	$\mathbb{Z}/9\mathbb{Z}$	$\mathrm{diag}(\zeta_9^1,\zeta_9^7,\zeta_9^4,\zeta_9^6,\zeta_9^6)$	0	0	Yes
32	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1,\zeta_3^1,1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,1,1)$	1	1	Yes
33	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1,\zeta_3^1,1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,1,\zeta_3^1)$	0	0	Yes
34	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1,\zeta_3^1,1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,1,\zeta_3^2)$	2	2	Yes
35	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1,\zeta_3^1,1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1)$	1	1	Yes
36	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1,\zeta_3^1,1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,\zeta_3^1,\zeta_3^2)$	2	2	Yes
37	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1,\zeta_3^1,1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,\zeta_3^2,\zeta_3^2)$	2	2	Yes
38	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1,\zeta_3^1,1,1,\zeta_3^1)$, $\mathrm{diag}(1,1,\zeta_3^1,1,1)$	0	0	Yes
39	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1,\zeta_3^1,1,1,\zeta_3^1)$, $\mathrm{diag}(1,1,\zeta_3^1,1,\zeta_3^1)$	0	1	No
40	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1,\zeta_3^1,1,1,\zeta_3^1)$, $\mathrm{diag}(1,1,\zeta_3^1,1,\zeta_3^2)$	0	1	No
41	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1,\zeta_3^1,1,1,\zeta_3^1)$, $\mathrm{diag}(1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1)$	0	1	No

42	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1, \zeta_3^1, 1, 1, \zeta_3^1)$, $\mathrm{diag}(1, 1, \zeta_3^1, \zeta_3^1, \zeta_3^2)$	0	1	No
43	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1, \zeta_3^1, 1, 1, \zeta_3^1)$, $\mathrm{diag}(1, 1, \zeta_3^1, \zeta_3^2, \zeta_3^2)$	1	1	Yes
44	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1, \zeta_3^1, 1, 1, \zeta_3^2)$, $\mathrm{diag}(1, 1, \zeta_3^1, 1, 1)$	2	2	Yes
45	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1, \zeta_3^1, 1, 1, \zeta_3^2)$, $\mathrm{diag}(1, 1, \zeta_3^1, 1, \zeta_3^1)$	0	1	No
46	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1, \zeta_3^1, 1, 1, \zeta_3^2)$, $\mathrm{diag}(1, 1, \zeta_3^1, 1, \zeta_3^2)$	1	3	No
47	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1, \zeta_3^1, 1, 1, \zeta_3^2)$, $\mathrm{diag}(1, 1, \zeta_3^1, \zeta_3^1, \zeta_3^1)$	0	1	No
48	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1, \zeta_3^1, 1, 1, \zeta_3^2)$, $\mathrm{diag}(1, 1, \zeta_3^1, \zeta_3^1, \zeta_3^2)$	2	2	Yes
49	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1, \zeta_3^1, 1, 1, \zeta_3^2)$, $\mathrm{diag}(1, 1, \zeta_3^1, \zeta_3^2, \zeta_3^2)$	2	3	No
50	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1, \zeta_3^1, 1, \zeta_3^1, \zeta_3^1)$, $\mathrm{diag}(1, 1, \zeta_3^1, 1, 1)$	1	1	Yes
51	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1, \zeta_3^1, 1, \zeta_3^1, \zeta_3^1)$, $\mathrm{diag}(1, 1, \zeta_3^1, 1, \zeta_3^1)$	0	1	No
52	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1, \zeta_3^1, 1, \zeta_3^1, \zeta_3^1)$, $\mathrm{diag}(1, 1, \zeta_3^1, 1, \zeta_3^2)$	0	1	No
53	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1, \zeta_3^1, 1, \zeta_3^1, \zeta_3^1)$, $\mathrm{diag}(1, 1, \zeta_3^1, \zeta_3^1, \zeta_3^1)$	0	1	No
54	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1, \zeta_3^1, 1, \zeta_3^1, \zeta_3^1)$, $\mathrm{diag}(1, 1, \zeta_3^1, \zeta_3^1, \zeta_3^2)$	1	1	Yes
55	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1, \zeta_3^1, 1, \zeta_3^1, \zeta_3^1)$, $\mathrm{diag}(1, 1, \zeta_3^1, \zeta_3^2, \zeta_3^2)$	0	1	No
56	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1, \zeta_3^1, 1, \zeta_3^1, \zeta_3^2)$, $\mathrm{diag}(1, 1, \zeta_3^1, 1, 1)$	2	2	Yes
57	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1, \zeta_3^1, 1, \zeta_3^1, \zeta_3^2)$, $\mathrm{diag}(1, 1, \zeta_3^1, 1, \zeta_3^1)$	0	1	No
58	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1, \zeta_3^1, 1, \zeta_3^1, \zeta_3^2)$, $\mathrm{diag}(1, 1, \zeta_3^1, 1, \zeta_3^2)$	2	2	Yes
59	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1, \zeta_3^1, 1, \zeta_3^1, \zeta_3^2)$, $\mathrm{diag}(1, 1, \zeta_3^1, \zeta_3^1, \zeta_3^1)$	1	1	Yes
60	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1, \zeta_3^1, 1, \zeta_3^1, \zeta_3^2)$, $\mathrm{diag}(1, 1, \zeta_3^1, \zeta_3^1, \zeta_3^2)$	2	3	No

61	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1,\zeta_3^1,1,\zeta_3^1,\zeta_3^2)$, $\mathrm{diag}(1,1,\zeta_3^1,\zeta_3^2,\zeta_3^2)$	1	1	Yes
62	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1,\zeta_3^1,1,\zeta_3^2,\zeta_3^2)$, $\mathrm{diag}(1,1,\zeta_3^1,1,1)$	2	2	Yes
63	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1,\zeta_3^1,1,\zeta_3^2,\zeta_3^2)$, $\mathrm{diag}(1,1,\zeta_3^1,1,\zeta_3^1)$	1	1	Yes
64	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1,\zeta_3^1,1,\zeta_3^2,\zeta_3^2)$, $\mathrm{diag}(1,1,\zeta_3^1,1,\zeta_3^2)$	2	3	No
65	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1,\zeta_3^1,1,\zeta_3^2,\zeta_3^2)$, $\mathrm{diag}(1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1)$	0	1	No
66	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1,\zeta_3^1,1,\zeta_3^2,\zeta_3^2)$, $\mathrm{diag}(1,1,\zeta_3^1,\zeta_3^1,\zeta_3^2)$	1	1	Yes
67	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(1,\zeta_3^1,1,\zeta_3^2,\zeta_3^2)$, $\mathrm{diag}(1,1,\zeta_3^1,\zeta_3^2,\zeta_3^2)$	2	2	Yes
68	$\mathbb{Z}/11\mathbb{Z}$	$\mathrm{diag}(\zeta_{11}^1,\zeta_{11}^9,\zeta_{11}^4,\zeta_{11}^3,\zeta_{11}^5)$	0	0	Yes
69	$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(\zeta_4^1,-1,1,1,1)$, $\mathrm{diag}(1,1,1,\zeta_3^1,1)$	1	1	Yes
70	$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(\zeta_4^1,-1,1,1,1)$, $\mathrm{diag}(1,1,1,\zeta_3^1,\zeta_3^1)$	0	0	Yes
71	$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(\zeta_4^1,-1,1,1,1)$, $\mathrm{diag}(1,1,1,\zeta_3^1,\zeta_3^2)$	1	2	No
72	$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(\zeta_4^1,-1,1,1,-1)$, $\mathrm{diag}(1,1,1,\zeta_3^1,\zeta_3^1)$	0	1	No
73	$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(\zeta_4^1,-1,1,1,\zeta_4^3)$, $\mathrm{diag}(1,1,1,\zeta_3^1,1)$	1	1	Yes
74	$(\mathbb{Z}/2\mathbb{Z})^2\times\mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(-1,1,1,1,1)$, $\mathrm{diag}(1,1,-1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,\zeta_3^1,1)$	1	2	No
75	$(\mathbb{Z}/2\mathbb{Z})^2\times\mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(-1,1,1,1,1)$, $\mathrm{diag}(1,1,-1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1)$	1	2	No
76	$(\mathbb{Z}/2\mathbb{Z})^2\times\mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(-1,1,1,1,1)$, $\mathrm{diag}(1,1,-1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,\zeta_3^1,\zeta_3^2)$	1	2	No
77	$(\mathbb{Z}/2\mathbb{Z})^2\times\mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(-1,1,1,1,1)$, $\mathrm{diag}(1,1,-1,1,1)$, $\mathrm{diag}(1,1,1,\zeta_3^1,1)$	2	2	Yes

78	$(\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(-1,1,1,1,-1)$, $\mathrm{diag}(1,1,-1,1,-1)$, $\mathrm{diag}(1,1,1,\zeta_3^1,1)$	1	1	Yes
79	$\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(\zeta_5^1,\zeta_5^3,\zeta_5^4,\zeta_5^2,1)$, $\mathrm{diag}(1,1,1,1,\zeta_3^1)$	0	0	Yes
80	$\mathbb{Z}/16\mathbb{Z}$	$\mathrm{diag}(\zeta_{16}^1,\zeta_{16}^{14},\zeta_{16}^4,\zeta_{16}^8,1)$	0	0	Yes
81	$\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathrm{diag}(\zeta_9^1,\zeta_9^7,\zeta_9^4,1,1)$, $\mathrm{diag}(1,1,1,-1,1)$	0	0	Yes
82	$\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathrm{diag}(\zeta_9^1,\zeta_9^7,\zeta_9^4,\zeta_9^3,\zeta_9^3)$, $\mathrm{diag}(1,1,1,-1,1)$	0	0	Yes
83	$\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathrm{diag}(\zeta_9^1,\zeta_9^7,\zeta_9^4,\zeta_9^6,\zeta_9^6)$, $\mathrm{diag}(1,1,1,-1,1)$	0	0	Yes
84	$\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(-1,1,1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,1,1)$, $\mathrm{diag}(1,1,1,\zeta_3^1,1)$	1	1	Yes
85	$\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(-1,1,1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,1,1)$, $\mathrm{diag}(1,1,1,\zeta_3^1,\zeta_3^1)$	0	0	Yes
86	$\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(-1,1,1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,1,1)$, $\mathrm{diag}(1,1,1,\zeta_3^1,\zeta_3^2)$	1	1	Yes
87	$\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(-1,1,1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,1,\zeta_3^1)$, $\mathrm{diag}(1,1,1,\zeta_3^1,\zeta_3^1)$	0	1	No
88	$\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(-1,1,1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,1,\zeta_3^2)$, $\mathrm{diag}(1,1,1,\zeta_3^1,\zeta_3^2)$	1	2	No
89	$\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(\zeta_8^1,\zeta_8^6,\zeta_8^4,1,1)$, $\mathrm{diag}(1,1,1,1,\zeta_3^1)$	0	0	Yes
90	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(\zeta_4^1,-1,1,1,1)$, $\mathrm{diag}(1,1,1,-1,1)$, $\mathrm{diag}(1,1,1,\zeta_3^1,\zeta_3^1)$	0	0	Yes
91	$\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(\zeta_9^1,\zeta_9^7,\zeta_9^4,1,1)$, $\mathrm{diag}(1,1,1,\zeta_3^1,1)$	0	0	Yes
92	$\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(\zeta_9^1,\zeta_9^7,\zeta_9^4,1,\zeta_9^3)$, $\mathrm{diag}(1,1,1,\zeta_3^1,1)$	0	0	Yes
93	$\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	$\mathrm{diag}(\zeta_9^1,\zeta_9^7,\zeta_9^4,1,\zeta_9^6)$, $\mathrm{diag}(1,1,1,\zeta_3^1,1)$	0	0	Yes
94	$(\mathbb{Z}/3\mathbb{Z})^3$	$\mathrm{diag}(1,\zeta_3^1,1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,1,1)$, $\mathrm{diag}(1,1,1,\zeta_3^1,1)$	0	0	Yes

95	$(\mathbb{Z}/3\mathbb{Z})^3$	$\mathrm{diag}(1, \zeta_3^1, 1, 1, 1)$, $\mathrm{diag}(1, 1, \zeta_3^1, 1, 1)$, $\mathrm{diag}(1, 1, 1, \zeta_3^1, \zeta_3^1)$	0	0	Yes
96	$(\mathbb{Z}/3\mathbb{Z})^3$	$\mathrm{diag}(1, \zeta_3^1, 1, 1, 1)$, $\mathrm{diag}(1, 1, \zeta_3^1, 1, 1)$, $\mathrm{diag}(1, 1, 1, \zeta_3^1, \zeta_3^2)$	1	1	Yes
97	$(\mathbb{Z}/3\mathbb{Z})^3$	$\mathrm{diag}(1, \zeta_3^1, 1, 1, 1)$, $\mathrm{diag}(1, 1, \zeta_3^1, 1, \zeta_3^1)$, $\mathrm{diag}(1, 1, 1, \zeta_3^1, \zeta_3^1)$	0	0	Yes
98	$(\mathbb{Z}/3\mathbb{Z})^3$	$\mathrm{diag}(1, \zeta_3^1, 1, 1, 1)$, $\mathrm{diag}(1, 1, \zeta_3^1, 1, \zeta_3^1)$, $\mathrm{diag}(1, 1, 1, \zeta_3^1, \zeta_3^2)$	0	0	Yes
99	$(\mathbb{Z}/3\mathbb{Z})^3$	$\mathrm{diag}(1, \zeta_3^1, 1, 1, 1)$, $\mathrm{diag}(1, 1, \zeta_3^1, 1, \zeta_3^2)$, $\mathrm{diag}(1, 1, 1, \zeta_3^1, \zeta_3^2)$	1	1	Yes
100	$(\mathbb{Z}/3\mathbb{Z})^3$	$\mathrm{diag}(1, \zeta_3^1, 1, 1, \zeta_3^1)$, $\mathrm{diag}(1, 1, \zeta_3^1, 1, \zeta_3^1)$, $\mathrm{diag}(1, 1, 1, \zeta_3^1, \zeta_3^1)$	0	1	No
101	$(\mathbb{Z}/3\mathbb{Z})^3$	$\mathrm{diag}(1, \zeta_3^1, 1, 1, \zeta_3^1)$, $\mathrm{diag}(1, 1, \zeta_3^1, 1, \zeta_3^1)$, $\mathrm{diag}(1, 1, 1, \zeta_3^1, \zeta_3^2)$	0	0	Yes
102	$(\mathbb{Z}/3\mathbb{Z})^3$	$\mathrm{diag}(1, \zeta_3^1, 1, 1, \zeta_3^1)$, $\mathrm{diag}(1, 1, \zeta_3^1, 1, \zeta_3^2)$, $\mathrm{diag}(1, 1, 1, \zeta_3^1, \zeta_3^2)$	0	1	No
103	$(\mathbb{Z}/3\mathbb{Z})^3$	$\mathrm{diag}(1, \zeta_3^1, 1, 1, \zeta_3^2)$, $\mathrm{diag}(1, 1, \zeta_3^1, 1, \zeta_3^2)$, $\mathrm{diag}(1, 1, 1, \zeta_3^1, \zeta_3^2)$	0	1	No
104	$\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(\zeta_4^1, -1, 1, 1, 1)$, $\mathrm{diag}(1, 1, 1, \zeta_3^1, 1)$, $\mathrm{diag}(1, 1, 1, 1, \zeta_3^1)$	0	0	Yes
105	$(\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z})^2$	$\mathrm{diag}(-1, 1, 1, 1, 1)$, $\mathrm{diag}(1, -1, 1, 1, 1)$, $\mathrm{diag}(1, \zeta_3^1, 1, \zeta_3^1, 1)$, $\mathrm{diag}(1, 1, 1, 1, \zeta_3^1)$	0	0	Yes
106	$\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^3$	$\mathrm{diag}(-1, 1, 1, 1, 1)$, $\mathrm{diag}(1, 1, \zeta_3^1, 1, 1)$, $\mathrm{diag}(1, 1, 1, \zeta_3^1, 1)$, $\mathrm{diag}(1, 1, 1, 1, \zeta_3^1)$	0	0	Yes
107	$(\mathbb{Z}/3\mathbb{Z})^3 \times \mathbb{Z}/2\mathbb{Z}$	$\mathrm{diag}(1, \zeta_3^1, 1, 1, 1)$, $\mathrm{diag}(1, 1, -1, 1, 1)$, $\mathrm{diag}(1, 1, 1, \zeta_3^1, 1)$, $\mathrm{diag}(1, 1, 1, 1, \zeta_3^1)$	0	0	Yes

108	$(\mathbb{Z}/3\mathbb{Z})^4$	$\mathrm{diag}(1,\zeta_3^1,1,1,1)$, $\mathrm{diag}(1,1,\zeta_3^1,1,1)$, $\mathrm{diag}(1,1,1,\zeta_3^1,1)$, $\mathrm{diag}(1,1,1,1,\zeta_3^1)$	0	0	Yes

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