THE FANO VARIETY OF LINES

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1. Introduction

In this bachelor thesis we aim at understanding the Fano variety of lines, that parameterizes the lines contained in a fixed cubic hypersurface in $\mathbb{P}^m_k$. We start to do so by exploring the functorial description of the Grassmannian scheme. This includes the so called Plücker embedding, which embeds the Grassmannian $\mathrm{Grass}_{d,n}$ into the projective space $\mathbb{P}(\bigwedge^d k^n)$ as well as a study of its automorphism group. Note that any $f \in \mathrm{Aut}(\mathbb{P}(V))$ induces an automorphism $\bigwedge^d f$ of $\mathbb{P}(\bigwedge^d V)$ where V is any k-vector space. It turns out that all automorphisms of $\mathrm{Grass}_{d,n}$ arise as the restriction of such an induced automorphism.
The following section 3 establishes the Fano variety of lines. Our approach follows [AK77]. It goes like this: We start by parameterizing r-planes in $\mathbb{P}^m_k$, then degree d hypersurfaces in $\mathbb{P}^m_k$ and as a last step we parameterize r-planes that are contained in such hypersurfaces. Each time we exhibit an universal family. In the case of the r-planes this family is defined over the Grassmannian, in the case of the degree d hypersurfaces over $\mathbb{P}(\mathrm{Sym}_d(k^{m+1})^\vee)$. Finally the Fano variety F attached to a cubic hypersurface $Y \subseteq \mathbb{P}^m_k$ comes up as the representing scheme of the contravariant functor

$$\begin{aligned}(\mathrm{Sch}/k)^{\mathrm{o}} &\longrightarrow (\mathrm{Sets})\\ T &\longmapsto \{L \subseteq Y \times T \mid L \text{ is a flat family of lines over } T\}\end{aligned}$$

where a family of lines designates a closed subscheme $L \subseteq \mathbb{P}^m_k \times T$ such that for all $t \in T$ the fiber L_t is a line in $\mathbb{P}^m_{\kappa(t)}$. Besides we prove that F can be embedded into the Grassmannian $\mathrm{Grass}_{2,m+1}$ and is the zero scheme of global section of the locally free sheaf $\mathrm{Sym}_3(\mathcal{Q})$, where $\mathcal{Q}$ denotes the universal quotient bundle on $\mathrm{Grass}_{2,m+1}$.
The motivation for our previous study can be found at the end of this thesis (section 4), where we prove the following result.

Theorem 1.1. [Ch12, Proposition 4] *Let k be a field of characteristic different from 3 and V the standard k-vector space of dimension $m+1 \geq 5$. Let Y and Y' be cubic hypersurfaces in $\mathbb{P}(V)$ with at most isolated singularities and let F and F' denote the corresponding Fano varieties of lines considered as subschemes of $\mathbb{P}(\bigwedge^2 V)$ using the Plücker embedding.*
Given an automorphism $g\colon \mathbb{P}(\bigwedge^2 V) \to \mathbb{P}(\bigwedge^2 V)$ that restricts to an isomorphism $F \to F'$ then there exists an automorphism $f\colon \mathbb{P}(V) \to \mathbb{P}(V)$ that restricts to an isomorphism $Y \to Y'$ and induces g.

The theorem says, that studying the Fano variety yields information about the given cubic. For this purpose it arises in [Ch12], where Charles proves a Torelli theorem for cubic fourfolds. However he gives a proof that does not rely on the study of a specific cubic but uses that the Fano variety is an irreducible symplectic variety. Precisely he deduces a Torelli theorem for cubic fourfolds from a global Torelli theorem for irreducible symplectic varieties.

Notations. Fix a field k. Throughout the thesis we will concentrate on the category of k-schemes. All schemes are assumed to be locally noetherian.
For every k-scheme X we denote by h_X the functor given by

$$S \mapsto h_X(S) := \mathrm{Hom}_k(S, X).$$

Sometimes we also write $X(S)$ instead of $h_X(S)$. By convention all functors are covariant and contravariant functors are hallmarked by the use of the opposite category of a category $\mathcal{C}$, which we denote by $\mathcal{C}^o$.
We denote the projective m-space $\mathbb{P}^m_k$ by P and its structure morphism by $f\colon P \to \operatorname{Spec} k$. If X is a scheme and $\mathscr{A}$ is a graded quasi-coherent $\mathcal{O}_X$-algebra, $\mathrm{Proj}_X(\mathscr{A})$ denotes the projective spectrum of $\mathscr{A}$. The scheme

$$g\colon H := \mathbb{P}(\mathrm{Sym}_d(k^{m+1})^\vee) \to \operatorname{Spec} k$$

is the scheme parameterizing degree d hypersurfaces of P and

$$q\colon \mathrm{Grass}_{r+1,m+1} \to \operatorname{Spec} k$$

parametrizes r-planes as we prove in section 3. We often abbreviate G for the considered Grassmannian; its universal quotient bundle is denoted by $\mathcal{Q}$.
Let $h\colon T \to \operatorname{Spec} k$ be any scheme. For $t \in T$ we denote by $t\colon \operatorname{Spec} \kappa(t) \to T$ the induced morphism and if $h'\colon T' \to \operatorname{Spec} k$ is a second k-scheme the projection $T' \times T \to T$ is denoted by h'_T and analogously we treat similar situations. E.g. here is a situation we encounter several times:

$$\begin{array}{ccccc} P_{\kappa(t)} & \longrightarrow & P_T & \xrightarrow{h_P} & P \\ \downarrow{\scriptstyle f_t} & & \downarrow{\scriptstyle f_T} & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} \kappa(t) & \xrightarrow{t} & T & \xrightarrow{h} & \operatorname{Spec} k. \end{array}$$

In the case that we have additionally two sheaves $\mathscr{F}$ and $\mathscr{F}'$ on T resp. T' we write $\mathscr{F} \boxtimes \mathscr{F}'$ for the tensor product taken over $T \times T'$.
Furthermore let X and Y be schemes and $\mathscr{F}$ resp. $\mathscr{G}$ be sheaves on X resp. Y. For any $x \in X$ we write $\mathscr{F}(x) = \mathscr{F} \otimes \kappa(x) = \mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$. If $p\colon Y \to X$ is a morphism, we write $Y_x = p^{-1}(x)$ for the fiber over x and $\mathscr{G}(x) = \mathscr{G}|_{Y_x}$ for the restriction of $\mathscr{G}$ to this fiber and analogously $s(x)\colon \mathscr{G}(x) \to \mathscr{G}'(x)$ for the restriction of any given sheaf homomorphism $s\colon \mathscr{G} \to \mathscr{G}'$ over Y.
For a closed immersion $W \hookrightarrow X$ the ideal of W in X is denoted by $\mathscr{I}_W$.
For a morphism of schemes $p\colon X \to Y$ and an $\mathcal{O}_X$-module homomorphism $u\colon p^*\mathscr{E} \to \mathscr{F}$ we denote the adjoint $\mathcal{O}_Y$-module homomorphism by $u^\flat\colon \mathscr{E} \to p_*\mathscr{F}$. Similarly we use the notation $v^\sharp\colon p^*\mathscr{E} \to \mathscr{F}$ whenever an $\mathcal{O}_Y$-module homomorphism $v\colon \mathscr{E} \to p_*\mathscr{F}$ is given. We have $(u^\flat)^\sharp = u$ and $(v^\sharp)^\flat = v$.

2. The Grassmannian

The reader may already be familiar with the concept of the Grassmannian $\mathrm{Grass}_{n-d,n}(\mathbb{C})$ parameterizing d-dimensional subspaces of $\mathbb{C}^n$. Characterizing $\mathrm{Grass}_{n-d,n}$ as a quotient of $\mathrm{GL}_n(\mathbb{C})$ after $\mathrm{GL}_{n-d}(\mathbb{C})$ it inherits the structure of a manifold. In this section we use an analogous approach to construct the Grassmannian scheme. It turns out that there is a useful description as a representable functor.

2.1. **Construction of the Grassmannian.** In this section we will follow [FGA05, 5.1.6]. Let $n \geq d \geq 1$ and A any $d \times n$ matrix. For a subset $I \subseteq \{1, \ldots, n\}$ with d elements we call the matrix A_I consisting of the columns with index in I the I-th minor of A. Define

$$U^I := \operatorname{Spec} k[x^I_{p,q} \mid p = 1, \ldots, d\ ;\ q \in \{1, \ldots, n\} \setminus I\ ].$$

Now let $X^I \in \mathrm{Mat}_{d\times n}(\Gamma(U^I, \mathcal{O}_{U^I}))$ be the matrix whose I-th minor is the $d \times d$ identity matrix and with other entries $x^I_{p,q}$. If $J \subseteq \{1, \ldots, n\}$ is a second subset with $|J| = d$ we define

$$U^I_J = \operatorname{Spec} k[x^I_{p,q}, \det(X^I_J)^{-1}] \subseteq U^I.$$

Note that U^I may be identified with $\mathbb{A}^{d(n-d)}_k$ and if X is any scheme the X-valued points of U^I are given by

$$\{A \in M_{d\times n}(\Gamma(X, \mathcal{O}_X))|\ A_I = \mathbb{I}_{d\times d}\}$$

and under this identification we have

$$U^I_J(X) = \{A \in U^I(X)|\ A_J \text{ is invertible}\} \subseteq U^I(X).$$

Now we define a map $U^I_J(X) \to U^J_I(X)$ by $A \mapsto A_J^{-1}A$. One verifies that $(A_J^{-1}A)_J = \mathbb{I}_{d\times d}$ and $(A_J^{-1}A)_I = A_J^{-1}$ is invertible. By the Yoneda lemma this defines a morphism of schemes

$$\theta_{J,I} \colon U^I_J \to U^J_I.$$

We claim that this gives a gluing datum, i.e. that for any three subsets I, J and K of $\{1, \ldots, n\}$ of cardinality d the cocycle condition $\theta_{I,K} = \theta_{I,J}\theta_{J,K}$ is satisfied. Indeed, for $A \in \mathrm{Mat}_{d\times n}(\Gamma(X, \mathcal{O}_X))$ with $A_K = \mathbb{I}_{d\times d}$ and both A_J and A_I invertible this is the matrix equality $A_I^{-1}A = (A_J^{-1}A)_I^{-1}A_J^{-1}A$. Now define $\mathrm{Grass}_{d,n}$ to be the resulting scheme after gluing the schemes U^I where I varies over all the $\binom{n}{d}$ subsets of $\{1, \ldots, n\}$ with d elements by the cocyle $(\theta_{J,I})$.
By construction $\mathrm{Grass}_{d,n} \to \operatorname{Spec} k$ is smooth and has the relative dimension $d(n-d)$.

Remark 2.1. In an analogous manner the Grassmannian scheme can be constructed over an arbitrary base scheme.

Remark 2.2. The case $d = 1$ is the construction of $\mathbb{P}^{n-1}_k$ by gluing the open sets $D_+(T_i) \cong \operatorname{Spec} k\left[\frac{T_0}{T_i}, \ldots, \frac{T_{n-1}}{T_i}\right]$. Hence

$$\mathrm{Grass}_{1,n} = \mathbb{P}^{n-1}_k.$$

2.2. **Universal quotient.** We will define a locally free sheaf $\mathcal{Q}$ of rank d on $G = \mathrm{Grass}_{d,n}$ together with a surjective homomorphism $\pi\colon \mathcal{O}_G^n \twoheadrightarrow \mathcal{Q}$ as follows: By abuse of notation we denote by U^I also the image of the open immersion $U^I \hookrightarrow G$. On each U^I the matrix X^I defines a surjection $\pi^I\colon \mathcal{O}_{U^I}^n \to \mathcal{O}_{U^I}^d$. Denoting the morphism given by $X_J^I \in \mathrm{GL}_d(\Gamma(U_J^I, \mathcal{O}_{U_J^I}))$ by ϑ_J^I we have the following commuting square

$$\begin{array}{ccc} \mathcal{O}^n_{U^I_J} & \xrightarrow{\pi^I|_{U^I_J}} & \mathcal{O}^d_{U^I_J} \\ \| & & \downarrow{\scriptstyle (\vartheta^I_J)^{-1}} \\ \theta^*_{J,I}\mathcal{O}^n_{U^J_I} & \xrightarrow{\theta^*_{J,I}(\pi^I|_{U^I_J})} & \theta^*_{J,I}\mathcal{O}^d_{U^J_I}. \end{array}$$

One verifies that (ϑ_J^I) satisfies the cocycle condition and enables us to glue $\mathcal{O}_{U^I}^d$ as well as π^I on $U^I \subseteq G$. Let this be the definition of $\pi\colon \mathcal{O}_G^n \twoheadrightarrow \mathcal{Q}$. In Proposition 2.3 we prove that this surjection is universal for surjections $\mathcal{O}_X^n \twoheadrightarrow \mathcal{E}$ where $\mathcal{E}$ is locally free of rank d and X any scheme.

2.3. **Functorial description.** The following is based on [GW10, Chapter 8].

Proposition 2.3. *The Grassmannian* $\mathrm{Grass}_{d,n}$ *represents the functor*

$$F\colon (\mathrm{Sch}/k)^{\mathrm{o}} \longrightarrow (\mathrm{Sets})$$

given by

$$F(X) := \{\mathcal{O}_X^n \twoheadrightarrow \mathcal{E} \mid \mathcal{E} \text{ locally free } \mathcal{O}_X\text{-module of rang } d\}/\sim$$

where $\varphi\colon \mathcal{O}_X^n \twoheadrightarrow \mathcal{E} \sim \varphi'\colon \mathcal{O}_X^n \twoheadrightarrow \mathcal{E}'$ *if there is an isomorphism* $\psi\colon \mathcal{E} \to \mathcal{E}'$ *such that* $\psi \circ \varphi = \varphi'$. *For a morphism of schemes* $f\colon Y \to X$ *and every equivalence class* $[\varphi] \in F(X)$ *representing* $\varphi\colon \mathcal{O}_X^n \twoheadrightarrow \mathcal{E}$ *one defines*

$$F(f)([\varphi]) := [f^*\varphi\colon \mathcal{O}_Y^n \twoheadrightarrow f^*\mathcal{E}] \in F(Y)$$

Proof. For a start note that the definition of F on morphisms is well-defined. Now let X be a scheme and let $\varphi\colon \mathcal{O}_X^n \twoheadrightarrow \mathcal{E}$ be a representative of the class of surjections $[\varphi]$. We have to define a morphism of schemes $f_{[\varphi]}\colon X \to \mathrm{Grass}_{d,n}$ such that $[\varphi] = F(f_{[\varphi]})([\pi])$. To this end let $I \in \{1, \ldots, n\}$ with $|I| = d$ and set

$$V^I = \{x \in X \mid (\varphi_x)_I\colon \kappa(x)^I \hookrightarrow \kappa(x)^n \xrightarrow{\varphi_x} \mathcal{E} \otimes \kappa(x) \text{ is an isomorphism}\}.$$

The V^I with I as above form an open cover of X. First of all $V^I \subseteq X$ is open (see [GW10, Proposition 7.29]). Secondly let $x \in X$ then φ_x is a surjective homomorphism of $\kappa(x)$-vector spaces with d-dimensional target. Hence there must be d vectors of the standard basis whose images are linearly independent. Take the indices of these basis vectors to be the set I and it follows that $x \in V^I$.
Now fix the subset I and consider $\varphi|_{V^I}\colon \mathcal{O}_{V^I}^n \twoheadrightarrow \mathcal{E}|_{V^I}$ by definition the composition

$$(\varphi|_{V^I})_I\colon \mathcal{O}_{V^I}^I \hookrightarrow \mathcal{O}_{V^I}^n \xrightarrow{\varphi|_{V^I}} \mathcal{E}|_{V^I}$$

is an isomorphism and therefore yields the composition

$$\mathcal{O}^n_{V^I} \xrightarrow{\varphi|_{V^I}} \mathcal{E}|_{V^I} \xrightarrow{(\varphi|_{V^I})_I^{-1}} \mathcal{O}^I_{V^I} \cong \mathcal{O}^d_{V^I}$$

which is given by a $d \times n$ matrix with entries in $\Gamma(V^I, \mathcal{O}_{V^I})$ and trivial I-th minor. Thus it corresponds to a morphism of schemes

$$f^I \colon V^I \to U^I.$$

Note that f^I only depends on the equivalence class of surjections. In a next step, we want to glue the f^I's to the desired morphism $f_{[\varphi]} \colon X \to \mathrm{Grass}_{d,n}$. For any I, J we write $f^I_J = f^I|_{V^I \cap V^J}$ and $f^J_I = f^J|_{V^I \cap V^J}$. We have to verify that

$$\theta_{J,I} \circ f^I_J = f^J_I. \tag{2.1}$$

Let $A^{IJ} \in \mathrm{Mat}_{d \times n}(V_I \cap V_J, \mathcal{O}_X)$ be the matrix that corresponds to f^I_J and A^{JI} the matrix that corresponds to f^J_I. Then (2.1) is the matrix equality

$$(A^{IJ})^{-1}_J A^{IJ} = A^{JI}$$

one reads of the commuting square

$$\begin{array}{ccccc}
\mathcal{O}^n_{V^I \cap V^J} & \xrightarrow{\varphi|_{V^I \cap V^J}} & \mathcal{E}|_{V^I \cap V^J} & \xrightarrow{(\varphi|_{V^I \cap V^J})_I^{-1}} & \mathcal{O}^I_{V^I \cap V^J} \\
\Vert & & \Vert & & \downarrow{\scriptstyle ((\varphi|_{V^I \cap V^J})_I^{-1} \circ (\varphi|_{V^I \cap V^J}))_J^{-1}} \\
\mathcal{O}^n_{V^I \cap V^J} & \xrightarrow{\varphi|_{V^I \cap V^J}} & \mathcal{E}|_{V^I \cap V^J} & \xrightarrow{(\varphi|_{V^I \cap V^J})_I^{-1}} & \mathcal{O}^I_{V^I \cap V^J}.
\end{array}$$

It remains to verify, that we have $[\varphi] = F(f_{[\varphi]})([\pi])$. On each V^I the pull-back $(f^I)^*\pi^I$ is given by the matrix whose I-th minor is the identity and whose other entries are the images of $x^I_{p,q}$ under the homomorphism of rings $k[x^I_{p,q}] \to \Gamma(V^I, \mathcal{O}_{V^I})$ that corresponds to $f^I \colon V^I \to U^I$. By definition of f^I this is the matrix representation of $(\varphi|_{V^I})_I^{-1} \circ \varphi|_{V^I}$ and we have the commuting triangle

$$\begin{array}{ccc}
\mathcal{O}^n_{V^I} & \xrightarrow{\varphi|_{V^I}} & \mathcal{E}|_{V^I} \\
& {\scriptstyle (f^I)^*\pi^I} \searrow & \downarrow{\scriptstyle (\varphi|_{V^I})_I^{-1}} \\
& & \mathcal{O}^I_{V^I} \cong \mathcal{O}^d_{V^I}.
\end{array}$$

However, this is compatible with the gluing data $(\theta_{J,I})$ and (ϑ^I_J) as the diagram

$$\begin{array}{ccc}
\mathcal{O}^n_{V^I \cap V^J} & \xrightarrow{(f^I)^*\pi^I_J} & \mathcal{O}^I_{V^I \cap V^J} = f^{I*}_J \ \mathcal{Q}|_{U^I_J} \\
\Vert & & \downarrow{\scriptstyle (f^I_J)^*((\vartheta^I_J)^{-1})} \\
\mathcal{O}^n_{V^J \cap V^I} & \xrightarrow{(f^J)^*\pi^J_I} & \mathcal{O}^J_{V^J \cap V^I} = f^{I*}_J \theta^*_{J,I} \ \mathcal{Q}|_{U^J_I}
\end{array}$$

commutes.

At last we also have $f_{f^*[\pi]} = f$ for any morphism $f \colon X \to G$ as can locally and easily be verified. $\square$

Remark 2.4. In particular we have

$$\mathbb{P}^n_k(X) = \{\mathcal{O}_X^{n+1} \twoheadrightarrow \mathscr{L} \mid \mathscr{L} \text{ line bundle on } X\}/\sim$$
$$= \left\{ (\mathscr{L}, s_1, \dots s_{n+1}) \;\middle|\; \begin{array}{c} \mathscr{L} \text{ line bundle and } s_i \in \Gamma(X,\mathscr{L}) \\ \text{such that the } s_i \text{ generate } \mathscr{L} \end{array} \right\}/\sim .$$

This can be generalized to the notion of projective bundles we introduce in the following section.

The universal bundle of $\mathbb{P}^n_k$ is given by $\mathcal{O}_{\mathbb{P}^n_k}(1)$ i.e.

$$\begin{aligned} \operatorname{Hom}_k(\mathbb{P}^n_k, \mathbb{P}^n_k) &\longrightarrow \{\mathcal{O}_{\mathbb{P}^n_k}^{n+1} \twoheadrightarrow \mathscr{L}\}/\sim \\ \operatorname{id}_{\mathbb{P}^n_k} &\longmapsto (\mathcal{O}_{\mathbb{P}^n_k}^{n+1} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^n_k}(1)) \end{aligned}$$

where the universal surjection $\mathcal{O}_{\mathbb{P}^n_k}^{n+1} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^n_k}(1)$ is on $D_+(T_i)$ given by

$$\begin{aligned} k\left[\tfrac{T_0}{T_i}, \dots, \tfrac{T_n}{T_i}\right]^{n+1} &\longrightarrow (k[T_0, \dots, T_n]_{T_i})_1 \\ e_k &\longmapsto T_k. \end{aligned}$$

2.4. **Projective bundles.** Let X be a scheme and $\mathscr{E}$ any coherent $\mathcal{O}_X$-module. Recall that we have the notion of the *projective bundle defined by* $\mathscr{E}$ given by

$$\mathbb{P}(\mathscr{E}) := \operatorname{Proj}_X(\operatorname{Sym} \mathscr{E})$$

and it is $\mathbb{P}^n_X = \operatorname{Proj}_X(\mathcal{O}_X[T_0, \dots, T_n]) = \mathbb{P}((\mathcal{O}_X^{n+1})^\vee)$.

Remark 2.5 (universal property of $\mathbb{P}(\mathscr{E})$). Let X and $\mathscr{E}$ be as above. The projective bundle $\mathbb{P}(\mathscr{E})$ represents the following functor

$$\begin{aligned} F\colon (\mathrm{Sch}/X)^{\mathrm{o}} &\longrightarrow (\mathrm{Sets}) \\ (f\colon T \to X) &\longmapsto \{f^*\mathscr{E} \twoheadrightarrow \mathscr{L} \mid \mathscr{L} \text{ line bundle on } X\}/\sim \end{aligned}$$

where the equivalence relation and the definition of F on morphisms is analogous to Proposition 2.3. For details see [GW10, Section (13.8)].

Lemma 2.6. [GW10, Remark 13.36] *Let X be a scheme and $\mathscr{E}$ a locally free $\mathcal{O}_X$-module of finite rank. Let $d \in \mathbb{N}$ and denote the structure morphism by $p\colon \mathbb{P}(\mathscr{E}) \to X$. Then the canonical morphism*

$$\operatorname{Sym}_d(\mathscr{E}) \xrightarrow{\cong} p_*\mathcal{O}_{\mathbb{P}(\mathscr{E})}(d)$$

is an isomorphism.

Proof. The desired canonical morphism is the adjoint morphism of

$$p^* \operatorname{Sym}_d(\mathscr{E}) = \operatorname{Sym}_d(p^*\mathscr{E}) \longrightarrow \mathcal{O}_{\mathbb{P}(\mathscr{E})}(d)$$

obtained by factorizing the d-fold tensor product of the universal surjection $p^*\mathscr{E} \twoheadrightarrow \mathcal{O}_{\mathbb{P}(\mathscr{E})}(1)$ over $(p^*\mathscr{E})^{\otimes d} \to \operatorname{Sym}_d(p^*\mathscr{E})$. Now the question is local on X. Hence we can assume that $X = \operatorname{Spec} A$ and $\mathscr{E} = \tilde{E}$ where $E = (A^{n+1})^\vee$ for some n. Then $p\colon \mathbb{P}^n_A \to \operatorname{Spec} A$ corresponds to $A \to \Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A})$ and

$$\operatorname{Sym}_d(\mathscr{E}) = A[T_0, \dots, T_n]_d \xrightarrow{\cong} p_*\mathcal{O}_{\mathbb{P}^n_A}(d) = \Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(d))^\sim$$

is an isomorphism. And one easily verifies that this is the morphism described above. $\square$

2.5. **The Plücker embedding.** The Plücker embedding allows us to view the Grassmannian as a closed subscheme of the projective space. On X-valued points its definition is the following

$$\begin{aligned} \iota\colon \operatorname{Grass}_{d,n}(X) &\longrightarrow \mathbb{P}(\bigwedge^d k^n)(X) \\ (\mathcal{O}_X^n \twoheadrightarrow \mathscr{E}) &\longmapsto (\bigwedge^d \mathcal{O}_X^n \twoheadrightarrow \bigwedge^d \mathscr{E}). \end{aligned}$$

Let $e_1, \ldots, e_n$ be the standard basis of k^n. The elements $e_{i_1} \wedge \ldots \wedge e_{i_d}$ with $i_1 < \ldots < i_d$ form a basis of $\bigwedge^d k^n$, that we use ordered lexicographically whenever we identify $\bigwedge^d k^n$ with $k^{\binom{n}{d}}$.

Proposition 2.7. [GW10, Proposition 8.23] *The Plücker map defined above gives rise to a closed immersion*

$$\operatorname{Grass}_{d,n} \longrightarrow \mathbb{P}_k^N$$

where $N = \binom{n}{d} - 1$.

Proof. Set $G = \operatorname{Grass}_{d,n}$ and $\mathbb{P} = \mathbb{P}(\bigwedge^d k^n)$. We prove the result locally on the target. Let $I \subseteq \{1, \ldots, n\}$ be a subset with d elements and denote by $J = \{1, \ldots, n\} \setminus I$ its complement. Recall that we defined an open subscheme U^I of G to be the representing scheme of the subfunctor given by

$$\begin{aligned} (\mathrm{Sch}/k)^{\mathrm{o}} &\longrightarrow (\mathrm{Sets}) \\ X &\longmapsto \{[\mathcal{O}_X^n \twoheadrightarrow \mathscr{E}] \in G(X) \mid \mathcal{O}_X^I \xrightarrow{i^I} \mathcal{O}_X^n \twoheadrightarrow \mathscr{E} \text{ is an isomorphism}\} \\ &\cong \{A \in \mathrm{Mat}_{d\times n}(\Gamma(X, \mathcal{O}_X)) \mid A_I = \mathbb{I}_{d\times d}\} \\ &\cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^J, \mathcal{O}_X^I) \end{aligned}$$

In the same manner we now define an open subfunctor of $\mathbb{P}$ by

$$\mathbb{P}^I(X) := \left\{ [\bigwedge^d \mathcal{O}_X^n \twoheadrightarrow \mathscr{L}] \in \mathbb{P}(X) \;\middle|\; \begin{array}{c} \bigwedge^d \mathcal{O}_X^I \hookrightarrow \bigwedge^d \mathcal{O}_X^n \twoheadrightarrow \mathscr{L} \\ \text{is an isomorphism} \end{array} \right\}.$$

We have seen that $(U^I)_I$ is an open covering of G and similarly $(\mathbb{P}^I)_I$ is an open covering of $\mathbb{P}$. Now as for any $\varphi\colon \mathcal{O}_X^n \to \mathscr{E}$ the composition $\varphi \circ i^I$ is an isomorphism if and only if $\bigwedge^d(\varphi \circ i^I) = \iota(X)(\varphi) \circ \bigwedge^d i^I$ is an isomorphism (cf. [GW10, Corollary 8.12]), we see that $\iota^{-1}(\mathbb{P}^I) = U^I$. Therefore it suffices to show that $\iota^I\colon U^I \to \mathbb{P}^I$ is a closed immersion.
To begin with, we notice that there is an isomorphism

$$\bigwedge^d \mathcal{O}_X^n \cong \bigoplus_{p+q=d} \bigwedge^p \mathcal{O}_X^I \otimes \bigwedge^q \mathcal{O}_X^J \cong \bigwedge^d \mathcal{O}_X^I \oplus (\bigoplus_{q=1}^{d} \underbrace{\bigwedge^{d-q} \mathcal{O}_X^I \otimes \bigwedge^q \mathcal{O}_X^J}_{:=\mathscr{E}_q})$$

induced by

$$(x_1, \ldots, x_p) \otimes (y_1, \ldots, y_q) \mapsto x_1 \wedge \ldots \wedge x_p \wedge y_1 \wedge \ldots \wedge y_q$$

for every pair (p, q) with $p + q = d$. (We set $\mathscr{E}_q = 0$ in case $q > n - d$.) Therefore

$$\mathbb{P}^I(X) \cong \bigoplus_{q=1}^{d} \operatorname{Hom}_{\mathcal{O}_X}(\mathscr{E}_q, \bigwedge^d \mathcal{O}_X^I)$$

and we conclude that ι^I is given by the following map

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^J, \mathcal{O}_X^I) &\longrightarrow \bigoplus_{q=1}^{d} \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{E}_q, \bigwedge^d \mathcal{O}_X^I) \\ \varphi &\longmapsto (f_q \colon x \otimes y \mapsto x \wedge (\bigwedge^q \varphi)(y))_{1 \le q \le d} \end{aligned}$$

where x and y are such that the above is defined. It turns out that f_1 determines uniquely all other entries due to the fact that

$$\begin{aligned} \alpha\colon \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^J, \mathcal{O}_X^I) &\longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\bigwedge^{d-1} \mathcal{O}_X^I \otimes \mathcal{O}_X^J, \bigwedge^d \mathcal{O}_X^I) \\ \varphi &\longmapsto f_1 \end{aligned}$$

is an isomorphism, as we have

$$\varphi(e_j) = \sum_{k=1}^{d} f_1(b_k \otimes e_j) e_{i_k} \quad \text{for all } j \in J$$

where $I = \{i_1, \ldots, i_d\}$ and $b_k = e_{i_1} \wedge \ldots \wedge \widehat{e_{i_k}} \wedge \ldots \wedge e_{i_d}$ is the k-th basis vector of $\bigwedge^{d-1} \mathcal{O}_X^I$. Hence we find that $f_q \colon \mathscr{E}_q \to \bigwedge^d \mathcal{O}_X^I$ for $2 \le q \le d$ is given by

$$x \otimes y \mapsto x \wedge (\bigwedge^q \alpha^{-1}(f_1))(y).$$

Moreover let $\varphi = \alpha^{-1}(f_1)$ relative to the standard basis $e_1, \ldots, e_n$ of $\mathcal{O}_X^n$ be given by the $d \times (n-d)$ matrix $A = (a_{ij})_{i \in I, j \in J}$. If $K = \{k_1, \ldots, k_{d-q}\} \subseteq I$ with $k_1 < \ldots < k_{d-q}$ we denote the basis vector $e_{k_1} \wedge \ldots \wedge e_{k_{d-q}}$ of $\bigwedge^{d-q} \mathcal{O}_X^I$ by e_K and use e_L analogously for any $L \subseteq J$ with $|L| = q$. Then

$$f_q(e_K \otimes e_L) = \det(A_{I \setminus K, L})(e_{i_1} \wedge \ldots \wedge e_{i_d}) \in \bigwedge^d \mathcal{O}_X^I$$

where $\det(A_{I \setminus K, L})$ is the minor of A that consists of the rows with index in $I \setminus K$ and columns with index in L.
This means that we found polynomial relations, which are independent of X and exhibited $U^I(X) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^J, \mathcal{O}_X^I) \cong \Gamma(X, \mathcal{O}_X)^{n(n-d)}$ as the subset of $\mathbb{P}^I(X) \cong \bigoplus_{q=1}^d \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{E}_q, \bigwedge^d \mathcal{O}_X^I) \cong \Gamma(X, \mathcal{O}_X)^N$ where these relations are satisfied. In other words $\iota^I \colon U^I \to \mathbb{P}^I$ is a closed immersion. □

Example 2.8. Consider the case $d = 2$ and $n = 4$. We have

$$\mathrm{Grass}_{2,4}(k) \cong M := \{A \in \mathrm{Mat}_{4 \times 2}(k) \mid \mathrm{rk}\, A = 2\} / \mathrm{GL}_2(k)$$

and the Plücker embedding is given by

$$\begin{aligned} M &\longrightarrow \mathbb{P}(\bigwedge^2 k^4)(k) \cong (k^6 \setminus \{0\})/k^\times \\ [(a_{ij})] &\longmapsto [a_{11}a_{22} - a_{12}a_{21} : a_{11}a_{23} - a_{13}a_{21} : \ldots : a_{13}a_{24} - a_{14}a_{23}]. \end{aligned}$$

Let $c_{ij} = a_{i1}a_{j2} - a_{i2}a_{j1}$ for $1 \le i < j \le 4$ be the (ij)-th coordinate. One verifies that the image of $\mathrm{Grass}_{2,4}(k)$ is determined by the equation

$$c_{12}c_{34} - c_{13}c_{24} + c_{14}c_{23} = 0.$$

In other words $\mathrm{Grass}_{2,4}(k) \subset \mathbb{P}_k^5(k)$ is a quadric hypersurface.

Remark 2.9. The calculation of the ideal $\mathscr{I}_G$ in the general case can be found in [KL77]. It is given as follows: Let X be any k-scheme and write

$$\mathbb{P}(\bigwedge^d k^n)(X) = \operatorname{Proj}\big(\Gamma(X, \mathcal{O}_X)[T_I \mid I \subseteq \{1, \dots, n\}, |I| = d]\big).$$

Then $\operatorname{Grass}_{d,n}(X) \subseteq \mathbb{P}^N(X)$ is given by the quadratic equations

$$\sum_{k=0}^{d} (-1)^k T_{\{i_0,\dots,i_{d-1},j_k\}} T_{\{j_0,\dots,\hat{j}_k,\dots,j_d\}} = 0$$

with $1 \leq i_l, j_m \leq n$. In the case $d = 2$ we see that $\mathscr{I}_G$ is generated by the relations

$$T_{ij}T_{kl} - T_{ik}T_{jl} + T_{il}T_{jk} \quad \text{where } 1 \leq i < j < k < l \leq n.$$

2.6. **Automorphisms of the Grassmannian.** Every $f \in \operatorname{Aut}(\mathbb{P}^n_k)$ induces a well-defined automorphism $\bigwedge^d f \in \operatorname{Aut}(\mathbb{P}(\bigwedge^d k^{n+1}))$ as follows: We have

$$\operatorname{Aut}(\mathbb{P}^n_k) = \operatorname{PGL}_n(k)$$

and equally $\operatorname{Aut}(\mathbb{P}(\bigwedge^d k^{n+1})) = \operatorname{PGL}(\bigwedge^d k^{n+1})$ (cf. [Ha77, II Example 7.1.1]). Therefore let $\tilde{f} \in \operatorname{GL}_{n+1}(k)$ be a lift of f and define $\bigwedge^d f$ to be the equivalence class of the linear map

$$v_1 \wedge \ldots \wedge v_d \longmapsto \tilde{f}(v_1) \wedge \ldots \wedge \tilde{f}(v_d).$$

Note that this definition is independent of the choice of $\tilde{f}$ and that $\bigwedge^d f$ fixes the Grassmannian $G = \operatorname{Grass}_{d,n+1} \subseteq \mathbb{P}(\bigwedge^d k^{n+1})$. We denote by

$$\operatorname{Aut}(G, \mathbb{P}(\bigwedge^d k^n)) = \{g \in \operatorname{Aut}(\mathbb{P}(\bigwedge^d k^n)) \mid \; g|_G \in \operatorname{Aut}(G)\}.$$

The following theorem of Chow (first proven 1949 in [Ch49]) states that essentially all automorphisms of the Grassmannian are of this type and moreover occur as induced morphisms, in the sense we described above.

Theorem 2.10. [Ha92, Theorem 10.19] *Let $n \geq d \geq 1$. For $n \neq 2d$*

$$\operatorname{Aut}(\operatorname{Grass}_{d,n}) \cong \operatorname{Aut}(\operatorname{Grass}_{d,n}, \mathbb{P}(\bigwedge^d k^n)) \cong \operatorname{PGL}_{n-1}(k).$$

In case $n = 2d > 2$ we have

$$\operatorname{Aut}(\operatorname{Grass}_{d,n}) \cong \operatorname{Aut}(\operatorname{Grass}_{d,n}, \mathbb{P}(\bigwedge^d k^n)) \cong \mathbb{Z}/2\mathbb{Z} \times \operatorname{PGL}_{n-1}(k).$$

Remark 2.11. The case $n = 2d$ is special because in this case the isomorphism

$$\begin{aligned} *\colon \bigwedge^d k^n &\longrightarrow \bigwedge^{n-d} k^n \\ e_I &\longmapsto \epsilon_{IJ} e_J \end{aligned}$$

where $I \subseteq \{1, \dots, n\}$ with $|I| = d$, $J = \{1, \dots, n\} \setminus I$ and $\epsilon_{IJ} \in \{\pm 1\}$ is such that $e_I \wedge e_J = \epsilon_{IJ}(e_1 \wedge \ldots \wedge e_n)$ yields an endomorphism of $\bigwedge^d k^n$. It turns out that if $g \in \operatorname{Aut}(\operatorname{Grass}_{d,n}(k), \mathbb{P}(\bigwedge^d k^n))$ then either g or $* \circ g$ are induced by an automorphism of k^n. We want to illustrate this by an example.

Example 2.12. Let $d = 2$, $n = 4$ and $G = \mathrm{Grass}_{2,4}$. Recall from Example 2.8 that $G \subseteq \mathbb{P}^5_k = \mathrm{Proj}_k(k[T_{12}, T_{13}, T_{14}, T_{23}, T_{24}, T_{34}])$ is given by the homogeneous polynomial

$$T_{12}T_{34} - T_{13}T_{24} + T_{14}T_{23}. \tag{2.2}$$

Let g be induced by the endomorphism of $\bigwedge^2 k^4$ with matrix

$$A = \begin{pmatrix} 1 & 0 & & & & \\ 0 & 1 & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & 1 & 0 \\ & & & & 0 & 1 \end{pmatrix}$$

i.e. A exchanges $e_1 \wedge e_4$ and $e_2 \wedge e_3$ and fixes all other basis vectors. In view of (2.2) we have

$$g|_G : G \xrightarrow{\sim} G$$

and hence $g \in \mathrm{Aut}(G, \mathbb{P}(\bigwedge^2 k^4))$. However, we will show that g is not induced by an automorphism of k^4. Suppose this was the case, i.e. there is a matrix $B = (b_{ij})_{1 \le i,j \le 4} \in \mathrm{GL}_4(k)$ such that

$$\det \begin{pmatrix} b_{ik} & b_{il} \\ b_{jk} & b_{jl} \end{pmatrix} = A_{(ij),(kl)}$$

for all $1 \le i < j \le 4$ and $1 \le k < l \le 4$. Expanding the determinant of the 3×3-minor $\widehat{B}$ consisting of the first three rows and all columns but the third of B, produces the contradiction

$$\det \widehat{B} = b_{11} \det \begin{pmatrix} b_{22} & b_{24} \\ b_{32} & b_{34} \end{pmatrix} - b_{21} \det \begin{pmatrix} b_{12} & b_{14} \\ b_{32} & b_{34} \end{pmatrix} + b_{31} \det \begin{pmatrix} b_{12} & b_{14} \\ b_{22} & b_{24} \end{pmatrix} = -b_{21}$$

and

$$\det \widehat{B} = -b_{12} \det \begin{pmatrix} b_{21} & b_{24} \\ b_{31} & b_{34} \end{pmatrix} + b_{22} \det \begin{pmatrix} b_{11} & b_{14} \\ b_{31} & b_{34} \end{pmatrix} - b_{32} \det \begin{pmatrix} b_{11} & b_{14} \\ b_{21} & b_{24} \end{pmatrix} = 0.$$

On the other hand $*$ has the matrix

$$\begin{pmatrix} & & & & & 1 \\ & & & & -1 & \\ & & & 1 & & \\ & & 1 & & & \\ & -1 & & & & \\ 1 & & & & & \end{pmatrix}$$

and elementary calculation yields that $* \circ g$ is induced by the endomorphism with matrix

$$\begin{pmatrix} & & & -1 \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}.$$

From now on let k be V be the standard $m+1$-dimensional k-vector space. We use the dual basis to identify $V^\vee$ and k^{m+1} and along these lines we obtain the identification

$$\mathrm{Grass}_{d,m+1}(k) \cong \{W \subseteq V \mid \dim W = d\}$$

i.e. the elements of $\mathrm{Grass}_{d,m+1}(k)$ are taken to be d-dimensional vector subspaces of V, which we call d-planes or in the case $d=2$ simply planes. We will write $\mathrm{Grass}_d(V)$ instead of $\mathrm{Grass}_{d,m+1}(k)$ in order to indicate this identification. Analogously we consider the elements of $\mathbb{P}(V)(k)$ to be the lines in V.
For the moment we consider $\mathbb{P}(V)(k)$ and $\mathrm{Grass}_d(V)$ as a classical variety in the sense of [Ha77, Chapter I]. Therefore take k to be algebraically closed for the rest of this section unless specified otherwise. We only prove a special case of Theorem 2.10. Namely the following

Proposition 2.13. *Let $G = \mathrm{Grass}_2(V)$ and $m \geq 4$.*
For all $g \in \mathrm{Aut}(G, \mathbb{P}(\bigwedge^2 V)(k))$ there exists an automorphism of classical varieties $f\colon \mathbb{P}(V)(k) \to \mathbb{P}(V)(k)$ such that

$$f \wedge f|_G = g|_G\,.$$

Our proof follows [Co89]. The idea is to characterize a line in V (i.e. an element of $\mathbb{P}(V)$) as the intersection of two 2-dimensional subspaces (i.e. elements of $\mathrm{Grass}_2(V)$). This requires some preparations. To start with some more notations: For any $W \in \mathrm{Grass}_d(V)$ we denote by $[w] \in \mathbb{P}(\bigwedge^d V)(k)$ the image of W under the Plücker embedding and by $w \in \bigwedge^d V$ what we call an corresponding vector in $\bigwedge^d V$, i.e. a representative of $[w]$ under the identification $\mathbb{P}(\bigwedge^d V)(k) = \big(\bigwedge^d V \setminus (0)\big)/k^\times$. Note that a corresponding vector $w \in \bigwedge^d V$ is always decomposable, i.e. can be written in the form $v_1 \wedge \ldots \wedge v_d$ with $v_i \in V$ for $i = 1\ldots, d$. For any family $(v_i \mid i \in I) \subseteq V$ we denote its vector space span by $< v_i \mid i \in I >$. Finally we equip V with the standard basis $\{e_0, \ldots, e_m\}$ and $\bigwedge^2 V$ with the basis $\{e_i \wedge e_j \mid 0 \leq i < j \leq m\}$.

Definition 2.14. Two planes U and W in V are called *adjacent* if

$$\dim(U \cap W) = 1.$$

Lemma 2.15. *Two distinct planes U and W are adjacent if and only if the sum $u + w$ of any corresponding vectors $u, w \in \bigwedge^2 V$ is decomposable.*
In particular the property of being decomposable does not depend on the chosen representatives u and w.

Proof. First suppose that U and W are adjacent and let $v_1 \in U \cap W \setminus (0)$. Given any corresponding vector $u = u_1 \wedge u_2 \in \bigwedge^2 V$ we can find $\lambda \in k^\times$ such that $u = \lambda u_1 \wedge v_1$ or $u = \lambda u_2 \wedge v_1$. Hence we can assume that

$$u + w = (u_1 \wedge v_1) + (w_1 \wedge v_1) = (u_1 + w_1) \wedge v_1$$

with $u_1 \in U$ and $w_1 \in W$. This shows that $u + w$ is decomposable.
Conversely let $u + w$ be decomposable and suppose that $U \cap W = (0)$. Since $u + w$ is an decomposable element of $\bigwedge^2(U + W)$, it is of the form

$(u_1 + w_1) \wedge (u_2 + w_2)$ with $u_i \in U$ and $w_i \in W$ for $i = 1, 2$. As at most two vectors in U are linearly independent we find the following equality in $\bigwedge^4(U + W)$

$$u \wedge w = u \wedge (u + w) = u \wedge (u_1 + w_1) \wedge (u_2 + w_2) = u \wedge (w_1 \wedge w_2)$$

and therefore $u \wedge (w_1 \wedge w_2 - w) = 0 \in \bigwedge^4(U + W)$. However as we assumed that $U \cap W = (0)$ this implies $w_1 \wedge w_2 = w$ and equally one deduces that $u_1 \wedge u_2 = u$. In particular it follows that u_1, u_2, w_1 and w_2 are linearly independent. On the other hand it also follows that

$$(u_1 \wedge u_2) + (w_1 \wedge w_2) = (u_1 + w_1) \wedge (u_2 + w_2)$$

as both sides are equal to $u + w$ and consequently $u_1 \wedge w_2 + w_1 \wedge u_2 = 0$. This contradicts the linear independence. □

Corollary 2.16. *Any endomorphism of* $\mathrm{Grass}_2(V)$ *that is induced by a linear map* $\bigwedge^2 V \to \bigwedge^2 V$ *preserving decomposable vectors, preserves adjacency.*

If L is any line and W any 3-plane in V we write

$$\sigma(L) := \{U \in \mathrm{Grass}_2(V) \mid L \subseteq U\}$$
$$\Sigma(W) := \{U \in \mathrm{Grass}_2(V) \mid U \subseteq W\}.$$

Proposition 2.17. *Let* $g\colon \mathrm{Grass}_2(V) \to \mathrm{Grass}_2(V)$ *be a bijective map preserving adjacency. Let* L *be a line in* V *and* W_1 *and* W_2 *two distinct planes containing* L. *We denote by* $f^1(L)$ *the line* $g(W_1) \cap g(W_2)$ *and by* $f^3(L)$ *the 3-plane* $g(W_1) + g(W_2)$.
Then either

$$\text{(i)} \quad g(\sigma(L)) \subseteq \sigma(f^1(L)) \qquad \textit{or} \qquad \text{(ii)} \quad g(\sigma(L)) \subseteq \Sigma(f^3(L)).$$

In other words either f^1 *or* f^3 *is independent of the choice of* W_1 *and* W_2

Proof. If (i) does not hold, there is a plane U_0 such that $L = W_1 \cap W_2 \subseteq U_0$ but $f^1(L) = g(W_1) \cap g(W_2) \not\subseteq g(U_0)$. Hence $g(U_0) \cap g(W_1) \neq g(U_0) \cap g(W_2)$ (otherwise by dimension reasons $g(U_0) \cap g(W_1) \cap g(W_2) = g(W_1) \cap g(W_2)$) and we have

$$g(U_0) \subseteq g(W_1) + g(W_2)$$

i.e. $g(U_0) \in \Sigma(f^3(L))$. Now if U is any plane containing L, then the intersection of $g(U)$ with $g(U_0), g(W_1)$ or $g(W_2)$ respectively is a line as g preserves adjacency. But the lines can't all be the same. Hence $g(U) \in \Sigma(f^3(L))$ i.e. (ii) holds. □

Corollary 2.18. *Let* $g\colon \mathrm{Grass}_2(V) \to \mathrm{Grass}_2(V)$ *be induced by the vector space isomorphism* $\tilde{g}\colon \bigwedge^2 V \to \bigwedge^2 V$ *preserving decomposable vectors. If* $m \geq 4$ *there is a map*

$$f\colon \mathbb{P}(V)(k) \longrightarrow \mathbb{P}(V)(k)$$

such that $g(\sigma(L)) \subseteq \sigma(f(L))$.

Proof. The above conditions (i) and (ii) are equivalent to

$$\text{(i)}' \ \tilde{g}(L \wedge V) \subseteq f^1(L) \wedge V \quad \text{and} \quad \text{(ii)}' \ \tilde{g}(L \wedge V) \subseteq f^3(L) \wedge f^3(L).$$

Now (ii)$'$ implies that

$$\dim(L \wedge V) = m \leq 3 = \dim(f^3(L) \wedge f^3(L))$$

and is therefore impossible by assumption. This means we deal with case (i) and can take $f = f^1$.

□

Proof of Proposition 2.13. Let $g \in \mathrm{Aut}(\mathrm{Grass}_2(V), \mathbb{P}(\bigwedge^2 V)(k))$. Hence g is induced by an $\tilde{g} \in \mathrm{GL}(\bigwedge^2 V)$. By Corollary 2.18 the assignment

$$L = U \cap W \longmapsto g(U) \cap g(W)$$

with $U, W \in \mathrm{Grass}_2(V)$ gives a well-defined map $f\colon \mathbb{P}(V)(k) \to \mathbb{P}(V)(k)$. We have to show that f is a morphism of projective spaces and that f induces g.

f is an automorphism of classical varieties. Let $L \in \mathbb{P}(V)(k)$ be the line spanned by $z = (z_0, \ldots, z_m) \in V$. Without loss of generality we can suppose that $z_0 = 1$ otherwise we can permute the basis vectors of V appropriately. Hence

$$L = <z, e_1> \cap <z, e_2> .$$

Now let W_i be the plane that corresponds to $[\tilde{g}(<z, e_i>)]$ for $i = 1, 2$. We have to show that the local coordinates of $f(L) = W_1 \cap W_2 \in \mathbb{P}(V)(k)$ are polynomials in the z_i's. To this end write

$$\tilde{g}(z \wedge e_1) = (a_{ij}(1, z_1, \ldots, z_m))_{0 \leq i < j \leq m} \in \bigwedge^2 V$$

where $a_{ij} \in k[X_0, \ldots, X_m]_1$ for all $0 \leq i < j \leq n$. After reordering the basis we assume that $a_{01}(z) \neq 0$. We claim that W_1 is the span of

$$\begin{aligned} a^{(1)} &:= (a_{01}(z), 0, -a_{12}(z), \ldots, -a_{1m}(z)) \\ \text{and} \quad a^{(2)} &:= (0, a_{01}(z), \quad a_{02}(z), \ldots, \quad a_{0m}(z)). \end{aligned}$$

Indeed let the image of $W' := < a^{(1)}, a^{(2)} >$ under the Plücker embedding into $\mathbb{P}(\bigwedge^2 V)(k)$ have the homogeneous coordinates $[c_{01} : c_{02} : \ldots : c_{m-1m}]$. By Remark 2.9 we have $c_{01}c_{ij} - c_{0i}c_{1j} + c_{0j}c_{1i} = 0$ for all $1 < i < j \leq m$ and therefore it is enough to calculate the first $2(m-1)$ coordinates. These are

$$c_{0j} = \begin{cases} a_{01}(z)^2 & \text{if } j = 1 \\ \det\left(\begin{smallmatrix} a_{01}(z) & 0 \\ -a_{1j}(z) & a_{2j}(z) \end{smallmatrix}\right) = a_{01}(z)a_{2j}(z) & \text{if } 2 \leq j \leq m \end{cases}$$

$$\text{and} \quad c_{1j} = \quad \det\left(\begin{smallmatrix} 0 & a_{01}(z) \\ -a_{1j}(z) & a_{2j}(z) \end{smallmatrix}\right) = a_{01}(z)a_{1j}(z) \text{ if } 2 \leq j \leq m$$

i.e. $a_{01}(z)^{-1}c_{ij} = a_{ij}(z)$ for all $0 \leq i < j \leq m$ with $i \in \{0, 1\}$ and therefore

$$[c_{ij}] = [\tilde{g}(<z, e_1>)] \in \mathbb{P}(\bigwedge^2 V)(k)$$

or in other words $W_1 = W'$ as claimed.
Analogously we find

$$b^{(j)} = (b_{ij}(z))_{0 \leq i \leq m} \in V \text{ with } b_{ij} \in k[X_0, \ldots, X_m]_1 \text{ for } j \in \{1, 2\}$$

such that $W_2 = < b^{(1)}, b^{(2)} >$. This means that $f(L) \in \mathbb{P}(V)(k)$ is the line generated by an $x \in V$ such that there are $\lambda_1, \lambda_2, \mu_1, \mu_2 \in k$ such that

$$x = \lambda_1 a^{(1)} + \lambda_2 a^{(2)} = \mu_1 b^{(1)} + \mu_2 b^{(2)}.$$

This is an system of linear equations where all coefficients are linear polynomials in the coordinates of z. By Corollary 2.16 its solution is one-dimensional. Using the Gauss algorithm we see that the solution is polynomial in the coordinates of z.
Finally f is an automorphism as the same proof applied to g^{-1} produces the inverse morphism f^{-1}.

f induces g. We have to show that

$$g([v_1 \wedge v_2]) = [\tilde{f}(v_1) \wedge \tilde{f}(v_2)] \in \mathbb{P}(\bigwedge^2 V)(k) \tag{2.3}$$

for all $v_1, v_2 \in V$ where $\tilde{f} \in \mathrm{GL}(V)$ is a representative of $f \in \mathrm{PGL}(V)$. Let $v_1, v_2 \in V$ and choose $v_3 \in V$ such that v_1, v_2 and v_3 are linearly independent. Moreover let $g([v_1 \wedge v_2]) = [\tilde{g}(v_1 \wedge v_2)]$ correspond to $W \in \mathrm{Grass}_2(V)$ and $g([v_i \wedge v_3])$ correspond to $W_i \in \mathrm{Grass}_2(V)$ for $i = 1, 2$. As

$$< v_1, v_2 > = < v_1, v_3 > \cap < v_2, v_3 >$$

we find $[\tilde{f}(v_i)] = W \cap W_i$ for $i = 1, 2$, i.e. the right hand side in (2.3) corresponds to the two-dimensional vector subspace of V containing the different lines $W \cap W_1$ and $W \cap W_2$. This is the left hand side, namely W.

□

Corollary 2.19. *Let k be any field and $G = \mathrm{Grass}_{2,m+1}$ with $m \geq 4$. For all $g \in \mathrm{Aut}(G, \mathbb{P}(\bigwedge^2 k^{m+1}))$ there is an isomorphism of k-schemes $f \colon P \to P$ such that*

$$f \wedge f|_G = g|_G \, .$$

Proof. If k is algebraically closed, there is a fully faithful embedding of the category of varieties over k into the category of k-schemes (see [Ha77, II Proposition 2.6 and 4.10]). In particular we have

$$\mathrm{Hom}_{\mathrm{Var}/k}(P(k), P(k)) = \mathrm{Hom}_{\mathrm{Sch}/k}(P, P)$$

where the claim follows from.
If k is not algebraically closed, we consider an algebraic closure $\bar{k}$ and the corresponding automorphism $\bar{g} \in \mathrm{Aut}_{\bar{k}}(G \times_k \bar{k}, \mathbb{P}(\bigwedge^2 \bar{k}^{m+1}))$. Then $\bar{g}$ is induced by some $\bar{f} \in \mathrm{PGL}_n(\bar{k})$. On the other hand the previous proof shows that $\bar{f}$ maps k-valued points to k-valued points, hence it is defined over k. □

3. The Fano variety of lines

Now we turn towards our main interest of study: The Fano variety of lines. Yet we will not start by studying the lines in a fixed cubic hypersurface, but exhibit such a thing as the 'universal Fano variety' parameterizing pairs (L, Y) where Y is a degree d hypersurface in P and L an r-plane contained in Y. Therefore our first goal is to parameterize the r-planes and the degree

d hypersurfaces in P. With this approach we follow the third chapter of [AK77]. Prior to that we need some preparation:

3.1. The zero scheme of a global section.

Definition 3.1. Let X be a scheme and $\mathscr{F}$ a locally free $\mathcal{O}_X$-module. For any global section $s \in \Gamma(X, \mathscr{F}) = \mathrm{Hom}(\mathcal{O}_X, \mathscr{F})$ the *subscheme of zeros of* s is defined to be the closed subscheme $Z(s)$ of X corresponding to the quasi-coherent sheaf of ideals, that is the image of $s^\vee : \mathscr{F}^\vee \longrightarrow \mathcal{O}_X$.

Remark 3.2. Let $\mathscr{F}$ and s be as above with $\mathscr{F}$ of rank n. Choose an open covering $X = \bigcup U_i$ with $U_i = \operatorname{Spec} A_i$ for all i and trivializations

$$\psi_i : \mathscr{F}|_{U_i} \xrightarrow{\cong} \mathcal{O}_{U_i}^n.$$

Then $\psi_i(s|_{U_i})$ is an element of $\Gamma(U_i, \mathcal{O}_{U_i}^n) = A_i^n$. Write

$$\psi_i(s|_{U_i}) = (f_1^{(i)}, \dots, f_n^{(i)}) \text{ with } f_k^{(i)} \in A_i.$$

Thus we find that $Z(s) \cap U_i =: Z_i$ is the vanishing scheme of the $f_k^{(i)}$ for $1 \leq k \leq n$ that is

$$Z_i = \operatorname{Spec}\big(A_i/(f_1^{(i)}, \dots, f_n^{(i)})\big).$$

Proposition 3.3 (Universal property of $Z(s)$ cf. [EGAI, 9.7.9.1])**.** *Let X be a scheme, $\mathscr{F}$ a locally free $\mathcal{O}_X$-module of finite rank and $s \in \Gamma(X, \mathscr{F})$ a global section. Any morphism of schemes $f\colon T \to X$ factors through $Z(s)$ if and only if $f^*s = 0$.*

Proof. The condition $f^*s = 0$ can be checked locally on X, thus we can assume that $\mathscr{F} = \mathcal{O}_X^n$ and even $n = 1$ since $Z(s) = \bigcap_{i=1}^n Z(p_i \circ s)$ and $f^*s = 0$ if and only if $f^*(p_i \circ s) = 0$ for all i where $p_i\colon \mathcal{O}_X^n \to \mathcal{O}_X$ is the i-th projection. That the latter is true can be checked assuming $X = \operatorname{Spec} A$. Then s corresponds to tuple $(f_1, \dots, f_n) \in A^n$ and we have indeed $A/(f_1, \dots, f_n) = \bigotimes_{i=1}^n A/f_i$. This means we are reduced to the case $s\colon \mathcal{O}_X \to \mathcal{O}_X$. Let $\mathscr{I}$ be the ideal defining $Z(s)$ i.e. the image of $s^\vee = s$ and consider the exact sequence

$$\mathcal{O}_X \xrightarrow{s} \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathscr{I} \longrightarrow 0.$$

Applying the functor f^* the sequence becomes

$$\mathcal{O}_T \xrightarrow{f^*s} \mathcal{O}_T \longrightarrow \mathcal{O}_T/f^{-1}(\mathscr{I}) \cdot \mathcal{O}_T \longrightarrow 0.$$

Thus $f^*s = 0$ is equivalent to $f^{-1}(\mathscr{I}) \cdot \mathcal{O}_T = 0$ as claimed. □

Remark 3.4. For an invertible $\mathcal{O}_X$-module $\mathscr{L}$, a locally free $\mathcal{O}_X$-module $\mathscr{F}$ and an $\mathcal{O}_X$-module homomorphism $s\colon \mathscr{L} \to \mathscr{F}$ one has $f^*s = 0$ if and only if $f^*(s \otimes \mathrm{id}_{\mathscr{L}^\vee}) = 0$. Due to this fact the identification of $\mathrm{Hom}(\mathscr{L}, \mathscr{F})$ with $\mathrm{Hom}(\mathcal{O}_X, \mathscr{F} \otimes \mathscr{L}^\vee)$ allows us to define the notion of the zero scheme $Z(s)$ in this situation, that satisfies the same universal property as above.

Definition 3.5. If $\mathscr{F}$ is a locally free $\mathcal{O}_X$-module of rank n we call a global section $s \in \Gamma(X, \mathscr{F})$ *regular* if for any open subset $U \subseteq X$ such that there is an isomorphism

$$\mathscr{F}|_U \cong \mathcal{O}_U^n$$

the images of s in $\Gamma(U, \mathcal{O}_U)^n$ form a regular sequence.

For the definition of a regular sequence see [Ma80, Section 12].

Remark 3.6. By [Ma80, Theorem 27(ii)] the definition of a regular section in this setting does not depend on the choice of the trivializations, i.e. a global section $s \in \Gamma(X, \mathscr{F})$ is regular if and only if there is an open covering $X = \bigcup U_i$ such that $\mathscr{F}$ is trivial over each U_i and s is regular as above.

Lemma 3.7. [Ha77, III Proposition 9.5] *Let $p\colon X \to Y$ be a flat morphism of schemes of finite type over a field k. Moreover let $x \in X$ and set $y = p(x)$. Then*

$$\dim_x(X_y) = \dim_x X - \dim_y Y$$

where $\dim_x X = \dim \mathcal{O}_{X,x}$.

Lemma 3.8. *Let X be a scheme and $\mathscr{F}$ a locally free $\mathcal{O}_X$-module on X. If $s \in \Gamma(X, \mathscr{F})$ is a section over X. Then*

(i) $\dim Z(s) \geq \dim X - \operatorname{rk} \mathscr{F}$
(ii) $\dim Z(s) = \dim X - \operatorname{rk} \mathscr{F}$ *if s is regular*
(iii) *The converse of* (ii) *holds if X is Cohen-Macaulay.*

Proof. For (i) and (ii) use Krull's principal ideal theorem [Ma80, Theorem 18]. (iii) is [Ma80, Theorem 31]. □

We need another technical result.

Lemma 3.9. *Let $p\colon X \to S$ be a morphism of schemes and let $u\colon p^*\mathscr{E} \to \mathscr{F}$ be an $\mathcal{O}_X$-module homomorphism. For every base change $g\colon T \to S$ let $b\colon g^*p_*\mathscr{F} \to p_{T*}g_X^*\mathscr{F}$ denote the base change map. Then*

(i) *Adjunction commutes with base change up to the base change map. In other words we have the commutative triangle*

$$\begin{array}{ccc} g^*\mathscr{E} & & \\ {\scriptstyle g^*(u^\flat)}\downarrow & \searrow^{(g_X^* u)^\flat} & \\ g^*p_*\mathscr{F} & \xrightarrow{\;b\;} & p_{T*}g_X^*\mathscr{F}. \end{array}$$

(ii) *The adjunction map $\sigma(\mathscr{F}) = (\mathrm{id}_{p_*\mathscr{F}})^\sharp\colon p^*p_*\mathscr{F} \to \mathscr{F}$ commutes with base change up to the base change map.*

$$\begin{array}{ccc} p_T^*g^*p_*\mathscr{F} & \xrightarrow{\;p_T^*b\;} & p_T^*p_{T*}g_X^*\mathscr{F} \\ \Vert & & \downarrow{\scriptstyle \sigma(g_X^*\mathscr{F})} \\ g_Y^*p^*p_*\mathscr{F} & \xrightarrow{\;g_X^*(\sigma(\mathscr{F}))\;} & g_X^*\mathscr{F} \end{array}$$

(iii) *Base change and composition are compatible, i.e. if $g'\colon T' \to T$ is a second base change with associated base change map*

$$b'\colon g'^*p_{T*}g_X^*\mathscr{F} \to p_{T*}'(g'_{X \times_S T'})^*g_X^*\mathscr{F}$$

of the $\mathcal{O}_{X\times_S T}$-module $g_X^*\mathscr{F}$ then $b'\circ g'^*b$ is the base change map of $\mathscr{F}$ in the outer cartesian square of

$$\begin{array}{ccccc} X\times_S T' & \xrightarrow{g'_{X\times_S T'}} & X\times_S T & \xrightarrow{g_X} & X \\ \downarrow{\scriptstyle p'_T} & & \downarrow{\scriptstyle p_T} & & \downarrow{\scriptstyle p} \\ T' & \xrightarrow{g'} & T & \xrightarrow{g} & S. \end{array}$$

Proof. The first claim follows from [EGAI, 9.3.1] and [EGAI, 0_I 3.5.3-3.5.5[1]]. Then (ii) follows from (i) with $u=\sigma(\mathscr{F})$ and from [EGAI, 0_I 3.5.4.2]. The third claim can be found in [AHK73, (6.5)]. □

3.2. Families of r-planes.

Definition 3.10. Let $T\to \operatorname{Spec} k$ be a scheme. A *family of r-planes* in P over T is a closed subscheme $L\subseteq P\times T=P_T$ such that $L\to T$ is flat and for every $t\in T$ the fiber $L_t\hookrightarrow P_{\kappa(t)}$ is an r-dimensional linear subspace.

Example 3.11. The projective bundle $\mathbb{P}(\mathscr{E})$ associated to any T-valued point $\mathcal{O}_T^{m+1}\twoheadrightarrow\mathscr{E}$ of $\mathrm{Grass}_{r+1,m+1}$ defines a family of r-planes over T in the following way: The surjection $\mathcal{O}_T^{m+1}\twoheadrightarrow\mathscr{E}$ yields a closed immersion

$$\operatorname{Proj}_T(\operatorname{Sym}\mathscr{E})=\mathbb{P}(\mathscr{E})\hookrightarrow \operatorname{Proj}_T(\operatorname{Sym}(\mathcal{O}_T^{m+1}))=P_T.$$

and $\mathbb{P}(\mathscr{E})$ is flat over T. Now we verify that the fibers are r-planes in the ambient projective space. Let $t\in T$ and consider the fiber $\mathbb{P}(\mathscr{E})_t=\mathbb{P}(\mathscr{E}(t))$. We have the commutative diagram

$$\begin{array}{ccc} \operatorname{Proj}_T(\operatorname{Sym}\mathscr{E}) & \hookrightarrow & \operatorname{Proj}_T(\operatorname{Sym}(\mathcal{O}_T^{m+1})) \\ \uparrow & & \uparrow \\ \operatorname{Proj}(\operatorname{Sym}\mathscr{E}\otimes\kappa(t)) & \overset{(*)}{\hookrightarrow} & \operatorname{Proj}(\operatorname{Sym}(\mathcal{O}_T^{m+1}\otimes\kappa(t))). \end{array}$$

The lower line is induced by a surjection of $\kappa(t)$-vector spaces

$$\mathcal{O}_T^{m+1}\otimes\kappa(t)\twoheadrightarrow\mathscr{E}\otimes\kappa(t)$$

with $\dim(\mathscr{E}\otimes\kappa(t))=r+1$. After choosing suitable bases of $\mathcal{O}_T^{m+1}\otimes\kappa(t)$ and $\mathscr{E}\otimes\kappa(t)$ we can thus assume that $(*)$ is the inclusion of $\mathbb{P}(\mathscr{E}(t))\subset P_{\kappa(t)}$ as the vanishing set of the last $m-r$ coordinates, i.e. $\mathbb{P}(\mathscr{E}(t))\subset P_{\kappa(t)}$ is an r-plane as claimed.

The example applies in particular to $\mathcal{O}_G^{m+1}\twoheadrightarrow\mathcal{Q}$ from section 2.2. Hence $\mathbb{P}(\mathcal{Q})\subseteq P_G$ is a family of r-planes. The next proposition shows that every family of r-planes is a pullback of $\mathbb{P}(\mathcal{Q})$. Its proof will use the following consequence of Nakayama's lemma.

Lemma 3.12. *Let $p\colon S\to T$ be a morphism of noetherian schemes and let $\mathscr{F}$ be a coherent $\mathcal{O}_S$-module. Then $\mathscr{F}=0$ if $\mathscr{F}(t)=0$ for all $t\in T$.*

[1]Note that in 3.5.3.4 v should be replaced by w and in 3.5.3 one should read $v\colon\psi^{-1}(\mathscr{H})\to\psi^{-1}(\mathscr{G})$ and $w\colon\psi^{-1}(\mathscr{G})\to\mathscr{F}$.

Proof. By Nakayama's lemma it is enough to show that $\mathscr{F}(s) = 0$ for all $s \in S$. On the other hand

$$\mathscr{F}(s) = \mathscr{F}_s \otimes_{\mathcal{O}_{S,s}} \kappa(s) = (\mathscr{F}_s \otimes_{\mathcal{O}_{T,t}} \kappa(t)) \otimes_{\mathcal{O}_{S,s} \otimes_{\mathcal{O}_{T,t}} \kappa(t)} \kappa(s)$$

where $p(s) = t$ and we are reduced to show that $\mathscr{F}_s \otimes_{\mathcal{O}_{T,t}} \kappa(t)$ vanishes. However, $\mathscr{F}_s \otimes_{\mathcal{O}_{T,t}} \kappa(t)$ is isomorphic to the stalk of $\mathscr{F}(t)$ at the point s. □

Proposition 3.13. *The Grassmannian $G = \mathrm{Grass}_{r+1,m+1}$ parameterizes the r-planes of P and $\mathbb{P}(\mathcal{Q}) \subseteq P_G$ is the universal family of r-planes.*

Proof. Let $L \subseteq P_T$ be any flat family of r-planes over a k-scheme T and denote the projection by $u\colon L \to T$. We set

$$\mathscr{R} := u_* \mathcal{O}_L(1)$$

where $\mathcal{O}_L(1) = \mathcal{O}_L \otimes_{\mathcal{O}_{P_T}} h_T^* \mathcal{O}_P(1)$. By cohomology and base change (cf. [Ha77, II Theorem 12.11]) $\mathscr{R}$ is locally free of rank $r+1$ and its formation commutes with base change. Now consider the following pullback of the universal surjection $\beta\colon \mathcal{O}_L^{m+1} = u^*\mathcal{O}_T^{m+1} \twoheadrightarrow \mathcal{O}_L(1)$ and its adjoint

$$\beta^\flat\colon \mathcal{O}_T^{m+1} \longrightarrow \mathscr{R}$$

We claim two things: First that $\beta^\flat$ is surjective and thus defines a map $\lambda\colon T \to G$ and secondly that L is equal to $T \times_G \mathbb{P}(\mathcal{Q}) = \mathbb{P}(\mathscr{R})$. Fix $t \in T$ and denote $u_t\colon L_t \to \operatorname{Spec} \kappa(t)$ by Lemma 3.9(i) there is a commutative diagram

$$\begin{array}{ccc} \kappa(t)^{m+1} & & \\ {\scriptstyle \beta^\flat(t)}\downarrow & \searrow^{\beta(t)^\flat} & \\ \mathscr{R}(t) & \xrightarrow{\cong} & u_{t*}\mathcal{O}_{L_t}(1) \end{array}$$

where the horizontal arrow is a base change map, that is an isomorphism as $H^1(L_{t'}, \mathcal{O}_{L_{t'}}(1))$ vanishes for all t' in some neighborhood of t. For the latter note that $L_{t'}$ is isomorphic to $\mathbb{P}^r_{\kappa(t')}$ and $\mathcal{O}_{L_{t'}}(1)$ is just the twisting sheaf on this projective space. Since $L_t \hookrightarrow P_{\kappa(t)}$ is an r-plane we know that $\beta(t)^\flat$ is surjective for all $t \in T$ and consequently $\beta^\flat(t)$ is. Using Nakayama's Lemma we conclude that $\beta^\flat$ is surjective, as claimed. Now set $\sigma = (\mathrm{id}_{\mathscr{R}})^\sharp\colon u^*u_*\mathcal{O}_L(1) \to \mathcal{O}_L(1)$ and consider the natural commutative diagram (cf. [EGAI, 0_I 3.5.4.2])

$$\begin{array}{ccc} \mathcal{O}_L^{m+1} & \xrightarrow{\beta} & \mathcal{O}_L(1). \\ {\scriptstyle u^*(\beta^\flat)}\downarrow & \nearrow_{\sigma} & \\ u^*\mathscr{R} & & \end{array}$$

We have seen above that every morphism in this diagram is surjective thus induces a well-defined morphism on the corresponding projective bundles.

We obtain

$$\begin{tikzcd} P_T & P_T \times_T L = P_L \arrow[l, "\mathrm{pr}"'] & & L = \mathrm{Proj}_L(\mathrm{Sym}\,\mathcal{O}_L(1)) \arrow[ll, "\mathbb{P}(\beta)"'] \arrow[dll, "\mathbb{P}(\sigma)"] \\ \mathbb{P}(\mathscr{R}) \arrow[u, "\mathbb{P}(\beta^\flat)"] & \mathbb{P}(\mathscr{R}) \times_T L = \mathbb{P}(u^*\mathscr{R}) \arrow[l] \arrow[u, "\mathbb{P}(u^*\beta^\flat)"'] & & \end{tikzcd}$$

where $\mathbb{P}(\beta^\flat)$ is a closed immersion. Moreover $\mathbb{P}(\beta)$ is a morphism of P_T-schemes by [EGAII, 3.7.1]. Therefore the upper row of the diagram, i.e. $\theta(\beta) := \mathrm{pr} \circ \mathbb{P}(\beta) \colon L \to P_T$ is the closed immersion from the beginning and we are in the situation

$$\begin{tikzcd} P_T & L \arrow[l, hook', "\theta(\beta)"'] \arrow[dl, "\theta(\sigma)"] \\ \mathbb{P}(\mathscr{R}) \arrow[u, hook, "\mathbb{P}(\beta^\flat)"] & \end{tikzcd}$$

Therefore $\theta(\sigma)$ is a closed immersion, however we want to prove that it is an isomorphism. Once more this will be done by checking the property fiberwise and using that L is flat over T. Let $t \in T$, by the method of Example 3.11 the fibers $\mathbb{P}(\mathscr{R})_t \subseteq P_{\kappa(t)}$ are r-planes. This leads to the closed immersion

$$\theta(\sigma)(t) \colon L_t \hookrightarrow \mathbb{P}(\mathscr{R})_t$$

where both sides are r-planes in $P_{\kappa(t)}$. Thus $\theta(\sigma)(t)$ must be an isomorphism and we claim $\theta(\sigma)$ is also an isomorphism, i.e. $\mathscr{I}_L = 0$. As L is flat over T it follows that $\mathscr{I}_L(t) \cong \mathscr{I}_{L_t}$ and the latter one vanishes as $\theta(\sigma)(t)$ is an isomorphism. We conclude by Lemma 3.12 that $\mathscr{I}_L = 0$. This shows our second claim and finishes the proof.

□

3.3. **Families of hypersurfaces.**

Definition 3.14. Let $T \to \operatorname{Spec} k$ be a scheme. A *family of degree d hypersurfaces* in P over T is a closed subscheme $Y \subseteq P \times T = P_T$ such that $Y \to T$ is flat and for every $t \in T$ the fiber $Y_t \hookrightarrow P_{\kappa(t)}$ is a hypersurface of degree d.

Remark 3.15. This is the analog of Definition 3.10.

Recall that we write $f \colon P \to \operatorname{Spec} k$ for the structure map of P. Moreover let for any scheme $h \colon T \to \operatorname{Spec} k$

$$b^T \colon \mathrm{Sym}_d(\mathcal{O}_T^{m+1}) \longrightarrow f_{T*} h_P^* \mathcal{O}_P(d)$$

be the base change map of $\mathcal{O}_P(d)$ where we identify $\mathrm{Sym}_d(k^{m+1}) = f_*\mathcal{O}_P(d)$. Furthermore let $g \colon H = \mathbb{P}(\mathrm{Sym}_d(k^{m+1})^\vee) \to \operatorname{Spec} k$ be the structure map of H and

$$\alpha \colon \mathrm{Sym}_d(\mathcal{O}_H^{m+1})^\vee \twoheadrightarrow \mathcal{O}_H(1)$$

the natural surjection. Finally set

$$s = (b^H \circ \alpha^\vee)^\# \colon f_H^* \mathcal{O}_H(-1) \longrightarrow g_P^* \mathcal{O}_P(d).$$

Lemma 3.16. *The formation of s commutes with arbitrary base change $\lambda\colon T \to H$ over k. In other words one has the equality*

$$s_T := \lambda^*_{H\times P} s = (b^T \circ \lambda^* \alpha^\vee)^\sharp. \tag{3.1}$$

Proof. With Lemma 3.9(i) we have

$$(\lambda^*_{H\times P} s)^\flat = b \circ \lambda^*(s^\flat) = b \circ \lambda^* b^H \circ \lambda^*(\alpha^\vee)$$

where b denotes the base change map of $g_P^*\mathcal{O}_P(d)$ along λ and $b \circ \lambda^* b^H$ is equal to b^T by part (iii). □

Proposition 3.17. *There is a universal flat family W over H of degree d hypersurfaces in P. In other words H represents the functor*

$$\begin{aligned}(\mathrm{Sch}/k)^{\mathrm{o}} &\longrightarrow (\mathrm{Sets})\\ T &\longmapsto \left\{ Y \subseteq P \times T \;\middle|\; \begin{array}{c} Y \textit{ is a flat family of degree} \\ d \textit{ hypersurfaces in } P \end{array} \right\}\end{aligned}$$

Proof. We will show that there is an exact sequence on P_H

$$0 \longrightarrow f_H^*\mathcal{O}_H(-1) \xrightarrow{\;s\;} g_P^*\mathcal{O}_P(d) \longrightarrow \mathcal{O}_W(d) \longrightarrow 0 \tag{3.2}$$

or rather we use this sequence in order to define W.
Our first step is to see that s is injective. By definition we have

$$s = (b^H \circ \alpha^\vee)^\#\colon f_H^*\mathcal{O}_H(-1) \longrightarrow\ g_P^*\mathcal{O}_P(d)$$

and by Lemma 3.16 it commutes with base change, thus if we consider the restriction to a fiber over $h \in H$ we find by (3.1) that

$$s(h) = (b^{\kappa(h)} \circ \alpha(h)^\vee)^\sharp\colon \mathcal{O}_{P_{\kappa(h)}} \longrightarrow g_{P\circ i}^*\mathcal{O}_P(d).$$

where $i\colon P_{\kappa(h)} \hookrightarrow P_H$ is the inclusion. Now $b^{\kappa(h)}$ is an isomorphism, because $\kappa(h)$ is flat over k, and α is surjective. Thus $\alpha(h)$ is surjective and consequently $\alpha(h)^\vee \neq 0$. Hence $s(h) \neq 0$ and thus $s(h)_x$ is non-zero for all $x \in P_{\kappa(h)}$. However since every fiber $P_{\kappa(h)}$ is irreducible and reduced this shows that $s(h)$ is injective. In order to see that s is note that there is the equality

$$s(h)_x = s_{i(x)} \otimes_{\mathcal{O}_{H,h}} \mathrm{id}_{\kappa(h)}$$

as one verifies in the affine case. Thus $s_{i(x)} \neq 0$ and as P_H is irreducible and reduced this implies injectivity.
Define W by $\mathcal{O}_W := \mathrm{coker}(s) \otimes g_P^*\mathcal{O}_P(-d)$ i.e. W is defined to be the scheme theoretic support of $\mathrm{coker}\, s$. Then the sequence (3.2) is exact by definition. Note that equivalently we could define W to be the zero scheme of s in the sense of Remark 3.4. We verify that W has the desired properties: Due to [EGAIV$_3$, 11.3.8 implication *c*) ⇒ *b*)] one has that $\mathrm{coker}(s)$ is flat over H, hence W is. Furthermore the fibers W_h are hypersurfaces of degree d since by definition we have that W_h is the scheme theoretic support of $(\mathrm{coker}\, s)(h) = \mathrm{coker}(s(h))$ and this is nothing different but the vanishing locus of $s(h) \in \mathrm{Hom}(\mathcal{O}_{P_{\kappa(h)}}, \mathcal{O}_{P_{\kappa(h)}}(d)) = \Gamma(P_{\kappa(h)}, \mathcal{O}_{P_{\kappa(h)}}(d))$.
It remains to show that W is the universal flat family. Let Y be any flat family of hypersurfaces of degree d over a k-scheme T. We denote its ideal $\mathscr{I}_Y$ by $\mathscr{I}$. Using that Y is flat over T, we find for all $t \in T$ that the restriction

$\mathscr{I}_t$ of the ideal to the fiber over t is equal to the ideal $\mathscr{I}_{Y_t} \subseteq \mathcal{O}_{P_{\kappa(t)}}$. However Y_t is a hypersurface of degree d in $P_{\kappa(t)}$ and hence

$$\mathscr{I}_t \cong \mathcal{O}_{P_{\kappa(t)}}(-d).$$

Now set $\mathscr{L} := f_{T*}\mathscr{I}(d)$ by cohomology and base change (cf. [Ha77, II Theorem 12.11]) this defines is an invertible $\mathcal{O}_T$-module and there is an isomorphism

$$\mathscr{I} \cong h_P^*\mathcal{O}_P(-d) \otimes f_T^*\mathscr{L}.$$

Twisting the inclusion $\mathscr{I} \hookrightarrow \mathcal{O}_{P_T}$ with $h_P^*\mathcal{O}_P(d)$ we obtain an injection

$$s' \colon f_T^*\mathscr{L} \longrightarrow h_P^*\mathcal{O}_P(d)$$

and Y equals $Z(s')$. Consider the adjoint morphism $s'^\flat \colon \mathscr{L} \to f_{T*}h_P^*\mathcal{O}_P(d)$. We claim that it is fiberwise injective. To see this let $t \in T$ and $x \in P_{\kappa(t)}$ be a point in the fiber over t and consider the short exact sequence of $\mathcal{O}_{T,t}$-modules

$$0 \longrightarrow (f_T^*\mathscr{L})_x \xrightarrow{(s')_x} (h_P^*\mathcal{O}_P(d))_x \longrightarrow (\mathcal{O}_Y(d))_x \longrightarrow 0.$$

Since $(\mathcal{O}_Y(d))_x$ is flat as $\mathcal{O}_{T,t}$-module this leads via tensoring with $\kappa(t)$ to the exact sequence

$$0 \longrightarrow \mathcal{O}_{P_{\kappa(t)},x} \xrightarrow{(s'(t))_x} (\mathcal{O}_{P_{\kappa(t)}}(d))_x \longrightarrow (\operatorname{coker} s'(t))_x \longrightarrow 0$$

In particular $s'(t) \neq 0$ and thus $s'^\flat(t) \colon \kappa(t) \to f_{t*}\mathcal{O}_{P_{\kappa(t)}}(d)$ is non-zero, hence injective. With [EGAI, O_I 5.5.5] this implies that $s'^\flat$ has a retract on stalks and so does the composition

$$\mathscr{L} \xrightarrow{(s')^\flat} f_{T*}h_P^*\mathcal{O}_P(d) \xrightarrow{(b^T)^{-1}} \operatorname{Sym}_d(\mathcal{O}_T^{m+1}).$$

Hence its dual $h^*(\operatorname{Sym}_d(k^{m+1})^\vee) \to \mathscr{L}^\vee$ is surjective. By the universal property of the projective space it corresponds thus to a unique k-morphism $\lambda \colon T \to H$ such that $\lambda^*\alpha^\vee = (b^T)^{-1} \circ s'^\flat$. In (3.1) we computed that $\lambda^*\alpha^\vee = (b^T)^{-1} \circ (\lambda^*s)^\flat$, hence $s' = \lambda^*s$ and we find $T \times_H W = T \times_H Z(s) = Z(\lambda^*s) = Z(s') = Y$. This concludes the proof. □

3.4. **Families of r-planes in hypersurfaces.** As before let G denote the Grassmannian $\mathrm{Grass}_{r+1,m+1}$ and $H = \mathbb{P}(\operatorname{Sym}_d(k^{m+1})^\vee)$.

Theorem 3.18.

(i) *The functor*

$(\mathrm{Sch}/k)^{\mathrm{o}} \to (\mathrm{Sets})$

$$T \mapsto \left\{ (Y, L) \;\middle|\; \begin{array}{c} Y \text{ a family of degree } d \text{ hypersurfaces over } T, \\ L \text{ a family of } r\text{-planes over } T \text{ such that } L \subseteq Y \end{array} \right\}$$

is represented by a closed subscheme $Z \subseteq H \times G$, that is given as the scheme of zeros of a regular section $v \in \Gamma(H \times G, \mathcal{O}_H(1) \boxtimes \operatorname{Sym}_d(\mathcal{Q}))$.

(ii) *Let $\mathscr{K}$ be the sheaf on G that is defined by the following short exact sequence*

$$0 \to \mathscr{K} \to \operatorname{Sym}_d(\mathcal{O}_G^{m+1}) \to \operatorname{Sym}_d(\mathcal{Q}) \to 0. \tag{3.3}$$

Then $\mathscr{K}$ is a vector bundle and there is a canonical isomorphism of G-schemes

$$Z \cong \mathbb{P}(\mathscr{K}^\vee).$$

Proof. Let T be a k-scheme and (Y, L) a pair consisting of a flat family of hypersurfaces of degree d and a flat family of r-planes over T. Now $L \subseteq Y$ if and only if the composition

$$\mathscr{I}_Y \longrightarrow \mathcal{O}_{P_T} \longrightarrow \mathcal{O}_L \tag{3.4}$$

is equal to zero. By Proposition 3.17 and Proposition 3.13 there is a morphism $\lambda\colon T \to H \times G$ such that $Y = T \times_H W$ and $L = T \times_G \mathbb{P}(\mathcal{Q})$ where $q\colon G \to \operatorname{Spec} k$ is the structure morphism of the Grassmannian. From the proof of Proposition 3.17 we also know that the twist of $\mathscr{I}_Y \to \mathcal{O}_{P_T}$ by $h_P^*\mathcal{O}_P(d)$ is the pullback of $s_{H\times G} = q_{P_H}^* s = (b^{H\times G} \circ \lambda^*\alpha^\vee)^\sharp$ along λ, where the last equality follows from (3.1). Using [EGAI, O_I 3.5.3.2] we compute that $s_{H\times G}$ equals the composition

$$f_{G\times H}^* q_H^* \mathcal{O}_H(-1) \xrightarrow{\alpha^\vee_{P\times H\times G}} \operatorname{Sym}_d(\mathcal{O}_{P_{H\times G}}^{m+1}) \xrightarrow{(b^{H\times G})^\sharp} q_{P_H}^* g_P^* \mathcal{O}_P(d).$$

On the other hand $\mathcal{O}_{P_T} \to \mathcal{O}_L$ is obtained as pullback of

$$g_{P_G}^* \mathcal{O}_{P_G} \longrightarrow g_{P_G}^* \mathcal{O}_{\mathbb{P}(\mathcal{Q})}.$$

Putting all this together, the twist of (3.4) with $h_P^*\mathcal{O}_{P_T}(d)$ is equal to the pullback $\lambda^* u$ of the composition

$$u\colon f_{G\times H}^* q_H^* \mathcal{O}_H(-1) \xrightarrow{\alpha^\vee_{P\times H\times G}} f_{G\times H}^* \operatorname{Sym}_d(\mathcal{O}_{H\times G}^{m+1}) \to g_{P_G}^* \mathcal{O}_{\mathbb{P}(\mathcal{Q})}(d) \tag{3.5}$$

with adjoint

$$u^\flat\colon q_H^* \mathcal{O}_H(-1) \xrightarrow{q_H^*\alpha^\vee} \operatorname{Sym}_d(\mathcal{O}_{H\times G}^{m+1}) \to f_{G\times H*} g_{P_G}^* \mathcal{O}_{\mathbb{P}(\mathcal{Q})}(d).$$

Note that $\mathcal{O}_{\mathbb{P}(\mathcal{Q})}(d) = \mathcal{O}_{P\times G}(d)|_{\mathbb{P}(\mathcal{Q})}$ and there is an isomorphism

$$f_{G\times H*} g_{P\times G}^* \mathcal{O}_{\mathbb{P}(\mathcal{Q})}(d) \cong g_G^* f_{G*} \mathcal{O}_{\mathbb{P}(\mathcal{Q})}(d) \cong g_G^* \operatorname{Sym}_d \mathcal{Q}.$$

where the first isomorphism is obtained by base change, using that H is flat, and the second isomorphism is Lemma 2.6. By abuse of notation we also write $u^\flat$ for the composition of $u^\flat$ with this isomorphism. Hence, since $h_P^*\mathcal{O}_{P_T}(d)$ is locally free and with Lemma 3.9(i) we find that

$$L \subseteq Y \iff \lambda^*(u^\flat) = 0$$

By the universal property of the scheme of zeros (Proposition 3.3) and Remark 3.4 this is the case if and only if λ factors through $Z(v)$ where we set $v := u^\flat \otimes \mathrm{id}_{q_H^*\mathcal{O}_H(1)}$. Conversely every morphism $\lambda : T \to Z(v) \to H \times G$ defines a pair (Y, L) via $Y := T \times_H W$ and $L := T \times_G \mathbb{P}(\mathcal{Q})$. Thus $Z := Z(v)$ is the desired closed subscheme and its ideal of is given by the image of $v^\vee$ i.e. the image of the composition

$$\mathcal{O}_H(-1) \boxtimes \operatorname{Sym}_d \mathcal{Q}^\vee \to \mathcal{O}_H(-1) \boxtimes \operatorname{Sym}_d(\mathcal{O}_G^{m+1})^\vee \xrightarrow{q_H^*\alpha(-1)} \mathcal{O}_{H\times G}. \tag{3.6}$$

In order to finish the proof of part (i) it is left to show that v is a regular section. As $H \times G$ is smooth hence Cohen-Macaulay by Lemma 3.8 this is the case if and only if

$$\begin{aligned}\dim Z(v) &= \dim(H \times G) - \operatorname{rk}(\mathcal{O}_H(-1) \boxtimes \operatorname{Sym}_d \mathcal{Q}^\vee) \\ &= \dim G + \dim \operatorname{Sym}_d V + \operatorname{rk} \operatorname{Sym}_d \mathcal{Q} - 1\end{aligned}$$

and therefore follows once we have proven part (ii). This will be done calculating the ideal of $\mathbb{P}(\mathscr{K}^\vee)$. First note that $\mathscr{K}$ is locally free as the other terms of the short exact sequence (3.3) are. The dual sequence of (3.3) is

$$0 \to \operatorname{Sym}_d \mathcal{Q}^\vee \xrightarrow{w'} \operatorname{Sym}_d(\mathcal{O}_G^{m+1})^\vee \to \mathscr{K}^\vee \to 0.$$

We set $\mathscr{E} = \operatorname{Sym}_d \mathcal{Q}^\vee$ and $\mathscr{F} = \operatorname{Sym}_d(\mathcal{O}_G^{m+1})^\vee$ and consider the natural exact sequence of $\operatorname{Sym} \mathscr{F}$-modules

$$(\operatorname{Sym} \mathscr{F} \otimes \mathscr{E})[-1] \xrightarrow{w} \operatorname{Sym} \mathscr{F} \to \operatorname{Sym}(\mathscr{K}^\vee) \to 0 \qquad (3.7)$$

where $[-1]$ means that the natural grading of $\operatorname{Sym} \mathscr{F} \otimes \mathscr{E}$ is shifted by -1 and w is defined to be the composition

$$(\operatorname{Sym} \mathscr{F} \otimes \mathscr{E})[-1] \xrightarrow{\operatorname{id} \otimes w'} (\operatorname{Sym} \mathscr{F} \otimes \mathscr{E})[-1] \to \operatorname{Sym} \mathscr{F}. \qquad (3.8)$$

Applying the functor tilde to (3.7) yields by [AK75, A1.2] the exact sequence of $\operatorname{Proj}_G(\operatorname{Sym} \mathscr{F}) = H \times G$-modules

$$\mathcal{O}_H(-1) \boxtimes \mathscr{E} \xrightarrow{\widetilde{w}} \mathcal{O}_{H \times G} \to \mathcal{O}_{\mathbb{P}(\mathscr{K}^\vee)}.$$

This shows that the ideal of $\mathbb{P}(\mathscr{K}^\vee)$ is given by $\operatorname{im} \widetilde{w}$. Comparing (3.6) and (3.8) we see that $\widetilde{w} = v^\vee$ and thus $Z = \mathbb{P}(\mathscr{K}^\vee)$.

□

Proposition 3.19. *In the situation of the above theorem let F be the fiber of $Z \to H$ through $z \in Z$. The following assertions are equivalent:*

(a) *Z is flat over H in z.*
(b) *F is the zero scheme of the section $v \otimes 1$ of $g_G^* \operatorname{Sym}_d(\mathcal{Q})(y)$ that is regular in z where y is the image of z in H.*
(c) $\dim_z F = (r+1)(m-r) - \binom{d+r}{r}$
(d) $\dim_z F \leq (r+1)(m-r) - \binom{d+r}{r}$

Proof. Let $z \in Z$ with image y in H and F be the fiber over y, i.e.

$$F = Z_y = Z \cap (\{y\} \times G).$$

Then $F = Z(v \otimes 1) \subseteq \{y\} \times G$, where $v \otimes 1$ is the restriction of v to $\{y\} \times G$. Moreover, $\dim G = (r+1)(m-r)$ and $\operatorname{rk} \operatorname{Sym}_d \mathcal{Q} = \binom{d+r}{r}$. Now, as the Grassmannian is smooth and hence Cohen-Macaulay, Lemma 3.8 gives the equivalence of part b), c) and d). If Z is flat over H at the point z, then the dimension of F in z is given by $\dim Z - \dim H$ (see Lemma 3.7). As $\dim Z = \dim(H \times G) - \operatorname{rk}(\mathcal{O}_H(1) \boxtimes \operatorname{Sym}_d \mathcal{Q})$ this yields a) $\Rightarrow$ c) $\Rightarrow$b). The only implication that is left to show is b) $\Rightarrow$ a). This follows from [EGAIV$_3$, 11.3.7]. □

3.5. **The Fano scheme of lines in a cubic hypersurface.** We are ready to reap the fruits of our labor. Almost all the work needed in order to give an explicit description of the Fano scheme has been done.
In this section we fix a smooth cubic hypersurface $Y \subseteq P$. Furthermore we fix $r = 1$ and denote by G the Grassmannian $\mathrm{Grass}_{2,m+1}$.

Remark 3.20. All the below statements also hold for a hypersurface with at most isolated singularities.

Definition 3.21. The *Fano scheme* F of Y is the k-scheme parameterizing the lines in P contained in Y.

Theorem 3.22. [AK77, Theorem 1.3] *The Fano scheme F of Y exists. It is equal to the zero scheme $Z(s)$ of a regular section s of the locally free $\mathcal{O}_G$-module $\mathrm{Sym}_3(\mathcal{Q})$. Each component of F has dimension $2(m-3)$.*

Proof. By Proposition 3.19 it is enough to show that F is non-empty has the right dimension. In [AK77] non-emptiness follows from a computation of the global sections of the structure sheaf (cf. [AK77, Theorem 5.1]). By [AK77, Theorem 4.2] the Fano scheme is smooth and has the right dimension. This is shown by a computation of the conormal sheaf. Compare also [BV78, Section 3] for a more explicit computation of the map on tangent spaces induced by $Z \to H$. □

Remark 3.23. If Y is a surface i.e. $m = 3$ the dimension of F equals zero. In fact F consists exactly of 27 points. This is the famous case of the 27 lines on a cubic surface (for example see [Ha77, V Theorem 4.9]).

From now on we embed $F \subset G \subset \mathbb{P}^N$ with $N = \binom{m+1}{2}$. The following result builds on a cohomological study of $\mathrm{Sym}_d\, \mathcal{Q}^\vee$.

Proposition 3.24. [AK77, Proposition 1.15] *Let k be a field of characteristic different from 3. The canonical map*

$$\Gamma(G, \mathcal{O}_G(2)) \to \Gamma(F, \mathcal{O}_F(2))$$

is injective. Hence every quadric containing F, contains G.

Proof. Set $\mathscr{E} := \mathrm{Sym}_3(\mathcal{Q})^\vee$ and let $s \in \Gamma(G, \mathscr{E}^\vee)$ be the regular section from Theorem 3.22. Then $\mathscr{E}$ is a locally $\mathcal{O}_G$-module of rank 10 and s yields the following Koszul complex, that we have tensored with $\mathcal{O}_G(2)$

$$0 \to \bigwedge^{10} \mathscr{E}(2) \xrightarrow{f_{10}} \ldots \to \bigwedge^{2} \mathscr{E}(2) \to \mathscr{E}(2) \xrightarrow{f_1} \mathcal{O}_G(2) \xrightarrow{f_0} \mathcal{O}_F(2) \to 0. \tag{3.9}$$

with $f_1 = s^\vee(2)$. We want to prove that $H^0(f_0)$ is injective. By [Ei04, Corollary 17.5]) the Koszul complex completed by $\mathcal{O}_F$ is exact due to the fact that s is a regular section. As $\mathcal{O}_G(2)$ is a line bundle, tensoring with $\mathcal{O}_G(2)$ preserves exactness, i.e. (3.9) is exact. We split the long sequence into short exact ones by defining the $\mathcal{O}_G$-modules $M_i := \ker f_{i-1} = \mathrm{im}\, f_i$. This gives

$$0 \to \quad M_1 \quad \to \ \mathcal{O}_G(2) \ \to \mathcal{O}_F \to 0 \tag{3.10}$$

$$0 \to \quad M_{i+1} \quad \to \bigwedge^i \mathscr{E}(2) \to M_i \to 0 \quad \text{for } i = 1, \dots, 8 \tag{3.11}$$

$$0 \to \bigwedge^{10} \mathscr{E}(2) \to \bigwedge^9 \mathscr{E}(2) \to M_9 \to 0. \tag{3.12}$$

Taking global sections of (3.10) yields

$$0 \to H^0(M_1) \to H^0(\mathcal{O}_G(2)) \xrightarrow{H^0(f_0)} H^0(\mathcal{O}_F(2)) \to \dots$$

hence we want to show that $H^0(M_1) = 0$. Considering (3.11) with $i = 1$ gives the long exact sequence

$$0 \to H^0(M_2) \to H^0(\mathscr{E}(2)) \to H^0(M_1) \to H^1(M_2) \to \dots$$

hence it is enough to show $H^0(\mathscr{E}(2)) = 0$ and $H^1(M_2) = 0$. Continuing this way we see, that it suffices to show that

$$H^i(\bigwedge^{i+1} \mathscr{E}(2)) = 0 \text{ for } i = 0, \dots, 10.$$

Fortunately this was done in the literature. However $\operatorname{char} k \neq 3$ is needed. For example see [AK77, Theorem 5.1]. □

4. Proof of Theorem 1.1

Before giving a complete proof of Theorem 1.1 we need one more result.

Theorem 4.1. *Let Y be a cubic hypersurface in $P = \mathbb{P}_k^m$ with $m \geq 4$. Then any regular point $x \in Y$ lies on a line of P that lies entirely in Y.*

Proof. We choose coordinates such that $x = [1 : 0 : \dots : 0]$. The lines of P passing through x can be identified with the unique intersection point with any projective $m-1$ plane in P not containing x. We fix $P_0 := V_+(X_0)$. Moreover let $Y = V_+(f)$ for $f \in k[X_0, \dots, X_m]_3$ (i.e. a homogeneous polynomial of degree 3) and define

$$g := \frac{\partial f}{\partial X_0}(0, X_1, \dots, X_m) \in k[X_1, \dots, X_m].$$

Since $m - 4 \geq 0$ we find that

$$P_0 \cap Y \cap T_x Y \cap V_+(g) \neq \emptyset$$

where $T_x Y$ is embedded into P as the vanishing set of $\sum_{i=0}^m \frac{\partial f}{\partial X_i}\Big|_x X_i$ (see [Ku97, VII.§1]). Let $y = [0 : y_1 : \dots : y_m]$ be an element in this intersection. In particular we have $y \in Y$. Now consider the line L through x and y. We claim that it lies entirely in $T_y Y = V_+(\sum_{i=0}^m \frac{\partial f}{\partial X_i}\Big|_y X_i)$. Indeed we find that

$$L(k) = \{[t : sy_0 : \dots : sy_m] \mid t, s \in k\} \subseteq P(k)$$

and therefore

$$L \subseteq T_y Y \Leftrightarrow t \frac{\partial f}{\partial X_0}(y_0, \dots, y_m) + s \sum_{i=1}^m \frac{\partial f}{\partial X_i}(y_0, \dots, y_m) y_i = 0 \text{ for all } s, t \in k.$$

Using Euler's relation we see that the right hand side is equal to

$$tg(y) - s\frac{\partial f}{\partial X_0}(y_0, \dots, y_m)y_0$$

hence zero.
Alternatively we could restrict to an affine neighborhood of y isomorphic to some $\mathbb{A}^m_k$ (take $D_+(T_j)$ for j such that $y_j \neq 0$) and choose coordinates such that $y = (0, \dots, 0)$. The line through x and y is then given by

$$L(k) = \{t(1, 0, \dots, 0) \mid t \in k\}) \subseteq \mathbb{A}^m_k(k)$$

and $T_yY \cap \mathbb{A}^m_k = V(\sum_{i=0}^m \frac{\partial f(T_0, T_1+y_1, \dots, 1, \dots, T_m+y_m)}{\partial T_i}\Big|_0 T_i)$. Hence

$$L \subseteq T_yY \cap \mathbb{A}^m_k \Leftrightarrow \frac{\partial f(T_0, T_1 + y_1, \dots, 1, \dots, T_m + y_m)}{\partial T_0}\Big|_0 = 0.$$

However the right hand side is $g(y)$, hence zero.
But this means that we found two distinct points x, y in the intersection of $L \cap Y$ with intersection multiplicity greater than two, i.e. L intersects Y with multiplicity greater than 4. Therefore $L \subseteq Y$; otherwise we would have produced a contradiction to Bézout's theorem ([Ha77, I Theorem 7.7]). □

Remark 4.2. The theorem is wrong if $m = 3$ (cf. Remark 3.23).

Proof of Theorem 1.1. Again let $G = \mathrm{Grass}_{2,m+1}$, consider F and F' as subschemes of G and G as a subscheme of $\mathbb{P}(\bigwedge^2 V) \cong \mathbb{P}^N_k$ where $V = k^{m+1}$ and $N = \binom{m+1}{2} - 1$ as in Proposition 2.7. By Remark 2.9 the image of G in $\mathbb{P}^N_k$ is given by quadratic equations. Therefore G is the intersection of some quadrics in $\mathbb{P}^N_k$. On the other hand by Proposition 3.24 every quadric containing F (or F' respectively) contains G. Hence G is the intersection of all quadrics in $\mathbb{P}^N_k$ that contain F (resp. F'). This implies that any given automorphism $g\colon \mathbb{P}^N_k \to \mathbb{P}^N_k$ restricts to an automorphism of the Grassmannian, as it sends quadrics containing F to quadrics containing F'.
Now Chow's theorem (cf. Theorem 2.10 or Corollary 2.19) gives the existence of an automorphism f of P such that $(f \wedge f)\,|_G = g\,|_G$. This means that for every line L in P with $L \subseteq Y$, and therefore corresponding to a point $\ell \in F$, we have that $f(L)$ corresponds to the point $g(\ell) \in F'$ and is thus a line in Y'. As by Theorem 4.1 any point of Y lies on a line, we conclude that f maps Y to Y' as desired. □

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