Mirror symmetry for K3 surfaces

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$\mathbb{R}$-planes versus $\mathbb{C}$-lines

$W = \mathbb{R}$-vector space, $\langle\ ,\ \rangle$ definite symmetric bilinear, of $\mathrm{sign} = (k, \ell)$, $k \geq 2$, $\ell > 0$.

$$\mathrm{Gr}_2^{\mathrm{po}}(W) := \{P \subset W \mid \dim P = 2,\ \text{oriented},\ \langle\ ,\ \rangle|_P \text{ positive}\}$$

$$Q_W := \{\varphi \mid \langle \varphi, \varphi \rangle = 0,\ \langle \varphi, \bar{\varphi} \rangle > 0\} \subset \mathbb{P}(W_{\mathbb{C}})$$

Then: $\varphi \longmapsto P_\varphi := \mathrm{Re}(\varphi)\mathbb{R} \oplus \mathrm{Im}(\varphi)\mathbb{R}$ induces

$$Q_W \xrightarrow{\sim} \mathrm{Gr}_2^{\mathrm{po}}(W).$$

(Inverse: Pick oriented ON basis $\alpha, \beta \in P$ and $P \longmapsto \alpha + i\beta$.)

Connected for $k \geq 3$. Two connected components for $k = 2$.

Apply to $W = \bigwedge^{2*} V$ where $V = \mathbb{R}$-vector space, $\dim V = 4$. Fix $\bigwedge^4 V = \mathbb{R}$ and let $\langle \ , \ \rangle$ be the Mukai pairing:

$$\langle \varphi_0 + \varphi_2 + \varphi_4, \psi_0 + \psi_2 + \psi_4 \rangle = \varphi_2 \wedge \psi_2 - \varphi_0 \wedge \psi_4 - \varphi_4 \wedge \psi_0.$$

Recall: $\varphi \in \bigwedge^{2*} V_{\mathbb{C}}$ is gen. CY structure if $\varphi \in Q := Q_{\bigwedge^{2*} V}$.

Corollary: $\varphi \longmapsto P_\varphi$ induces

$$\{\mathbb{C}\varphi \mid \varphi \text{ gen. CY}\} \simeq Q \simeq \mathrm{Gr}_2^{\mathrm{po}}\left(\bigwedge^{2*} V\right).$$

HK pair: $\varphi, \varphi' \in \bigwedge^{2*} V_{\mathbb{C}}$ gen. CY structures such that

$$P_\varphi \perp P_{\varphi'} \ \text{ and } \ \langle \varphi, \bar{\varphi} \rangle = \langle \varphi', \bar{\varphi}' \rangle.$$

Then $\Pi_{\varphi,\varphi'} := P_\varphi \oplus P_{\varphi'} \subset \bigwedge^{2*} V$ oriented positive four-space.

$V = T_x^* M$, $M = K3$

HK pair on M: $\varphi, \varphi' \in \mathcal{A}^{2*}(M)_{\mathbb{C}}$ gen. CY structures, which form HK pair in every point $x \in M$.

Examples:

i) $(\varphi = \sigma, \varphi' = \exp(B + i\omega))$ with σ holomoprhic two-form on $X = (M, I)$, $B \in \mathcal{A}^{1,1}(M)_{\mathrm{cl}}$ real, and ω HK-form with $2\omega^2 = \sigma \wedge \bar{\sigma}$.

ii) $\exists$ HK pairs of the form $(\exp(B + i\omega), \exp(B' + i\omega'))$.

iii) If (φ, φ') HK pair, $B \in \mathcal{A}^2(M)$ closed, then also $(\exp(B) \cdot \varphi, \exp(B) \cdot \varphi')$ HK pair.

iv) If (φ, φ') HK pair, then $\exists$ $B \in \mathcal{A}^2(M)$ closed st. $\Pi_{\varphi,\varphi'} = \exp(B) \cdot \Pi_{\sigma,\exp(i\omega)}$ with $(\sigma, \exp(i\omega))$ as in i).

Moduli and periods

Moduli space: $\mathfrak{M} := \mathfrak{M}_{(2,2)} := \{(\varphi, \varphi') \mid \mathrm{HK\ pair}\}/\simeq$, where $(\varphi_1, \varphi_1') \simeq (\varphi_2, \varphi_2')$ if $\exists\ f \in \mathrm{Diff}_*(M)$, $B \in \mathcal{A}^2(M)_{\mathrm{ex}}$, st. $(\varphi_1, \varphi_1') = \exp(B) \cdot f^*(\varphi_2, \varphi_2')$.

Recall: $\widetilde{\mathfrak{N}} = \{\mathbb{C}\varphi\}/\simeq$. Thus,

$$\mathfrak{M}/(\mathbb{C}^* \times S^1) \hookrightarrow \widetilde{\mathfrak{N}} \times \widetilde{\mathfrak{N}}.$$

Period map: Recall $\varphi \longmapsto [\varphi] \longmapsto P_{[\varphi]} := [\mathrm{Re}(\varphi)]\mathbb{R} \oplus [\mathrm{Im}(\varphi)]\mathbb{R}$ yields

$$\widetilde{\mathfrak{N}} \longrightarrow \widetilde{Q} \simeq \mathrm{Gr}_2^{\mathrm{po}}(\widetilde{H}(M, \mathbb{R}))$$

which is essentially bijective.
Similarly, $(\varphi, \varphi') \longmapsto (P_{[\varphi]}, P_{[\varphi']})$ defines

$$\mathfrak{M}/(\mathbb{C}^* \times S^1) \hookrightarrow \mathrm{Gr}_{2,2}^{\mathrm{po}}(\widetilde{H}(M, \mathbb{R})).$$

Here $\mathrm{Gr}_{2,2}^{\mathrm{po}}(\widetilde{H}(M, \mathbb{R})) = \{(R, R') \mid R \perp R'\} \subset \mathrm{Gr}_2^{\mathrm{po}} \times \mathrm{Gr}_2^{\mathrm{po}}$.

$\Gamma := H^2(M,\mathbb{Z})$, $\widetilde{\Gamma} := \widetilde{H}(M,\mathbb{Z}) = \Gamma \oplus U$

Example: If $\sigma \in H^{2,0}(X)$ and $\alpha \in \mathcal{C}_X$ orthogonal to (-2)-curve, then $(P_{[\sigma]}, P_{\exp(i\alpha)}) \notin \mathfrak{M}$.
Note that $\mathfrak{M}$ contains complement of $\bigcup \delta^\perp$ with (-2)-class $\delta \in \widetilde{H}(M,\mathbb{Z})$ and $\delta^\perp := \{(R, R') \mid R, R' \subset \delta^\perp\}$.

Recall: $\widetilde{\mathrm{O}} := \mathrm{O}(\widetilde{\Gamma})$ acts on $\widetilde{Q} \subset \mathbb{P}(\widetilde{H}(M,\mathbb{C}))$ but also on $\mathrm{Gr}^{\mathrm{po}}_{2,2}(\widetilde{H}(M,\mathbb{R}))$ by

$$(P, P') \longmapsto (gP, gP').$$

Easy: The $\widetilde{\mathrm{O}}$-action on $\mathrm{Gr}^{\mathrm{po}}_{2,2}(\widetilde{H}(M,\mathbb{R}))$ preserves $\mathfrak{M} \setminus \bigcup \delta^\perp$.

Why $\widetilde{\mathrm{O}}$?

i) C.T.C. Wall: $\widetilde{\mathrm{O}}$ generated by:

- $\mathrm{O} := \mathrm{O}(\Gamma) \leftrightsquigarrow$ isomorphisms of K3 surfaces (Global Torelli)
- $\mathrm{O}(U) = \langle -\mathrm{id}, -T_{\mathcal{O}} : e_1 \leftrightarrow e_2 \rangle \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
- $\{\exp(B) \mid B \in \Gamma\}$.

ii) $\widetilde{\mathrm{O}} \subset \mathrm{O}(\widetilde{\Gamma}_{\mathbb{R}})$ maximal.

Mirror symmetry for two HK pairs $(\varphi_j, \varphi'_j) \in \mathfrak{M}$ with periods (P_j, P'_j), $j = 1, 2$ is an isometry $g \in \widetilde{\mathrm{O}}$ such that

$$g(P_1, P'_1) = (P'_2, P_2).$$

(φ_1, φ'_1) and (φ_2, φ'_2) are *mirror partners*.

Classical case: $(\varphi_j = \sigma_j, \varphi'_j = \exp(i\omega_j))$, $j = 1, 2$:

$$g : \begin{cases} \sigma_1 & \leftrightarrow & exp(i\omega_2) \\ \exp(i\omega_1) & \leftrightarrow & \sigma_2 \end{cases}$$

Complex and symplectic structures are interchanged.

Projections: $(\varphi, \varphi') \longmapsto \varphi$ (resp. $\longmapsto \varphi'$) yield $\widetilde{\mathrm{O}}$-equivariant maps:

$$\widetilde{\mathfrak{N}} \xleftarrow{p_1} \mathfrak{M} \xrightarrow{p_2} \widetilde{\mathfrak{N}}.$$

Remarks:

- If (φ_1, φ_1'), (φ_2, φ_2') are mirrors, then so are (φ_1, φ_1'), $(h\varphi_2, h\varphi_2')$ for all $h \in \widetilde{\mathrm{O}}$.
- If (φ_1, φ_1'), (φ_2, φ_2') are mirrors, then $\varphi_2' \in \widetilde{\mathrm{O}}\varphi_1$ and $\varphi_2 \in \widetilde{\mathrm{O}}\varphi_1'$.
- If $\varphi_2' \in \widetilde{\mathrm{O}}\varphi_1$, then $\exists\ \varphi_1', \varphi_2$ st. (φ_1, φ_1'), (φ_2, φ_2') are mirrors.

HMS: Want to associate to $\varphi \in \widetilde{\mathfrak{N}}$ a certain category ($\mathbb{C}$-linear, triangulated, A_∞, ...) $D(\varphi)$ st.

$$(\varphi_1, \varphi_1'), (\varphi_2, \varphi_2') \text{ mirrors} \quad \Leftrightarrow \quad D(\varphi_1) \simeq D(\varphi_2'),\ D(\varphi_1') \simeq D(\varphi_2)$$

In other words: $\varphi \in \widetilde{\mathrm{O}}\psi \Leftrightarrow D(\varphi) \simeq D(\psi)$.

Problem: What is $D(\varphi)$?

Naive answers:

i) $\varphi = \sigma \in H^{2,0}(X)$ with $X = (M, I)$ projective:
$D(\varphi) := \mathrm{D}^{\mathrm{b}}(X)$.

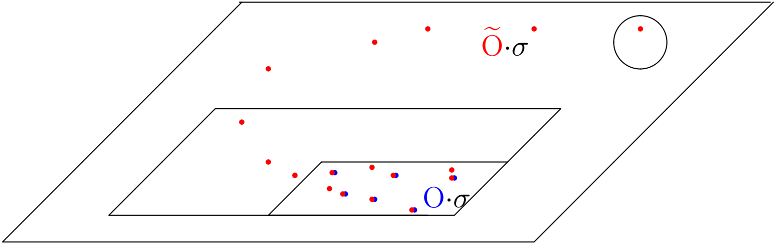
ii) $\varphi = \exp(B) \cdot \sigma$ with σ as in i), $\alpha_B := \exp(B^{0.2}) \in \mathrm{Br}(X)$:
$D(\varphi) := \mathrm{D}^{\mathrm{b}}(X, \alpha_B)$.

iii) $\varphi = \exp(i\omega)$: $D(\varphi) := \mathrm{D}^{\pi} \mathit{Fuk}(M, i\omega)$.

Warning: If X is not projective or α not torsion, then $\mathrm{D}^{\mathrm{b}}(X, \alpha)$ is too small.

GT and HMS

$\widetilde{\mathrm{O}}$-orbits



Recall: Derived GT for projective K3s: $\mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{D}^{\mathrm{b}}(X')$
$\Leftrightarrow \widetilde{H}(X, \mathbb{Z}) \simeq \widetilde{H}(X', \mathbb{Z})$
$\Leftrightarrow \sigma \in \widetilde{\mathrm{O}}\sigma'$, where $\sigma \in H^{2,0}(X)$, $\sigma' \in H^{2,0}(X')$.

Conclusion: Derived GT confirms HMS. Similarly for derived GT for twisted K3s.

$\Gamma := H^2,\ \widetilde{\Gamma} := \Gamma \oplus U \simeq \widetilde{H},\ U := -(H^0 \oplus H^4)$

Assumption: $N \subset \Gamma$, $N' \subset \Gamma$ st. $N^\perp = N' \oplus U'$, $\mathrm{sign}(N) = (1, \quad)$.
($\Rightarrow \Gamma = U'^\perp \oplus U'$ and $N \oplus N' \subset U'^\perp$ has finite index.)

Example: Write $\Gamma = 2(-E_8) \oplus U_1 \oplus U_2 \oplus U'$ and let $\alpha = e_1 + 2e_2 \in U_1$. Then $(\alpha, \alpha) = 4$. Choose $N' = \mathbb{Z}\alpha$ and $N = 2(-E_8) \oplus \mathbb{Z}(e_1 - 2e_2) \oplus U_2$.

Introduce: $\mathfrak{N}_N \subset \mathfrak{N} \subset \widetilde{\mathfrak{N}}$ and $\mathfrak{M}_{N,N'} \subset \mathfrak{M}$.

$$\mathfrak{N}_N := \{\varphi = \sigma \mid \sigma \in (N' \oplus U')_{\mathbb{C}}\} = \{(X, \eta) \mid N \subset \eta(Pic(X))\}$$

$$\mathfrak{M}_{N,N'} := \{(\varphi, \varphi') \mid \varphi \in \mathfrak{N}_N, \varphi' \in (N \oplus U)_{\mathbb{C}}\}.$$

Study projection $p_N : \mathfrak{M}_{N,N'} \longrightarrow \mathfrak{N}_N$!

$\widetilde{\Gamma} = \Gamma \oplus U,\ N \oplus N' \oplus U' \subset \Gamma$

Suppose $\sigma = X \in \mathfrak{N}_N$. Then

$$\begin{aligned} p_N^{-1}(X) &\simeq \{\varphi' \in (N \oplus U)_{\mathbb{C}} \cap \sigma^{\perp} \text{ gen. CY}\} \\ &\simeq \{\lambda \exp(B + i\omega) \in (N \oplus U)_{\mathbb{C}} \cap \sigma^{\perp},\ \omega = \text{ Kähler}\} \\ &\simeq \mathbb{C}^* \times N_{\mathbb{R}} \times i \left(N_{\mathbb{R}} \cap (\mathcal{C}_X \setminus \bigcup_{C \simeq \mathbb{P}^1} [C]^{\perp}) \right). \end{aligned}$$

More natural: $p_N^{-1}(X)_0 := p_N^{-1}(X) \setminus \bigcup_{C \simeq \mathbb{P}^1} [C]^{\perp}$.

Recall: If $\mathrm{Pic}(X) = N$ (i.e. $X \in \mathfrak{N}_N$ generic), then

$$\mathcal{P}^+(X) := \{\psi \in (N \oplus U)_{\mathbb{C}} \mid \langle \mathrm{Re}(\psi), \mathrm{Im}(\psi) \rangle \text{ positive}\}^{\circ}.$$

Section of the $\mathrm{Gl}_2^+(\mathbb{R})$-action is given by

$$\mathcal{Q}^+(X) = \{\exp(B + i\omega) \mid \omega \in N_{\mathbb{R}} \cap \mathcal{C}_X,\ B \in N_{\mathbb{R}}\}.$$

Thus,

$$p_N^{-1}(X)_0 \subset p_N^{-1}(X) \subset \mathbb{C}^* \cdot \mathcal{Q}^+(X) \subset \mathcal{P}^+(X).$$

$\mathcal{P}_0^+(X) \leftrightsquigarrow p_N^{-1}(X)$, $X \in \mathfrak{M}_N$ general

Recall: Period domain for distinguished stability conditions

$$\mathcal{P}_0^+(X) := \mathcal{P}^+(X) \setminus \bigcup \delta^\perp,$$

where $\delta \in N \oplus U$ runs through all (-2)-classes.

Thus

$$\mathcal{P}_0^+(X) = \left(\mathrm{Gl}_2^+(\mathbb{R}) \cdot p_N^{-1}(X)_0\right) \setminus \bigcup_{\delta \notin N} \delta^\perp \subset \mathrm{Gl}_2^+(\mathbb{R}) \cdot p_N^{-1}(X)_0.$$

Since $\pi_1(\mathrm{Gl}_2^+(\mathbb{R})) = \pi_1(\mathbb{C}^*) = \mathbb{Z}$ and $\mathbb{C}^* \cdot p_N^{-1}(X)_0 = p_N^{-1}(X)_0$, the map

$$\pi_1(\mathcal{P}_0^+(X)) \twoheadrightarrow \pi_1(p_N^{-1}(X)_0)$$

forgets only the loops around all $\delta^\perp$ for $\delta \notin N$.

$(N, N') \leftrightarrow (N', N)$, $g : U \leftrightarrow U'$

Fix: N, N', U' as before and $g \in \widetilde{\mathrm{O}}$ extending $U' \simeq U$ st. $g|_{N \oplus N'} = \mathrm{id}$.

Study the particular mirror symmetry map

$$\iota \circ g : \mathfrak{M}_{N,N'} \dashrightarrow \mathfrak{M}_{N',N},$$

where $\iota(\varphi, \varphi') = (\varphi', \varphi)$.

Note $\varphi \in (N' \oplus U')_{\mathbb{C}} \Leftrightarrow g\varphi \in (N' \oplus U)_{\mathbb{C}}$.

Study image of

$$\mathfrak{M}_{N,N'} \supset p_N^{-1}(X)_0 \xrightarrow{\iota \circ g} \mathfrak{M}_{N',N} \xrightarrow{p_{N'}} \mathfrak{N}_{N'}.$$

$p_N^{-1}(X) \longrightarrow \mathfrak{N}_{N'}$

Suppressing $\iota \circ g$ in the notation:

$$\begin{array}{ccccc} \mathcal{P}_0^+(X) & \hookrightarrow & \mathrm{Gl}_2^+(\mathbb{R}) \cdot p_N^{-1}(X)_0 & \hookleftarrow & p_N^{-1}(X)_0 \\ & {\scriptstyle q}\searrow & \downarrow & \swarrow{\scriptstyle q'}\ {\scriptstyle \mathbb{C}^*} & \\ & & \mathfrak{N}_{N'} & & \end{array}$$

Image of $\mathcal{P}_0^+(X)$: All $\tau = Y \in \mathfrak{N}_{N'}$ st. there is no (-2)-class in $H^{1,1}(Y, \mathbb{Z})$ which is orthogonal to N'.

Image of $p_N^{-1}(X)_0$: All $\tau = Y \in \mathfrak{N}_{N'}$ st. there is no (-2)-class in $H^{1,1}(Y, \mathbb{Z})$ which is orthogonal to N' *and which is contained in* N.

$$\begin{array}{ccc} \pi_1(\mathcal{P}_0^+(X)) & \twoheadrightarrow & \pi_1(p_N^{-1}(X)_0) \\ \mathbb{Z}\ \big\downarrow & & \big\downarrow\ \mathbb{Z} \\ \pi_1(q(\mathcal{P}_0^+(X))) & \twoheadrightarrow & \pi_1(q(p_N^{-1}(X)_0)). \end{array}$$

Symplectic monodromies

Easiest case: Let $N' = \mathbb{Z}\alpha$ with $(\alpha, \alpha) = 4$. Generic $\tau = Y \in \mathfrak{N}_{N'} =: \mathfrak{N}_\alpha$ is quartic $Y \subset \mathbb{P}^3$ with Picard group $\simeq \mathbb{Z}$ generated by $\alpha = \mathrm{c}_1(\mathcal{O}(1))$.

The class α represents an ample class as long as $(\alpha . C) \neq 0$ for all $\mathbb{P}^1 \simeq C \subset Y$. Thus

$$\mathfrak{N}_4 := \text{moduli space of marked quartics} \simeq q(\mathcal{P}_0^+(X)) \subset \mathfrak{N}_\alpha.$$

Consider universal family $\mathcal{Y} \longrightarrow \mathfrak{N}_4$ as a family of symplectic manifolds (use restriction of Fubini–Study Kähler form). Pick special fibre $Y \subset \mathcal{Y}$ and consider monodromy operation.

Seidel: Symplectic monodromy operation yields

$$\pi_1(\mathfrak{N}_4) \longrightarrow \pi_0(\mathrm{Sympl}(Y)) \longrightarrow \mathrm{Aut}(\mathrm{D}^\pi \mathit{Fuk}(Y, \alpha))/[2].$$

Complex mirror

Consider: $Z_q \subset \mathbb{P}^3$, $x_0, \ldots, x_3$ coordinates:

$$\prod x_i + q \sum x_i^4 = 0$$

as family $\mathcal{Z} \longrightarrow \mathbb{A}^1$, smooth over $\mathbb{A}^1 \setminus \{0\}$.

Fibres:

- $1/q = 0$: Fermat quartic $\sum x_i^4 = 0$.
- $q = 0$: maximal degeneration $\bigcup H_i$ =union of four hyperplanes.
- General fibre Z_K over $K = \overline{\mathbb{C}((t))}$.

G-**action:** $G = \{(a_i) \in (\mathbb{Z}/4\mathbb{Z})^4 \mid \sum a_i = 0\}/(\mathbb{Z}/4\mathbb{Z}) \subset \mathrm{PSl}(4)$ acts on Z_q and $\mathcal{Z}$.

Quotient, resolution: Consider quotient $\mathcal{Z}/G \longrightarrow \mathbb{A}^1$ and its general fibre Z_K/G. Let $X \longrightarrow Z_K/G$ be the minimal resolution (over K).

Seidel: Up to an automorphism of $\mathbb{C}[[t]]$:

$$\mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{D}^{\pi} Fuk(Y)$$

as K-linear triangulated categories (Recall: $Y \subset \mathbb{P}^3$ quartic).

Fukaya category: $\mathrm{D}^{\pi} Fuk(Y)$ split-closed derived Fukaya category. ($K \leftrightarrow$ area of pseudo-holomorphic curves).

Split generators: Restriction of $\Omega^i_{\mathbb{P}^3_K}(i)$, $i = 0, \ldots, 3$ to $Z_K \subset \mathbb{P}^3_K$ together with 16 linearizations split generate $\mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{D}^{\mathrm{b}}_G(Z_K)$ (Kapranov, Vasserot).

Seidel: Find similar generators on Fukaya side.

The two categories are shown to be deformations of one explicit category, which has only one non-trivial deformation. The automorphism of $\mathbb{C}[[t]]$ comes in here.