Autoequivalences of K3 surfaces

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Chern and Todd

Chern character: If $E \in \mathrm{D}^{\mathrm{b}}(X)$, then

$$\mathrm{ch}(E) := \sum (-1)^i \mathrm{ch}(\mathcal{H}^i(E)) \in H^{*,*}(X) \cap H^*(X, \mathbb{Q}).$$

For line bundles L_i one has $\mathrm{ch}(\bigoplus L_i) = \sum e^{\mathrm{c}_1(L_i)}$. Use the splitting principle and locally free resolutions for the general case.Explicitly:

$$\mathrm{ch} = \mathrm{rk} + \mathrm{c}_1 + \frac{1}{2}(\mathrm{c}_1^2 - 2\mathrm{c}_2) + \frac{1}{6}(\mathrm{c}_1^3 - 3\mathrm{c}_1\mathrm{c}_2 + 3\mathrm{c}_3) + \dots$$

Todd character: For line bundles $\mathrm{td}(\bigoplus L_i) = \prod \frac{\mathrm{c}_1(L_i)}{1 - e^{-\mathrm{c}_1(L_i)}}$.

$$\mathrm{td}(X) = 1 + \frac{1}{2}\mathrm{c}_1(X) + \frac{1}{12}(\mathrm{c}_1^2(X) + \mathrm{c}_2(X)) + \frac{1}{24}\mathrm{c}_1(X)\mathrm{c}_2(X) + \dots$$

HHR and GRR

Hirzebruch–Riemann–Roch: $\chi(E) = \int_X \mathrm{ch}(E) \cdot \mathrm{td}(X)$.

Grothendieck–Riemann–Roch: For $p : X \longrightarrow Y$ and $E \in \mathrm{D^b}(X)$:

$$\mathrm{ch}(Rp_* E).\mathrm{td}(Y) = p_* \left(\mathrm{ch}(E).\mathrm{td}(X)\right).$$

Example: For $X = Y \times Z$ and $E \in \mathrm{D^b}(Z)$ one has $\mathrm{ch}(Rp_* q^* E).\mathrm{td}(Y) = p_*(\mathrm{ch}(q^* E).\mathrm{td}(Y \times Z))$ and hence $\mathrm{ch}(Rp_* q^* E) = \chi(Z, E)[Y]$.

Mukai vector

Since $\mathrm{td} = 1 + \ldots$, the square root $\sqrt{\mathrm{td}(X)} = 1 + \frac{1}{4}\mathrm{c}_1(X) + \cdot$ exists.

K3 surface: $\mathrm{td}(X) = 1 + 0 + 2[x]$ and $\sqrt{\mathrm{td}(X)} = 1 + 0 + [x]$.

For any $E \in \mathrm{D}^{\mathrm{b}}(X)$ one defines the *Mukai vector*

$$v(E) := \mathrm{ch}(E) \cdot \sqrt{\mathrm{td}(X)} \in H^{*,*}(X) \cap H^*(X, \mathbb{Q}).$$

Note: $v(E[1]) = -v(E)$.

Let $\Phi_{\mathcal{E}} : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(X')$ with FM kernel $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times X')$. The induced *cohomological FM transform* is

$$\Phi^H_{\mathcal{E}} : H^*(X, \mathbb{Q}) \longrightarrow H^*(X', \mathbb{Q}), \quad \alpha \longmapsto p_*(q^*\alpha \cdot v(\mathcal{E})).$$

How FM acts on H^*

Then:

- $v(\Phi_{\mathcal{E}}(E)) = \Phi_{\mathcal{E}}^H(v(E)) \in H^*(X', \mathbb{Q})$ for all $E \in \mathrm{D}^{\mathrm{b}}(X)$. (Use GRR.)
- If $\Phi_{\mathcal{E}}$ is an equivalence, then $\Phi_{\mathcal{E}}^H$ is bijective. (Use $\Phi_{\mathcal{O}_\Delta}^H = \mathrm{id}$.)
- Hodge structure: $H^n = \bigoplus_{p+q=n} H^{p,q}$. But

$$\Phi_{\mathcal{E}}^H(H^{p,q}(X)) \subset \bigoplus_{p-q=r-s} H^{r,s}(X').$$

Corollary: If $\mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{D}^{\mathrm{b}}(X')$ and $X = $ K3 surface, then $X' = $ K3 surface (and not abelian).

Mukai pairing

Mukai, Caldararu: For $v = \sum v_j \in H^*(X, \mathbb{Q})$ define $v^\vee := \sum \sqrt{-1}^j v_j \in H^*(X, \mathbb{C})$ and the *Mukai pairing*:

$$\langle v, w \rangle := -\int_X \exp(\mathrm{c}_1(X)/2) \cdot v^\vee \cdot w.$$

HRR again: If $\chi(E, F) := \sum (-1)^i \dim \mathrm{Ext}^i(E, F) = \chi(E^\vee \otimes F)$ for $E, F \in \mathrm{D^b}(X)$ and $E^\vee$ the derived dual, then

$$\chi(E, F) = -\langle v(E), v(F) \rangle.$$

Then: For any equivalence $\Phi_{\mathcal{E}} : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X')$

$$\Phi_{\mathcal{E}}^H : H^*(X, \mathbb{Q}) \xrightarrow{\sim} H^*(X', \mathbb{Q})$$

is an isomorphism of vector spaces compatible with the Mukai pairing on X and X', resp.

Special case: K3 surfaces

$\langle\ ,\ \rangle =$ Mukai pairing seen already for generalized CY structures.

Mukai: Let $\Phi_{\mathcal{E}} : \mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(X')$ with $X =$ K3 surface. Then

- $\Phi_{\mathcal{E}}^{H}$ is an isometry defined over $\mathbb{Z}$.
- $\Phi_{\mathcal{E}}^{H}$ respects the natural weight two Hodge structure $\widetilde{H}$ on $H^*(X, \mathbb{Z})$. ($\Leftrightarrow \Phi_{\mathcal{E}}^{H}(H^{2,0}(X)) = H^{2,0}(X')$)

Corollary: For K3 surface X one has representation

$$\rho : \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \longrightarrow \mathrm{O}(\widetilde{H}(X, \mathbb{Z})).$$

Here, $\mathrm{O}(\widetilde{H}(X, \mathbb{Z}))$ denotes the group of *Hodge isometries*.

Autoequivalences for K3s: easy ones

- Automorphism $f : X \xrightarrow{\sim} X$:
  $\rightsquigarrow \Phi_{\mathcal{E}} := f_* : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X)$, $\mathcal{E} = \mathcal{O}_{\mathrm{Graph}(f)}$
  $\rightsquigarrow \Phi_{\mathcal{E}}^H = f_*$.
  Torelli: $f_* = \mathrm{id} \Leftrightarrow f = \mathrm{id}$.
- Shift: $\Phi_{\mathcal{E}} : F \longmapsto F[1]$, $\mathcal{E} = \mathcal{O}_\Delta[1]$, $\Delta \subset X \times X$ diagonal.
  $\rightsquigarrow \Phi_{\mathcal{E}}^H = -\mathrm{id}$.
  $\Phi_{\mathcal{E}}^{H^2} = \mathrm{id}$.
- Tensor product with line bundle $L \in \mathrm{Pic}(X)$:
  $\rightsquigarrow \Phi_{\mathcal{E}} : F \longmapsto F \otimes L$, $\mathcal{E} = \iota_* L$, $\iota : X \simeq \Delta \subset X \times X$.
  $\rightsquigarrow \Phi_{\mathcal{E}}^H = \mathrm{ch}(L) \cdot \quad = \exp(\mathrm{c}_1(L)) \cdot \quad .$
  $\Phi_{\iota_* L} = \mathrm{id} \Leftrightarrow L \simeq \mathcal{O}_X$.

Autoequivalences for K3s: interesting ones

- Spherical twist: $E \in \mathrm{D}^{\mathrm{b}}(X)$ with $\mathrm{Ext}^*(E,E) = H^*(S^2,\mathbb{C})$
  $\rightsquigarrow \Phi_{\mathcal{E}} = T_E$, $\mathcal{E} = \mathrm{C}(E^\vee \boxtimes E \longrightarrow \mathcal{O}_\Delta)$.
  $\rightsquigarrow T_E^H = s_{v(E)} =$ reflection in hyperplane $v(E)^\perp$.
  $(T_E^2)^H = \mathrm{id}$, but $T_E^2 \neq \mathrm{id}$: $T_E^2(E) = E[-2]$.

Examples: i) $\mathbb{P}^1 \simeq C \subset X$, $E = \mathcal{O}_C(i)$ spherical with $v(\mathcal{O}_C(i)) = (0,[C], i+1)$ and $T^H_{\mathcal{O}_C(i)} = s_{[C]}$ on H^2.
ii) $T^H_{\mathcal{O}}(r,\ell,s) = (-s,\ell,-r)$.

- Universal family $\mathcal{E} \in \mathrm{Coh}(X \times M)$ of stable sheaves with dim $M = 2$ and M projective. Sometimes $X \simeq M$.

Definitions

Stability: $\mathcal{O}(1) \in \mathrm{Pic}(X)$ ample. Then $E \in \mathrm{Coh}(X)$ is *stable* if $\forall\ 0 \neq F \subsetneq E$, $n \gg 0$:

$$\chi(F(n)) < \frac{\mathrm{rk}(F)}{\mathrm{rk}(E)} \cdot \chi(E(n)).$$

Moduli functor: $\mathcal{M}(v) : \mathrm{Sch}_{\mathbb{C}} \longrightarrow \mathrm{Set}$, $T \longmapsto \mathcal{M}(v)(T)$ with

$$\mathcal{M}(v)(T) := \{\mathcal{E} \in \mathrm{Coh}(X \times T) \mid T\text{−flat}, \ \mathcal{E}_t \text{ stable}, \ v(\mathcal{E}_t) = v\}/\sim .$$

$\mathcal{E} \sim \mathcal{E}'$ if $\exists\ L \in \mathrm{Pic}(T)$ with $\mathcal{E} \simeq \mathcal{E}' \otimes p_T^* L$.

Fine moduli space: $M(v)$ such that $\mathcal{M}(v) = \mathrm{Mor}(\ \ , M(v))$.

Warning: Does not always exist. Can be singular and even non-reduced. Can be empty. Can be non-projective. Can be reducible and even disconnected.

Existence:

Then:

- $\{E \in \mathrm{Coh}(X) \mid \text{stable},\ v(E) = v\} = \mathrm{Mor}(\mathrm{Spec}(\mathbb{C}), M(v)) = M(v)(\mathbb{C})$.
- $\mathrm{id} \in \mathrm{Mor}(M(v), M(v))$ corresponds to $\mathcal{E} \in \mathrm{Coh}(X \times M(v))$, the *universal family*, which is unique up to $\mathrm{Pic}(M(v))$.
- $\mathcal{E}|_{X \times [E]} \simeq E$ for all $[E] \in M(v)$.

Facts: If $\exists\ v'$ with $\langle v, v' \rangle = 1$, then $\exists\ \mathcal{O}(1)$ ample with

- Fine moduli space $M(v)$ exists.
- $M(v) \neq \varnothing$ if $\langle v, v \rangle \geq -2$.
- $\dim M(v) = 2 + \langle v, v \rangle$.
- $M(v)$ smooth.

Moduli spaces as FM partners

Corollary: If $\langle v, v \rangle = 0$ and $\langle v, v' \rangle = 1$, then $\exists\ \mathcal{O}(1)$ ample such that $M(v)$ exists and universal family $\mathcal{E}$ induces

$$\Phi_{\mathcal{E}} : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(M(v)).$$

Moreover, $M(v)$ is a K3 surface.

Bondal, Orlov: $\Phi_{\mathcal{E}}^{-1}$ fully faithful:
i) $\mathrm{End}(\mathcal{E}_{[E]}) = \mathrm{End}(E) = \mathbb{C}$.
ii) $\chi(\mathcal{E}_{[E]}, \mathcal{E}_{[F]}) = -\langle v(E), v(F) \rangle = 0$. If $[E] \neq [F]$, then $\mathrm{Hom}(E, F) = \mathrm{Ext}^2(E, F) = 0$ and thus $\mathrm{Ext}^1(E, F) = 0$.
iii) $\mathrm{Ext}^i(\mathcal{E}_{[E]}, \mathcal{E}_{[F]}) = 0$ if $i \neq 0, 1, 2$.

Since $\omega = \mathcal{O}$, also equivalence.

Derived GT: $\mathrm{D^b}(X) \simeq \mathrm{D^b}(X') \Leftrightarrow \widetilde{H}(X,\mathbb{Z}) \simeq \widetilde{H}(X',\mathbb{Z})$

Aim: For any Hodge isometry $g : \widetilde{H}(X,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X',\mathbb{Z})$ there exists a FM equivalence $\Phi_{\mathcal{E}} : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X')$ with

$$\Phi_{\mathcal{E}}^{H} = g \quad \text{or} \quad \Phi_{\mathcal{E}}^{H} = g \circ \eta,$$

where $\eta := \mathrm{id}_{H^0 \oplus H^4} \oplus (-\mathrm{id}|_{H^2})$ (cf. [M], [O], [HLOY], [P]).

Rough idea: Write g as composition of $\Phi_{\mathcal{E}}^{H}$ of our list of examples.

i) Suppose $g(0,0,1) = \pm(0,0,1)$. Set $(r,\ell,s) := \pm g(1,0,0)$. Then $r = 1$ and $s = (\ell,\ell)/2$, i.e. $g(1,0,0) = \exp(\mathrm{c}_1(L))$ for some $L \in \mathrm{Pic}(X')$.
Let $\Phi := L^* \otimes$ resp. $(L^* \otimes \ \ ) \circ [1]$ and

$$g' := \Phi^H \circ g.$$

Then $g' = \mathrm{id}_{H^0} \oplus g'_2 \oplus \mathrm{id}_{H^4}$ (graded!).

Torelli: $\exists\ \mathbb{P}^1 \simeq C_i \subset X'$ and $f \in \mathrm{Aut}(X')$ such that $\prod s_{[C_i]} \circ g' = f_*$ or $\prod s_{[C_i]} \circ g' = f_* \circ \eta$, but f_* and $s_{[C_i]}$ lift.

ii) Suppose $g(0,0,1) = \pm(r,\ell,s) =: v$ with $r > 0$. Then $\langle v, v\rangle = 0$. Let $M := M(v)$ and $\mathcal{E}$ universal family on $X' \times M$. Define

$$g' := \Phi^H \circ g$$

with Φ FM transform with FM kernel $\mathcal{E}$ resp. $\mathcal{E}[1]$. Then $g'(0,0,1) = (0,0,1) \rightsquigarrow$ **i)**.

iii) Suppose $g(0,0,1) = (0,\ell,s)$ with $\ell \neq 0$. Define

$$g' := T_{\mathcal{O}}^H \circ (L \otimes \quad)^H \circ g,$$

where $L \in \mathrm{Pic}(X')$ st. $s + (\mathrm{c}_1(L).\ell) \neq 0$. Then g' as in **ii)**.

Positive four-spaces

Know: $\widetilde{H}(X, \mathbb{Z}) \simeq 2(-E_8) \oplus 4U$. Bilinear extension yields symmetric bilinear form on $\widetilde{H}(X, \mathbb{R})$ of signature $(4, 22)$.

Positive directions: Write $H^{2,0}(X) = \mathbb{C}\sigma$ and pick Kähler (or ample) class $\omega \in H^{1,1}(X) \cap H^2(X, \mathbb{R})$. Then

- $\mathrm{Re}(\sigma), \mathrm{Im}(\sigma), 1 - \omega^2/2, \omega \in \widetilde{H}(X, \mathbb{R})$ span four dimensional space P_ω.
- P_ω gets natural orientation by choice of basis.
- Restriction of $\langle \ , \ \rangle$ to P_ω is positive definite.

Remark: Note that $\varphi := \sigma$ and $\psi := \exp(i\omega)$ are two generalized CY structures whose real and imaginary parts are orthogonal to each other.

Orientation of (*the four positive directions* in) $\widetilde{H}(M,\mathbb{Z})$ is: a subspace $P \subset \widetilde{H}(M,\mathbb{R})$ with an orientation st

$$\text{i)}\quad \dim P = 4 \qquad \text{ii)}\quad \langle\ ,\ \rangle|_P \text{ positive.}$$

Two orientations given by two oriented subspaces $P, P' \subset \widetilde{H}(M,\mathbb{R})$ are equal if the orthogonal projection $P \hookrightarrow \widetilde{H}(M,\mathbb{R}) \twoheadrightarrow P'$ (which is an isomorphism!) preserves the orientation.

Exercise: For two different Kähler classes ω, ω' the associated spaces $P_\omega, P_{\omega'}$ define the same orientation.

Similarly: An isometry $g : \widetilde{H}(M,\mathbb{R}) \xrightarrow{\sim} \widetilde{H}(M,\mathbb{R})$ is *orientation preserving* if for an oriented positive four space $P \subset \widetilde{H}(M,\mathbb{R})$

$$P \hookrightarrow \widetilde{H}(M,\mathbb{R}) \xrightarrow{g} \widetilde{H}(M,\mathbb{R}) \twoheadrightarrow P$$

preserves the orientation. (This is independent of the choice of P!)
More generally, an isometry $g : \widetilde{H}(X,\mathbb{R}) \xrightarrow{\sim} \widetilde{H}(X',\mathbb{R})$ is *orientation preserving* if for Kähler classes ω and ω' on X resp. X' the induced $P_\omega \xrightarrow{\sim} P_{\omega'}$ is orientation preserving.

Example: A diffeomorphism $f \in \mathrm{Diff}(M)$ induces an isometry f_* of $\widetilde{H}(M,\mathbb{Z})$ which is id on $H^0 \oplus H^4$. Then f_* is orientation preserving if and only if $f_*\mathcal{C}_M = \mathcal{C}_M$, where $\mathcal{C}_M \subset H^2(M,\mathbb{R})$ is one connected component of $\{\alpha \in H^2(M,\mathbb{R}) \mid (\alpha,\alpha) > 0\}$.

Definition: $\mathrm{O}_+(\widetilde{H}(X,\mathbb{Z}))$ = group of orientation preserving Hodge isometries.

Easy: f_* for $f \in \mathrm{Aut}(X)$ (use $f_*\mathcal{K}_X = \mathcal{K}_X$), $[1]^H$, $(L \otimes \ \_\ )^H$, and T_E^H are orientation preserving, i.e. are contained in $\mathrm{O}_+(\widetilde{H}(X,\mathbb{Z}))$.

Check: Also $\Phi_{\mathcal{E}}^H$ is orientation preserving for $\mathcal{E}$ the universal family of stable sheaves on $X \times M$. (H., Stellari)

Corollary: $\mathrm{O}_+(\widetilde{H}(X,\mathbb{Z})) \subset \mathrm{Im}(\rho)$.

H., Macrì, Stellari: $\eta \notin \mathrm{Im}(\rho)$ and hence

$$\mathrm{O}_+(\widetilde{H}(X,\mathbb{Z})) = \mathrm{Im}(\rho).$$

Let's twist

Let $X =$ K3 surface, $\alpha = \{\alpha_{ijk} \in \Gamma(U_{ijk}, \mathcal{O}^*)\} \in H^2(X, \mathcal{O}_X^*)_{\mathrm{tor}}$.
Then an α-twisted sheaf is: $(E_i \in \mathrm{Coh}(U_i), \varphi_{ij} : E_i|_{U_{ij}} \xrightarrow{\sim} E_j|_{U_{ij}})$ st.

$$\mathrm{i)}\ \ \varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot \mathrm{id}, \quad \mathrm{ii)} \quad \ldots \quad , \mathrm{iii)} \quad \ldots$$

This yields the abelian category $\mathrm{Coh}(X, \alpha)$.

Derived GT for twisted K3s: Let X, X' be K3 surfaces with Brauer classes $\alpha \in \mathrm{Br}(X)$ and $\alpha' \in \mathrm{Br}(X')$. Then (H., Stellari):

- If $\mathrm{D^b}(X, \alpha) \simeq \mathrm{D^b}(X', \alpha')$, then $\exists$ Hodge isometry $\widetilde{H}((X, \alpha), \mathbb{Z}) \simeq \widetilde{H}((X', \alpha'), \mathbb{Z})$.
- Conversely, any orientation preserving Hodge isometry lifts to a derived equivalence.

Problems: o) '$\Rightarrow$' more involved in the twisted case.
i) Need Hodge structure $\widetilde{H}((X,\alpha),\mathbb{Z})$. (Use B-field lift of α.)
ii) Need Chern character and Mukai vector for twisted sheaves.

iii) For $\alpha \neq 1$: $\mathcal{O}_X \notin \mathrm{Coh}(X,\alpha)$. Needed the spherical twist $T_{\mathcal{O}}$ for the proof. Also $L \notin \mathrm{Coh}(X,\alpha)$, but $L \otimes \quad$ is still defined.
iv) Need moduli spaces of twisted sheaves. Existence and non-emptiness.

v) One clearly expects that any equivalence induces an *orientation preserving* Hodge isometry. The proof of [HMS] does not apply directly, as $\eta = \mathrm{id}_{H^0 \oplus H^4} \oplus (-\mathrm{id}_{H^2})$ does not define a Hodge isometry in the general twisted case.