Birational geometry of moduli spaces of stable objects on Enriques surfaces

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1. Introduction

Moduli spaces of stable sheaves on surfaces are much studied objects. As stability depends on the choice of a polarization, it is interesting to study the dependence of the geometry of the moduli spaces on this choice. The introduction of Bridgeland stability conditions [14] prompted new techniques, which can be applied to study this question. In [5, 6], Bayer and Macrì have analyzed in detail the birational geometry of moduli spaces on projective K3 surfaces X. In particular, they proved that crossing a wall induces a birational transformation and that every smooth K-trivial birational model of a moduli space can be obtained by varying stability conditions in $\mathrm{Stab}^\dagger(X)$.

The purpose of this thesis is to prove analogous results for moduli spaces of stable objects on Enriques surfaces Y. The main technique is to consider the covering K3 surface $\widetilde{Y}$ and use the already established results for $\widetilde{Y}$.

More precisely, using the pullback along the covering map $\pi\colon \widetilde{Y} \to Y$, we get a 2:1 morphism of the moduli spaces $M_\sigma^Y(v)$ of stable objects on Y onto Lagrangian subvarieties of the moduli spaces $M_{\widetilde{\sigma}}^{\widetilde{Y}}(\pi^*(v))$ on $\widetilde{Y}$ [27, 45]. Applying a result by Marian and Zhao [38], we conclude

Proposition 1.1 (see Proposition 4.2)**.** *Let $v \in H^*_{\mathrm{alg}}(Y, \mathbb{Z})$ be a Mukai vector such that $\pi^*(v) \in H^*_{\mathrm{alg}}(\widetilde{Y}, \mathbb{Z})$ is primitive and $\sigma \in \mathrm{Stab}^\dagger(Y)$ a generic stability condition. The image of the morphism*

$$\pi^*\colon M_\sigma^Y(v) \to M_{\widetilde{\sigma}}^{\widetilde{Y}}(\pi^*(v))$$

is a constant cycle Lagrangian.

Recall that a subvariety is called constant cycle if all its points become rationally equivalent in the ambient variety.

It turns out that this cycle-theoretic property is enough to deduce birational equivalence of moduli spaces with different stability conditions.

Theorem 1.2 (see Theorem 4.5). *Let Y be a generic Enriques surface and $v \in H^*_{\mathrm{alg}}(Y, \mathbb{Z})$ a Mukai vector such that $\pi^*(v) \in H^*_{\mathrm{alg}}(\widetilde{Y}, \mathbb{Z})$ is primitive. Then, for generic stability conditions $\sigma, \tau \in \mathrm{Stab}^\dagger(Y)$ the moduli spaces $M_\sigma(v)$ and $M_\tau(v)$ are birationally equivalent.*

More precisely, we show that the birational transformation obtained by Bayer and Macrì [5, Thm. 1.1] for moduli spaces on K3 surfaces restricted to the moduli spaces on Enriques surfaces induces a birational transformation. To prove this, we use that these are constant cycle Lagrangians and, therefore, cannot be contained in the exceptional locus, cf. Proposition 2.6.

This enables us to apply Fourier–Mukai transforms to the moduli space and change the stability condition without changing its birational type. Using this technique, one relates moduli spaces for different Mukai vectors.

Corollary 1.3 (see Proposition 4.8). *Let Y be an Enriques surface and $v \in H^*_{\mathrm{alg}}(Y, \mathbb{Z})$ a primitive Mukai vector of odd rank. Then, for generic polarization $H \in \mathrm{Amp}(Y)$ the moduli spaces $M_H(v)$ are birationally equivalent to some* $\mathrm{Hilb}^n(Y)$.

If the rank is even, we deduce that the moduli spaces are Calabi–Yau manifolds employing results by Saccà [48].

As a final application of the birational equivalence of wall-crossing we consider the natural nef divisor classes associated to a stability condition [6, Sec. 4]. These behave well under fixed-point-free actions [45, Prop. 10.2]. Since the birational transformation of Theorem 1.2 can be obtained as the restriction of the birational transformation of the moduli spaces on the K3 surface, we can furthermore prove the following statement about minimal models of the moduli spaces.

Theorem 1.4 (see Theorem 4.14). *Let Y be a generic Enriques surface and $v \in H^*_{\mathrm{alg}}(Y, \mathbb{Z})$ a Mukai vector such that $\pi^*(v) \in H^*_{\mathrm{alg}}(\widetilde{Y}, \mathbb{Z})$ is primitive. Consider a generic stability condition $\sigma \in \mathrm{Stab}^\dagger(Y)$. Assume that the classes that are mapped to big and movable divisors under the Mukai morphism have positive square with respect to the Mukai pairing. Then, every smooth K-trivial birational model M of $M_\sigma(v)$ appears as a moduli space, i.e. $M \cong M_\tau$ for some $\tau \in \mathrm{Stab}^\dagger(Y)$ generic.*

We expect the assertion about the positive square of classes which are mapped to big and movable divisors to always hold. See Section 4.3 for further details.

Outline. At first, we will consider constant cycle subvarieties and present some known results and examples. We conclude the section with properties of these subvarieties needed in our investigation of the birational geometry of the moduli spaces.

Next, we will review briefly moduli spaces of coherent sheaves and present the notion of stability conditions introduced by Bridgeland. Concluding, we state relevant facts about Enriques surfaces before we give a broad overview of what is known for moduli spaces of stable sheaves and objects on K3 and Enriques surfaces.

The final chapter is devoted to the presentation of the author's results and their proofs. After we observe that moduli spaces on Enriques surfaces give rise to constant cycle Lagrangians, we continue to prove Theorem 1.2. This is the main result of this thesis. Thereafter, we determine the birational type of the moduli spaces of stable objects on Enriques surfaces for odd and even rank Mukai vectors and finish with the result on their minimal models.

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Notations and conventions. We will work over the complex numbers. An Enriques surface is called generic if the Picard rank of its covering K3 surface is 10. The bounded derived category of a smooth projective variety X is denoted by $\mathrm{D}^{\mathrm{b}}(X)$. Throughout, Chow groups are the groups of cycles modulo rational equivalence.

2. Constant cycle subvarieties

We are particularly interested in the following class of subvarieties.

Definition 2.1. Let X be a variety and $Y \subset X$ a subvariety. Then Y is called a *constant cycle subvariety* if all points in Y are rationally equivalent as points in X. If, moreover, X is a symplectic variety and Y is a Lagrangian subvariety, then Y is called *constant cycle Lagrangian.*

Observe that if X is symplectic, then every constant cycle subvariety is isotropic, which follows from Roitman's Theorem [51, Prop. 10.24]. Thus, in the definition of constant cycle Lagrangians one may replace the Lagrangian assumption with a condition on the dimensions $2\dim(Y) = \dim(X)$.

The starting point of this thesis was the investigation of constant cycle subvarieties. Therefore, we want to elaborate on these subvarieties and present some known results.

The notion was introduced in [22] for the case X a K3 surface by Hubyrechts. More precisely, he defined two notions such that the one above would be called a pointwise constant cycle curve. The second definition allows to attribute to a constant cycle curve an invariant called the *order*. Take a point $x_0 \in X$ such that its class satisfies $[x_0] = o_X$, where $o_X \in \mathrm{CH}_0(X)$ is the canonical zero-cycle introduced by Beauville and Voisin [8]. Consider the cycle $\kappa_C \coloneqq \Gamma_C - C \times x_0 \in \mathrm{CH}^2(C \times X)$ where $C \subset X$ is a curve inside a K3 surface and Γ_C the graph of the inclusion $C \hookrightarrow X$. We then say that C is a constant cycle curve of order n if the image of the cycle κ_C under the specialization map $\mathrm{CH}^2(C \times X) \to \mathrm{CH}^2(k(\eta_C) \times X)$ becomes n-torsion. Over $\mathbb{C}$ these two definitions coincide [22, Prop. 3.7].

Obvious examples of constant cycle subvarieties are given by CH_0-trivial subvarieties inside a variety or by subvarieties in a CH_0-trivial variety. Here, a variety is called CH_0-trivial if the group of zero cycles modulo rational equivalence maps isomorphically onto $\mathbb{Z}$ via the degree map. In particular, rational curves in K3 surfaces are constant cycle curves of order one. The converse does not hold.

We want to sketch two geometric examples of constant cycle curves on K3 surfaces following [22]. The first series is obtained from elliptic K3 surfaces $X \to \mathbb{P}^1$ with a zero-section $C_0 \subset X$. The set C_n defined as the closure of n-torsion points in the smooth fibres is easily seen to be a constant cycle curve since $\mathrm{CH}_0(X)$ is torsion-free [47]. The second one uses a non-symplectic automorphism $f\colon X \xrightarrow{\sim} X$ of finite order n. An automorphism is called *non-symplectic* if its action on $H^{2,0}(X)$ is not trivial. Curves contained in the fixed locus of f are constant cycle curves of order $d|n$. The crucial ingredient in the proof is Bloch's conjecture applied to the singular quotient $X/\langle f\rangle$ [11]. Every complex K3 surface contains infinitely many constant cycle curves.

The higher dimensional analogues of K3 surfaces are hyperkähler varieties.

Definition 2.2. A smooth projective variety X is called *hyperkähler* or *irreducible holomorphic symplectic*, if X is symplectic, simply connected and its space of holomorphic two-forms is spanned by a nowhere degenerate form.

In this thesis we are almost exclusively concerned with the study of constant cycle subvarieties inside hyperkähler varieties.

In higher dimensions very few examples of constant cycle subvarieties are known. Nevertheless, the two geometric examples from K3 surfaces have their analogues for hyperkähler varieties. Lin constructed constant cycle Lagrangians for hyperkähler varieties admitting a Lagrangian fibration [34]. The construction again uses the torsion points in the fibers, which are abelian varieties. If we consider a hyperkähler variety with a non-symplectic automorphism, subvarieties in the fixed locus are constant cycle provided the quotient satisfies the generalized Bloch's conjecture, cf. [10].

Example 2.3. We illustrate an example of this last principle. Consider the following ten-dimensional family

$$X = \{f(x_0,x_1,x_2,x_3,x_4) + x_5^3 = 0\} \subset \mathbb{P}^5$$

of cubic fourfolds, where $Y = \{f(x_0,x_1,x_2,x_3,x_4) = 0\} \subset \mathbb{P}^4$ defines a smooth cubic threefold, cf. [1,13]. Beauville and Donagi showed that the Fano variety of lines $F(X)$ of a cubic fourfold is a four-dimensional hyperkähler variety [7]. The inclusion $Y \subset X$ yields an inclusion $F(Y) \subset F(X)$ of the Fano variety of lines. On X we have an involution induced by the following automorphism of the projective space

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \mapsto [x_0 : x_1 : x_2 : x_3 : x_4 : \xi x_5],$$

where ξ is a primitive third root of unity. This automorphism of X induces a non-symplectic automorphism σ on $F(X)$ and $F(Y)$ is exactly its fixed locus [12, Ex. 6.4]. Laterveer [30,32] studied this example and proved using Chow motives that the singular quotient $F(X)/\langle\sigma\rangle$

has a trivial CH_0-group. Thus, this is an example of smooth constant cycle Lagrangians of non-negative Kodaira dimension inside the Fano variety of lines of a cubic fourfold. In [31, 32] one can find two more examples of this type.

A possible generalization of constant cycle Lagrangians has been defined by Voisin in [52]. Recall that a variety $Z \subset X$ inside a symplectic variety (X, σ) is called coisotropic if for each regular point $z \in Z_{\mathrm{sm}}$ the symplectic complement of the tangent space $T_{Z,z}^{\perp\sigma}$ is contained in $T_{Z,z}$.

Definition 2.4. A subvariety $Z \subset X$ of codimension $i \leq n$ inside a hyperkähler variety is called *algebraically coisotropic* if there exists a rational map $\phi\colon Z \dashrightarrow B$ onto a variety B of dimension $2n - 2i$ such that $\sigma|_Z = \phi^*\sigma_B$, where σ_B is a $(2,0)$-form on B.

One can furthermore require the general fibers of the rational map ϕ to be constant cycle subvarieties. Then, for $i = n$, we recover the notion of constant cycle Lagrangians. Voisin conjectured that every algebraic hyperkähler variety contains algebraic coisotropic subvarieties of every codimension $0 \leq i \leq n$ with constant cycle fibers. Furthermore, the cycle class map $\mathrm{cl}\colon \mathrm{CH}^i(X) \to H^{2i}(X, \mathbb{Z})$ is conjectured to be injective on these subvarieties, so in particular on constant cycle Lagrangians. For more results and conjectures we refer the reader to [52].

For our investigation of the birational geometry of the moduli spaces we only need the following properties.

Lemma 2.5. *Let X be a projective variety, Y a smooth projective variety and consider a birational morphism $f\colon X \to Y$. Then a subvariety $Z \subset Y$ is constant cycle if and only if $f^{-1}(Z) \subset X$ is a constant cycle subvariety.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}_0(f^{-1}(Z)) \arrow[r, "\iota_*"] \arrow[d, "f_*"] & \mathrm{CH}_0(X) \arrow[d, "f_*"] \\
\mathrm{CH}_0(Z) \arrow[r, "\iota_*"] & \mathrm{CH}_0(Y),
\end{tikzcd}$$

where ι denotes the inclusion of both subvarieties. If X is smooth, the birational map f_* induces an isomorphism between $\mathrm{CH}_0(X)$ and $\mathrm{CH}_0(Y)$ and the assertion follows from the commutativity of the diagram. For arbitrary X, use a resolution of singularities $\pi\colon \widetilde{X} \to X$ and argue as above using the above diagram and the corresponding one for π. $\square$

The following result is the main ingredient for our proof of Theorem 1.2.

Proposition 2.6. *Let X and Y be smooth varieties and $f\colon X \dashrightarrow Y$ a birational map. Suppose X is symplectic and consider a constant cycle Lagrangian $Z \subset X$ of Kodaira dimension* $\operatorname{kod}(Z) \geq 0$*. Assume that a general point of the proper transform of Z is a fundamental point of the birational map f^{-1}. Then, Z cannot be contained in the exceptional locus.*

Proof. Consider the diagram

$$\begin{tikzcd} & \Gamma \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[rr, dashed, "f"] & & Y, \end{tikzcd}$$

where Γ denotes the closure of the graph of f and p_i the projections from $X \times Y$. Applying Lemma 2.5, the variety

$$T \coloneqq p_1\left(p_2^{-1}\left(p_2\left(p_1^{-1}(Z)\right)\right)\right) \subset X$$

is a constant cycle subvariety containing Z. The fiber $p_2^{-1}(a) \subset \Gamma$ for a fundamental point a of f^{-1} is uniruled [29, IV. Prop. 1.5]. Thus, the images of the fibers under the morphism p_1 are non-trivial uniruled varieties. If for generic $x \in Z$ there exists a fiber $p_2^{-1}(a)$ with $x \in p_1(p_2^{-1}(a))$ and a rational curve $x \in C_x \subset p_1(p_2^{-1}(a))$ such that C_x is contained in Z, then Z would be uniruled, in contradiction to $\operatorname{kod}(Z) \geq 0$.

Hence, we may assume that for generic $x \in Z$ the rational curves $C_x \subset p_1(p_2^{-1}(a))$ through x are not contained inside Z. However, then the constant cycle subvariety T strictly contains Z and, thus, is of larger dimension. Since the dimension of a constant cycle subvariety inside a symplectic variety is bounded by half the dimension of the ambient space, we derive a contradiction. □

In a similar vein one can show that constant cycle Lagrangians $Z \subset X$ of non-negative Kodaira dimension inside a smooth symplectic variety cannot be contained inside a uniruled variety $W \subset X$. Indeed, if Z would be contained in W, there is again through every point of Z a rational curve. These are also generically not contained inside Z. However, they must be contained in the rational orbit

$$O_z = \{x \in X \mid [x] = [z] \text{ in } \mathrm{CH}_0(X)\}$$

of a point $z \in Z$ which is a countable union of constant cycle subvarieties. This yields a contradiction.

Since the exceptional locus of a birational map between hyperkähler varieties is uniruled [29, Thm. VI.1.10], we could as well use this observation in our proof of Theorem 1.2.

3. Review: Moduli spaces and stability conditions

3.1. **Moduli spaces of coherent sheaves.** A key aspect of algebraic geometry is the study of coherent sheaves on varieties. The goal then is to construct moduli spaces which parametrize sheaves with certain properties, for example a given Hilbert polynomial or Mukai vector. As these spaces should inherit a geometric structure themselves, one needs the notion of stability to ensure their existence as a reasonable space. A reference for a detailed discussion of moduli spaces is [24].

Let X be a surface and H an ample line bundle on X. Recall that for a coherent sheaf E of pure dimension 2 the Hilbert polynomial $P(E)$ is defined as $P(E,m)=\chi(E\otimes H^m)$, where χ denotes the Euler characteristic. The reduced Hilbert polynomial $p(E)$ is the quotient of $P(E)$ by the rank of E.

Definition 3.1. A coherent sheaf E of pure dimension 2 is called *(H-)Gieseker semistable* if for all proper subsheaves $0\neq F\subset E$ one has $p(F)\leq p(E)$. The sheaf E is called *(H-)Gieseker stable* if strict inequality holds.

It is easy to verify that stable sheaves are in fact simple. If we fix the Mukai vector $v(E)=\mathrm{ch}(E)\sqrt{\mathrm{td}(X)}\in H^*(X,\mathbb{Q})$ (or the topological invariants $\mathrm{rank}(E)$ and $c_i(E)$), the moduli functor of interest to us is

$$\mathcal{M}_H(v)\colon (\mathrm{Sch}/\mathbb{C})^{\mathrm{op}}\to \mathrm{Set}$$

that associates to any scheme T of finite type over $\mathbb{C}$ the set of equivalence classes of all T-flat coherent sheaves $\mathcal{E}$ on $T\times X$ such that the fibers $\mathcal{E}_t$ for all closed points $t\in T$ are (H-)Gieseker semistable with Mukai vector v. Two such sheaves are called equivalent if they differ by the pullback of a line bundle on T via the first projection.

Using GIT one can show the existence of a projective variety $M_H(v)$ which is a coarse moduli space parametrizing S-equivalence classes of semistable sheaves with Mukai vector v [24, Sec. 4]. Recall that two semistable sheaves are called S-equivalent if their Jordan–Hölder filtrations have the same graded objects. The $\mathbb{C}$-valued points of $M_H(v)$ are exactly the S-equivalence classes. If $[E]\in M_H(v)$ is a stable sheaf, then $T_{[E]}M_H(v)\cong \mathrm{Ext}^1(E,E)$. Furthermore, $M_H(v)$ is smooth at E if $\mathrm{Ext}^2(E,E)=0$ [24, Thm. 4.5.3].

The ample cone $\mathrm{Amp}(X)$ admits for given Mukai vector v a wall and chamber decomposition [24, App. 4.C]. We call $H \in \mathrm{Amp}(X)$ *(v-)generic* if H does not lie on any of the walls. If v is primitive and H is generic, then semistability and stability coincide.

3.2. **Bridgeland stability conditions.** Motivated by M. Douglas's work on Π-stability, Bridgeland [14] introduced the notion of a stability condition on a triangulated category.

Definition 3.2. Let $\mathcal{D}$ be a triangulated category. A *stability condition* $\sigma = (Z, \mathcal{P})$ consists of a group homomorphism $Z\colon K(\mathcal{D}) \to \mathbb{C}$, called the *central charge*, and for each $\phi \in \mathbb{R}$ a full additive subcategory $\mathcal{P}(\phi) \subset \mathcal{D}$ satisfying the following conditions

- If $0 \neq E \in \mathcal{P}(\phi)$, then $Z(E) \in \mathbb{R}_{>0} e^{i\pi\phi}$.
- If $\phi_1 > \phi_2$ and $E_i \in \mathcal{P}(\phi_i)$, then $\mathrm{Hom}(E_1, E_2) = 0$.
- $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1)$.
- Every $0 \neq E \in \mathcal{D}$ has a *categorical Harder–Narasimhan* filtration, i.e. there exists a finite sequence of real numbers $\phi_1 > \phi_2 > \cdots > \phi_n$ and a (unique) collection of triangles

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \dots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

  such that $A_i \in \mathcal{P}(\phi_i)$.

The collection of subcategories $\mathcal{P}$ with the above properties is called a *slicing* of $\mathcal{D}$. Equivalently, one can use bounded t-structures and their hearts $\mathcal{A} = \mathcal{P}((0, 1])$ to define Bridgeland stability conditions [14, Prop. 5.3]. We refer to [9] for an introduction to t-structures. With this notion the first property encodes two positivity conditions. The imaginary part of the central charge can be thought of as a rank function which is additive on short exact sequences on the abelian category $\mathcal{A}$. Furthermore, the real part adjusted by a sign $-\mathrm{Re}(Z)$ defines the degree of an object such that non-zero objects in $\mathcal{A}$ of rank 0 have strictly positive degree. With this in mind we can define a notion of slope stability on the abelian category $\mathcal{A}$ via $\mu(E) = -\mathrm{Re}(Z(E))/\mathrm{Im}(Z(E))$. Then E is called *semistable* (resp. *stable*) if for all proper subobjects $0 \neq F \subset E$ we have the (strict) inequality $\mu(F) \leq \mu(E)$ ($\mu(F) < \mu(E)$). If we use the notion of a slicing, objects in $\mathcal{P}(\phi)$ are semistable of phase ϕ and simple objects in $\mathcal{P}(\phi)$ are called stable.

Bridgeland constructed on the set of stability conditions $\mathrm{Stab}(\mathcal{D})$ a generalised metric to show the following

Theorem 3.3 (Bridgeland)**.** *The space of stability conditions* $\mathrm{Stab}(\mathcal{D})$ *is a complex manifold. A local homeomorphism is given by sending* $\sigma = (Z, \mathcal{P})$ *to* $Z \in \mathrm{Hom}(K(\mathcal{D}), \mathbb{C})$*.*

Usually one restricts to *full numerical* stability conditions. These are stability conditions with two further properties. Firstly, one requires the central charge to factorize through the numerical Grothendieck group $K_{\mathrm{num}}(\mathcal{D}) = K(\mathcal{D})/\chi(-,-)$, where $\chi(E, F)$ denotes the alternating sum of the dimensions of the Ext groups $\sum_i (-1)^i \dim \mathrm{Ext}^i(E, F)$. Secondly, the image of the local homeomorphism from Theorem 3.3 has the maximal dimension possible. From now on we will only consider full numerical stability conditions and the space $\mathrm{Stab}(X)$ of those on the bounded derived category $\mathrm{D}^{\mathrm{b}}(X)$ of a smooth projective surface X.

The space of stability conditions comes naturally with two group actions. The first one is the group of exact autoequivalences $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ acting via $\psi.(Z, \mathcal{P}) = (Z \circ \psi_*, \psi(\mathcal{P}))$. The universal cover $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ of $\mathrm{GL}^+(2, \mathbb{R})$ acts on the right via $(Z, \mathcal{P}).(A, f) = (A^{-1} \circ Z, \mathcal{P}')$, where $\mathcal{P}'(\phi) = \mathcal{P}(f(\phi))$.

For curves the stability manifolds have been completely determined [14, 35, 46]. For surfaces we will list some of the results in the next chapter. Constructing stability conditions in higher dimensions is in general very difficult.

Similiar to the case of Gieseker stability, the space of stability conditions $\mathrm{Stab}(X)$ admits for a Mukai vector v a wall and chamber decomposition, see [15, Sec. 9] and [4, Prop. 3.3].

Proposition 3.4 (Bridgeland)**.** *There exists a locally finite collection of real codimension one submanifolds* $W_i \subset \mathrm{Stab}(X)$*, only depending on the Mukai vector* v*, with the following properties*

(i) *Each wall* W_i *is a real codimension one closed submanifold with boundary.*

(ii) *When* σ *varies within a chamber, the set of* σ*-semistable and* σ*-stable objects with Mukai vector* v *does not change.*

(iii) *If* $\sigma \in W_i$ *for some* i*, then there is a* σ*-semistable object that is semistable in one of the adjacent chambers and unstable in the other adjacent chamber.*

We call a stability condition *generic* (for v) if it does not lie on any of the walls. If v is primitive and σ is generic, then σ-stability and σ-semistability coincide. In particular, the set of stability conditions for which a given object is stable is open in $\mathrm{Stab}(X)$.

A question that arises is the relationship between Gieseker stability and stability conditions. Bridgeland obtained a result for (β, ω)-twisted Gieseker stability on K3 surfaces [15, Prop. 14.2]. We refer to [39] for details on twisted stability. For $\beta = 0$ this notion reduces to classical Gieseker stability. He proved that one can recover (β, ω)-twisted stability as the large volume limit of the geometric stability condition associated to the pair (β, ω) if the rank of the Mukai vector is not zero. In fact, it is not necessary to make this assumption on the rank [5, Rem. 2.14] and the statement can be modified for arbitrary smooth projective surfaces [37, Exc. 6.27]. For our purposes the following is enough.

Proposition 3.5 (Bridgeland)**.** *Let X be a K3 or Enriques surface and $v = (r, c_1, s)$ a positive, primitive Mukai vector. For a generic ample H satisfying $c_1 H > 0$ there exists a stability condition σ_H such that the sets of σ_H-semistable and σ_H-stable objects with class v are the same as the sets of H-Gieseker semistable and H-Gieseker stable sheaves.*

A Mukai vector $v = (r, c_1, s)$ is called *positive* if $r > 0$ or $r = 0$ and c_2 effective and $s \neq 0$ or $r = c_1 = 0$ and $s > 0$.

As in the sheaf case one would like to construct moduli spaces of semistable objects. Unfortunately, as these are not naturally associated to a GIT problem, there is up to now no general result on the existence of a coarse moduli space. More precisely, one considers the 2-functor

$$\mathfrak{M}_X \colon (\mathrm{Sch}/\mathbb{C})^{\mathrm{op}} \to \mathrm{Grp}$$

which associates to a scheme T the groupoid $\mathfrak{M}_X(T)$ whose objects $\mathcal{E}$ consist of unbounded T-perfect complexes of quasi coherent sheaves on $T \times X$ such that $\mathrm{Ext}^i(\mathcal{E}_t, \mathcal{E}_t) = 0$ for $i < 0$ and all $t \in T$. Inaba [26] and Lieblich [33] then proved that $\mathfrak{M}_X$ is an Artin stack locally of finite type over $\mathbb{C}$. Toda [50] improved this result for K3 surfaces and showed amongst other things that σ-stability is an open property and that the moduli stack satisfies the valuative criterion of universal closedness. Nevertheless, in the cases which are of interest to us, coarse moduli spaces are known to exist.

3.3. **Enriques surfaces.** We assume that the reader is familiar with K3 surfaces. For a broad introduction to K3 and Enriques surfaces we refer to [23] and [2].

Let Y be a smooth projective surface over the complex numbers. It is called an Enriques surface, if its canonical bunde ω_Y is non-trivial and has 2-torsion and $h^{1,0}(Y) = 0$. The Hodge diamond of an Enriques surface is concentrated in the middle column and the second integral cohomology decomposes into $H^2(Y,\mathbb{Z}) = \mathbb{Z}^{\oplus 10} \oplus \mathbb{Z}/2\mathbb{Z}$. If we divide out by torsion, the intersection form on the second cohomology is isometric to $E_8(-1) \oplus U$, where U is the hyperbolic plane and E_8 is the even positive definite lattice with Dynkin diagram E_8.

The fundamental group of Y is $\mathbb{Z}/2\mathbb{Z}$ and the universal cover of an Enriques surfaces $\pi\colon \widetilde{Y} \to Y$ is isomorphic to a projective K3 surface and we denote by $i \in \operatorname{Aut}(\widetilde{Y})$ the covering involution. Note that there is a bijection between Enriques surfaces and projective K3-surfaces with a fixed-point-free involution. An Enriques surface Y with universal cover $\widetilde{Y}$ of Picard rank 10 is called *generic*. Observe that as $h^1(\mathcal{T}_Y) = 10$ and $h^0(\mathcal{T}_Y) = h^2(\mathcal{T}_Y) = 0$, the moduli space of Enriques surfaces is smooth of dimension 10. The generic Enriques surfaces are dense in this moduli space.

The covering map induces a morphism $\pi^*\colon H^*(Y,\mathbb{Z}) \to H^*(\widetilde{Y},\mathbb{Z})$ whose image is an index 2 sublattice of the i^*-invariant part of the cohomology [43]. However, if we consider the torsion free Néron–Severi group $\mathrm{NS}_f(Y)$, the embedding $\mathrm{NS}_f(Y) \hookrightarrow \mathrm{NS}(\widetilde{Y})$ is primitive and for generic Y even onto. For $\alpha, \beta \in \mathrm{NS}_f(Y)$ we have $(\pi^*\alpha, \pi^*\beta) = 2(\alpha,\beta)$.

3.4. **Moduli spaces on K3 and Enriques surfaces.** In this section we want to give an overview about what is known for moduli spaces of stable sheaves and objects on K3 and Enriques surfaces.

Let X be a K3 or an Enriques surface.

Definition 3.6. Denote by $H^*_{\mathrm{alg}}(X,\mathbb{Z})$ the image of the map

$$v = \mathrm{ch}(\_)\sqrt{\mathrm{td}(X)}\colon K_{\mathrm{num}}(X) \to H^*(X,\mathbb{Q}).$$

We call $H^*_{\mathrm{alg}}(X,\mathbb{Z})$ together with the *Mukai pairing*

$$(v,w) := \int_X -v_0w_4 + v_2w_2 - v_4w_0$$

the *Mukai lattice* of X.

If X is a K3 surface, the image of the Mukai vector is in fact contained in the integral cohomology. For Enriques surfaces we have to allow rational coefficients since the square root of the Todd class is $\sqrt{\mathrm{td}(X)} = (1, 0, \frac{1}{2})$. Using the Hirzebruch–Riemann–Roch formula we observe for $E, F \in K_{\mathrm{num}}(X)$ the relation $\chi(E,F) = -(v(E), v(F))$.

We will first consider the case of moduli spaces of stable sheaves on K3 surfaces. The low-dimensional examples have been worked out by Mukai [42].

Theorem 3.7 (Mukai)**.** *Let v be a primitive Mukai vector together with a generic polarization $H \in \mathrm{Amp}(X)$.*

(i) *If $v^2 = -2$, then $M_H(v)$ is either empty or a single point.*
(ii) *If v is isotropic, then $M_H(v)$ is empty or a K3 surface.*

The K3 surface appearing in part (ii) of the theorem is in general not isomorphic to X. Yoshioka, building upon previous work by Mukai and Kuleshov, established numerical conditions for the non-emptiness of the moduli spaces [54].

Proposition 3.8 (Yoshioka)**.** *If v is primitive, positive and satisfies $v^2 \geq -2$, then the moduli space $M_H(v)$ is non-empty.*

In higher dimensions, the moduli spaces share a certain geometric structure.

Theorem 3.9 (Huybrechts, Mukai, O'Grady, Yoshioka)**.** *If v is positive, primitive with $v^2 > 0$ and H is generic, the moduli space $M_H(v)$ is a projective hyperkähler variety of dimension $2n = v^2 + 2$ deformation equivalent to $\mathrm{Hilb}^n(X)$. The symplectic form at a point $[E]$ is given by the Yoneda product*

$$\mathrm{Ext}^1(E, E) \times \mathrm{Ext}^1(E, E) \to \mathrm{Ext}^2(E, E) \cong \mathrm{Hom}(E, E) = \mathbb{C}.$$

Let us now pass to the derived category $\mathrm{D^b}(X)$ and stability conditions. In [15], Bridgeland constructed explicitly stability conditions $\sigma_{\beta,\omega}$ for given $\beta, \omega \in \mathrm{NS}(X) \otimes \mathbb{R}$ with $\omega \in \mathrm{Amp}(X)$. The central charge $Z = Z_{\beta,\omega}$ is given by $Z(E) = (\exp(\beta + i\omega), v(E))$. The heart $\mathcal{A}(\beta, \omega)$ of the bounded t-structure is obtained via tilting with respect to a torsion pair, see [15, Sec. 6]. We will call these stability conditions *geometric stability conditions* and denote the connected component of the stability manifold containing them by $\mathrm{Stab}^\dagger(X)$.

One defines a map

$$\mathcal{Z} \colon \mathrm{Stab}^\dagger(X) \to H^*_{\mathrm{alg}}(X, \mathbb{Z}) \otimes \mathbb{C}$$

sending a stability condition $\sigma = (Z, \mathcal{P})$ to the unique vector Ω_Z satisfying $Z(v) = (\Omega_Z, v)$ for all v. To describe the image of $\mathcal{Z}$ let us first define $\mathcal{P}(X)$ to be the set of all vectors whose real and imaginary part span a positive definite two-dimensional plane. This set has two connected components. We denote by $\mathcal{P}^+(X)$ the connected component containing vectors of the form $\exp(\beta + i\omega)$ for ample classes $\omega \in \mathrm{NS}(X) \otimes \mathbb{R}$. Then, $\mathcal{P}^+_0(X)$ arises from $\mathcal{P}^+(X)$ by removing the orthogonal complement of all -2-classes in $H^*_{\mathrm{alg}}(X, \mathbb{Z})$.

Theorem 3.10 (Bridgeland)**.** *The map $\mathcal{Z}\colon \mathrm{Stab}^\dagger(X) \to \mathcal{P}_0^+(X)$ is a covering map. The subgroup of homologically trivial autoequivalences which preserves the connected component $\mathrm{Stab}^\dagger(X)$ acts freely and is the group of deck transformations of $\mathcal{Z}$.*

Using the theorem a conjectural picture of the group of autoequivalences of $\mathrm{D^b}(X)$ has been put forward. It has been verified in the case of K3 surfaces with Picard rank one [25].

Minamide, Yanagida, and Yoshioka [41] proved the existence of coarse moduli spaces parametrizing Bridgeland stable complexes for K3 surfaces with Picard rank one. Bayer and Macrì [6] used their technique and extended it to arbitrary algebraic K3 surfaces.

Theorem 3.11 (Bayer, Macrì)**.** *For a projective K3 surface X with primitive Mukai vector v and generic stability condition $\sigma \in \mathrm{Stab}^\dagger(X)$ the coarse moduli space $M_\sigma(v)$ of σ-semistable objects exists as a projective hyperkähler variety.*

Moreover, they described nef divisors ℓ_σ associated to a stability condition $\sigma = (Z, \mathcal{P})$. They can be defined as the composition

$$\mathrm{Stab}^\dagger(X) \xrightarrow{\ \mathcal{Z}\ } \mathcal{P}_0^+(X) \xrightarrow{\ I\ } v^\perp \xrightarrow{\ \theta_\sigma\ } \mathrm{NS}(M_\sigma(v)),$$

where $\mathcal{Z}\colon \mathrm{Stab}^\dagger(X) \to \mathcal{P}_0^+(X)$ is the map from Theorem 3.10, $I(w) = -\mathrm{Im}(\frac{w}{(w,v)})$ and θ_σ is the Mukai morphism, cf. [24, Sec. 8.1]. For a generic stability condition σ the divisor ℓ_σ is ample on $M_\sigma(v)$ [6, Thm. 1.3].

In a subsequent paper [5], Bayer and Macrì studied in detail the birational geometry of moduli spaces of Bridgeland stable objects. In particular, they described the relationship of moduli spaces for different stability conditions and obtained the Minimal Model Program for these moduli spaces.

Theorem 3.12 (Bayer, Macrì)**.** *Let X be a K3 surface, v a primitive Mukai vector and σ and τ generic stability conditions in $\mathrm{Stab}^\dagger(X)$. Assume $v^2 > 0$. Then, the two moduli spaces $M_\sigma(v)$ and $M_\tau(v)$ are birational to each other.*

Furthermore, every smooth K-trivial birational model of $M_\sigma(v)$ appears as a moduli space $M_\tau(v)$ of Bridgeland stable objects for some generic $\tau \in \mathrm{Stab}^\dagger(X)$.

The proof is based on a detailed study of an hyperbolic lattice associated to each wall. They describe under which condition (e.g. whether there are spherical or isotropic classes contained in this lattice) which type of birational transformation occurs [5, Thm. 5.7]. The statement about the minimal models is achieved by computing the image of the map $\ell\colon \mathrm{Stab}^\dagger(X) \to \mathrm{NS}(M_\sigma(v))$.

Remark 3.13. We can apply this theorem in the following way. In the next chapter we will be interested in moduli spaces up to birational equivalence. For primitive v and generic σ we consider $M_\sigma(v)$. Applying a Fourier–Mukai transform Φ yields an isomorphism $\Phi\colon M_\sigma(v) \xrightarrow{\sim} M_{\Phi.\sigma}(\Phi^H(v))$, where Φ^H denotes the induced cohomological Fourier–Mukai transform. Observe that σ may not be generic for $\Phi^H(v)$. However, $\Phi.\sigma$ is again generic for $\Phi^H(v)$. If Φ preserves the distinguished connected component $\mathrm{Stab}^\dagger(X)$, we get birationally equivalent moduli spaces $M_\sigma(v) \sim M_{\sigma'}(\Phi^H(v))$ for a generic stability condition σ' using the theorem. Thus, to prove statements about the birational type of the moduli spaces, we can apply cohomological Fourier–Mukai transforms to modify the Mukai vector given that the transform preserves the distinguished component. Hartmann proved that actually every cohomological Fourier–Mukai transform is induced by an autoequivalence respecting the distinguished component [18, Prop. 7.9]. Since there is a stability condition σ_H which coincides with H-Gieseker stability, this allows us to deduce the results as well for moduli spaces of stable sheaves.

Now we pass to Enriques surfaces Y and their moduli spaces. Throughout this thesis, the universal cover will be denoted $\widetilde{Y}$. For better understanding, the moduli spaces on the Enriques surface will be denoted by $M_H^Y(v)$ respectively $M_\sigma^Y(v)$ and the moduli spaces on its universal cover by $M_{\pi^*H}^{\widetilde{Y}}(\pi^*(v))$ respectively $M_{\widetilde{\sigma}}^{\widetilde{Y}}(\pi^*(v))$.

Since the canonical divisor ω_Y of an Enriques surface is 2-torsion, the moduli space for the Mukai vector v decomposes into

$$M_H^Y(v) \cong M_H^Y(v, L) \sqcup M_H^Y(v, L \otimes \omega_Y),$$

where we furthermore fix the determinant line bundle L respectively $L \otimes \omega_Y$. If the rank of the Mukai vector is odd, the two components are isomorphic.

Kim [27, 28] was the first one to study moduli spaces of stable sheaves on Enriques surfaces.

Proposition 3.14 (Kim)**.** *Given an Enriques surfaces Y with its covering $\pi\colon \widetilde{Y} \to Y$, the morphism*

$$\pi^*\colon M_H^Y(v) \to M_{\pi^*H}^{\widetilde{Y}}(\pi^*(v))$$

has degree 2 and it is étale onto its image away from all points $[E]$ satisfying $E \cong E \otimes \omega_Y$. Its image is the fixed locus of the action given by the covering involution $i^* \in \mathrm{Aut}(M_{\pi^*H}^{\widetilde{Y}}(\pi^*(v)))$ and is a Lagrangian subvariety.

The first step towards criteria to determine when the moduli spaces are non-empty was again taken by Yoshioka [54, Thm. 4.6]. He used the virtual Hodge polynomial

$$e(X) \coloneqq \sum_{p,q} e^{p,q}(X)x^p y^q$$

for a variety X over $\mathbb{C}$, where $e^{p,q}(X) \coloneqq \sum_k (-1)^k h^{p,q}(H^k_c(X))$. Here one uses that for a variety over $\mathbb{C}$ the cohomology with compact support carries a natural mixed Hodge structure. The main ingredient in his result is the invariance of this polynomial under a certain Fourier–Mukai transform for moduli spaces of stable sheaves on Enriques surfaces [54, Prop. 4.5]. The proof then reduces to a purely lattice-theoretic computation using cohomological Fourier–Mukai transforms.

Theorem 3.15 (Yoshioka)**.** *Let $v \in H^*_{\mathrm{alg}}(Y, \mathbb{Z})$ be a primitive, positive Mukai vector of odd rank on an Enriques surface Y and take a generic polarization H. Then*

$$e(M^Y_H(v, L)) = e(\mathrm{Hilb}^n(Y)),$$

where $2n = v^2 + 1$ and L is one of the two possible determinants. In particular, $M^Y_H(v, L)$ is irreducible and non-empty if and only if $v^2 \geq -1$.

The above strategy was employed by Hauzer [19] to show that in the even rank case the virtual Hodge polynomial of a moduli space of semistable sheaves is the same as for a moduli space parametrizing sheaves of rank 2 or 4. In his article one also finds an explicit description of certain moduli spaces of rank two sheaves using the elliptic fibration of an Enriques surface. Finally, Nuer [44, 45] settled the question of non-emptiness completely. We only present the result in the primitive case for generic Enriques surfaces [44, Thm. 5.1].

Theorem 3.16 (Nuer)**.** *Consider a primitive Mukai vector $v = (r, c_1, \frac{s}{2}) \in H^*_{\mathrm{alg}}(Y, \mathbb{Z})$ of even rank on a generic Enriques surface and a generic polarization H. Take a line bundle L satisfying $c_1(L) = c_1$.*

- (i) *If $\pi^*(v)$ is primitive and $v^2 \geq 0$, the moduli space $M^Y_H(v, L)$ is non-empty, irreducible and smooth of dimension $v^2 + 1$ with torsion canonical bundle.*
- (ii) *If $2 \mid \pi^*(v)$ and $v^2 \geq 1$, the moduli space $M^Y_H(v, L)$ is non-empty and irreducible of dimension $v^2 + 1$.*
- (iii) *If $v^2 = 0$ and $2 | \pi^*(v)$, then $M^Y_H(v) \cong Y$.*

Note that the above theorem in particular implies that in case (iii) the moduli space has only one component whereas in (i) and (ii) it has two components.

Independently, Saccà [48] studied the geometry of moduli spaces of pure dimension one sheaves on Enriques surfaces. The main tool is the support morphism whose fibers are the Jacobians of the curves. She was able to compute the canonical bundle, fundamental group, second Betti number and certain Hodge numbers.

Theorem 3.17 (Saccà)**.** *Let Y be a generic Enriques surface, $C \subset Y$ a curve of arithmetic genus $g \geq 2$ and $v = (0, [C], s)$ a primitive Mukai vector. For a generic polarization H the moduli space $M \coloneqq M_H^Y(v, L)$ is a smooth projective Calabi–Yau variety of dimension $2g-1$, i.e.*

$$\omega_M \cong \mathcal{O}_M \text{ and } h^{p,0}(M) = 0 \text{ for } p \neq 0, 2g-1.$$

In addition, there is a surjection $\mathbb{Z}/2\mathbb{Z} \twoheadrightarrow \pi_1(M)$. For $g = 2$, the moduli space is a threefold with the following Hodge diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 10 & & 0 & \\ 1 & & 10 & & 10 & & 1 \end{array}$$

For low values of g she shows that the above surjection onto the fundamental group is actually an isomorphism. She conjectures it to be the case for all g and verifies it under an extra assumption [48, As. 4.1.5]. This assumption then also allows her to compute the second Betti number of the moduli spaces $h^2(M_H^Y(v, L)) = 11$ for $g \geq 3$.

We also want to mention results for Bridgeland stability conditions on Enriques surfaces and the associated moduli spaces.

The category of coherent sheaves $\mathrm{Coh}(Y)$ on an Enriques surface Y is naturally isomorphic to the category of coherent $\langle i^* \rangle$-sheaves $\mathrm{Coh}_{i^*}(\widetilde{Y})$. This yields a natural equivalence between the bounded derived categories $\mathrm{D}^{\mathrm{b}}(Y)$ and $\mathrm{D}^{\mathrm{b}}_{i^*}(\widetilde{Y})$. In [36], the authors describe stability conditions on Enriques surfaces using i^*-invariant stability conditions $\Gamma_{\widetilde{Y}} \coloneqq \{\widetilde{\sigma} \in \mathrm{Stab}^\dagger(\widetilde{Y}) \mid i^*\widetilde{\sigma} = \widetilde{\sigma}\}$ on $\mathrm{Stab}^\dagger(\widetilde{Y})$. They obtain two continuous maps using the functors π^* and π_*. The first one

$$(\pi^*)^{-1} \colon \Gamma_{\widetilde{Y}} \to \mathrm{Stab}(Y)$$

sends $\widetilde{\sigma} = (\widetilde{Z}, \widetilde{\mathcal{P}})$ to $(\widetilde{Z} \circ \pi^*, \widetilde{\mathcal{P}}')$, where $\widetilde{\mathcal{P}}'(\phi) = \{E \in \mathrm{D}^{\mathrm{b}}(Y) \mid \pi^* E \in \widetilde{\mathcal{P}}(\phi)\}$. The map

$$(\pi_*)^{-1} \colon (\pi^*)^{-1}(\Gamma_{\widetilde{Y}}) \to \mathrm{Stab}^\dagger(\widetilde{Y})$$

is defined analogously using π_*.

Theorem 3.18 (Macrì, Mehrotra, Stellari)**.** *The non-empty subset $(\pi^*)^{-1}(\Gamma_{\widetilde{Y}})$ of the stability manifold of an Enriques surface embeds via $(\pi_*)^{-1}$ into the distinguished component of its universal cover $\mathrm{Stab}^\dagger(\widetilde{Y})$ as a closed submanifold. If Y is generic, the components coincide.*

For a generic Enriques surface Y we denote by $\widetilde{\sigma} \in \mathrm{Stab}^\dagger(\widetilde{Y})$ the stability condition corresponding to $\sigma \in \mathrm{Stab}^\dagger(Y)$, i.e. $(\pi^*)^{-1}(\widetilde{\sigma}) = \sigma$.

We denote by $\mathrm{Stab}^\dagger(Y)$ the connected component of $(\pi^*)^{-1}(\Gamma_{\widetilde{Y}})$ containing the images of the invariant geometric stability conditions. Note that one could as well repeat the same construction of Bridgeland in [15] for Enriques surfaces. The authors also define the subset $\mathcal{P}_0^+(Y)$ as the image of the corresponding one for the universal cover $\mathcal{P}_0^+(\widetilde{Y})$ and obtain a covering $\mathcal{Z}\colon \mathrm{Stab}^\dagger(Y) \to \mathcal{P}_0^+(Y)$.

Nuer [45] established the existence of projective coarse moduli spaces for Bridgeland stability conditions on Enriques surfaces. The idea is to use the result of Bayer and Macrì for the covering K3 surface. For primitive Mukai vector these are also smooth projective K-trivial varieties and the results on the non-emptiness and smoothness from Theorem 3.15 and 3.16 remain valid. Furthermore, the morphism

$$\pi^*\colon M_\sigma^Y(v) \to M_{\widetilde{\sigma}}^{\widetilde{Y}}(\pi^*(v))$$

is 2:1 onto the fixed locus of the action given by the covering involution. The divisor ℓ_σ associated to a stability condition $\sigma \in \mathrm{Stab}^\dagger(Y)$ satisfies the following compatibility [45, Prop. 10.2].

Proposition 3.19 (Nuer)**.** *For the stability conditions $\sigma \in \mathrm{Stab}^\dagger(Y)$ and $\widetilde{\sigma} \in \mathrm{Stab}^\dagger(\widetilde{Y})$ the associated divisors satisfy*

$$(\pi^*)^*\ell_{\widetilde{\sigma}} = \ell_\sigma,$$

where $(\pi^*)^*\colon \mathrm{NS}(M_{\widetilde{\sigma}}^{\widetilde{Y}}(\pi^*(v))) \to \mathrm{NS}(M_\sigma^Y(v))$ is the induced map on the Néron–Severi groups.

4. Geometry of moduli spaces on Enriques surfaces

We first observe that the image of the moduli space of stable objects on an Enriques surface is a constant cycle Lagrangian. Shen, Yin, and Zhao [49] studied the group of zero-cycles on moduli spaces of stable objects on K3 surfaces. They formulated a conjecture, which was later proven by the third author and Marian [38].

Theorem 4.1 (Marian, Shen, Yin, Zhao)**.** *Let X be an algebraic K3 surface and $v \in H^*_{alg}(X,\mathbb{Z})$ a primitive Mukai vector. For a generic stability condition $\sigma \in \mathrm{Stab}^\dagger(X)$ consider the moduli space $M_\sigma(v)$ of stable complexes with Mukai vector v. For $E, F \in M_\sigma(v)$ we have*

$$[E] = [F] \text{ in } \mathrm{CH}_0(M_\sigma(v)) \iff c_2(E) = c_2(F) \text{ in } \mathrm{CH}_0(X).$$

By [11], Enriques surfaces Y satisfy Bloch's conjecture. Since their Albanese variety is trivial, we get $\mathrm{CH}_0(Y) = \mathbb{Z}$. Thus, we conclude the following

Proposition 4.2. *Assume that $\pi^*(v) \in H^*_{alg}(\widetilde{Y},\mathbb{Z})$ is primitive. Then, the image of the morphism $\pi^*\colon M^Y_\sigma(v) \to M^{\widetilde{Y}}_{\widetilde{\sigma}}(\pi^*(v))$ is a constant cycle Lagrangian.* □

If $v = (r, c_1, \frac{s}{2}) \in H^*_{\mathrm{alg}}(Y,\mathbb{Z})$ is primitive and r is even, the hypothesis of the proposition is fulfilled if and only if 2 does not divide $\gcd(r, c_1)$ [44, Lem. 2.1].

As we will see later, this is a useful result to gain new insights since up to now rather little is known about the geometry of these moduli spaces.

Observe that this argument does not solely work for Enriques surfaces. For example, one may take a K3 surface X given as a 2:1 cover $X \to \mathbb{P}^2$. Similarly, one may consider K3 surfaces with a non-symplectic automorphism of finite order. The quotient also satisfies Bloch's conjecture.

The moduli space $M^Y_\sigma(v)$ itself is not always CH_0-trivial. Indeed, these moduli spaces have Kodaira dimension zero which implies that the one-dimensional moduli spaces are elliptic curves. However, as we will see later, moduli spaces parametrizing odd rank Mukai vectors are always CH_0-trivial.

4.1. **Wall-crossing for Enriques surfaces.** Now we want to focus on the moduli space of stable objects on generic Enriques surfaces. Our goal in this section is to prove that for a generic Enriques surface Y the birational type of the moduli space of stable objects with respect to a generic stability condition $\sigma \in \mathrm{Stab}^\dagger(Y)$ does not depend on σ. The main idea is to use the result for the covering K3 surface and observe what happens with the moduli space of objects on the Enriques surface under the birational transformations appearing in Theorem 3.12. In this section all Enriques surfaces are generic.

Recall that we have an action of $i^* \in \mathrm{Aut}(\mathrm{D^b}(\widetilde{Y}))$ on $\mathrm{Stab}^\dagger(\widetilde{Y})$. Since Y is generic, the morphism $\pi^*\colon \mathrm{NS}(Y) \to \mathrm{NS}(\widetilde{Y})$ is surjective identifying the image with the eigenspace associated to the eigenvalue 1 of the map i^* acting on $H^2(\widetilde{Y},\mathbb{Z})$. Hence, the map induced by the covering involution acts trivially on the Mukai lattice. We use this to prove

Lemma 4.3. *The action of* $i^* \in \mathrm{Aut}(\mathrm{D^b}(\widetilde{Y}))$ *on the connected component of the stability manifold* $\mathrm{Stab}^\dagger(\widetilde{Y})$ *is trivial.*

Proof. We know already that the action on the Mukai lattice is trivial. Furthermore, the covering map $\mathcal{Z}\colon \mathrm{Stab}^\dagger(\widetilde{Y}) \to \mathcal{P}_0^+(\widetilde{Y}) \subset H^*_{\mathrm{alg}}(\widetilde{Y},\mathbb{Z}) \otimes \mathbb{C}$ from Theorem 3.10 is equivariant. Therefore, the homeomorphism i^* acts as a deck transformation for $\mathcal{Z}$ on $\mathrm{Stab}^\dagger(\widetilde{Y})$. However, the action is trivial on geometric stability conditions $\sigma_{\beta,\omega}$. Thus, i^* must be the trivial deck transformation. □

The lemma also immediately follows from the work of Macrì, Mehrotra, and Stellari [36, Prop. 3.12].

Corollary 4.4. *The autoequivalence* $i^* \in \mathrm{Aut}(\mathrm{D^b}(\widetilde{Y}))$ *acts on the moduli space, i.e. for all* $E \in M^{\widetilde{Y}}_{\widetilde{\sigma}}(\widetilde{v})$ *we have* $i^*E \in M^{\widetilde{Y}}_{\widetilde{\sigma}}(\widetilde{v})$. *In particular, if* $S \in \mathrm{D^b}(\widetilde{Y})$ *is a spherical object with primitive Mukai vector* $v(S)$ *such that* S *is stable for some* $\sigma \in \mathrm{Stab}^\dagger(\widetilde{Y})$*, then* $i^*S \cong S$. □

The moduli space decomposes into $M^Y_\sigma(v) \cong M^Y_\sigma(v,L) \sqcup M^Y_\sigma(v, L\otimes\omega_Y)$ and we will concentrate only on one component omitting the determinant in the notation. For an odd rank vector the two components are exchanged via tensoring with ω_Y and each component embeds into $M^{\widetilde{Y}}_{\widetilde{\sigma}}(\pi^*(v))$ under the morphism π^*. In the even rank case, π^* is an étale cover on each component.

Consider now two stability conditions $\sigma_+, \sigma_- \in \mathrm{Stab}^\dagger(Y)$. Inside one chamber the moduli spaces $M^Y_{\sigma_+}(v)$ and $M^Y_{\sigma_-}(v)$ stay the same. Hence, we only need to study the relationship of these two moduli spaces for σ_+ and σ_- in adjacent chambers. We can assume that the corresponding stability conditions $\widetilde{\sigma}_+$ and $\widetilde{\sigma}_-$ in $\mathrm{Stab}^\dagger(\widetilde{Y})$ are also generic with respect to $\pi^*(v)$ and lie in adjacent chambers since Y is generic. We are now ready to prove the main result of this thesis.

Theorem 4.5. *Let* Y *be a generic Enriques surface and* $v \in H^*_{\mathrm{alg}}(Y,\mathbb{Z})$ *a Mukai vector such that* $\pi^*(v) \in H^*_{\mathrm{alg}}(\widetilde{Y},\mathbb{Z})$ *is primitive. Then, for generic stability conditions* $\sigma,\tau \in \mathrm{Stab}^\dagger(Y)$ *the moduli spaces* $M^Y_\sigma(v)$ *and* $M^Y_\tau(v)$ *are birationally equivalent.*

Proof. To ease notation, we will prove the statement at first only for odd rank Mukai vectors and describe at the end how to deduce the result in the even rank case. Recall that under this assumption $M^Y_\sigma(v) \hookrightarrow M^{\widetilde{Y}}_{\widetilde{\sigma}}(\pi^*(v))$ is an embedding of a constant cycle Lagrangian since we only consider one component.

Consider the moduli spaces $M^Y_{\sigma_+}(v)$ and $M^Y_{\sigma_-}(v)$ in adjacent chambers and embed them inside $M^{\widetilde{Y}}_{\widetilde{\sigma}_+}(\pi^*(v))$ and $M^{\widetilde{Y}}_{\widetilde{\sigma}_-}(\pi^*(v))$ respectively. We know the assertion for the two moduli spaces of stable objects on the K3 surface $\widetilde{Y}$. For our purpose we need an additional property of the birational map. Observe that, since i^* acts on the moduli spaces $M^{\widetilde{Y}}_{\widetilde{\sigma}_+}(\pi^*(v))$ and $M^{\widetilde{Y}}_{\widetilde{\sigma}_-}(\pi^*(v))$, it makes sense to ask whether the birational map is i^*-equivariant.

The occuring birational map depends on the wall in the sense of [5, Thm. 5.7]. There are three different types. The first type induces a divisorial contraction of the moduli space. The contraction map contracts curves of stable objects that become S-equivalent for a stability condition on the wall. The second type is a wall inducing a flopping contraction. The remaining case is a fake wall, i.e. there are no curves in $M^{\widetilde{Y}}_{\widetilde{\sigma}_+}(\pi^*(v))$ and $M^{\widetilde{Y}}_{\widetilde{\sigma}_-}(\pi^*(v))$ that become S-equivalent with respect to a stability condition on the wall.

We now treat each of these cases and show that the corresponding birational map is equivariant.

In case of a flopping contraction or a fake wall there either exists a common open subset whose complement has at least codimension two or the birational map is induced by the composition of spherical twists. The spherical objects involved are all stable for some stability condition. Using Lemma 4.4 we see that in both cases the map is equivariant.

The case of a wall inducing a divisorial contraction is divided into three subcases. If we are in the Brill–Noether case, the birational map is again defined on an open subset to be a sequence of spherical twists associated to stable spherical objects. The second type is the Hilbert–Chow case, where the moduli spaces are even isomorphic via the derived anti-autoequivalence $(\_)^\vee[2]$. The last occurring type is called Li–Gieseker–Uhlenbeck. In this case the birational map is given on an open subset via the functor $(\_)^\vee \otimes L[2]$, where L is a line bundle. Since $\pi^*\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(\widetilde{Y})$ is surjective for generic Enriques surfaces, we get that all the possible birational maps are equivariant with respect to the action of i^*.

Denote by $f\colon M^{\widetilde{Y}}_{\widetilde{\sigma}_+}(\pi^*(v)) \dashrightarrow M^{\widetilde{Y}}_{\widetilde{\sigma}_-}(\pi^*(v))$ the biratonal map and U the biggest open subset where f is an isomorphism. The intersection $U \cap M^Y_{\sigma_+}(v) \subset M^{\widetilde{Y}}_{\widetilde{\sigma}_+}(\pi^*(v))$ is non-empty by Proposition 2.6. Since $M^Y_{\sigma_\pm}(v)$ can be identified with the fixed set of the involution i^* and f is equivariant, the induced map $f|_{M^Y_{\sigma_+}(v)}\colon M^Y_{\sigma_+}(v) \dashrightarrow M^Y_{\sigma_-}(v)$ gives the desired birational transformation. This finishes the proof for Mukai vectors of odd rank.

If the rank is even, we can still deduce that the image of both components of the moduli space $M^Y_{\sigma_\pm}(v)$ under π^* intersects the open set U. Indeed, the pullback is étale onto its image and, therefore, the image is still of Kodaira dimension 0. Thus, there are i^*-equivariant Fourier–Mukai transforms inducing a birational map between $\pi^*(M^Y_{\sigma_+}(v))$ and $\pi^*(M^Y_{\sigma_-}(v))$. To deduce the result for the moduli space we just observe that π^* corresponds to forgetting the equivariant structure. Thus, the isomorphism on the image can be used to obtain an isomorphism of equivariant sheaves [16, Thm. 4.5]. □

Nuer already observed the birational equivalence of the moduli spaces in the case that the wall is not totally semistable. Note that we have proven more precisely that the birational map for the moduli spaces on a K3 surface from Theorem 3.12 restricts to a birational map for the moduli spaces on an Enriques surface.

An idea to prove the result in the non-generic case would be to use a specialization argument similiar to the one we will use in Proposition 4.8. For this one needs to establish the existence of a family of moduli spaces of stable objects on generic and non-generic Enriques surfaces.

4.2. **Birational moduli spaces.** In this section we study the birational geometry of $\pi^*\colon M^Y_\sigma(v) \to M^{\widetilde{Y}}_{\widetilde{\sigma}}(\pi^*(v))$ and the two moduli spaces involved. We start with the hyperkähler manifold. Since the torsion-free part of the second cohomology of an Enriques surface is isometric to $U \oplus E_8(-1)$, we only have to focus on Mukai vectors $\widetilde{v} = (r, c_1, s)$ with $c_1 \in U(2) \oplus E_8(-2)$.

Proposition 4.6. *Let $\widetilde{Y}$ be a K3 surface with a fixed-point-free involution and $\widetilde{v} = (r, c_1, s)$ a primitive Mukai vector with $c_1 \in U(2) \oplus E_8(-2) \subset \operatorname{Pic}(\widetilde{Y})$, r odd and $\widetilde{v}^2 > 0$. Then, for generic $\widetilde{\sigma}$ the moduli space $M^{\widetilde{Y}}_{\widetilde{\sigma}}(\widetilde{v})$ is birationally equivalent to some Hilbert scheme of points on $\widetilde{Y}$.*

Proof. Although the proof only differs slightly from [54, Thm. 4.6], we explain it in full.

By Remark 3.13, since we are only interested in the birational type of the moduli spaces, we will freely apply autoequivalences to modify the Mukai vector. First, assume that $c_1 \in E_8(-2)$ and set $k = \gcd(r, c_1)$. Then, we can find $x \in E_8(-2)$ such that $(c_1 + rx)/k$ is primitive in $H^2(\widetilde{Y}, \mathbb{Z})$, see [19, Lem. 2.1]. Now consider the Fourier–Mukai transform given by tensoring with a line bundle $\mathcal{L}$ whose first Chern class satisfies $c_1(\mathcal{L}) = x$. This autoequivalence induces an isometry of Mukai lattices given by multiplication with $\exp(x)$ [21, Exc. 5.37]. Thus, we can assume that c_1/k is primitive.

Applying the spherical twist associated to the structure sheaf the Mukai vector v gets sent to $(-s, c_1, -r)$. Since v is by assumption primitive, we have $\gcd(k, s) = 1$. Employing the same technique as above we can assume that c_1 itself is primitive. Hence, we are in the case where $v = (r, c_1, s)$ with c_1 primitive.

Choose $\xi \in E_8(-2)$ with $c_1\xi = -1-s$ and consider $\eta = e - \frac{\xi^2}{4} f + \xi$, where e and f denote the standard basis of $U(2)$. From the definition of η we conclude $\eta^2 = 0$ and $\eta c_1 = -1 - s$. Using the autoequivalence given by twisting with a line bundle whose first Chern class is η, the action on cohomology sends v to $(r, c_1 + r\eta, s + \eta^2/2 + c_1\eta) = (r, c_1 + r\eta, -1)$. Finally, applying once more the spherical twist associated to $\mathcal{O}_{\widetilde{Y}}$ we conclude that in the case $c_1 \in E_8(-2)$ the moduli space $M_{\widetilde{\sigma}}^{\widetilde{Y}}(\widetilde{v})$ is birationally equivalent to $M_H^{\widetilde{Y}}(1, c_1', s')$ which itself is isomorphic to $\mathrm{Hilb}^n(\widetilde{Y})$.

Using the standard autoequivalences for the general case we may assume $r > 0$. We proceed via induction over r. Write $c_1 = a_1 e + a_2 f + \eta$ with $\eta \in E_8(-2)$. Applying the autoequivalence given by twisting with the line bundle $\mathcal{L}^k$, where $c_1(\mathcal{L}) = e$ and $k \in \mathbb{Z}$, we can assume $0 \leq |a_1| < r/2$. If a_1 is zero, we do the same with a_2. If a_2 is also zero, we are in the previous case. Hence, we may assume $|a_1| > 0$. Choose $l \in \mathbb{Z}$ such that $0 < r - la_1 < r$. Applying the spherical twist associated to the structure sheaf and the Fourier–Mukai transform given by $\_ \otimes \mathcal{M}$, where $c_1(\mathcal{M}) = lf$, we modified the Mukai vector to $(-s, c_1 + lf, -r + lc_1 f)$. Using one last time the spherical twist, we lowered the rank to $0 < r - la_1 < r$ which finishes the proof. $\square$

Thus, in the case of a K3 surface $\widetilde{Y}$ covering an Enriques surface Y and a Mukai vector v of odd rank, the image of $M_\sigma^Y(v)$ embeds into a hyperkähler variety birationally equivalent to a Hilbert scheme of points on $\widetilde{Y}$.

Now we want to use the established wall-crossing for moduli spaces of stable objects on Enriques surfaces and deduce some consequences.

On generic Enriques surfaces there are no spherical objects [36, Prop. 3.17]. Consequently, there are no spherical twists available in $\mathrm{Aut}(\mathrm{D^b}(Y))$. Instead, one can use exceptional objects to construct non-trivial autoequivalences. Recall that an object $E \in \mathrm{D^b}(Y)$ is called exceptional if $\mathrm{Hom}^i(E, E)$ is zero except for $i = 0$, where it is one-dimensional. The following was already observed by Yoshioka, cf. [54, Sec. 4].

Lemma 4.7. *Let Y be an Enriques surface and $E \in \mathrm{D}^{\mathrm{b}}(Y)$ an exceptional object. The spherical twist associated to π^*E preserves $\mathrm{D}^{\mathrm{b}}_{i^*}(\widetilde{Y}) \cong \mathrm{D}^{\mathrm{b}}(Y)$ and, therefore, descends to an element in $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(Y))$.*

Proof. Serre duality and adjunction of π^* and π_* yield that π^*E is indeed a spherical object. The spherical twist associated to π^*E is the Fourier–Mukai transform with kernel $\mathcal{P}_{\pi^*E}$ given by the distinguished triangle

$$p^*\pi^*E \otimes q^*\pi^*E^\vee \to \mathcal{O}_\Delta \to \mathcal{P}_{\pi^*E} \to p^*\pi^*E \otimes q^*\pi^*E^\vee[1],$$

where p and q denote the two projections from $\widetilde{Y} \times \widetilde{Y}$ onto its factors. Applying the corresponding Fourier-Mukai transforms to an element $F \in \mathrm{D}^{\mathrm{b}}_{i^*}(\widetilde{Y})$ yields the triangle

$$\pi^*E \otimes \mathrm{Hom}^\bullet(\pi^*E, F) \to F \to \Phi_{\mathcal{P}_{\pi^*E}}(F) \to \pi^*E \otimes \mathrm{Hom}^\bullet(\pi^*E, F)[1].$$

Since the first two factors and the morphism between them are in $\mathrm{D}^{\mathrm{b}}_{i^*}(\widetilde{Y})$, their cone $\Phi_{\mathcal{P}_{\pi^*E}}(F)$ is also invariant under the action of i^*. □

If we denote $G := p^*E \otimes q^*E^\vee \oplus p^*E(\omega_Y) \otimes q^*E(\omega_Y)^\vee$, the autoequivalence of $\mathrm{D}^{\mathrm{b}}(Y)$ associated in this way to E can alternatively be described as the Fourier–Mukai transform associated to the kernel K completing the triangle

$$G \to \mathcal{O}_\Delta \to K \to G[1]$$

in $\mathrm{D}^{\mathrm{b}}(Y \times Y)$. The associated cohomological Fourier–Mukai functor sends $v \in H^*_{\mathrm{alg}}(Y, \mathbb{Z})$ to $v + 2(v(E), v)v(E)$. Taking the structure sheaf $\mathcal{O}_Y$ we obtain the isometry of Mukai lattices $(r, c_1, \frac{s}{2}) \mapsto (-s, c_1, -\frac{r}{2})$. We call an equivalence arising in this way a *weakly spherical twist.*

Proposition 4.8. *Let Y be a (not necessarily generic) Enriques surface and $v = (r, c_1, \frac{s}{2})$ be a primitive Mukai vector with r odd. Then, for generic H the moduli space $M_H(v)$ is birationally equivalent to the Hilbert scheme of points $\mathrm{Hilb}^n(Y)$.*

Proof. The proof in the generic case is similiar to the one for Proposition 4.6. Instead of spherical twists associated with the structure sheaf we will use the weakly spherical ones from Lemma 4.7.

We again first assume that c_1 lies inside $E_8(-1)$. With the same argument using the weakly spherical twist for $\mathcal{O}_Y$ we can assume that c_1 is primitive. We now choose $\xi \in E_8(-1)$ satisfying $c_1\xi = \frac{1-s}{2}$ and consider $\eta = e - \frac{\xi^2}{2} f + \xi$. The rest of the argument as well as the general case copies without change.

Now we consider the non-generic case. Consider a family $\pi\colon \mathcal{Y} \to B$ of Enriques surfaces with non-generic central fiber and generic general fiber. We construct two families. The first one is the relative Hilbert scheme $\phi\colon \mathrm{Hilb}^n_B(\mathcal{Y}) \to B$ whose fibers are the Hilbert scheme of points on $\mathcal{Y}_b$. The second is the relative moduli space of stable sheaves $\psi\colon M_{\mathcal{Y}/B}(v) \to B$. All fibers of both families are smooth K-trivial varieties. By the above we know for general $b \in B$ that the fibers $\phi^{-1}(b)$ and $\psi^{-1}(b)$ are birationally equivalent. Thus, there is at least one component of the relative Hilbert scheme $\mathrm{Hilb}_B(\mathrm{Hilb}^n_B(\mathcal{Y}) \times_B M_{\mathcal{Y}/B}(v))$ that contains uncountably many birational correspondences between the fibers. The universal subvariety parametrized by this component then induces a birational correspondence on the special fiber, cf. [40, Thm. 1]. $\square$

In fact, the birational equivalence is a K-equivalence. Since the above construction is compatible with π^*, the inclusion of moduli spaces of stable sheaves is up to K-equivalence the inclusion of the Hilbert scheme of points inside the Hilbert scheme of the covering K3 surface. Hence, the constant cycle Lagrangians are already CH_0-trivial. If relative moduli spaces of Bridgeland stability conditions over a curve are constructed, the same proof gives the above result for moduli spaces of stable complexes, see [3].

The above statement will definitely fail for even rank vectors for dimension reasons. Furthermore, we cannot hope that they are always CH_0-trivial since for example the one-dimensional moduli spaces are elliptic curves.

Next, we want to determine the birational type of the moduli spaces in the even rank case.

Proposition 4.9. *Let Y be a generic Enriques surface and v an even rank Mukai vector such that $\pi^*(v)$ is primitive and $v^2 > 0$. Then, $M^Y_\sigma(v)$ is birational to $M^Y_{\sigma'}(v')$, where the Mukai vector $v' = (0, c_1, \frac{s}{2})$ has primitive and effective c_1 and σ' is generic for v'.*

Proof. We again make use of Remark 3.13. Suppose $v = (r, c_1, \frac{s}{2})$ has strictly positive rank $r > 0$. Using the techniques of the proofs of Hauzer and Nuer, we know that every moduli space $M^Y_\sigma(v)$ is birational to a moduli space where $r = 2$. Since $\pi^*(v)$ is primitive, 2 does not divide c_1. Hence, we can write $c_1 = a_1 e + a_2 f + \xi$, where e and f denote the standard basis of the hyperbolic plane U, $a_i \in \{0, 1\}$ and $\xi \in E_8(-1)$ is primitive.

If $a_1 = a_2 = 0$, choose $\eta \in E_8(-1)$ satisfying $\eta\xi = -\frac{s}{2}$ and consider $D \coloneqq e - \frac{\eta^2}{2} f + \eta$. Then $D^2 = 0$ and $D\xi = -\frac{s}{2}$. Applying $\exp(D)$ and the weakly spherical twist associated to the structure sheaf yields the assertion.

If $a_1 = 1$ or $a_2 = 1$, use $D = -\frac{s}{2}f$ respectively $D = -\frac{s}{2}e$ and conclude as above. $\square$

The proposition enables us to apply the results of Saccà.

Corollary 4.10. *Let $v \in H^*_{\mathrm{alg}}(Y,\mathbb{Z})$ be an even rank Mukai vector such that $\pi^*(v)$ is primitive and $v^2 > 0$. Then $M^Y_\sigma(v)$ is a smooth projective Calabi–Yau variety, i.e.*

$$\omega_{M^Y_\sigma(v)} \cong \mathcal{O}_{M^Y_\sigma(v)} \text{ and } h^{p,0}(M^Y_\sigma(v)) = 0 \text{ for } p \neq 0, v^2+1. \qquad \square$$

4.3. **Minimal models of moduli spaces.** As a last application of Theorem 4.5, we show that under an extra assumption all birational smooth K-trivial models of $M^Y_\sigma(v)$ for generic Y and $\pi^*(v)$ primitive can be obtained as a moduli space $M_\tau(v)$ for some stability condition $\tau \in \mathrm{Stab}^\dagger(Y)$.

The argument uses the nef divisor ℓ_σ associated to a stability condition. In our arguments we will often use the following compatibility result.

Lemma 4.11. *The diagram*

$$\begin{tikzcd}
\mathrm{Stab}^\dagger(\widetilde{Y}) \arrow[r, "\mathcal{Z}"] \arrow[d, "(\pi^*)^{-1}"] & \mathcal{P}^+_0(\widetilde{Y}) \arrow[r, "I"] \arrow[d, "\pi_*"] & \pi^*(v)^\perp \arrow[r, "\theta_{\widetilde{\sigma}}"] \arrow[d, "\pi_*"] & \mathrm{NS}(M^{\widetilde{Y}}_{\widetilde{\sigma}}(\pi^*(v))) \arrow[d, "(\pi^*)^*"] \\
\mathrm{Stab}^\dagger(Y) \arrow[r, "\mathcal{Z}"] & \mathcal{P}^+_0(Y) \arrow[r, "I"] & v^\perp \arrow[r, "\theta_\sigma"] & \mathrm{NS}(M^Y_\sigma(v))
\end{tikzcd}$$

commutes.

Proof. The commutativity of the left square is proven in [36, Prop. 3.1]. The proof for the right square is analogous to the proof of [45, Prop. 10.2] and the commutativity of the middle square is immediate. $\square$

For two adjacent chambers $\mathcal{C}^+, \mathcal{C}^-$ we pick again stability conditions $\sigma_\pm \in \mathcal{C}^\pm$ and a stability condition σ_0 on the wall $\mathcal{W}$ separating the two chambers. We identify $\mathrm{NS}(M^Y_{\sigma_+}(v))$ and $\mathrm{NS}(M^Y_{\sigma_-}(v))$ using the restriction of the birational transformation between $M^{\widetilde{Y}}_{\widetilde{\sigma}_+}(\pi^*(v))$ and $M^{\widetilde{Y}}_{\widetilde{\sigma}_-}(\pi^*(v))$. This gives us two maps

$$\ell^\pm \colon \mathcal{C}^\pm \to \mathrm{NS}(M^Y_{\sigma_+}(v)).$$

We now study the behaviour at the wall $\mathcal{W}$. We denote by $\widetilde{\mathcal{W}}$ the corresponding wall in $\mathrm{Stab}^\dagger(\widetilde{Y})$. The stability condition $\widetilde{\sigma_0}$ produces nef and big divisor classes $\ell_{\widetilde{\sigma_0},\pm}$ on $M^{\widetilde{Y}}_{\widetilde{\sigma}_\pm}(\pi^*(v))$ which give rise to birational contraction morphisms

$$\pi_{\widetilde{\sigma}_\pm} \colon M^{\widetilde{Y}}_{\widetilde{\sigma}_\pm}(\pi^*(v)) \to \widetilde{M}_\pm.$$

Lemma 4.12. *The maps ℓ^+ and ℓ^- agree on the wall $\mathcal{W}$ when extended by continuity.*

(i) *If $\pi_{\widetilde{\sigma}_+}$ is an isomorphism or a small contraction, then the maps ℓ^+ and ℓ^- are analytic continuations of each other.*

(ii) *If $\pi_{\widetilde{\sigma}_+}$ contracts a divisor $\widetilde{D}$, then the maps ℓ^+ and ℓ^- differ in $\mathrm{NS}(M^Y_{\sigma_+}(v))$ by the reflection ρ_D, where $D = (\pi^*)^*\widetilde{D}$.*

Proof. The assertion follows from the corresponding statement for the covering K3 surface [5, Lem. 10.1] and Lemma 4.11. □

Nuer [45] studied the nef divisors $\ell_{\sigma_0,\pm}$ and was able to transfer results from [6] to moduli spaces of Enriques surfaces and showed that these are as well semiample. However, the question whether or not these divisors are big remained open.

Proposition 4.13. *The nef and semiample divisors $\ell_{\sigma_0,\pm} \in \mathrm{NS}(M^Y_{\sigma_\pm}(v))$ are big and induce birational contraction morphisms*

$$\pi_{\sigma_\pm} \colon M^Y_{\sigma_\pm}(v) \to M_\pm.$$

Proof. Consider the images of the moduli spaces $\pi^*(M^Y_{\sigma_\pm}(v)) \subset M^{\widetilde{Y}}_{\widetilde{\sigma}_\pm}(\pi^*(v))$. Using the projection formula a curve $C \subset M^Y_{\sigma_\pm}(v)$ gets contracted to a point under the morphism associated to the linear system $|\ell_{\sigma_0,\pm}|$ if and only if its image $\pi^*(C) \subset M^{\widetilde{Y}}_{\widetilde{\sigma}_\pm}(\pi^*(v))$ gets contracted to a point.

There are three possible cases for the contraction on the K3 side [6, Thm. 1.4]. If the wall is a fake wall, then no curve gets contracted and the same remains true on the Enriques side. If the morphism induced by the big line bundle has an exceptional locus B that is contracted, there are two possibilities. Firstly, the induced morphism is a divisorial contraction. In this case the divisor B has negative square with respect to the Beauville–Bogomolov form and is therefore uniruled [20]. Secondly, the codimension of the subvariety B that is contracted is greater than one. In this case B is even rationally chain connected [17].

In either case the image $\pi^*(M^Y_{\sigma_\pm}(v))$ cannot be contained in the uniruled variety B since it is a constant cycle Lagrangian of non-negative Kodaira dimension. □

We now come to our final result about the minimal models. Unfortunately, we have to impose an extra assumption in our theorem.

Consider the Mukai morphism

$$\theta_\sigma \colon v^\perp \to \mathrm{NS}(M_\sigma(v)).$$

In the case of a K3 surface and moduli of stable sheaves, Yoshioka [53] has proven that this morphism induces an isometry between $v^\perp$ with the Mukai pairing and the Néron–Severi group with the Beauville–Bogomolov form on the hyperkähler variety. This result has been generalized to Bridgeland stability conditions by Bayer and Macrì [6, Thm. 6.10]. Thus, for $a \in v^\perp$ to be mapped to a big and movable divisor its square under the Mukai pairing has to be positive.

To the best of my knowledge, there are no results on the relationship in the case of moduli spaces on Enriques surfaces. In particular, it is unclear whether the square of a class mapping to a big and movable divisor is positive. Small evidence for this assertion is that the ample divisors constructed by Huybrechts and Lehn [24, Thm. 8.1.11] and Nuer [45, Cor. 12.6] all satisfy this property. We denote by ($\star$) the assumption that for the moduli space of stable objects on an Enriques surface classes which are mapped under θ_σ to a big and movable divisor have positive square with respect to the Mukai pairing.

The following has been proven by Bayer and Macrì for moduli spaces of stable objects on K3 surfaces. The result implies that all birational models of a moduli space of stable complexes are again moduli spaces.

Theorem 4.14. *Let Y be generic and fix a basepoint $\sigma \in \mathrm{Stab}^\dagger(Y)$. Assume ($\star$) holds. Then, the image of the map*

$$\ell\colon \mathrm{Stab}^\dagger(Y) \to \mathrm{NS}(M_\sigma^Y(v))$$

is the intersection of the movable cone and the big cone.

Proof. Take a big and movable divisor D and consider a class a such that $\theta_\sigma(a) = D$. The arguments from the proof of [5, Thm. 1.2(b)] apply to our situation. Hence, there is a path $\gamma\colon [0,1] \to \mathcal{P}_0^+(Y)$ starting at $\gamma(0) = \mathcal{Z}(\sigma)$ and ending at (possibly a slight modification of) $\gamma(1) = ia - \frac{v}{v^2} \in H^*_{\mathrm{alg}}(Y,\mathbb{Z}) \otimes \mathbb{C}$. Furthermore, for all $t \in [0,1]$ the class $\theta_\sigma(I(\gamma(t)))$ is contained in the movable cone of $M_\sigma^Y(v)$. Since the morphism $\mathcal{Z}\colon \mathrm{Stab}^\dagger(Y) \to \mathcal{P}_0^+(Y)$ is a covering, we can lift the path γ to obtain a path $\delta\colon [0,1] \to \mathrm{Stab}^\dagger(Y)$ starting at $\delta(0) = \sigma$.

To conclude the proof we want to apply Lemma 4.12. Therefore, we have to show that the path $\widetilde{\delta}$ in $\mathrm{Stab}^\dagger(\widetilde{Y})$ does not hit a wall corresponding to a divisorial contraction. However, a divisorial contraction on the K3 side would induce a divisorial contraction on the Enriques side since $(\pi^*)^*\colon \mathrm{NS}(M_{\widetilde{\sigma}}^{\widetilde{Y}}(\pi^*(v))) \to \mathrm{NS}(M_\sigma^Y(v))$ is injective. This is not possible because at each point of the path δ the image under the map ℓ is contained in the movable cone. $\square$

Another way to achieve this statement would be to show that if a divisor $D \in \mathrm{NS}(M_\sigma^Y(v))$ is movable and big, then so is $\widetilde{D} \in \mathrm{NS}(M_{\widetilde{\sigma}}^{\widetilde{Y}}(\pi^*(v)))$, where $(\pi^*)^*\widetilde{D} = D$. Then one can simply apply the result for moduli spaces of stable objects on K3 surfaces and find a stability condition $\widetilde{\tau} \in \mathrm{Stab}^\dagger(\widetilde{Y})$ which maps to $\widetilde{D}$ under the morphism $\widetilde{\ell}$. By Lemma 4.11, the induced stability condition $\tau \in \mathrm{Stab}^\dagger(Y)$ then maps to D.

An approach of the author was to study what happens at the boundary of the movable cones. To establish the inclusion of the big and movable cones one tries to show that a divisor on the boundary of the movable cone of $\mathrm{NS}(M_{\widetilde{\sigma}}^{\widetilde{Y}}(\pi^*(v)))$ gets mapped to a divisor on the boundary of the movable cone of $\mathrm{NS}(M_\sigma^Y(v))$. There are two types of divisors that can occur on the boundary, cf. [5, Thm. 12.3].

If the divisor $\widetilde{D}$ on the moduli space of stable objects on a K3 surface induces a birational divisorial contraction, then we can use the same arguments from the proof of Proposition 4.13 to show that the same is true for the divisor D on the Enriques side. The other case is that the square of the divisor $\widetilde{D}$ under the Beauville–Bogomolov form tends to zero. If $\widetilde{D}$ is rational, then a positive multiple induces a birational Lagrangian fibration. Thus, it remains to show that in this case the induced map also has generically positive dimensional fibers. We believe that this is the case. However, due to the time limitation of this thesis, we were not able to prove this yet. We hope to establish Theorem 4.14 in the near future without assumption $(\star)$.

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