Problems on global aspects of the Langlands Program solution guide

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This is a solution guide to the problem set for Gaëtan Chenevier's course on *global aspects of the Langlands program* at the Sheffield Langlands Program School in July 2026. It leads the reader through the solutions to the problems, and supplements it with more details and additional problems.

1 Lecture 1: modular forms, lattices, and the adèles

The first few exercises concern classical modular forms and their reinterpretation as lattice functions and as functions on $\mathrm{SL}_2(\mathbb{R})$.

Recall that the group $\mathrm{SL}_2(\mathbb{Z})$ acts on the upper half plane $\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ by Möbius transformations: for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we let $\gamma(z) = \frac{az+b}{cz+d}$.

Definition 1.1. A *modular form* of level N and weight $k \in \mathbb{Z}$ is a holomorphic function $f \colon \mathcal{H} \to \mathbb{C}$ such that:

(a) for any $\gamma \in \Gamma(N)$, $f(\gamma(z)) = (cz+d)^k f(z)$,

(b) for any $\gamma \in \Gamma(N)$, $(cz+d)^{-k} f(\gamma(z))$ is bounded as $\mathrm{Im}(z) \to \infty$.

Moreover, f is a *cusp form* if it satisfies the stronger condition:

(b') for any $\gamma \in \Gamma(N)$, $(cz+d)^{-k} f(\gamma(z)) \to 0$ as $\mathrm{Im}(z) \to \infty$.

We write $M_k(N)$ for the space of modular forms of level N and weight k, and $S_k(N) \subseteq M_k(N)$ for the subspace of cusp forms. When $N = 1$, we write simply $M_k := M_k(1)$ and $S_k := S_k(1)$.

On the other hand, we make the following definition. A *lattice* in $\mathbb{C}$ is a discrete cocompact rank 2 subgroup $L \subset \mathbb{C}$. Write $\mathcal{L}$ for the set of lattices in $\mathbb{C}$.

Definition 1.2. A *lattice function of weight* $k \in \mathbb{Z}$ is a function $F \colon \mathcal{L} \to \mathbb{C}$ such that $F(\lambda L) = \lambda^{-k} F(L)$ for all $\lambda \in \mathbb{C}^\times$.

1. **Modular forms as lattice functions.**
    (a) Show that for $z \in \mathcal{H}$, we have an associated lattice $L_z := \mathbb{Z}z \oplus \mathbb{Z} \subset \mathbb{C}$, and hence show that there is a well-defined function

$$\begin{aligned} \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H} &\to \mathcal{L}/\mathbb{C}^\times \\ [z] &\mapsto [L_z]. \end{aligned}$$

On the other hand, show that every $L \in \mathcal{L}$ has a basis (ω_1, ω_2) with $z := \omega_1/\omega_2 \in \mathcal{H}$, and that two such "positively oriented" bases of L differ by the action of $\mathrm{SL}_2(\mathbb{Z})$. Deduce that the above map is a bijection.

(b) Given a lattice function F of weight k, set $f(z) = F(L_z)$. Show that f satisfies the weight k transformation law (a) from Definition 1.1. Conversely, show that any $f\colon \mathcal{H} \to \mathbb{C}$ satisfying (a) arises from a unique weight-k lattice function F, via $F(L) = \omega_2^{-k} f(\omega_1/\omega_2)$ for any positively oriented basis. (Check this is well defined using part (a).)

(c) Show that under this correspondence, holomorphy of f on $\mathcal{H}$ is equivalent to holomorphy of F in the lattice variable (i.e. of $(\omega_1, \omega_2) \mapsto F(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$ on $\{\omega_1/\omega_2 \in \mathcal{H}\}$), and that boundedness condition (b) corresponds to a growth condition on F as L degenerates. Conclude that M_k is identified with the space of holomorphic weight-k lattice functions satisfying the growth condition.

2. **Modular forms as functions on** $\mathrm{SL}_2(\mathbb{R})$**.**

This problem realizes weight-k forms as functions on the group, preparing for the adèlic reformulation. Write $j(g, z) = cz + d$ for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, and

$$K_\infty = \mathrm{SO}(2) = \{r_\theta \mid \theta \in \mathbb{R}/2\pi\mathbb{Z}\} \subset \mathrm{SL}_2(\mathbb{R}), \qquad r_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right).$$

(a) Show that $\mathrm{SL}_2(\mathbb{R})$ acts transitively on $\mathcal{H}$ with stabilizer of i equal to $\{\pm 1\}K_\infty$, so $\mathcal{H} \cong \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}K_\infty$.

(b) Show that there is a natural isomorphism $\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{GL}_2(\mathbb{R}) \cong \mathcal{L}$, and deduce a relation between 1(a) and 2(a). (We will soon generalize this statement to GL_n and rank n lattices.)

(c) Verify the cocycle relation $j(gh, z) = j(g, hz)\, j(h, z)$. For $f \in M_k$ define

$$\begin{aligned} \varphi_f\colon\ \mathrm{SL}_2(\mathbb{R}) &\to \mathbb{C} \\ g &\mapsto j(g, i)^{-k} f(g \cdot i). \end{aligned}$$

Show that φ_f is left $\mathrm{SL}_2(\mathbb{Z})$-invariant and right K_∞-equivariant of weight k, i.e. $\varphi_f(g\, r_\theta) = e^{ik\theta}\varphi_f(g)$.

(d) Show $f \mapsto \varphi_f$ gives a bijection

$$\{f\colon \mathcal{H} \to \mathbb{C} \text{ satisfying } (a)\} \to \{\varphi\colon\ \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C} \mid \varphi(g r_\theta) = e^{ik\theta}\varphi(g)\}.$$

i. Recall that $X \in \mathfrak{sl}_2(\mathbb{R})$ acts $C^\infty(\mathrm{SL}_2(\mathbb{R}))$ by exponentiation and right translation:

$$(R_X\varphi)(g) := \frac{d}{dt}\varphi(g\exp(tX))\bigg|_{t=0}$$

and this action extends linearly to the action of $\mathfrak{sl}_2(\mathbb{C})$. Consider the weight-lowering operator $X_- := \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C})$. Prove that f is *holomorphic* if and only if φ_f is annihilated by the action of X_-.

ii. Show that f is *cuspidal* if and only if φ_f is cuspidal (in the sense introduced in the lectues) if and only if φ_f is square-integrable, i.e. $\varphi_f \in L^2(\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R}))$.

(e) Explain how, for $f \in S_k$ where $k \geq 2$, the right translates of φ_f generate a copy of the discrete series D_k, recovering the canonical identification $M(D_k) = S_k$ from the lectures.

(f) Show that φ_f descends to PSL_2, i.e. is a function on $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathrm{PSL}_2(\mathbb{R})$, if and only if k is even.

3. **Hecke operators for modular forms via lattices.**

In the lectures, we defined for a finite abelian group A, the Hecke operator on weight k lattice functions,

$$(T_A F)(L) \;=\; \sum_{\substack{L'\subseteq L \\ L/L'\cong A}} F(L').$$

Recall $f(z) = F(L_z)$ from Problem 1.

(a) Show that the index p sublattices of L are in bijection with the lines in $L/pL \cong (\mathbb{Z}/p)^2$; in particular there are $p+1$ of them. More generally, count the sublattices of index n and relate the answer to $\sigma_1(n) = \sum_{d|n} d$.

(b) Take $A = \mathbb{Z}/p$. Using the representatives $L_z \supset \mathbb{Z}(z+j)\oplus\mathbb{Z}p$ for $0 \le j \le p-1$, and $\mathbb{Z}pz \oplus \mathbb{Z}$, prove

$$(T_{\mathbb{Z}/p}F)(L_z) \;=\; f(pz) + p^{-k}\sum_{j=0}^{p-1} f\left(\tfrac{z+j}{p}\right).$$

Deduce that, after the normalization $T_p := p^{k-1}T_{\mathbb{Z}/p}$, one recovers the classical Hecke operator on M_k.

(c) Show that $T_{(\mathbb{Z}/p)^2}$ acts on weight-k lattice functions by the scalar p^{-k}. Interpret this central operator as the diamond operator $\langle p\rangle$ at level 1.

(d) Prove the multiplicativity relations: $T_{\mathbb{Z}/m}T_{\mathbb{Z}/n} = T_{\mathbb{Z}/mn}$ for $(m,n)=1$, and for a prime p,

$$T_{\mathbb{Z}/p}\,T_{\mathbb{Z}/p^r} \;=\; \begin{cases} T_{\mathbb{Z}/p^2} \;+\; (p+1)\,T_{(\mathbb{Z}/p)^2} & r=1, \\ T_{\mathbb{Z}/p^{r+1}} \;+\; p\,T_{(\mathbb{Z}/p)^2}\,T_{\mathbb{Z}/p^{r-1}} & r\ge 2. \end{cases}$$

Use this to give a definition of T_{p^r} in terms of these operators such that the classical relation $T_pT_{p^r} = T_{p^{r+1}} + p^{k-1}T_{p^{r-1}}$ holds.

The next series of questions is about the adèles, leading up to the adèlic interpretation of modular forms. We also generalize some of the results above from GL_2 to GL_n.

4. **The rational numbers $\mathbb{Q}$ are discrete and cocompact in $\mathbb{A}$.**

Recall that $\mathbb{Q}$ embeds diagonally in $\mathbb{A} = \mathbb{R}\times\mathbb{A}_f$.

(a) Show $\mathbb{A}_f = \mathbb{Q} + \widehat{\mathbb{Z}}$ and $\mathbb{Q}\cap\widehat{\mathbb{Z}} = \mathbb{Z}$. (Hint: by the Chinese remainder theorem reduce to one prime at a time; subtract a suitable rational with p-power denominator from a given $x \in \mathbb{A}_f$.)

(b) Deduce the "fundamental domain" statement from lectures $\mathbb{A} = \mathbb{Q} + \big([0,1]\times\widehat{\mathbb{Z}}\big)$ and $\mathbb{Q}\cap(\mathbb{R}\times\widehat{\mathbb{Z}}) = \mathbb{Z}$.

(c) Conclude that $\mathbb{Q}$ is discrete in $\mathbb{A}$ by exhibiting an open neighborhood of 0 meeting $\mathbb{Q}$ only in 0.

(d) Conclude that $\mathbb{A}/\mathbb{Q} \cong (\mathbb{R}\times\widehat{\mathbb{Z}})/\mathbb{Z}$ is compact (the adèlic solenoid).

5. $\mathrm{GL}_n(\mathbb{Q})$ **is discrete in** $\mathrm{GL}_n(\mathbb{A})$**.**

(a) Using Problem 4 coordinatewise, show that $M_n(\mathbb{Q})$ is discrete in $M_n(\mathbb{A}) = M_n(\mathbb{R}) \times M_n(\mathbb{A}_f)$.

(b) Recall that the topology on $\mathrm{GL}_n(\mathbb{A})$ is the one making $g \mapsto (g, g^{-1})$ a closed embedding into $M_n(\mathbb{A}) \times M_n(\mathbb{A})$; in particular the inclusion $\mathrm{GL}_n(\mathbb{A}) \hookrightarrow M_n(\mathbb{A})$ is continuous. Deduce that $\mathrm{GL}_n(\mathbb{Q}) = \mathrm{GL}_n(\mathbb{A}) \cap M_n(\mathbb{Q})$ is discrete in $\mathrm{GL}_n(\mathbb{A})$.

(c) Show that $\mathrm{GL}_n(\mathbb{Z}) = \mathrm{GL}_n(\mathbb{Q}) \cap \mathrm{GL}_n(\widehat{\mathbb{Z}})$.

6. **Rank n lattices and GL_n and strong approximation.**

We generalize the previous set of lattices as follows. Let $\mathcal{L}_{\mathbb{Q}}$ be the set of $\mathbb{Z}$-lattices in $\mathbb{Q}^n$ (free rank-n $\mathbb{Z}$-submodules spanning $\mathbb{Q}^n$), and let $\mathcal{L}_{\widehat{\mathbb{Z}}}$ be the set of $\widehat{\mathbb{Z}}$-lattices in $\mathbb{A}_f^n$ (compact open $\widehat{\mathbb{Z}}$-submodules spanning $\mathbb{A}_f^n$).

(a) Prove that the map

$$\begin{aligned} \mathcal{L}_{\mathbb{Q}} &\to \mathcal{L}_{\hat{\mathbb{Z}}} \\ L &\mapsto \prod_p (L \otimes_{\mathbb{Z}} \mathbb{Z}_p) \end{aligned}$$

is a bijection with inverse $M \mapsto \mathbb{Q}^n \cap M$.

(b) Show $g \mapsto g\widehat{\mathbb{Z}}^n$ identifies $\mathrm{GL}_n(\mathbb{A}_f)/K(1) \xrightarrow{\sim} \mathcal{L}_{\widehat{\mathbb{Z}}}$.

(c) Show $\mathrm{GL}_n(\mathbb{Q})$ acts transitively on $\mathcal{L}_{\mathbb{Q}}$ (any two bases differ by an element of $\mathrm{GL}_n(\mathbb{Q})$). Combine with (a),(b) to deduce the strong approximation theorem:

$$\mathrm{GL}_n(\mathbb{A}_f) = \mathrm{GL}_n(\mathbb{Q})\, K(1).$$

(d) Deduce that there is a homeomorphism:

$$\mathrm{GL}_n(\mathbb{Z}) \backslash \mathrm{GL}_n(\mathbb{R}) \cong \mathrm{GL}_n(\mathbb{Q}) \backslash \mathrm{GL}_n(\mathbb{A})/K(1) = \mathcal{L}_{\mathbb{R}},$$

where $\mathcal{L}_{\mathbb{R}}$ is the space of rank n lattices in $\mathbb{R}^n$. This generalizes Problem 2(b).

7. **Lattices with level structure and $\mathrm{GL}_n(\mathbb{Q}) \backslash \mathrm{GL}_n(\mathbb{A})/K(N)$.**

(a) Using Problem 6 and $\mathrm{GL}_n(\mathbb{Q}) \cap K(1) = \mathrm{GL}_n(\mathbb{Z})$, show that strong approximation reduces the double coset space to

$$\mathrm{GL}_n(\mathbb{Q}) \backslash \mathrm{GL}_n(\mathbb{A})/K(N) \;\cong\; \mathrm{GL}_n(\mathbb{Z}) \backslash \big(\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{Z}/N) \big),$$

where $\mathrm{GL}_n(\mathbb{Z})$ acts diagonally, on the right factor through reduction mod N.

(b) Compute the orbits of $\mathrm{GL}_n(\mathbb{Z})$ on $\mathrm{GL}_n(\mathbb{Z}/N)$ under left multiplication. Show that the image of $\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/N)$ is exactly $\{\bar{g} : \det \bar{g} \equiv \pm 1\}$ (recall $\det \mathrm{GL}_n(\mathbb{Z}) = \{\pm 1\}$), so the orbit set is $(\mathbb{Z}/N)^\times/\{\pm 1\}$ via the determinant.

(c) Show the stabilizer of each orbit is conjugate to $\Gamma(N)$, and conclude

$$\mathrm{GL}_n(\mathbb{Q}) \backslash \mathrm{GL}_n(\mathbb{A})/K(N) \;\cong\; \bigsqcup_{(\mathbb{Z}/N)^\times/\{\pm 1\}} \Gamma(N) \backslash \mathrm{GL}_n(\mathbb{R}).$$

(d) Interpret the left-hand side as the set of lattices $L \subset \mathbb{R}^n$ equipped with a level-N structure $L/NL \xrightarrow{\sim} (\mathbb{Z}/N)^n$, and explain the role of the determinant invariant in indexing the components.

2 Lecture 2: Automorphic forms for GL_n

1. **Generators of the spherical Hecke algebra.**

   Fix a prime p and let $\mathcal{H}_p = \mathcal{H}(\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{GL}_n(\mathbb{Z}_p))$ be the spherical Hecke algebra of $\mathrm{GL}_n(\mathbb{Z}_p)$-bi-invariant, compactly supported functions on $\mathrm{GL}_n(\mathbb{Q}_p)$, with convolution (Haar measure giving $\mathrm{GL}_n(\mathbb{Z}_p)$ volume 1). In this question, we will show that $\mathcal{H}_p$ is generated by the lattice Hecke operators T_A introduced in the lectures.

   For a finite abelian p-group $A \cong \bigoplus_{i=1}^{n} \mathbb{Z}/p^{a_i}$ with at most n cyclic factors ($a_1 \geq \cdots \geq a_n \geq 0$), let

   $$T_A := \mathbf{1}_{X_A} \in \mathcal{H}_p, \text{ where } X_A := \{g \in M_n(\mathbb{Z}_p) \cap \mathrm{GL}_n(\mathbb{Q}_p) \mid \mathbb{Z}_p^n / g\mathbb{Z}_p^n \cong A\}.$$

   (a) (*Cartan decomposition*) Show that there is a bijection:

   $$\begin{aligned} \{\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \in \mathbb{Z}^n\} &\xrightarrow{\cong} \mathrm{GL}_n(\mathbb{Z}_p) \backslash \mathrm{GL}_n(\mathbb{Q}_p) / \mathrm{GL}_n(\mathbb{Z}_p) \\ \lambda &\mapsto X_\lambda := \mathrm{GL}_n(\mathbb{Z}_p)\, p^\lambda\, \mathrm{GL}_n(\mathbb{Z}_p) \end{aligned}$$

   Moreover, check that for A as above, $X_A = X_\lambda$.

   Conclude the $\{T_A\}$ are exactly the indicators of the *integral* double cosets ($\lambda_n \geq 0$), and form a basis of the subalgebra $\mathcal{H}_p^{\geq 0}$ of functions supported on $M_n(\mathbb{Z}_p)$.

   (b) Let $z = T_{(\mathbb{Z}/p)^n} = \mathbf{1}_{X_{(1,\ldots,1)}}$, i.e. the indicator function for the double coset of pI. Show z is central, and that convolution by z corresponds to the shift $\lambda \mapsto \lambda + (1, \ldots, 1)$; in particular z is invertible in $\mathcal{H}_p$ with $z^{-1} = \mathbf{1}_{X_{(-1,\ldots,-1)}}$.

   (c) Deduce that $\mathcal{H}_p = \mathcal{H}_p^{\geq 0}[z^{-1}]$, i.e. the elements $\{T_A : |A| = p^m,\ m \geq 0\}$ generate $\mathcal{H}_p$ once the central element z is inverted. Explain why, in the action on automorphic forms (where z acts invertibly by rescaling lattices), it is accurate to say the action is generated by the T_A.

   (d) Show directly, by computing in coordinates, that $\mathcal{H}_p$ for GL_2 is generated by $T_p := T_{\mathbb{Z}/p}$ and $z^{\pm 1} = T^{\pm 1}_{(\mathbb{Z}/p)^2}$, and is the commutative ring $\mathbb{C}[T_p, z^{\pm 1}]$.

2. **Restricted tensor products and Flath's theorem.**

   Let $(\pi_p, V_p)_p$ be a collection of irreducible admissible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$, and suppose $\dim V_p^{\mathrm{GL}_n(\mathbb{Z}_p)} = 1$ for all $p \notin S_0$ for a fixed finite set S_0. For $p \notin S_0$, choose a generator $e_p \in V_p^{\mathrm{GL}_n(\mathbb{Z}_p)}$.

   (a) For finite $S \supseteq S_0$, put $V_S = \bigotimes_{p \in S} V_p$, and for $S' \supseteq S \supseteq S_0$, define transition maps

   $$\begin{aligned} V_S &\to V_{S'} \\ v &\mapsto v \otimes \bigotimes_{p \in S' \setminus S} e_p. \end{aligned}$$

   Show these form a direct system and that $V = \varinjlim_S V_S$ carries a smooth action of $\mathrm{GL}_n(\mathbb{A}_f)$.

   (b) Show V is admissible and irreducible.

   (c) Show that, up to isomorphism, V is independent of the choices of generators e_p. (Hint: rescaling $e_p \mapsto c_p e_p$ for p in a finite set extends to an isomorphism; for the infinitely many places, use that almost all choices agree.)

(d) (*Flath's theorem.*) Conversely, let π be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A}_f)$. Show $\pi^{\mathrm{GL}_n(\mathbb{Z}_p)} \neq 0$ for almost all p, recover the local components π_p, and prove that $\pi \cong \bigotimes'_p \pi_p$. Where is irreducibility and admissibility of π used?

3. **Satake isomorphism and Euler factors for GL_2.**

(a) Recall the Satake isomorphism $\mathcal{H}_p \xrightarrow{\sim} \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}]^{S_2}$ for GL_2. Show it sends $T_p \mapsto X_1+X_2$ and $z = T_{(\mathbb{Z}/p)^2} \mapsto X_1X_2$. (You may use the explicit double-coset computation of Problem 1(d).)

(b) For an unramified π_p, let $c(\pi_p) = \mathrm{diag}(\alpha, \beta)$ be its Satake parameter ($\{\alpha, \beta\}$ the values of X_1, X_2). Show that the spherical vector is an eigenvector for T_p and z, with eigenvalues equal to $\alpha + \beta$ and $\alpha\beta$, respectively.

4. **Modular forms as automorphic forms.**

Let $f \in S_k$. Using strong approximation $\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{Q})\,\mathrm{GL}_2(\mathbb{R})^+K(1)$, we define an associated function $\varphi_f\colon \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ by

$$\varphi_f(\gamma\, g_\infty\, k) \;=\; (\det g_\infty)^{k/2}\, j(g_\infty, i)^{-k}\, f(g_\infty \cdot i), \qquad \gamma \in \mathrm{GL}_2(\mathbb{Q}),\; g_\infty \in \mathrm{GL}_2(\mathbb{R})^+,\; k \in K(1).$$

(a) Show φ_f is well defined (check compatibility on the overlap $\mathrm{GL}_2(\mathbb{Q}) \cap (\mathrm{GL}_2(\mathbb{R})^+ \times K(1))$, using level 1 and that f is $\mathrm{SL}_2(\mathbb{Z})$-modular).

(b) Verify the automorphic form axioms: left $\mathrm{GL}_2(\mathbb{Q})$-invariance, right $K(1)$-invariance, right K_∞-equivariance of weight k at ∞, $\mathfrak{z}$-finiteness, and moderate growth.

(c) Show f is a cusp form if and only if φ_f satisfies the cuspidality condition

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \varphi_f\big(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) g\big)\, dx = 0.$$

(d) Suppose moreover f is a Hecke eigenform with $T_p f = a_p f$. Show that φ_f generates an irreducible cuspidal automorphic representation $\pi = \pi_\infty \otimes \bigotimes'_p \pi_p$ with $\pi_\infty \cong D_k$ (the weight-k discrete series of $\mathrm{GL}_2(\mathbb{R})$) and each π_p unramified.

(e) Let $c(\pi_p) = \mathrm{diag}(\alpha, \beta)$ be the Satake parameters of π_p. Use Problems 1, 3, and Problem 3 from Lecture 1 to show that:

$$\det(1 - p^{-s}c(\pi_p))^{-1} = \big((1-\alpha p^{-s})(1-\beta p^{-s})\big)^{-1} = \big(1 - a_p p^{-s} + p^{k-1}p^{-2s}\big)^{-1}.$$

In particular, the definition of the L-function of π and the standard representation ϱ of GL_2 agrees with the definition of an L-function of the modular form f. Deduce $\alpha+\beta = a_p$ and $\alpha\beta = p^{k-1}$.

(f) Show that the Ramanujan bound $|a_p| \le 2p^{(k-1)/2}$ is equivalent to $|\alpha| = |\beta| = p^{(k-1)/2}$, i.e. to temperedness of π_p after the unitary normalization $\alpha \mapsto \alpha p^{-(k-1)/2}$. This bound was proved by Deligne in 1974.

3 Lecture 3: the Langlands group $L_{\mathbb{Q}}$

Throughout, $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ carries its profinite topology. For a finite set S of primes, $\mathbb{Q}_S$ is the maximal algebraic extension of $\mathbb{Q}$ unramified outside S; thus $\mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$ is the quotient of $G_{\mathbb{Q}}$ by the closed normal subgroup topologically generated by the inertia groups $I_{\mathbb{Q}_p}$, $p \notin S$.

Before the Langlands group was introduced, Artin considered continuous finite-dimensional representations of this absolute Galois group, $r\colon G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{C})$, which we call *Artin representations*. However, these representations always factor through a finite quotient.

1. **Artin representations factor through a finite quotient.**
   (a) Show $\mathrm{GL}_n(\mathbb{C})$ has *no small subgroups*: there is an open neighborhood U of I containing no subgroup other than $\{I\}$.
   (b) Let $r\colon G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{C})$ be continuous. Show that $r^{-1}(U)$ contains an open subgroup $H \leq G_{\mathbb{Q}}$.
   (c) Deduce that $\ker r \supseteq H$ is open of finite index, and hence r has finite image. In particular, r and factors as $G_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(L/\mathbb{Q}) \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ for some finite Galois $L/\mathbb{Q}$.

There is a stark contrast between this and the ℓ-adic setting. The profinite group $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ *does* have small subgroups (e.g. the open subgroups $1+\ell^m M_n(\mathbb{Z}_\ell)$), so the argument above breaks down. In fact, Eichler–Shimura and Deligne attached ℓ-adic representations to weight $k \geq 2$ eigenforms, and these representations have infinite image! In this sense the complex representations of $G_{\mathbb{Q}}$ capture only the weight 1 modular forms: Deligne–Serre constructed odd 2-dimensional Artin representations associated with weight 1 modular forms. In fact, Khare–Wintenberger later proved that all odd 2-dimensional Artin representations arise this way. To study higher weight modular forms, one must either consider ℓ-coefficients, or posit the existence of the group $L_{\mathbb{Q}}$ as in the lectures.

2. **Artin L-functions.**

   Let $r\colon G_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{GL}(V)$ be an Artin representation. For each p let $I_p \subseteq D_p \subseteq \mathrm{Gal}(L/\mathbb{Q})$ be inertia inside a decomposition group, and set

$$
\begin{aligned}
L_p(s,r) &= \det\bigl(1 - p^{-s} r(\mathrm{Frob}_p) \bigm| V^{I_p}\bigr)^{-1},\\
L(s,r) &= \prod_p L_p(s,r).
\end{aligned}
$$

   (a) Show $L_p(s,r)$ is well defined: Frob_p acts on V^{I_p} and its conjugacy class there is independent of the choices of Frob_p, and of the embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.
   (b) Show that $L_p(s,r)$ does not depend on the choice of the field $L/\mathbb{Q}$ through which r factors.
   (c) If r is a character, show that $L(r,s)$ agrees with a Dirichlet L-function.
   (d) Show that all eigenvalues of $r(\mathrm{Frob}_p)$ have absolute value 1 (use Problem 1), and deduce that $L(s,r)$ converges absolutely and is nonzero for $\mathrm{Re}(s) > 1$.
   (e) Prove the formal properties:
      i. additivity $L(s, r_1 \oplus r_2) = L(s,r_1)L(s,r_2)$;
      ii. $L(s,\mathbf{1}) = \zeta(s)$;

In fact, Artin used the Brauer induction theorem to show that $L(s,r)$ admits a meromorphic continuation to the complex plane by expressing it as a quotient of products of Hecke L-functions. One of the biggest open problems in algebraic number theory is to prove that this function is holomorphic.

Conjecture 3.1 (Artin)**.** *For an irreducible non-trivial representation r of $G_{\mathbb{Q}}$, $L(s,r)$ is holomorphic.*

For odd two-dimensional representations, this follows from the theorem of Khare–Wintenberger mentioned above, because L-functions of modular forms are holomorphic.

Problem 1 and the discussion thereafter shows that considering Artin representations is not enough to even capture modular forms for weight $k > 1$, let alone all automorphic forms. To deal with this issue, the lectures introduced the (conjectural) *Langlands group* $L_{\mathbb{Q}}$, a *compact* topological group satisfying the following axioms:

(A1) for each place v, there is a distinguished $L_{\mathbb{Q}}$-conjugacy class of continuous morphisms

$$\mathrm{WD}_{\mathbb{Q}_v} \to L_{\mathbb{Q}},$$

with $\mathrm{WD}_{\mathbb{Q}_p} = W_{\mathbb{Q}_p} \times \mathrm{SU}(2)$ and $\mathrm{WD}_{\mathbb{Q}_\infty} = W_{\mathbb{R}}$; for a finite set S one sets

$$L_{\mathbb{Q},S} := L_{\mathbb{Q}} / \overline{\langle \operatorname{im}(I_{\mathbb{Q}_p} \times \mathrm{SU}(2)) : p \notin S \rangle},$$

which carries Frobenius classes Frob_p for $p \notin S$;

(A2) there is a continuous surjection

$$L_{\mathbb{Q}} \twoheadrightarrow G_{\mathbb{Q}}$$

with *connected* kernel, compatible with (A1);

(A3) (*Cebotarev*) for each finite S, the union of the conjugacy classes of the Frob_p ($p \notin S$) is dense in $L_{\mathbb{Q},S}$;

(A4) (*universal property*) for each $n \geq 1$, there is a bijection $\pi \mapsto r_\pi$ between $\Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$ and isomorphism classes of continuous irreducible representations $r\colon L_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{C})$, characterized by the property that r_π factors through $L_{\mathbb{Q},S}$ whenever π is unramified outside S, and $r_\pi(\mathrm{Frob}_p)$ is conjugate to $c(\pi_p)$ for $p \notin S$.

3. **The structure of the Langlands group $L_{\mathbb{Q}}$.**
   (a) (*Finite-level quotients.*) Assuming (A2), show that the composite $L_{\mathbb{Q}} \twoheadrightarrow G_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$ factors through $L_{\mathbb{Q},S}$, and that the resulting map $L_{\mathbb{Q},S} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$ is surjective with *connected* kernel. Deduce that $\pi_0(L_{\mathbb{Q},S}) = \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$, and that for $p \notin S$ the class $\mathrm{Frob}_p \in L_{\mathbb{Q},S}$ lifts the Frobenius of $\mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$.
   (b) (*Uniqueness in* (A4).)
      i. Show every continuous finite-dimensional representation of the compact group $L_{\mathbb{Q}}$ is unitarizable, hence semisimple, and is determined up to isomorphism by its character. Show moreover that it factors through some $L_{\mathbb{Q},S}$.
      ii. Let r, r' be two representations of $L_{\mathbb{Q}}$. Choose S so that r, r' both factor through $L_{\mathbb{Q},S}$, assume that

$$\operatorname{tr} r(\mathrm{Frob}_p) = \operatorname{tr} r'(\mathrm{Frob}_p) \qquad \text{for all } p \notin S.$$

      Prove that $r = r'$, and deduce that r_π in (A4) is unique when it exists.
   (c) (*Recovering $L_{\mathbb{Q}}$ from finite level.*) Show the natural map $L_{\mathbb{Q}} \to \varprojlim_S L_{\mathbb{Q},S}$ is an isomorphism of topological groups. (Hint: use the Peter–Weyl theorem.)

4. **Shadows of the $L_{\mathbb{Q}}$-formalism.**

   Each statement below gives an unconditional statement on the automorphic side, which is either conjectured or even proven in some cases.

   (a) (*Ramanujan conjecture*) Show that for $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$ and p unramified, every eigenvalue of $c(\pi_p) = r_\pi(\mathrm{Frob}_p)$ has absolute value 1. Recall from Lecture 2, Problem 4 that this is equivalent to the classical bound $|a_p| \leq 2p^{(k-1)/2}$ for automorphic representations associated with modular forms of weight k, proved by Deligne.

   (b) (*Strong multiplicity one; proved by Jacquet–Shalika.*)

      (i) Show that if $\pi, \pi' \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$ have $c(\pi_p) = c(\pi'_p)$ for almost all p, then $\pi \cong \pi'$.

      (ii) More generally, if cuspidal $\pi_1, \ldots, \pi_s$ and $\pi'_1, \ldots, \pi'_t$ (of possibly varying ranks) satisfy $\bigsqcup_i \{\text{eigenvalues of } c(\pi_{i,p})\} = \bigsqcup_j \{\text{eigenvalues of } c(\pi'_{j,p})\}$ (as multisets) for almost all p, then the multisets $\{\{\pi_i\}\}$ and $\{\{\pi'_j\}\}$ coincide.

   (c) (*Rankin–Selberg poles are given by Schur's lemma; proved by Jacquet–Piatetski-Shapiro–Shalika.*) For $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$ and $\pi' \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_m)$, put $L(s, \pi \times \pi'^{\vee}) := L(s, r_\pi \otimes r_{\pi'}^{\vee})$. Show that:

$$\mathrm{ord}_{s=1}\, L(s, \pi \times \pi'^{\vee}) \;=\; -\dim_{\mathbb{C}} \mathrm{Hom}_{L_{\mathbb{Q}}}(r_{\pi'}, r_\pi) \;=\; \begin{cases} -1, & \pi \cong \pi', \\ \;\;0, & \text{otherwise,} \end{cases}$$

   i.e. $L(s, \pi \times \pi'^{\vee})$ has a (simple) pole at $s = 1$ iff $\pi \cong \pi'$.

   (d) (*The Artin Conjecture.*) Using the surjection $L_{\mathbb{Q}} \twoheadrightarrow G_{\mathbb{Q}}$ of (A2), show that (A4) implies the Artin Conjecture 3.1. Hint: recall that Godement–Jacquet proved that $L(s, \pi)$ is entire.

   (e) (*Sato–Tate Conjecture.*) Use Weil's irreducibility criterion and the non-vanishing of $L(s, \pi)$ for $\mathrm{Re}(s) = 1$ to deduce the Sato–Tate Conjecture for $L_{\mathbb{Q}}$: the classes of $\mathrm{Frob}_p \in L_{\mathbb{Q},S}$ equidistribute with respect to the Haar measure. For non-CM modular forms, relate this to the classical Sato–Tate result.

We end this problem set with two explicit examples where the Artin Conjecture 3.1 is known; in fact, Problem 5 is a special case of Problem 6, which is a special case of the result of Khare–Wintenberger.

5. **Artin representations: the case of S_3.**

   Let $K/\mathbb{Q}$ be a Galois extension with $\mathrm{Gal}(K/\mathbb{Q}) \cong S_3$ and let $r\colon G_{\mathbb{Q}} \twoheadrightarrow S_3 \to \mathrm{GL}_2(\mathbb{C})$ be the unique irreducible 2-dimensional representation.

   (a) Show $r \cong \mathrm{Ind}_{A_3}^{S_3} \chi$ for either nontrivial character χ of $A_3 \cong C_3$ (and the two choices give isomorphic inductions).

   (b) Using inductivity (Problem 2(c)), show $L(s, r) = L(s, \chi)$, where χ is viewed via class field theory as a finite-order Hecke character of the quadratic field K^{A_3}. Hecke proved that $L(s, \chi)$ has analytic continuation to all of $\mathbb{C}$, a functional equation, and is *entire* — hence the Artin conjecture holds for r.

6. **A case of Langlands reciprocity: theta series of ray class characters.**

   In this problem, we will realize a simple instance of Langlands reciprocity. Let $F = \mathbb{Q}(\sqrt{D})$ be a quadratic field with discriminant D and ring of integers $\mathcal{O}_F$, and let χ be a finite-order Hecke character of F of conductor $\mathfrak{f}$ (a ray class character). By class field theory, χ is identified with a character of $\mathrm{Gal}(H/F)$ for some abelian extension H over F. Therefore, we can consider the Artin representation:

$$r := \mathrm{Ind}_F^{\mathbb{Q}} \chi \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C}).$$

The goal of the problem is to check that the theta series

$$\theta_\chi(z) \;=\; \sum_{\substack{\mathfrak{a}\subseteq\mathcal{O}_F\\ \text{coprime to } \mathfrak{f}}} \chi(\mathfrak{a})\, q^{N_{F/\mathbb{Q}}(\mathfrak{a})}, \qquad q = e^{2\pi i z},$$

is the modular form associated with r via the Langlands reciprocity conjecture.

(a) Show θ_χ is a weight 1 modular form on $\Gamma_0(|D|\mathrm{N}_{F/\mathbb{Q}}\mathfrak{f})$ with nebentypus $\varepsilon = \chi_{-D}\cdot\chi|_{\mathbb{Q}}$, where χ_D is the quadratic character of $F/\mathbb{Q}$ and $\chi|_{\mathbb{Q}}$ is the restriction of χ to a ray class character for $\mathbb{Q}$.

(b) Show θ_χ is a cusp form if and only if $\chi \neq \chi^\sigma$ (i.e. χ does not factor through the norm), where σ is complex conjugation. Check that this is equivalent to r being irreducible.

(c) Show θ_χ is a Hecke eigenform and with T_p-eigenvalues for p not dividing the level given by the formula:

$$a_p = \begin{cases} \chi(\mathfrak{p}) + \chi(\overline{\mathfrak{p}}) & \text{if } p = \mathfrak{p}\overline{\mathfrak{p}} \text{ splits} \\ 0 & \text{otherwise.} \end{cases}$$

(d) Deduce that $L(s,\theta_\chi) = L(s,\chi) = L(s,\mathrm{Ind}_F^{\mathbb{Q}}\chi)$, confirming that θ_χ corresponds to r via Langlands reciprocity.