# Problems on global aspects of the Langlands Program

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July 2026

This is a problem set for Gaëtan Chenevier's course on *global aspects of the Langlands program* at the Sheffield Langlands Program School in July 2026. There are, of course, more problems here than we expect any single person to solve, so please pick a few that sound interesting to you and attempt to solve those.

We have also posted a solution guide to this problem set, which contains more detailed instructions, leads the reader through the solutions, and also contain supplementary problems.

## 1 Lecture 1: modular forms, lattices, and the adèles

1. Reinterpret modular forms as functions on the set of lattices in $\mathbb{C}$.

2. Reinterpret modular forms as functions on $\mathrm{SL}_2(\mathbb{R})$, invariant under $\mathrm{SL}_2(\mathbb{Z})$.

3. Reinterpret Hecke operators on modular forms in terms of the lattice operators $T_A$ from lectures.

4. Check that $\mathbb{Q}$ is discrete and cocompact inside $\mathbb{A}$.

5. Show that $\mathrm{GL}_n(\mathbb{Q})$ is discrete in $\mathrm{GL}_n(\mathbb{A})$, where the topology on $\mathrm{GL}_n(\mathbb{A})$ is the finest topology making the embedding $\mathrm{GL}_n(\mathbb{A}) \to M_n(\mathbb{A}) \times M_n(\mathbb{A})$ given by $g \mapsto (g, g^{-1})$ closed.

6. Prove that lattices in $\mathbb{Q}^n$ are in bijection with $\hat{\mathbb{Z}}$-lattices in $\mathbb{A}_f^n$. Deduce that $\mathrm{GL}_n(\mathbb{A}_f) = \mathrm{GL}_n(\mathbb{Q})K(1)$.

7. Reinterpret the quotient $\mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A})/K(N)$ in terms of lattices.

## 2 Lecture 2: Automorphic forms for $\mathrm{GL}_n$

1. Show that the spherical Hecke algebra for $\mathrm{GL}_n$ is generated by the lattice Hecke operators $T_A$.

2. Check that the restricted tensor product is well-defined.

3. Reinterpret the Satake isomorphism for $\mathrm{GL}_2$ in terms of classical Hecke operators $T_p$ and $\langle p \rangle$.

4. Reinterpret modular forms as automorphic forms for $\mathrm{GL}_2$ and check that the local $L$-factor defined in terms of Satake parameters agrees with the classical definition of $L$-factors of modular forms.

## 3 Lecture 3: the Langlands group $L_{\mathbb{Q}}$

1. Show that Artin representations factor through a finite quotient of $G_{\mathbb{Q}}$.
2. Look up the definition of an Artin $L$-function and prove its basic properties: it is well-defined, recovers Dirichlet $L$-functions for characters of $G_{\mathbb{Q}}$, converges for $\mathrm{Re}(s) > 1$, is additive $L(s, r_1 \oplus r_2) = L(s, r_1)L(s, r_2)$, and $L(s, 1) = \zeta_K(s)$.
3. Deduce the following results about $L_{\mathbb{Q}}$ from its axioms:
   (a) Prove that $L_{\mathbb{Q},S} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$ is surjective with connected kernel.
   (b) Show that $r_\pi$ is uniquely determined by the automorphic representation $\pi$.
   (c) Show that the natural map $L_{\mathbb{Q}} \to \varprojlim_S L_{\mathbb{Q},S}$ is an isomorphism of topological groups.
4. Deduce the following results from the axioms of $L_{\mathbb{Q}}$:
   (a) The Ramanujan conjecture.
   (b) Strong multiplicity one.
   (c) For $\pi, \pi'$ cuspidal automorphic representations of $\mathrm{GL}_n$ and $\mathrm{GL}_m$,
   $$\mathrm{ord}_{s=1} L(s, \pi \times \pi'^{\vee}) = \begin{cases} -1 & \pi \cong \pi' \\ 0 & \text{otherwise.} \end{cases}$$
   (d) The Artin Conjecture.
   (e) The Sato–Tate Conjecture.