

ALGEBRAIC GROUPS
EXERCISE SHEET 2

Due Thursday, 29 October 2026

Throughout, k is an algebraically closed field and by an algebraic group we mean a group scheme of finite type over k .

1. (10 points) **Kernels and images.** Consider the homomorphism

$$\varphi : \mathbb{G}_m \times \mathrm{SL}_2 \longrightarrow \mathrm{GL}_2, \quad (t, A) \longmapsto tA.$$

- (a) (4 points) Prove that φ is faithfully flat, compute its kernel, and write down the resulting exact sequence of algebraic groups.
(b) (4 points) Compute the differential

$$d\varphi_{(1, I_2)} : T_1 \mathbb{G}_m \oplus T_{I_2} \mathrm{SL}_2 \rightarrow T_{I_2} \mathrm{GL}_2.$$

Determine exactly when φ is smooth, and compare this with smoothness of its kernel.

- (c) (2 points) Let $R = k(x)$. Does $\mathrm{diag}(x, 1)$ belong to the image of $\varphi(R)$? How is your answer compatible with faithful flatness?

2. (10 points) **Orbits and stabilizers.**

- (a) (3 points) Let $\mathbb{G}_m$ act on $\mathbb{A}^2$ by $t \cdot (x, y) = (tx, t^{-1}y)$. Compute the orbits. For each orbit, compute its dimension, closure, and the stabilizer subgroup.
(b) (7 points) Let a smooth connected algebraic group G act on a variety X . For $x \in X(k)$, let $H = \mathrm{Stab}_G(x)$ be its stabilizer, $O = G \cdot x$ be its orbit, with its reduced scheme structure, and $\mu_x : G \rightarrow O$ be its orbit map. Prove the following statements:
i. For every $g \in G(k)$, the scheme-theoretic fibre of μ_x over $g \cdot x$ is gH . Deduce that

$$\dim O = \dim G - \dim H.$$

- ii. The orbit O is smooth.
iii. The sequence

$$0 \longrightarrow T_e H \longrightarrow T_e G \xrightarrow{d(\mu_x)_e} T_x O$$

is exact. Deduce that the following are equivalent:

- $d(\mu_x)_e$ is surjective,
- H is smooth,
- μ_x is smooth.

3. (10 points) **Transporters, normalizers, and centralizers.**

- (a) (5 points) Let $G = \mathrm{GL}_2$, and let $T, U \subseteq G$ be the subgroup schemes given, for every commutative k -algebra R , by

$$T(R) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in R^\times \right\}, \quad U(R) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in R \right\}.$$

Consider the natural action of G on $\mathbb{A}^2$.

- i. Compute the transporters $T_G(V(x), V(x))$ and $T_G(V(x), V(y))$ as closed subschemes of G .
 - ii. Compute $C_G(U)$ and $N_G(U)$. Relate $N_G(U)$ to the stabilizer of the subscheme $V(x)$.
 - iii. Compute $C_G(T)$ and $N_G(T)$. Describe the connected components of $N_G(T)$ and their actions on T by conjugation.
- (b) (5 points) Let G be a smooth connected algebraic group, and let $N \trianglelefteq G$ be a finite normal subgroup scheme.
- i. If N is reduced, prove that N is central in G .
 - ii. Check that the reducedness hypothesis in part i cannot be omitted as follows. Assume $\text{char } k = p > 0$, and let G and N be given by

$$G(R) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in R^\times, b \in R \right\}, \quad N(R) = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : u^p = 0 \right\}.$$

Verify that G is smooth and connected and that N is finite and normal in G . Compute $C_G(N)$ to see that it is not equal to G .