GROUP ACTIONS AND CHROMATIC HOMOTOPY THEORY

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Abstract. We give a survey on the theory of group actions on finite complexes, with a focus on its homological aspects. We describe the passage from classical Smith theory on the mod p homology of fixed point spaces for periodic maps to recent refinements in terms of generalized homology theories, in particular the Morava K-theories from chromatic homotopy theory. We also discuss the relationship between these results and the structure of the moduli stack of equivariant formal groups via complex bordism theory.

1. Introduction

The study of the symmetry of finite complexes has a long history. In this survey we consider the following broad question:

> Assume given an action of a finite group G on a finite complex X. What can one say about the topology of the subspace of points X^G that are fixed by the G-action?

More generally, what can one say about the collection of fixed points X^H as H varies through all subgroups of G? The question whether it is possible that X^H is empty for all non-trivial subgroups H concerns the existence of free G-actions on X. A famous example of this kind is the problem of which finite groups can act freely on some sphere, which by a combination of work of Smith [Smi44], Milnor [Mil57], Swan [Swa60] and Madsen–Thomas–Wall [MTW76] was shown to be the case if and only if all abelian subgroups are cyclic and every involution is central.

More precisely, we focus on the homological aspects of the above question: Assuming we know the homology of X, what can we say about the homology of X^G? Here, 'homology' can be interpreted in several different ways. In the classical case of singular homology with $\mathbb{F}_p$ coefficients, P. A. Smith [Smi38, Smi39] showed, among other things, that if $G = C_p$ and X has the $\mathbb{F}_p$-homology of a sphere, the fixed points X^{C_p} also have the $\mathbb{F}_p$-homology of a sphere (of possibly different but not larger dimension). Smith's results were the beginning of the now well-developed field of transformation groups, with many applications in topology and other areas of mathematics.

The rational homology of X^G is less directly accessible. In particular it is not the case that the fixed points of a G-action on a rational homology sphere are again a rational homology sphere. However, recent results show that one can obtain information about the rational homology of X^G by considering the value of so-called *generalized*, or *extraordinary*,

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homology theories on X. For example, Kuhn–Lloyd [KL24] proved that if the mod p topological K-theory of X is that of a sphere, then the fixed points X^{C_p} of any C_p-action on X indeed form a rational homology sphere. More generally, if X has the nth Morava K-theory $K(n)_*(X)$ of a sphere, then the fixed points $X^{C_p^n}$ of any action by an elementary abelian p-group of rank n are a rational homology sphere. The theories $K(n)_*$ arise from the chromatic point of view on stable homotopy theory via the connection between complex bordism and the algebra of formal group laws [Qui69], which has shaped much of our understanding of the subject over the last 50 years.

The results of [KL24] are part of more general recent progress on the interplay of group actions and chromatic homotopy theory, which links classical Smith theory with the tensor-triangular classification problem of compact G-spectra [BS17, BHN$^+$19, BGH20, KL24] and the algebra of G-equivariant formal groups [CGK00, Str11, HW18, Hau22, HM25, BC25]. These connections provide new algebraic tools for studying G-spaces and equivariant maps between them, beyond the motivating question at the beginning of the introduction. The aim of this survey article is to give a brief introduction to this subject for non-experts.

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2. Classical Smith theory and $\mathbb{F}_p$-homology

We begin with a review of group actions and the classical theorems of P. A. Smith. For simplicity we focus on the case where G is a finite group.

2.1. **Recollections on group actions.** Let X be a G-space, i.e., a topological space equipped with a continuous G-action. Given a subgroup H of G, we denote the subspace of H-fixed points by

$$X^H = \{x \in X \mid hx = x \text{ for all } h \in H\}.$$

As explained in the introduction, one of the main problems discussed in this survey concerns the relationship between the various fixed point spaces X^H. In particular, given a space X equipped with a G-action, what can we say about the subspace of G-fixed points X^G?

Most theorems require some regularity conditions on the G-action on X. A convenient class of well-behaved G-spaces are the *G-CW complexes*. These are G-spaces X equipped with a CW structure such that G acts through cellular maps and satisfies the 'all-or-nothing' property: If a group element maps an n-cell to itself, it fixes it pointwise. The class of G-CW complexes includes the more rigid G-simplicial complexes as well as smooth manifolds with smooth G-actions [Ill78]. A G-CW complex X is called *finite* if it only has finitely many cells, which is equivalent to X being compact.

We will study the relationship between X and X^G mainly through homological invariants. Given an abelian group A, space Y and $n \in \mathbb{N}$ we write $H_n(Y, A)$ for the nth singular homology group of Y with coefficients in A. When $A = R$ is a ring, the groups $H_n(Y, R)$

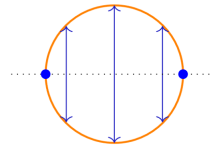

FIGURE 1. The sign sphere S^σ, with reflection action across the x-axis. It carries a C_2-CW structure with two C_2-fixed 0-cells, and two 1-cells which are permuted.

form an R-module. Most important for us will be the cases $R = \mathbb{F}_p$ and $R = \mathbb{Q}$. Later in Section 3 we will consider more general forms of homology.

Remark 2.1. While almost no homological information about X^G can be gleaned from that of X for general infinite dimensional G-CW complexes, many of the theorems we discuss have analogs for G-CW complexes which are merely finite-dimensional but not necessarily finite. For simplicity we focus on the case of finite G-CW complexes.

Remark 2.2. By Elmendorf's theorem [Elm83], every G-space can be reconstructed – up to equivariant (weak) homotopy-equivalence – from the diagram of its fixed point spaces. Hence in a homotopy-theoretic sense the study of G-spaces is equivalent to the study of fixed point spaces.

2.2. **Euler characteristic.** Perhaps the most elementary homological relationship between X and its fixed point subspace X^G arises from the Euler characteristic. Recall that the Euler characteristic $\chi(Y)$ of a finite CW complex Y is defined as the alternating sum

$$\chi(Y) = \dim_k(H_0(Y,k)) - \dim_k(H_1(Y,k)) + \dim_k(H_2(Y,k)) - \dots,$$

for any choice of field k, on which it does not depend. Equivalently, the Euler characteristic can be computed in terms of the CW-structure as the alternating sum of the number of n-dimensional cells:

$$\chi(Y) = \#\{\text{0-cells}\} - \#\{\text{1-cells}\} + \#\{\text{2-cells}\} - \dots.$$

Now let G be a p-group, i.e., the order of G is a power of p, and X a finite G-CW complex. Then an n-cell of X is either fixed, in which case it is also a cell of the fixed point space X^G, or its orbit of n-cells is isomorphic to G/H for some proper subgroup H of G. As G is a p-group, the cardinality of G/H is divisible by p. Hence the number of n-cells which are not fixed is divisible by p, and it follows that

$$\chi(X^G) \equiv \chi(X) \mod p.$$

Translating back to the homological formula for the Euler characteristic this describes a relation between the homology k-vector spaces of X and X^G.

Example 2.3. For the sign sphere from Figure 1, the Euler characteristic of the underlying space S^1 is $2 - 2 = 0$, while the Euler characteristic of the fixed point space $(S^\sigma)^{C_2} = S^0$ is 2.

2.3. **Classical Smith theory.** In a series of papers from the late 1930's, P. A. Smith proved further homological conditions for actions of p-groups.

Theorem 2.4 ([Smi38, Smi39]). *Let G be a p-group and X a finite G-CW complex. If $\tilde{H}_*(X, \mathbb{F}_p) = 0$, then also $\tilde{H}_*(X^G, \mathbb{F}_p) = 0$.*

Here, we wrote $\tilde{H}_*(-, \mathbb{F}_p)$ for the reduced homology groups as a graded $\mathbb{F}_p$-vector space. Implicit in the statement is the claim that X^G is non-empty, which follows from X and X^{C_p} having the same Euler characteristic modulo p. In words, Smith showed that if X is contractible through the lens of $\mathbb{F}_p$-homology, then the same is true for its subspace of G-fixed points. This was later refined to the following inequality by E. E. Floyd:

Theorem 2.5 ([Flo52]). *Let G be a p-group and X a finite dimensional G-CW complex. Then for all $n \in \mathbb{N}$ we have*

$$\sum_{k=0}^{\infty} \dim H_{n+k}(X^G, \mathbb{F}_p) \leq \sum_{k=0}^{\infty} \dim H_{n+k}(X, \mathbb{F}_p).$$

In particular, the case $n = 0$ yields an inequality

$$\dim_{\mathbb{F}_p} H_*(X^G, \mathbb{F}_p) \leq \dim_{\mathbb{F}_p} H_*(X, \mathbb{F}_p),$$

i.e., the total dimension of the $\mathbb{F}_p$-homology of the fixed points X is always bounded above by the total dimension of the $\mathbb{F}_p$-homology of X. For a fixed choice of k, however, it is possible that $H_k(X^G, \mathbb{F}_p)$ is larger than $H_k(X, \mathbb{F}_p)$. In these theorems it is crucial that the two primes match up: There is no analogous theorem for the dimensions of $\mathbb{F}_q$-homology of X^G for a p-group G when $p \neq q$. In Section 3 we discuss what happens for homology with coefficients in $\mathbb{Q}$, which will be the starting point for the interaction of Smith theory and chromatic homotopy theory.

Example 2.6 (Group actions on homology spheres). We can consider a C_p-CW complex X whose underlying space has the $\mathbb{F}_p$-homology of an n-dimensional sphere, i.e., $H_0(X, \mathbb{F}_p) \cong H_n(X, \mathbb{F}_p) \cong \mathbb{F}_p$ and $H_k(X, \mathbb{F}_p) = 0$ for all $k \neq 0, n$. Hence the total dimension of the $\mathbb{F}_p$-homology is 2 and its Euler characteristic is either 0 (if n is odd) or 2 (if n is even). Floyd's theorem implies that the total dimension of $H_*(X^{C_p}, \mathbb{F}_p)$ must be either 0 (in which case $X^{C_p} = \emptyset$ and the action is free), 1 or 2. By the Euler-characteristic condition, it cannot be 1. The other two cases depend on the prime:

(1) If $p = 2$, then the Euler characteristic condition gives no further restriction, and X^{C_2} can be either empty or a $\mathbb{F}_2$-homology sphere of dimension $0 \leq m \leq n$. Each case is realized by the unit sphere inside the C_2-representation $\sigma^{n-m} \oplus \mathbb{R}^{m+1}$, where σ denotes the sign representation and $\mathbb{R}$ denotes the trivial representation. Setting $m = -1$ covers the free case.

(2) If $p \neq 2$, the Euler characteristic condition shows that X^{C_p} can only be empty if n is odd. Otherwise, X^{C_p} has the $\mathbb{F}_p$-homology of an m-dimensional sphere with $0 \leq m \leq n$ of the same parity as n. Again, all cases are realized by unit spheres inside real C_p-representations.

Remark 2.7. Recall that every finite p-group G admits a subnormal tower of subgroups

$$e = H_0 \leq H_1 \leq H_2 \leq \cdots \leq H_n = G$$

with each subquotient H_i/H_{i-1} isomorphic to C_p. Since G-fixed points can be taken iteratively along this chain, the Euler characteristic condition as well as Theorem 2.4 and Theorem 2.5 can be derived from the special case $G = C_p$.

Remark 2.8. Theorem 2.4 implies as a special case that if a space Y can be written as the C_p-fixed points of a contractible C_p-CW complex X, the $\mathbb{F}_p$-homology of Y must be that of a point. In [Jon71] Lowell Jones showed that this condition is in fact sufficient.

2.4. **Smith theory via homotopy orbits.** In Borel's seminar on transformation groups [Bor60], one of the classics of the subject, Smith's ideas were developed further and put in a broader context. In particular, Borel gave new proofs of the theorems of Smith and Floyd by relating the cohomology of the fixed points X^G to that of the homotopy-orbits (also called Borel-construction) X_{hG} via the so-called *localization theorem.* The cohomology of X_{hG} in turn is directly related to the cohomology of X and the group cohomology of G via the Serre spectral sequence, providing a bridge between the cohomologies of X and X^G. For more information we recommend the books [tD87, AP93].

Remark 2.9. Later in [DW88] Bill Dwyer and Clarence Wilkerson showed that in fact the full $\mathbb{F}_p$-cohomology of X^{C_p} can be recovered from the cohomology of X_{hC_p} as a module over the Steenrod algebra (and more generally for C_p^n in place of C_p).

We do not develop the details of this point of view in this survey, but emphasize that similar ideas are regularly used also in the proofs of the chromatic versions of Smith theory discussed in the next sections.

Remark 2.10. A much-studied problem in transformation groups is the (non-)existence of free group actions on products of spheres, generalizing the motivating question at the beginning of the introduction. Conjecturally (e.g., [Ade04, Conjecture 2.1]), if an elementary abelian p-group C_p^r acts freely on some product of the form $X = S^{n_1} \times \cdots \times S^{n_k}$, then $r \leq k$. More precisely, for odd p, r is conjectured to be no larger than the number of odd dimensional factors. This conjecture has been verified for small k [Hel58, Car87], in the equidimensional case $n_1 = \cdots = n_k$ [AB88] (away from some exceptional dimensions when $p = 2$) and for primes $p > 3\dim(X)$ [Han09], but is open in general.

The Halperin–Carlsson conjecture (see, e.g., [AB88, Question 7.3]) predicts more generally that if C_p^r acts freely on a finite complex X, then the total dimension of the $\mathbb{F}_p$-homology of X must be at least 2^r, i.e., that of a product of r spheres. This conjecture, too, is wide open. An algebraic generalization was disproved by Iyengar–Walker [IW18].

We also want to mention the important work around the resolution of the Sullivan conjecture (in particular [Sul74, Mil84, Car91, Lan92]) relating the fixed points X^G and homotopy-fixed points X^{hG} of finite G-CW complexes X. Finally, mod p Smith theoretic ideas have led to advances also in areas other than topology, one recent example being Smith–Treumann theory for sheaves [Tre19], with applications to the Langlands program [TV16] and the representation theory of reductive algebraic groups [RW22].

3. Smith theory for rational homology and Morava K-theories

Classical Smith theory gives strong constraints on the $\mathbb{F}_p$-homology of the fixed point space X^G when G is a p-group. What can we say about the homology of X^G with $\mathbb{Q}$-coefficients? The first observation is that the direct analog of Smith's and Floyd's theorems do not hold, as the following example shows:

Example 3.1. Let C_2 act on projective space $\mathbb{R}\mathrm{P}^2$ via

$$\tau([x_0 \colon x_1 \colon x_2]) = [(-x_0) \colon x_1 \colon x_2].$$

The fixed point space $(\mathbb{R}\mathrm{P}^2)^{C_2}$ of this action is the disjoint union of one point (the line $\mathbb{R} \times 0 \times 0$) and $\mathbb{R}\mathrm{P}^1$ (the projective space of $0 \times \mathbb{R} \times \mathbb{R}$), see also Figure 2 below. The total rational homology of $\mathbb{R}\mathrm{P}^2$ is 1-dimensional, concentrated in degree 0, while the total rational homology of $(\mathbb{R}\mathrm{P}^2)^{C_2} = \{*\} \sqcup \mathbb{R}\mathrm{P}^1$ is 3-dimensional, so

$$\dim_{\mathbb{Q}} H_*((\mathbb{R}\mathrm{P}^2)^{C_2}, \mathbb{Q}) = 3 > 1 = \dim_{\mathbb{Q}} H_*(\mathbb{R}\mathrm{P}^2, \mathbb{Q}),$$

hence the rational homology of the fixed point space is not bounded by that of the underlying space. Here and below, $H_*(-)$ denotes the direct sum over all the homology groups $H_k(-)$.

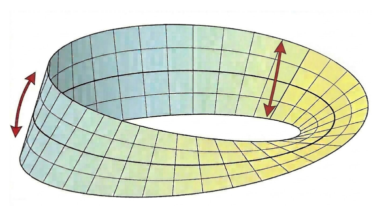

Figure 2. The reflection C_2-action on the Möbius strip. Collapsing the boundary to a point yields the action on the projective plane from Example 3.1, with fixed points the equator and the collapsed boundary.

Note that for any finite CW complex Y we have

$$\dim_{\mathbb{Q}} H_*(Y, \mathbb{Q}) \leq \dim_{\mathbb{F}_p} H_*(Y, \mathbb{F}_p)$$

by an elementary application of the universal coefficient theorem. When X is a finite C_p-CW complex this can be combined with Floyd's theorem for $\mathbb{F}_p$-homology to obtain the inequality

$$\dim_{\mathbb{Q}} H_*(X^{C_p}, \mathbb{Q}) \leq \dim_{\mathbb{F}_p} H_*(X^{C_p}, \mathbb{F}_p) \leq \dim_{\mathbb{F}_p} H_*(X, \mathbb{F}_p) \quad (1)$$

in the previous example. In particular this shows that 3 is in fact the maximal possible dimension of the rational homology of the fixed point space of a C_2-action on $\mathbb{R}\mathrm{P}^2$. For more complicated spaces the bound for $H_*(X^{C_p}, \mathbb{Q})$ via $H_*(X, \mathbb{F}_p)$ from (1) is not optimal, however, as we now explain.

3.1. **Morava K-theories.** It turns out that in order to tell the full story of the interplay of rational homology and group actions one needs to consider *generalized homology theories* h_*. Like ordinary homology, these assign a $\mathbb{Z}$-graded sequence of abelian groups to every space, which are homotopy-invariant, satisfy excision and participate in long exact sequences for well-behaved inclusions of subspaces. The main difference is that the homology groups $h_*(*)$ of the one-point space, called the *coefficients* of the theory, are no longer required to be concentrated in degree 0. If h_* is a multiplicative homology theory, the coefficients $h_*(*)$ inherit a graded ring structure and $h_*(X)$ forms a graded $h_*(*)$-module for every space X. It is customary to denote the coefficients simply by h_* instead of $h_*(*)$, and we will do so too.

The generalized homology theories that are relevant to us are the *Morava K-theories* $K(n)_*$, which depend on an implicit prime p and a natural number n, called the *height*. We will say more about their origin and role in stable homotopy theory in Section 4.1. For now we need to know that each of these is a field homology theory in the sense that the coefficient ring $K(n)_*$ is a graded field (i.e., every graded $K(n)_*$-module is a sum of free modules). The value $K(n)_*(X)$ on every finite CW complex X is then a finitely-generated free $K(n)_*$-module and it makes sense to speak of its dimension

$$\beta^{K(n)}(X) = \dim_{K(n)_*} K(n)_*(X).$$

The numbers $\beta^{K(n)}(X)$ are called the *Morava K-theory Betti numbers of the space X*.

Remark 3.2. While the definition of $K(n)_*$ relies on some choices, the dimension $\beta^{K(n)}(X)$ is independent of these, for all finite CW complexes X. Moreover, given any multiplicative generalized homology theory h_* whose coefficients form a graded field, the resulting dimension function $\dim_{h_*} h_*(-)$ agrees with that of one of the $K(n)_*$ (see Section 4.1.3 for more on this point).

Rational homology $H_*(-,\mathbb{Q})$ is an example of a theory of type $K(0)_*$, independent of p. One model for $K(1)_*$ is given by topological K-theory KU, defined in terms of complex vector bundles, with coefficients in $\mathbb{F}_p$. The higher theories $K(n)_*$ can be constructed as 'bordism theories of manifolds with singularities' [Baa73, Mor85, JW75]. Together they interpolate between $H_*(-,\mathbb{Q})$ and $H_*(-,\mathbb{F}_p)$. This can be made precise in various ways, one of which is a theorem of Ravenel:

Theorem 3.3 ([Rav84, Theorem 2.11])**.** *For every finite CW complex X, the sequence of Morava K-theory Betti numbers is weakly increasing, i.e.,*

$$\dim_{\mathbb{Q}} H_*(X,\mathbb{Q}) = \beta^{\mathbb{Q}}(X) \leq \beta^{K(1)}(X) \leq \beta^{K(2)}(X) \leq \ldots \leq \beta^{\mathbb{F}_p}(X) = \dim_{\mathbb{F}_p} H_*(X,\mathbb{F}_p).$$

Moreover, for every fixed X the Betti numbers $\beta^{K(n)}(X)$ agree with $\beta^{\mathbb{F}_p}(X)$ for sufficiently large n.

Example 3.4. For the projective space $\mathbb{R}\mathrm{P}^2$ from Example 3.1 we have

$$\dim_{K(0)_*} K(0)_*(\mathbb{R}\mathrm{P}^2) = \dim_{\mathbb{Q}} H_*(\mathbb{R}\mathrm{P}^2,\mathbb{Q}) = 1$$

$\beta^{\mathbb{Q}}$	$\beta^{K(1)}$	$\beta^{K(2)}$	$\beta^{K(3)}$	$\beta^{K(4)}$	$\beta^{K(5)}$	...	$\beta^{\mathbb{F}_2}$
1	3	7	15	31	33	...	33

FIGURE 3. The Morava K-theory Betti numbers for $\mathbb{R}\mathrm{P}^{32}$.

and

$$\dim_{K(n)_*} K(n)_*(\mathbb{R}\mathrm{P}^2) = 3$$

for all $n > 0$, i.e., the weakly monotone sequence of Betti numbers is given by $(1, 3, 3, 3, \dots)$. Higher dimensional real projective spaces provide examples where the sequence stabilizes late, see [KL24, Sec. 7]. For example, the Betti numbers for real projective space $\mathbb{R}\mathrm{P}^{32}$ are listed in Figure 3 below.

We now return to Smith theory and consider a finite C_p-CW complex X. While $\tilde{H}_*(X, \mathbb{Q}) = 0$ does not imply that $\tilde{H}_*(X^{C_p}, \mathbb{Q}) = 0$, it turns out that the stronger condition $\tilde{K}(1)_*(X) = 0$ *does*. More generally, the following is true for all $n \geq 0$:

Theorem 3.5 (Balmer–Sanders [BS17])**.** *For every $n \geq 0$ and all finite C_p-CW complexes X we have: If $\tilde{K}(n+1)_*(X) = 0$, then $\tilde{K}(n)_*(X^{C_p}) = 0$.*

See also Figure 4. Here, again, we use a tilde to refer to the reduced version of the respective homology theories, and implicit in the claim is that X^{C_p} is non-empty, which follows by a variation of the Euler characteristic argument from before. The condition that $\tilde{K}(n)_*(X) = 0$ is equivalent to $K(n)_*(X)$ being 1-dimensional over $K(n)_*$, similarly to the case of ordinary homology. The proof makes use of blueshift phenomena in Tate cohomology established in work of Greenlees–Sadofsky [HS96] and Kuhn [Kuh04]. Since $\tilde{H}_*(-, \mathbb{F}_p)$ vanishes on a finite CW complex if and only if all $\tilde{K}(n)_*$ vanish, this implies and refines Smith's classical theorem.

More generally for a p-group G of order p^r, an induction over a subnormal series with cyclic quotients of order p as in Remark 2.7 implies that for every finite G-CW complex X, if $\tilde{K}(n+r)_*(X) = 0$, then $\tilde{K}(n)_*(X^G) = 0$. However, it is not clear whether this yields the optimal 'slope', and in fact it turns out not to: In [BHN+19], Barthel–Hausmann–Naumann–Nikolaus–Noel–Stapleton show that when $G = C_{p^r}$ is a cyclic group one still has the implication

$$K(n+1)_*(X) \Rightarrow K(n)_*(X^{C_{p^r}}) = 0,$$

i.e., the slope is always 1, see Figure 5.

Similarly to the case of C_p, the slope is determined by the blueshift of generalized Tate constructions, see [BS17, BHN+19]. On the other hand, the slope for the elementary abelian p-group $C_p^r = C_p \times \cdots \times C_p$ of rank r is precisely r, i.e., for every $n \geq 0$ there exists a finite C_p^r-CW complex X such that $\tilde{K}(n+r-1)_*(X) = 0$ but $\tilde{K}(n)_*(X^{C_p^r}) \neq 0$. The first examples of such X were listed in [BHN+19]. They are closely related to Mitchell's first examples of non-equivariant type n complexes [Mit85], and make use of equivariant properties of partition complexes of finite sets as studied in work of Arone, Dwyer, Lesh, Mahowald and others (see, e.g., [Aro98, AM99, ADL16]). Kuhn–Lloyd [KL24] constructed

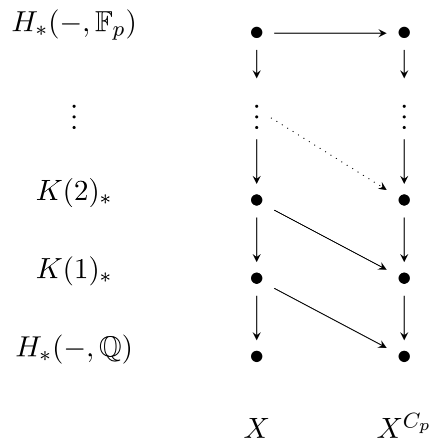

FIGURE 4. An illustration of the homological dependence between X and its fixed points X^{C_p}, with the vertical downward arrows corresponding to Ravenel's Theorem 3.3 and the slope 1 horizontal arrows to Theorem 3.5. The horizontal arrow on the top corresponds to Smith's classical Theorem 2.4 for $\mathbb{F}_p$-homology.

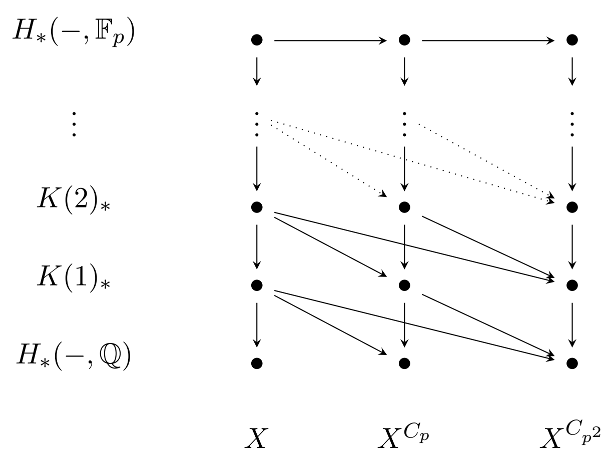

FIGURE 5. The analog of Figure 4 for the group C_{p^2}.

a further class of examples by splitting off summands from smash powers of projective spaces in linear representations (as in Example 3.7) via idempotents in the p-local group ring of symmetric groups. This technique was used previously by Mike Hopkins and Jeff Smith in the non-equivariant context [HS98] in their proof of the Periodicity Theorem.

These results demonstrate that the interplay of fixed points and Morava K-theories is sensitive to the structure of the p-group, which was invisible through the eyes of mod p homology where the slope is always 0. The two cases C_{p^r} and C_p^r can be combined to deduce that the slope for an abelian p-group G always equals the rank $\mathrm{rk}(G)$ of G [BHN+19], i.e., its minimal number of generators. Kuhn and Lloyd used their above-mentioned technique via idempotents in $\mathbb{Z}_{(p)}[\Sigma_m]$ to strengthen this result to a version of Floyd's theorem for Morava K-theories:

Theorem 3.6 ([KL24])**.** *Let G be an abelian p-group. Then for every $n \geq 0$ and all finite G-CW complexes X we have*

$$\beta^{K(n)}(X^G) \leq \beta^{K(n+\mathrm{rk}(G))}(X).$$

In particular this means –for abelian G – that the dimension of the rational homology of X^G is bounded above by the Betti number $\beta^{K(\mathrm{rk}(G))}(X)$ for Morava K-theory of height $\mathrm{rk}(G)$.

Example 3.7. Let $G = C_2$. We consider the C_2-action on $\mathbb{R}\mathrm{P}^{32}$ induced by viewing it as projective space in the C_2-representation $\sigma^{17} \oplus \mathbb{R}^{16}$. Then a point in $\mathbb{R}\mathrm{P}^{32}$, i.e., a line in $\sigma^{17} \oplus \mathbb{R}^{16}$, is fixed if and only if it is contained in either σ^{17} or $\mathbb{R}^{16}$. This shows that the fixed points decompose as a disjoint union $\mathbb{R}\mathrm{P}^{16} \sqcup \mathbb{R}\mathrm{P}^{15}$. The table of Betti numbers below illustrates the shift by 1. We note that Floyd's classical theorem together with the

	$\beta^{\mathbb{Q}}$	$\beta^{K(1)}$	$\beta^{K(2)}$	$\beta^{K(3)}$	$\beta^{K(4)}$	$\beta^{K(5)}$	...	$\beta^{\mathbb{F}_2}$
$\mathbb{R}\mathrm{P}^{32}$	1	3	7	15	31	33	...	33
$(\mathbb{R}\mathrm{P}^{32})^{C_2}$	3	7	15	31	33	33	...	33

inequality $\beta^{\mathbb{Q}} \leq \beta^{\mathbb{F}_2}$ only shows that the rational Betti number for the C_2-fixed points of any action on $\mathbb{R}\mathrm{P}^{32}$ is at most 33, a much weaker bound than the one implied by $\beta^{K(1)}(\mathbb{R}\mathrm{P}^{32}) = 3$ via Theorem 3.6. The C_2-action from this example shows that 3 is the optimal bound, and that more generally $(\mathbb{R}\mathrm{P}^{32})^{C_2}$ has the maximal possible Morava K-theory Betti numbers at all heights n.

As mentioned in the introduction, Theorem 3.6 also implies results about group actions on $K(n)_*$-homology spheres similar to the classical ones recalled in Example 2.6, see [KL24, Thm. 2.23].

Remark 3.8. As explained above, the results of [BHN+19] on the optimal slope for chromatic Smith theorems concern abelian finite groups. These results were slightly extended to a larger class of finite groups in [KL24], in particular including extraspecial 2-groups. In work-in-progress of the author together with Robert Burklund, Ishan Levy and Lennart Meier an equivariant form of the Periodicity Theorem is used to address optimal slopes for general finite groups. We return to this at the end of Section 4.2.

Remark 3.9. Most results of this section have analogs for actions by compact Lie groups, see [BGH20].

4. Chromatic homotopy theory and the stack of (equivariant) formal groups

The connection between the Morava K-theory Betti numbers of the underlying space X and those of the subspace of G-fixed points X^G is an incarnation of a deeper relationship between the topological theory of group actions on finite complexes and the algebra of the moduli stack of G-equivariant formal groups, summarized under the umbrella term *chromatic equivariant homotopy theory.* The goal of this final section is to give an overview of the nature and central features of this correspondence, to explain recent advances in the field, and to list some further applications and future directions.

4.1. **Chromatic homotopy theory.** We begin by recalling the non-equivariant story, which originated in a theorem of Quillen [Qui69] from 1969. We only give a short summary and refer the interested reader to the survey [BB20] or the books [Rav86, Rav92] for more details. The chromatic point of view has proved highly effective over the last 50+ years and underlies much of our current understanding of stable phenomena in homotopy theory.

4.1.1. *Complex bordism theory.* In short, chromatic homotopy theory studies spaces through the lens of a particularly powerful homology theory MU_* called *complex bordism*, introduced by René Thom in his seminal work [Tho54]. The theory is of geometric origin and can be described via bordism classes of maps from compact, smooth manifolds equipped with a complex structure on their stable tangent bundle. In particular, the coefficient ring MU_* is isomorphic to the complex bordism ring of stably almost complex manifolds. Thom showed that bordism theories can be interpreted in terms of homotopy classes of maps into (what are now called) Thom spaces of vector bundles, which allowed many explicit computations. In the complex case, Milnor [Mil60] proved an isomorphism

$$\mathrm{MU}_* \cong \mathbb{Z}[x_1, x_2, \dots],$$

with one polynomial generator in every positive even dimension.

Quillen showed that in addition to the geometric description, MU_* also has an algebraic interpretation. To explain this, we need to use that every generalized homology theory h_* has an associated generalized *co*homology theory h^*. The cohomology theory MU^* associated to complex bordism has a special property: It comes with a preferred *complex orientation.* This means that it allows a theory of Chern classes for complex vector bundles similar to those for ordinary cohomology $H^*(-,\mathbb{Z})$ or complex K-theory KU^*.

There is, however, one key difference: Assume given two complex line bundles ψ and ξ over a space X. We can form their tensor product $\psi \otimes \xi$, which is again a line bundle. Then each of the theories comes with a formula to express the first Chern class $c_1(\psi \otimes \xi)$ in terms of $c_1(\psi)$ and $c_1(\xi)$, but these formulas are different:

- For ordinary cohomology $H^*(-,\mathbb{Z})$ we have:

$$c_1^{H\mathbb{Z}}(\psi \otimes \xi) = c_1^{H\mathbb{Z}}(\psi) + c_1^{H\mathbb{Z}}(\xi),$$

- For complex K-theory KU^* we have:

$$c_1^{\mathrm{KU}}(\psi \otimes \xi) = c_1^{\mathrm{KU}}(\psi) + c_1^{\mathrm{KU}}(\xi) + b(c_1^{\mathrm{KU}}(\psi) \cdot c_1^{\mathrm{KU}}(\xi)),$$

where $b \in \mathrm{KU}_2$ is the Bott class.

Quillen's theorem [Qui69] says that for MU^*, the analogous formula is a power series in $c_1(\psi)$ and $c_1(\xi)$ which is maximally complicated with the constraints that it is (1) symmetric in ψ and ξ, (2) associative and (3) unital. Algebraically, Properties (1)–(3) say that the formula is encoded by a *formal group law* over the coefficient ring MU_*, and 'maximally complicated' means that this formal group law is *universal*: Every formal group law over a commutative ring k is the pushforward of the one for MU_* under a unique ring map $\mathrm{MU}_* \to k$. In other words, the map

$$L \to \mathrm{MU}_*$$

from the Lazard ring L, which is defined via this universal property, classifying the preferred complex orientation of complex bordism is an isomorphism.

4.1.2. $\mathrm{MU}_*(X)$ *as an algebraic approximation to a space* X. One consequence of Quillen's theorem, and an extension which computes the cooperations on complex bordism in terms of strict isomorphisms of formal group laws, is that for every space X, the MU_*-homology $\mathrm{MU}_*(X)$ forms a quasi-coherent sheaf over the moduli stack of formal groups $\mathcal{M}_{FG}$, see e.g. [Goe08]. This provides a bridge between topology and the algebra of formal groups which is central to chromatic homotopy theory.

The sheaf $\mathrm{MU}_*(X)$ turns out to be a strong algebraic approximation to X. More precisely, $\mathrm{MU}_*(X)$ approximates the suspension spectrum $\Sigma^\infty X$. This means that it detects properties and structure of X that are stable under the suspension operator Σ.

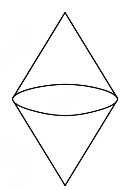

FIGURE 6. The suspension ΣX of a space X.

The study of such properties, informally described as inverting Σ, is called *stable homotopy theory*, which we do not attempt to summarize here (see, e.g., [Rav92]). We denote by $[\Sigma^\infty X, \Sigma^\infty Y]_*$ the graded abelian group of homotopy classes of stable maps between two spaces X and Y. When X is a finite CW complex every such homotopy class is represented by an actual continuous map $f\colon \Sigma^{n+k} X \to \Sigma^n Y$ for some $n \in \mathbb{N}, k \in \mathbb{Z}$. Understanding $[\Sigma^\infty X, \Sigma^\infty Y]_*$, or more generally understanding $[W, Z]_*$ for spectra W and Z, is one of the central problems in homotopy theory, with applications to geometric topology (e.g., via Thom spectra), algebraic geometry (e.g., via algebraic K-theory or topological Hochschild homology) and other areas of mathematics.

One example in what sense MU_* approximates spaces is that for every pair of spaces X, Y, with X a finite CW complex, there is the *Adams–Novikov spectral sequence* [Nov67] which takes the form:

$$E_2^{*,*} \cong \mathrm{Ext}^{*,*}_{\mathcal{M}_{FG}}(\mathrm{MU}_*(X), \mathrm{MU}_*(Y)) \Rightarrow [\Sigma^\infty X, \Sigma^\infty Y]_*.$$

The spectral sequence exists for more general classes of spectra in place of $\Sigma^\infty X$ and $\Sigma^\infty Y$, but for simplicity we restrict to the case of suspension spectra. The Ext groups are taken in quasi-coherent sheaves over $\mathcal{M}_{FG}$. Informally this means that there is a filtration on $[\Sigma^\infty X, \Sigma^\infty Y]_*$ whose subquotients are also subquotients of the groups

$$\mathrm{Ext}^{*,*}_{\mathcal{M}_{FG}}(\mathrm{MU}_*(X), \mathrm{MU}_*(Y)),$$

in a way that is controlled by a series of differentials via taking kernels and cokernels. As

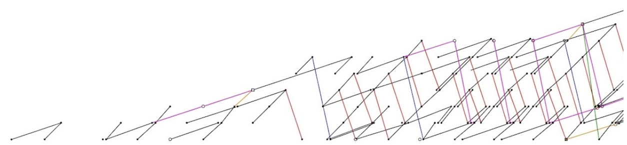

FIGURE 7. The first 45 degrees of the E_2-page of the Adams–Novikov spectral sequence for $X = Y = S^0$ at the prime 2, away from the α-family and its α_1-multiples [Isa14]. The coloured arrows pointing to the left indicate differentials.

is indicated in Figure 7 there are many differentials, so the passage from the algebraic Ext groups to the topological homotopy classes of stable maps still requires significant additional input which typically is only understood in a range of degrees. On the other hand, all the information encoded in the differentials, and in filtration degrees > 0 more generally, is in some sense nilpotent. This was envisioned by Ravenel [Rav84] and proved by Devinatz–Hopkins–Smith [DHS88]. The nilpotence property can be phrased in various different ways, for example the following: Assume that $X = Y$ is a finite CW complex, and hence $[\Sigma^\infty X, \Sigma^\infty X]_*$ denotes the stable self-maps of X which like ordinary maps can be composed. Then the nilpotence theorem says that such a stable self-map

$$f\colon \Sigma^{\infty+k} X \to \Sigma^\infty X$$

is nilpotent (i.e., $f^n = 0$ for some n) if and only if its induced map of sheaves

$$\mathrm{MU}_*(f)\colon\ \mathrm{MU}_*(X)[k] \to \mathrm{MU}_*(X)$$

is nilpotent. The periodicity theorem by Hopkins–Smith [HS98] strengthens this to the stronger statement that in a suitable sense the stable self-maps of X (or more precisely the *central ones*) agree with those of $\mathrm{MU}_*(X)$ asymptotically, i.e., that some power of every algebraic self-map of the sheaf $\mathrm{MU}_*(X)$ can be lifted to a topological one of $\Sigma^\infty X$.

FIGURE 8. A picture of the topological space underlying the moduli stack of formal groups over p-local rings, and via the thick subcategory theorem of the Balmer spectrum of compact spectra.

4.1.3. *Morava K-theories and the thick subcategory theorem.* We now return to the Morava K-theories from the previous section, which are closely related to the chromatic point of view: The $K(n)_*$ are designer (co)homology theories which implement specific formal group laws topologically, namely the Honda formal group law of height n defined over $\mathbb{F}_p$. Every $K(n)_*$ comes with a multiplicative map of homology theories

$$\mathrm{MU}_* \to K(n)_*.$$

For varying n and p, these can be thought of as the residue fields of complex bordism theory MU_*. From the point of view of stacks, the formal group laws over the varying $K(n)_*$ pick out the points of the topological space $|\mathcal{M}_{FG}|$ underlying the moduli stack of formal groups $\mathcal{M}_{FG}$. Localized at the implicit prime p, this space forms a tower, see Figure 8.

By the *thick subcategory theorem*, again due to Hopkins–Smith [HS98], the space $|\mathcal{M}_{FG}|$ also captures the global structure of finite CW complexes. The cleanest way to phrase this is to say that $|\mathcal{M}_{FG}|$ is homeomorphic to the *Balmer spectrum* of the triangulated category of compact spectra, with comparison map induced by the support of the sheaves $\mathrm{MU}_*(-)$. The Balmer spectrum encodes the global structure of a tensor-triangulated category and carries the universal support theory, playing a similar role to the Zariski spectrum of a commutative ring, see [Bal05, Bal20]. In simpler terms, the thick subcategory theorem says the following: For every generalized p-local homology theory h_* there exists an $n \in \mathbb{N} \cup \{\infty\}$ such that for all finite CW complexes X we have

$$\tilde{h}_*(X) = 0 \Longleftrightarrow \tilde{K}(n)_*(X) = 0.$$

In other words, the kernel of the reduced theory $\tilde{h}_*$ agrees with the kernel of $\tilde{K}(n)_*$ for some n. Here, we take $K(\infty)_*$ to mean ordinary homology with $\mathbb{F}_p$ coefficients. The thick subcategory theorem in particular implies that the Smith theory results from Section 3 give a full picture of the interplay of fixed points with generalized homology theories, as they cover all the kernels of such. Finally we note that the linear structure of the Balmer spectrum of compact spectra leads to an inductive approach for studying spaces or spectra by understanding their behavior at each of the points individually and then working up the tower, in a similar way to how one can study modules over a commutative ring by studying their localizations at prime ideals individually and then gluing the information via arithmetic fracture squares. This technique lies at the heart of chromatic homotopy theory.

4.2. **Chromatic equivariant homotopy theory.** We now return to group actions. Chromatic equivariant homotopy theory aims to develop a similarly strong dictionary between topology, bordism theory and the algebra of formal groups for studying the symmetry of spaces. Early papers in this direction, from the 60's and 70's, include the work of Conner–Floyd [CF64], Stong [Sto70] and tom Dieck and his collaborators [BtD70, tD70, tD71a, tD71b]. For *abelian* G the area has seen much progress over the last 15 years, as we explain below, and is closely related to the Morava K-theory versions of Smith theory from Section 3. For a recent account of the chromatic point of view on equivariant homotopy theory we recommend the paper [BC25].

Remark 4.1. There is another side to equivariant structures in chromatic homotopy theory coming from Real bordism theory $\mathrm{MU}_{\mathbb{R}}$ [Lan68, Ara79, HK01], a C_2-equivariant (co)homology theory which played a central role in the famous resolution of the Kervaire invariant 1 problem by Hill–Hopkins–Ravenel [HHR16] and has seen lots of research activity since then. We do not discuss this direction in this survey, but refer the interested reader to [Hil20] and the references therein.

As in the non-equivariant case, the passage to a stable category of 'equivariant spectra' is required for many of the constructions. There is a number of different choices for what this could mean. We will focus on the most studied one with the best properties called *genuine G-spectra Sp_G*, which is obtained by suspending Σ^V in the direction of all representation spheres S^V, i.e., one-point compactifications of linear G-representations, see, e.g. [Hil20]. This means that a stable map $\Sigma^\infty X \to \Sigma^\infty Y$ (of degree 0) for finite G-CW complexes X, Y is represented by an ordinary continuous map

$$f\colon \Sigma^V X \to \Sigma^V Y$$

which preserves the G-actions. To be precise, the suspension operator Σ^V requires a choice of G-fixed basepoint on X and Y, or alternatively the addition of a disjoint such basepoint.

4.2.1. *Chromatic Smith theory vs. the Balmer spectrum of compact G-spectra.* The pursuit of Smith theorems for Morava K-theories from Section 3 turns out to be equivalent to understanding the global structure of the compact objects in this category of G-spectra, i.e., its Balmer spectrum:

Theorem 4.2 ([BS17])**.** *Let G be a finite group. Then the following hold:*

i) The points in the Balmer spectrum $\mathrm{Spec}(Sp_G^\omega)$ of p-local finite G-spectra are in bijection with pairs $([H], n)$ of a conjugacy class of subgroups $[H]$ and an integer n.

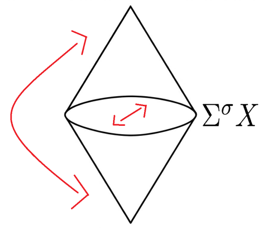

FIGURE 9. A picture of the suspension $\Sigma^\sigma X$ with respect to the sign sphere. The C_2 action on $\Sigma^\sigma X$ is the composition of the given action on X with the reflection action in the suspension direction.

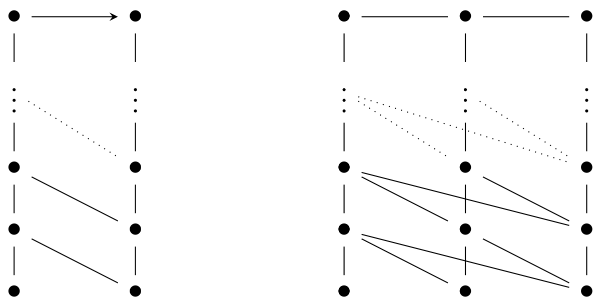

FIGURE 10. The Balmer spectra of compact p-local G-spectra, and by Theorem 4.5 the underlying space of the moduli stack of G-equivariant formal groups, for $G = C_p$ and $G = C_{p^2}$.

ii) A point $([H], n)$ lies in the closure of another point $([K], m)$ if and only if there is a Morava K-theory Smith theorem for the pair H, K and heights n, m, i.e., if every finite G-CW complex X with $\tilde{K}(n)_*(X^H) = 0$ satisfies $\tilde{K}(m)_*(X^K) = 0$.

iii) The topology on $\mathrm{Spec}(Sp_G^\omega)$ is an Alexandrov topology, i.e., every closed set is a union of closures of points.

Hence, the chromatic Smith theorems for G-actions describe the closures of points by *ii)*, which in turn determines the full topology on $\mathrm{Spec}(Sp_G^\omega)$ via *iii)*. In other words, Figures 4 and 5, also recalled below in Figure 10, are really a picture of the Balmer spectrum $\mathrm{Spec}(Sp_G^\omega)$. So the chromatic stratification for G-actions is precisely a stratification by this space.

Remark 4.3. The paper [BGH20] contains a version of Theorem 4.2 for compact Lie groups. In that setting the topology on $\mathrm{Spec}(Sp_G^\omega)$ also depends on the topology on the space of subgroups of G, in addition to the Smith theorems.

4.2.2. *Forms of equivariant complex bordism.* Much of the usefulness of chromatic homotopy theory comes not only from understanding its Balmer spectrum but from the connection to the moduli stack of formal groups via MU_*-homology. What is the analog of this for group actions? First, what is the correct analog of MU_*-homology? Recall from Section 4.1 that non-equivariant MU_* has several features:

(1) Its coefficients MU_* have a geometric interpretation in terms of bordism classes of stably almost complex manifolds,
(2) its associated cohomology theory MU^* is complex orientable, meaning it allows a theory of Chern classes for complex vector bundles, and
(3) its coefficients MU_* have an algebraic interpretations in terms of formal group laws.

It turns out that there is no analog of MU_* in the world of homology theories for G-spaces that satisfies all three properties. Instead, there is a geometric bordism theory denoted Ω_*^G implementing (1) for stably almost complex G-manifolds [CF64], and a different 'homotopical' bordism theory $(\mathrm{MU}_G)_*$ which satisfies (2) for G-equivariant complex vector bundles, introduced in [tD70, BH72]. The two theories are connected by a natural Thom–Pontryagin collapse map

$$\Omega_*^G \to (\mathrm{MU}_G)_*$$

which is only an isomorphism for the trivial group. Understanding the relationship between the geometric and the homotopical theory is an active area of research, see e.g. [Han05, AB24] or [HS24] for unoriented equivariant bordism.

4.2.3. *Connection to equivariant formal groups.* We now turn to the analog of (3). The connection to formal group laws for non-equivariant MU_* arises via the universal formula for the first Chern class of a tensor product of complex line bundles. This suggests that an analog of Quillen's theorem is more likely to hold for the homotopical theory $(\mathrm{MU}_G)_*$. For abelian groups G, the notion of a *G-equivariant formal group law*, capturing the structure of the formula for c_1 for tensor products of G-equivariant complex line bundles, was first formalized by Michael Cole, John Greenlees and Igor Kriz in [CGK00]. Roughly speaking, a G-equivariant formal group law is a commutative formal group scheme X over a commutative ring k equipped with

- a multiplicative map $G^* \to X$ from the dual group G^* of G, and
- a coordinate y cutting out the image of the neutral element of G^*,

satisfying certain properties. See also [Str11, HM25]. Unlike for non-equivariant formal group laws, the underlying formal scheme of X is not generally given by the formal spectrum of a power series ring over k but can carry a more complicated structure.

As shown in [CGK00], there exists a universal G-equivariant formal group law defined over the *G-equivariant Lazard ring L_G*, and a ring map

$$L_G \to (\mathrm{MU}_G)_*$$

classifying the preferred equivariant orientation. Greenlees [Gre01] showed that for finite groups this map is surjective with nilpotent kernel and conjectured that it is an isomorphism. The first case of this conjecture was proved for $G = C_2$ by Bernhard Hanke and Michael Wiemeler in [HW18], building on work of Strickland [Str01]. The general case was shown in [Hau22]:

Theorem 4.4 ([Hau22]). *The map $L_G \to (\mathrm{MU}_G)_*$ is an isomorphism for every abelian compact Lie group G.*

This generalizes Quillen's classical theorem for MU_* to the equivariant context. A main tool in the proof is the global structure of equivariant complex bordism in the sense of Schwede [Sch18]. While the non-equivariant Lazard ring L is isomorphic to a polynomial ring on infinitely many generators (cf. Section 4.1), its equivariant counterpart L_G is much more complicated. For example, L_G is not an integral domain when G is a non-trivial finite abelian group. Presentations in terms of generators and relations are given for $G = C_2$ in [Str01] and larger cyclic groups in [Hu25]. The correct definition of a G-equivariant formal group law – and in particular an analog of Quillen's theorem – for non-abelian groups is still wide open.

Similarly to the non-equivariant case, for abelian G there is a notion of strict isomorphism of G-equivariant formal group laws, giving rise to the *moduli stack of G-equivariant formal groups $\mathcal{M}_{FG}^G$*, see [HM25]. The $(\mathrm{MU}_G)_*$-homology of a G-space (or more generally G-spectrum) X forms a quasi-coherent sheaf over $\mathcal{M}_{FG}^G$. As in the non-equivariant case, the functor $(\mathrm{MU}_G)_*$ controls the global structure of compact G-spectra:

Theorem 4.5 ([HM25]). *For every abelian compact Lie group G, $(\mathrm{MU}_G)_*$-homology induces a homeomorphism*

$$|\mathcal{M}_{FG}^G| \xrightarrow{\cong} \mathrm{Spec}(Sp_G^\omega)$$

from the space underlying the moduli stack of G-equivariant formal groups to the Balmer spectrum of compact G-spectra.

This means that the chromatic Smith theorems allow an algebraic interpretation in terms of equivariant formal groups, see Figure 10. Moreover, Theorem 4.4 can be used to show that as in the non-equivariant case there exists a spectral sequence of the form

$$E_2^{*,*} \cong \mathrm{Ext}^{*,*}_{\mathcal{M}_{FG}^G}((\mathrm{MU}_G)_*(X), (\mathrm{MU}_G)_*(Y)) \Rightarrow [\Sigma^\infty X, \Sigma^\infty Y]_*^G,$$

called the *G-equivariant Adams–Novikov spectral sequence*. Here, X is a finite G-CW complex and Y is an arbitrary G-CW complex. Again, the spectral sequence exists for more general G-spectra.

An important special case is when $X = S^V$ and $Y = S^W$ are representation spheres, in which case

$$[\Sigma^\infty S^V, \Sigma^\infty S^W]_*^G$$

describes stable equivariant maps of the form $S^{V+k} \to S^W$. The study of such maps has a long history, going back at least to the Borsuk–Ulam Theorem [Bor33], see e.g. [Bre67, Lan69, tDP78, Iri89, BHZ24]. The G-equivariant Adams–Novikov spectral sequence allows to attack open questions on the structure of these maps through computations on the moduli stack of G-equivariant formal groups. Remarkably, the height 0 part of the G-equivariant computations is linked to the height $\mathrm{rk}(G)$ part of the non-equivariant stable self-maps of spheres via the blueshift phenomenon, the precise mechanism of which is not fully understood.

More generally, for a finite G-CW complex X, the G-equivariant Adams–Novikov spectral sequence expresses stable equivariant self-maps of the form $\Sigma^{\infty+V}X \to \Sigma^\infty X$ in terms of derived, graded self-maps of the quasicoherent sheaf $(\mathrm{MU}_G)_*(X)$. In forthcoming work of the author with Robert Burklund, Ishan Levy and Lennart Meier we show that as in the non-equivariant case the asymptotic, non-nilpotent part (more precisely the *p-perfection*) of the central stable self-maps of X and $(\mathrm{MU}_G)_*(X)$ agree and can be computed explicitly. This in turn feeds back into determining the Balmer spectrum and chromatic Smith theorems also for non-abelian finite groups. We refer to [BC25, BHZ26] for a general discussion of properties of periodic self-maps of G-spectra as well as explicit examples of such at small heights.

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