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Lectures on Cohomology of Arithmetic Groups I

Zu erscheinen 2023
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Preface

During the years 1980-2000 I gave various advanced courses on number theory, algebraic geometry and on Cohomology of Arithmetic Groups at the university of Bonn. I prepared some very informal notes for my students, I wanted the necessary prerequisites to be available for them at one place.

At some point I had the feeling that these notes could as a basis for a book. on the subject *Cohomology of arithmetic groups.*

The cohomology groups of arithmetic groups are sheaf cohomology groups. Hence we should provide some basic material on sheaves and homological algebra and cohomology of sheaves. On the other hand the theory of sheaves and sheaf cohomology are ubiquitous in algebraic geometry and some other branches of mathematics. I also gave lectures on algebraic geometry and turned my notes-into the two volumes [39], [40]..

The present volume is now part 1 of volume III. If I need a well known theorem, whose proof is given in one of these two volumes, then I take the freedom to refer to these volumes.

The subject has applications to number theory - actually it is part of number theory. The central theme is the relationship between special values of L-functions and the integral structure of the cohomology as module under the Hecke algebra. We can prove rationality results for special values of L-functions (Manin, Shimura and many others). On the other hand these special values tell us something about the denominators of the Eisenstein classes, and in some cases from here we get information about the structure of the Galois group. This relationship has been already discussed in the original notes, for the special case of $\mathrm{Sl}_2(\mathbb{Z})$ we have the culminating theorem 5.1.2. In the probably removed section I discuss - for a specific example - an application concerning to the structure of the Galois group. In theorem 5.1.5 I construct a normal extension $K/\mathbb{Q}$ of degree $690 \cdot 691^3$, which is unramified outside 691 and we have a partial decomposition law. This is really number theory.

The theorem 5.1.2 can be stated in elementary terms, we do not need any analysis (thanks to Euler, who taught us that the the equality numerator of $\zeta(-11) = 691$ is an elementary statement) and can be verified by an algorithm. To prove it we need some analysis. We need some tools from representation theory Lie groups and from the theory of automorphic forms.

I am convinced -and there is a lot of evidence for it - that theorem 5.1.2 is a special case of a much larger class of (conjectural) assertions about L-values and denominator of Eisenstein classes.

In section (3.3.9) we will see that denominators imply congruences between eigenvalues of Hecke operators acting the cohomology of arithmetic different groups. It was extremely important for me that these conjectures on congruences could be verified in some finite number of cases by experimental calculations [19]. The experimental support for these congruences had great influence on the content of this book. But these experimental verifications confirmed only the congruences for a finite number of Hecke operators, but not the denominators.

On the other hand it seems that our analytic method to prove 5.1.2 only works in a few other cases. In Chapter III I describe a toy model of an algorithm , which in a given simple case computes the cohomology and the action of a Hecke operator. Hence we can check our conjecture (theorem 5.1.2) experimentally for some small cases.

But to the best of my knowledge there are only very few other cases, where we have such an algorithm, which works in practice. It is outlined in Chapter III that we can write an algorithm which works in principle. But on the other hand there are abundantly many situations, where we can raise the denominator issue, some will be discussed in volume III part 2 .

Contents

0.1 Introduction

An arithmetic group Γ is a discrete subgroups of a Lie group $G(\mathbb{R}) \subset \mathrm{Gl}_n(\mathbb{R})$ whose matrix entries satisfy certain rationality and integrality condition. The most basic example of such a group is the group $\mathrm{Sl}_n(\mathbb{Z}) \subset \mathrm{Sl}_n(\mathbb{R})$. More generally we can start from an algebraic subgroup $G/\mathbb{Q} \subset \mathrm{Gl}_n/\mathbb{Q}$, for instance the orthogonal group of a quadratic form. Then we get arithmetic groups $\Gamma \subset \mathbb{G}(\mathbb{Q}) \subset G(\mathbb{R})$ if we impose certain integrality conditions on the matrix coefficients of the elements of Γ.

For any Γ- module $\mathcal{M}$ we can define the cohomology groups $H^\bullet(\Gamma, \mathcal{M}) = \bigoplus_q H^q(\Gamma, \mathcal{M})$. These cohomology groups are abelian groups, which are defined in terms of homological algebra, they are the derived functors of the functor $\mathcal{M} \to \mathcal{M}^\Gamma(=$ invariants under Γ. .

We are mainly interested in the cohomology of a very special class of Γ-modules. We consider rational representations $\rho : G/\mathbb{Q} \to \mathcal{M}_\mathbb{Q}$, where $\mathcal{M}_\mathbb{Q}$ is a finite dimensional $\mathbb{Q}$-vector space. Then we can find finitely generated $\mathbb{Z}$ modules $\mathcal{M}$ such that $\mathcal{M}_\mathbb{Q} = \mathcal{M} \otimes_\mathbb{Z} \mathbb{Q}$ which are Γ-invariant and hence Γ-modules.

Let $K_\infty \subset G(\mathbb{R})$ be a maximal compact subgroup, for example $\mathrm{SO}(n) \subset \mathrm{Sl}_n(\mathbb{R})$. The quotient $X = \mathbb{G}(\mathbb{R})/K_\infty$ is a symmetric space, it carries a Riemannian metric which is $G(\mathbb{R})-$ invariant under the left action, it may have finitely many connected components, each connected component is diffeomorphic to $\mathbb{R}^d$, hence contractible.

Our arithmetic group Γ acts properly discontinuously on X, we can form the quotient $\Gamma\backslash X$, this quotient is an orbifold. We can pass to a suitable subgroup of finite index $\Gamma' \subset \Gamma$ such that Γ' has no non trivial elements of finite order (i.e. is torsion free). Then $\Gamma'\backslash X$ is a Riemannian manifold, it is a so called locally symmetric space. The map $\Gamma'\backslash X \to \Gamma\backslash X$ is a finite covering with some ramifications. If Γ has elements of finite order then $\Gamma\backslash X$ is only a Riemannian orbifold. These spaces $\Gamma\backslash X$ provide a very interesting class of spaces, which are of interest for differential geometers, mathematicians interested in analysis on manifolds and topologists. But they are in a sense of arithmetic origin and therefore they are of interest for number theorists.

Our Γ module $\mathcal{M}$ endows the space $\Gamma\backslash X$ with a sheaf $\tilde{\mathcal{M}}$ with values in finitely generated abelian groups. If Γ is torsion free then $\tilde{\mathcal{M}}$ is a locally constant sheaf, or in other words a local system.

We introduce the sheaf- cohomology groups

$$H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) = \bigoplus_q H^q(\Gamma\backslash X, \tilde{\mathcal{M}})$$

these cohomology groups are "essentially" the same as the above group cohomology groups, these two versions of cohomology become equal, if X is connected and Γ is torsion free. We will see that these cohomology groups are finitely generated $\mathbb{Z}-$ modules.

We have some additional structure on these cohomology groups. In general the quotient space $\Gamma\backslash X$ is not compact. We have the Borel-Serre compactification $i : \Gamma\backslash X \hookrightarrow \Gamma\backslash \bar{X}$, where i is a homotopy equivalence and $\Gamma\backslash \bar{X}$ is a manifold (orbifold) with corners. The difference set $\partial(\Gamma\backslash X) := \Gamma\backslash \bar{X} \setminus \Gamma\backslash X$ is the boundary of the Borel-Serre compactification. Moreover we will construct a "tubular" neighbourhood $\overset{\bullet}{\mathcal{N}} (\Gamma\backslash X) \subset \Gamma\backslash X$ of "infinity" (see (1.2.8)). We may also consider the cohomology with compact supports $H_c^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})$. and we get the fundamental long exact sequence

$$\cdots \to H_c^q(\Gamma\backslash X, \tilde{\mathcal{M}}) \xrightarrow{i_c} H^q(\Gamma\backslash X, \tilde{\mathcal{M}}) \xrightarrow{r} H^q(\overset{\bullet}{\mathcal{N}} (\Gamma\backslash X), \tilde{\mathcal{M}}) \xrightarrow{\delta} H_c^{q+1}(\Gamma\backslash X, \tilde{\mathcal{M}}) \to \dots \tag{1}$$

We also introduce the "inner cohomology"

$$H_!^q(\Gamma\backslash X, \tilde{\mathcal{M}}) := \ker(r) = \mathrm{Im}(i_c).$$

A second structural ingredient is the Hecke algebra. We have an action of a big algebra of operators acting on all these cohomology groups and the action commutes with arrows in the fundamental exact sequence.

This is the so called Hecke algebra $\mathcal{H}($ or $\mathcal{H}_\Gamma)$, it originates from the structure of the arithmetic group Γ. The group Γ has many subgroups Γ' of finite index, for which we can construct two arrows

$$\Gamma'\backslash X \underset{p_2}{\overset{p_1}{\rightrightarrows}} \Gamma\backslash X. \tag{2}$$

Such a pair of arrows is also called a correspondence between on $\Gamma\backslash X$. A correspondence, together with a suitable map $u : p_1^*(\tilde{\mathcal{M}}) \to p_2^*(\tilde{\mathcal{M}})$, induces an endomorphism in the cohomology. These endomorphisms act on all the modules in the exact sequence above and are compatible with the arrows.

The basic objects of interest in this book are the various cohomology groups, which appear in the fundamental exact sequence, together with the action of the Hecke algebra $\mathcal{H}$ on them.

The central theme of this book is the understanding of the integral cohomology $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})$ as a module under the Hecke algebra, for instance we want to understand the denominators of the Eisenstein classes.

In Chapter 9 we formulate the general principle that under suitable conditions this denominator should be related (divisible?, equal ?) to a certain special value of an L-function, which occurs in the constant term of the Eisenstein series. The prototype of such a relationship occurs in [42], (actually the "abelian" case is discussed in chapter 5).

This principle (or conjecture) can be verified (or falsified) experimentally, on the other hand there is a strategy to prove assuming certain finiteness for mixed Grothendieck motives.

It is my intention is to keep the exposition as elementary as possible, the text should be readable by graduate students. We will need some background material from algebraic topology and from homological algebra (cohomology and homology of groups, spectral sequences, sheaf cohomology). This material is expounded in the first four chapters in [39], of course it can be found in many other textbooks.

In the later chapters (starting from chapter 6) we also need results and concepts from the theory of algebraic groups, the theory of symmetric spaces, arithmetic groups, and reduction theory for arithmetic groups. Furthermore we need results from the theory of representations of real semi-simple groups.

This material is somewhat more advanced, but in the in the first five chapters all these concepts and results are explained in terms in terms of special examples. Especially the sections on the general reduction theory and the Borel-Serre compactification (section (1.2.8)) could be skipped in a first reading.

For the Lie groups $\mathrm{Sl}_2(\mathbb{R})$ and $\mathrm{Sl}_2(\mathbb{C})$ and their arithmetic subgroups $\mathrm{Sl}_2(\mathbb{Z})$ and $\mathrm{Sl}_2(\mathbb{Z}[\sqrt{-1}])$ these prerequisite concepts are easy to explain and we will do so in this book. For instance if $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$ or more generally a congruence subgroup of finite index the symmetric space $\mathrm{Sl}_2(\mathbb{R})/K_\infty$ is the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) = y > 0\} = \mathrm{Sl}_2(\mathbb{R})/\mathrm{SO}(2)$. The quotient space $\Gamma\backslash\mathbb{H}$ is punctured Riemann surface. In this special case we have the Γ module $\mathcal{M}_n = \{\sum a_\nu X^\nu Y^{n-\nu} \mid a_\nu \in \mathbb{Z}\}$. We will study the cohomology groups $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n)$ and their module structure under the Hecke algebra in detail. We will prove some very specific results for these cohomology groups.

In Chapter four we discuss results from the theory of representations of the Lie- groups $\mathrm{Sl}_2(\mathbb{R})$ and $\mathrm{Sl}_2(\mathbb{C})$, and we explain the impact of these results on the cohomology. With these results at hand we formulate the famous Eichler-Shimura isomorphism, and we can sketch its proof. This Eichler-Shimura isomorphism also establishes the connection between $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n) \otimes \mathbb{C}$ and the space of modular forms of weight $n+2$. In the second half of this book in Chapter 8 we discuss what is called "Representation theoretic Hodge theory" and the Eichler-Shimura theorem becomes a special case of a much more general theorem.

On the other hand we will show that the results for the special groups $\mathrm{Sl}_2(\mathbb{Z}), \mathrm{Sl}_2(\mathbb{Z}[\sqrt{d}])$, or suitable subgroups of finite index of them, have deep and interesting consequences. We will discuss the Eisenstein cohomology for these special groups and explain the interaction between special values of L-functions and the structure of the cohomology. A prototype of such a result is the formula for the denominator of the Eisenstein class (Theorem 5.1.2). It is clear that this result should be a special case of a much more general theorem. At this moment it is not clear how far these generalisations reach (See section 3.3.9).

In Chapter 5 we discuss some applications of these results to number theory, and we have to accept some even more advanced topics. We concentrate on the case that $\Gamma \subset \mathrm{Sl}_2(\mathbb{Z})$ and we will use the fact that- with a grain of salt - the quotient $\Gamma\backslash\mathbb{H}$ is the set of $\mathbb{C}$-valued points the moduli space of elliptic curves (with some additional structure). This is also explained in [39],[40].

Then for any prime ℓ the cohomology groups $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n)\otimes\mathbb{Z}_\ell)$ are actually ℓ-adic etale cohomology groups, especially we get an action of the Galois-group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on these $\ell-$ adic cohomology groups. This action commutes with the action of the Hecke algebra. The insights into the structure of the cohomology groups as Hecke modules provides insights into the structure of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, for instance we discuss the theorem of Herbrand-Ribet ([25], [74])

In Chapter 6 we develop the analytic tools for the computation of the cohomology. Here we do not use the language of adeles. We assume that the Γ-module $\mathcal{M}$ is a $\mathbb{C}$-vector space and it is obtained from a rational representation of the underlying algebraic group. In this case one may interpret the sheaf $\tilde{\mathcal{M}}$ as the sheaf of locally constant sections in a flat bundle, and this implies that the cohomology is computable from the de-Rham-complex associated to this flat bundle. We could even go one step further and introduce a Laplace operator so that we get some kind of Hodge-theory and we can express the cohomology in terms of harmonic forms. Here we encounter serious difficulties since the quotient space $\Gamma\backslash X$ is not compact. But we will proceed in a slightly different way. Instead of doing analysis on $\Gamma\backslash X$ we work on $\mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R}))$. This space is a module under the group $G(\mathbb{R})$, which acts by right translations, but we rather consider it as a module under the Lie algebra $\mathfrak{g}$ of $G(\mathbb{R})$ on which also the group K_∞ acts, it is a $(\mathfrak{g}, K)$-module.

Since our module $\mathcal{M}$ comes from a rational representation of the underlying group G, we may replace the de-Rham-complex by another complex

$$H^\bullet(\mathfrak{g}, K_\infty, \mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R})) \otimes \tilde{\mathcal{M}}),$$

this complex computes the so called $(\mathfrak{g}, K)$-cohomology. The general principle will be to "decompose" the $(\mathfrak{g}, K)$-module $\mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R})$ into irreducible submodules and therefore to compute the cohomology as the sum of the contributions of the individual submodules. This is a group theoretic version of the classical approach by Hodge-theory. Again we have to overcome two difficulties. The first one is that the quotient $\Gamma\backslash G(\mathbb{R})$ is not compact and hence the above decomposition does not make sense.

The second problem is that we have to understand the irreducible $(\mathfrak{g}, K)$-modules and their cohomology.

The first problem is of analytical nature, we will give some indication how this can be solved by passing to certain subspaces of the cohomology the so called cuspidal or better the inner cohomology. The central result is the Theorem 6.1.1.

This result is a partial generalisation of the theorem of Eichler-Shimura, it describes the cuspidal part of the cohomology in terms of irreducible representations occurring in the space of cusp forms and contains some information on the discrete cohomology, which is slightly weaker. (See proposition **??**) We shall also give some indications how it can be proved.

We shall shall also state some general results concerning the second problem, we briefly recall the construction of the irreducible modules with non trivial $(\mathfrak{g}, K_\infty)$ cohomology.

We discuss some consequences of Theorem 6.1.1. It implies some vanishing theorems in cohomology, it implies that the inner cohomology is a semi simple module for the Hecke-algebra, and it helps to understand the $K-$-theory of algebraic number fields.

In the next section we use reduction theory-or better the description of $\overset{\bullet}{\mathcal{N}}$ $(\Gamma\backslash X), \tilde{\mathcal{M}})$- to discuss some growth conditions for cohomology classes, basically we show that cohomology classes which given by a weight can be represented by differential forms which have essentially the same weight.

In the second half of this chapter we will resume the discussion of modular symbols.

Chapter 1

Basic Notions and Definitions

$\boxed{\text{Affgr}}$

1.1 Affine algebraic groups over $\mathbb{Q}$.

A linear algebraic group $G/\mathbb{Q}$ is a subgroup $G \subset \mathrm{Gl}_n$, which is defined as the set of common zeroes of a set of polynomials in the matrix coefficients, where in addition these polynomials have coefficients in $\mathbb{Q}$. Of course we cannot take just any set of polynomials, the set has to be somewhat special before its common zeroes form a group. The following examples will clarify what I mean:

1.) The group GL_n is an algebraic group itself, the set of equations is empty. It has the subgroup $\mathrm{Sl}_n \subset \mathrm{Gl}_n$, which is defined by the polynomial equation

$$\mathrm{Sl}_n = \{x \in GL_n \mid \det(x) = 1\}$$

2.) Another example is given by the orthogonal group of a quadratic form

$$f(x_1, \ldots, x_n) = \sum_{i=}^{n} a_i x_i^2$$

where $a_i \in \mathbb{Q}$ and all $a_i \neq 0$ (this is actually not necessary for the following). We look at all matrices

$$\alpha = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}$$

which leave this form invariant, i.e.

$$f(\alpha \underline{x}) = f(\underline{x})$$

for all vectors $\underline{x} = (x_1, \ldots, x_n)$. This defines a set of polynomial equations for the coefficient a_{ij} of α. These α form a group, this is the linear algebraic group $SO(f)$.

3.) Instead of taking a quadratic form — which is the same as taking a symmetric bilinear form — we could take an alternating bilinear form

$$\begin{aligned}\langle \underline{x}, \underline{y}\rangle =&\langle x_1, \dots, x_{2n}, y_1, \dots y_{2n}\rangle = \\ &\sum_{i=1}^{n} \langle x_1 y_{i+n} - x_{i+n} y_i\rangle = f\langle \underline{x}, \underline{y}\rangle.\end{aligned}$$

This form defines the symplectic group:

$$Sp_n = \left\{\alpha \in GL_{2n} \mid \langle \alpha\underline{x}, \alpha\underline{y}\rangle = \langle \underline{x}, \underline{y}\rangle\right\}.$$

An Important remark: The reader may have observed tha we did not specify a field (or a ring) from which we take the entries of the matrices. This is done intentionally, because we may take the entries from any commutative ring R which contains the rational numbers $\mathbb{Q}$ and for which $1 \in \mathbb{Q}$ is the identity element (this means that R is a $\mathbb{Q}-$ algebra). In other words: for any algebraic group $G/\mathbb{Q} \subset GL_n$ and any $\mathbb{Q}$ algebra R we may define

$$G(R) \subset \mathrm{Gl}_n(R)$$

as the group of those matrices whose coefficients satisfy the required polynomial equations. Adopting this point of view we can say that

A linear algebraic group $G/\mathbb{Q}$ defines a functor from the category of $\mathbb{Q}$-algebras (i.e. commutative rings containing $\mathbb{Q}$) into the category of groups.

4.) Another example is obtained by the so-called ırestriction of scalars. Let us assume we have a finite extension $K/\mathbb{Q}$, we consider the vector space $V = K^n$. This vector space may also be considered as a $\mathbb{Q}$-vector space V_0 of dimension $n[K : \mathbb{Q}] = N$. We consider the group

$$GL_N/\mathbb{Q}.$$

Our original structure as a K-vector space may be considered as a map

$$\Theta : K \longrightarrow \mathrm{End}_{\mathbb{Q}}(V_0),$$

and the group $GL_n(K)$ is then the subgroup of elements in $GL_N(\mathbb{Q})$ which commute with all the elements of $\Theta(x), x \in K$. Hence we define the subgroup

$$G/\mathbb{Q} = R_{K/\mathbb{Q}}(\mathrm{Gl}_n) = \{\alpha \in \mathrm{Gl}_N \mid \alpha \text{ commutes with } \Theta(K)\}. \tag{1.1}$$

Then $G(\mathbb{Q}) = \mathrm{Gl}_n(K)$. For any $\mathbb{Q}$-algebra R we get

$$G(R) = \mathrm{Gl}_n(K \otimes_{\mathbb{Q}} R).$$

This can also be applied to an algebraic subgroup $H/K \hookrightarrow \mathrm{Gl}_n/K$, i.e. a subgroup that is defined by polynomial equations with coefficients in K.

Our definition of a linear algebraic group is a little bit provisorial. If we consider for instance the two linear algebraic groups

$$\begin{aligned} G_1/\mathbb{Q} &= \left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right\} \subset \mathrm{Gl}_2 \\ G_2/\mathbb{Q} &= \left\{\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\} \subset GL_3 \end{aligned}$$

then we would like to say, that these two groups are isomorphic. They are two different "realizations" of the *additive group* $G_a/\mathbb{Q}$. We see that the same linear algebraic group may be realized in different ways as a subgroup of different Gl_N's.

Of course there is a concept of linear algebraic group which does not rely on embeddings. The understanding of this concept requires a little bit of affine algebraic geometry. The drawback of our definition here is that we cannot define morphism between linear algebraic group. Especially we do not know when they are isomorphic.

We assert the reader that the general theory implies that a morphism between two algebraic groups is the same thing as a morphism between the two functors form $\mathbb{Q}$-algebras to groups. In some sense it is enough to give this functor. For instance, we have the *multiplicative group* $\mathbb{G}_m/\mathbb{Q}$ given by the functor

$$R \longrightarrow R^{\times}$$

and the additive group $G_a/\mathbb{Q}$ given by $R \to R^{+}$.

We can realise (represent is the right term) thegroup $\mathbb{G}_m/\mathbb{Q}$ as

$$\mathbb{G}_m/\mathbb{Q} = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right\} \subset \mathrm{Gl}_2$$

$\boxed{\text{AGS}}$

1.1.1 Affine group schemes

We say just a few words concerning the systematic development of the theory of linear algebraic groups. This is not directly used in the next few chapters but it will be useful later.

For any field k an affine k-algebra is a finitely generated algebra A/k, i.e. it is a commutative ring with identity, containing k, the identity of k is equal to the identity of A, which is finitely generated over k as an algebra. In other words

$$A = k[x_1, x_2, \dots, x_n] = k[X_1, X_2, \dots, X_n]/I,$$

where they X_i are independent variables and where I is a finitely generated ideal of polynomials in $k[X_1, \dots, X_n]$.

Such an affine k-algebra defines a functor from the category of $k-$ algebras to the category of sets, namely $B \mapsto \mathrm{Hom}_k(A, B)$.

A structure of an *affine group scheme scheme* on A/k consists of the following data:

a) A k homomorphism $m : A \to A \otimes_k A$ (the comultiplication)
b) A k-valued point $e : A \to k$ (the identity element)
c) An inverse $inv : A \to A$,

which satisfy the following requirement: For any k-algebra B our homomorphism m induces a map

$${}^t m : \mathrm{Hom}_k(A \otimes_k A, B) = \mathrm{Hom}_k(A, B) \times \mathrm{Hom}_k(A, B) \to \mathrm{Hom}_k(A, B)$$

and we require that this induces a group structure on $\mathrm{Hom}_k(A, B)$. We also require that the k valued point e is the identity and that inv yields the inverse.

We leave it to the reader to figure out what this means for m, e, inv, especially what does associativity mean (Hint: Choose $B = A$).

An affine k-algebra A together with such a collection m, e, inv is called an *affine group scheme* $G/k = (A, m, e, inv)$. The k-algebra A is the coordinate ring, or the ring of *regular functions* of the group scheme. We will denote it by $A(G)$. The group of B/k valued points will be denoted by $G(B) = \mathrm{Hom}_k(A(G), B)$. For $g \in G(B)$ and $f \in A(G) \otimes B$ we write $g(f) = f(g)$, we evaluate the regular function at the point $g \in G(B)$.

The group $\mathbb{G}_m$ has the coordinate ring $A(\mathbb{G}_m) = k[t, t^{-1}], m(t) = t \otimes t, e(t) = 1, \mathrm{inv}(t) = t^{-1}$ and the coordinate ring of the additive group $\mathbb{G}_a$ is $A(\mathbb{G}_a) = k[x]$ and $m(x) = x \otimes 1 + 1 \otimes x, e(x) = 0, \mathrm{inv}(x) = -x$.

The group scheme Gl_n/k has the coordinate ring

$$A = k[\ldots, x_{i,j}, \ldots, y]/(\det(x_{i,j})y - 1);\ 1 \leq i, j, \leq n$$

and the comultiplication is given by

$$m(x_{i,j}) = \sum_{\nu=1}^{n} x_{i,\nu} \otimes x_{\nu,j} \tag{1.2}$$

Now we know what a homomorphism between affine group schemes is. This is a homomorphism ${}^t\phi$ between the affine algebras $A(H)$ and $A(G)$ which is compatible with the respective maps in a),b),c). A homomorphism $\phi : G \to H$ is surjective (resp. injective) if the homomorphism ${}^t\phi : A(H) \to A(G)$ is injective (resp.) surjective.

A *rational representation* of G/k is a homomorphism of group schemes $\rho : G/k \to \mathrm{Gl}_n/k$.

If for instance V/k is a vector space of dimension n then we can define the group scheme $\mathrm{Gl}(V)$, if we choose a k-basis on V, then we can identify $\mathrm{Gl}(V)/k = \mathrm{Gl}_n/k$. If G/k is any affine group scheme, we say that V/k is a G-module if we have a homomorphism $\rho : G/k \to \mathrm{Gl}(V)$. Hence we know that for any k -algebra B/k we have a homomorphism $\rho(B) : G(B) \to \mathrm{Gl}(V \otimes_k B)$. Of course this is functorial in B/k, i.e. a homomorphism $\psi : B/k \to B'/k$ induces a homomorphism $G(B) \to G(B')$.

We may also consider actions of G/k on vector spaces W/k which are not of finite dimension, here we require a certain finiteness condition. As before we have an action

$$\rho_B : G(B) \times (W \otimes B) \to W \otimes B \tag{1.3}$$

which is functorial in B/k. But now we assume in addition that for any $w \in W$ there is a finite set of elements $w_1, w_2, \ldots, w_d$ such that for any B/k and any $g \in G(B)$

$$\rho_B(g)w = \sum_{i=1}^{d} w_i \otimes b_i(g) \text{ with } b_i \in A(G).$$

It suffices to check this for the "universal" element $\mathrm{Id} \in \mathrm{Hom}_k(A(G), A(G)) = G(A(G))$, this means we have to find $w_1, w_2, \dots, w_d \in W$ such that

$$\rho_{A(G)}(\mathrm{Id})w = \sum_{i=1}^{d} h_i \otimes w_i \text{ with } h_i \in A(G).$$

This implies of course that the k-subspace $W' = \sum k w_i$ which is generated by these w_i is invariant under the action ρ and it contains w. Hence we see that our k-vector space W is a union of finite dimensional subspaces which are invariant under the action of G/k.

Therefore we say that a vector space W/k with an action of G/k is a *G-module* if it satisfies the above finiteness condition. The category of these modules will be called Mod_G..

The ring of regular functions $A(G)$ is a $G \times_k G$ module: For $(g_1, g_2) \in G \times_k G(B) = G(B) \times G(B)$ the action and $f \in A(G), x \in G(B)$ the action is defined by

$$\rho(g_1, g_2)f(x) = f(g_1^{-1} x g_2).$$

We have to verify the finiteness condition. To do this we write a formula for $\rho(g_1, g_2)f \in A(G) \otimes B$. We have the comultiplication $m : A(G) \to A(G) \otimes_k A(G)$, we apply it to the first factor on the right hand side and get $m_{1,2} \circ m : A(G) \to A(G) \otimes_k A(G) \otimes_k A(G)$. Then

$$m_{1,2} \circ m(f) = \sum_{\mu} h'_\mu \otimes h_\mu \otimes h''_\mu$$

Then by definition

$$\rho(g_1, g_2)f = \sum_{\mu} h_\mu \otimes \mathrm{inv}(h'_\mu)(g_1) h''_\mu(g_2)$$

and this says that $\rho(g_1, g_2)f$ lies in the submodule generated by the h_μ.

Of course we may restrict the action to each the two factors, we simply choose $g_1 = e$,-we get the action by right translations- or we choose $g_2 = e$, this gives the action by left translations.

It is not difficult to show that for an affine group scheme we can find a collection of elements $e_0, e_1, \dots, e_r \in A(G)$ such that $e_i^2 = e_i \, \forall i, e_i e_j = 0 \, \forall i \neq j$ such that $1_A = \sum_i e_i$ and such that the subalgebras $A(G)e_i$ are integral. Then there is exactly one element (say e_0) such that $e(e_0) = 1$. Then $A(G)e_0$ is a subgroup scheme, it is called the *connected component of the identity* (See for instance [40], Chap. 7 , 7.2)

A group scheme G/k is *connected*, if its affine algebra $A(G) = A(G)e_0$ is integral, i.e. it does not have zero divisors.

Base change

If we have a field $L \supset k$ and a linear group G/k then the group $G/L = G \times_k L$ is the group over L where we forget that the coefficients of the equations are contained in k. The group $G \times_k L$ is the *base extension* from G/k to L.

$\boxed{\text{Charmod}}$

1.1.2 Tori, their character module,..

A special class of algebraic groups is given by the *tori.* We briefly recall the results of T. Ono [69].

An algebraic group T/k over a field k is called a *split torus* if it is isomorphic to a product of $\mathbb{G}_m/k$-s,

$$T/k \xrightarrow{\sim} \mathbb{G}_m^d.$$

The algebraic group T/k is called a *torus* if it becomes a split torus after a suitable finite extension of the ground field, i.e we have $T \times_k L \xrightarrow{\sim} \mathbb{G}_m^r/L$.

If we take an arbitrary separable finite field extension L/k we may consider the functor

$$R \to (L \otimes_k R)^\times.$$

It is not hard to see that this functor can be represented by an algebraic group over k, which is denoted by $R_{L/k}(\mathbb{G}_m/L)$ and called the *Weil restriction* of $\mathbb{G}_m/L$. We propose the notation

$$R_{L/k}(\mathbb{G}_m/L) = \mathbb{G}_m^{L/k} \tag{1.4}$$

The reader should try to prove that for a finite extension $\tilde{L}/L$ which is normal over $\mathbb{Q}$ we have

$$\mathbb{G}_m^{L/k} \times_k \tilde{L} \xrightarrow{\sim} (\mathbb{G}_m/\tilde{L})^{[L:k]}$$

and this shows that $\mathbb{G}_m^{L/k}$ is a torus .

A torus T/k is called *anisotropic* if is does not contain a non trivial split torus. Any torus C/k contains a maximal split torus S/k and a maximal anisotropic torus C_1/k. The multiplication induces a map

$$m : S \times C_1 \to C$$

this is a surjective (in the sense of algebraic groups) homomorphism whose kernel is a finite algebraic group. We call such map an *isogeny* and we write that $C = S \cdot C_1$, we say that C is the product of S and C_1 up to isogeny.

We give an example. Our torus $R_{L/\mathbb{Q}}(\mathbb{G}_m/L)$ contains $\mathbb{G}_m/k$ as a subtorus: For any ring R containing k we have $R^\times = \mathbb{G}_m(R) \subset (R \otimes L)^\times$. On the other and we have the norm map $N_{L/k} : (R \otimes L)^\times \to R^\times$ and the kernel defines a subgroup

$$R^{(1)}_{L/k}(\mathbb{G}_m/L) \subset R_{L/k}(\mathbb{G}_m/L)$$

and it is clear that

$$m : \mathbb{G}_m \times R^{(1)}_{L/k}(\mathbb{G}_m/L) \to R_{L/k}(\mathbb{G}_m/L)$$

has a finite kernel which is the finite algebraic group of $[L : k]$-th roots of unity.

For any torus T we define the character module as the group of homomorphisms

$$X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m).. \tag{1.5}$$

If the torus is split, i.e. $T = \mathbb{G}_m^r$ then $X^*(T) = \mathbb{Z}^r$ and the identification is given by $(n_1, n_2, \ldots, n_r) \mapsto \{(x_1, x_2, \ldots, x_r) \mapsto x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}\}$. We write the group structure on $X^*(T)$ additively, this means that $(\gamma_1 + \gamma_2)(x) = \gamma_(x)\gamma_2(x)$.

It is a theorem that for any torus T/k we can find a finite, separable, normal extension L/k such that $T \times_k L$ splits. Then it is easy to see that we have an action of the Galois group $\mathrm{Gal}(L/k)$ on $X^*(T \times_k L) = \mathbb{Z}^r$. If we have two tori $T_1/K, T_2/K$ which split over L

$$\mathrm{Hom}_k(T_1, T_2) \xrightarrow{\sim} \mathrm{Hom}_{\,\mathrm{Gal}(L/k)}(X^*(T_2 \times_k L), X^*(T_1 \times_k L)) \tag{1.6}$$

To any $\mathrm{Gal}(L/k)-$ action on $\mathbb{Z}^n$ we can find a torus T/k which splits over L and which realises this action.

A homomorphism $\phi : T_1/k \to T_2/k$ is called an *isogeny* if $\dim(T_1) = \dim(T_2)$ and if ${}^t\phi : X^*(T_2) \to X^*(T_1)$ is injective. Then the kernel $\ker(\psi)$ is a finite group scheme of multiplicative type. If $Y \subset X^*(T_1)$ is a submodule of finite index the $Y = X^*(T_2)$ and the inclusion provides an isogeny $\psi : T_1 \to T_2$. The quotient $X^*(T_1)/Y$ is a finite constant group scheme and $\ker(\psi)$ the*dual* of his quotient.

)We also define the *cocharacter module* $\mathrm{Hom}(\mathbb{G}_m, T)$. If the torus $/k = \mathbb{G}_m^r$ then every cocharacter is the form $x \mapsto (x^{n_1}, x^{n_2}, \ldots, x^{n_r})$ It is clear that we have a pairing

$$< \, , \, >: X_*(T) \times X^*(T) \to \mathbb{Z} \text{ which is defined by } \gamma(\chi(t)) = t^{<\chi,\gamma>} \tag{1.7}$$

A very prominent torus is the torus $\mathbb{S}^1/\mathbb{R}$, this the one dimensional torus whose character module $X^*(\mathbb{S}^1 \times \mathbb{C}) = \mathbb{Z}$ and the complex conjugation acts by -1.

1.1.3 Semi-simple groups, reductive groups,.

An important class of linear algebraic groups is formed by the *semisimple* and the *reductive* groups. (For a general reference [84].) We do not want to give the precise definition here. Roughly, a linear group is *reductive* if it is connected and if it does not contain a non trivial normal subgroup which is isomorphic to a product of groups of type G_a. A group is called semisimple, if it is reductive and does not contain a non trivial torus in its centre.

A semi-simple group G/k is simple, if it does not contain any normal subgroup of dimension > 0. Any semi-simple group is up to isogeny a product of simple groups. Any semi simple group $G/\mathbb{Q}$ contains a maximal torus $T/\mathbb{Q} \subset G/\mathbb{Q}$ such a maximal torus is equal to its own centraliser. A semi simple group is split if it contains a split maximal torus T_0/k, i.e. a maximal torus which is split. If $T/k \subset G/k$ is any (maximal) torus, then there is a finite extension $L/\mathbb{Q}$ such that $T \times_{\mathbb{Q}} L$ is split, and hence $G \times_{\mathbb{Q}} L$ is also split.

For example the groups Sl_n, Sp_n are (split) semi simple, the groups $\mathrm{SO}(f)$ are semi-simple provided $n \geq 3$. (See next subsection 1.1.5). The groups Gl_n and especially the multiplicative group $\mathrm{Gl}_1/\mathbb{Q} = \mathbb{G}_m/\mathbb{Q}$ are reductive. Any reductive group $G/\mathbb{Q}$ (or over any field of characteristic zero) has a central torus

$C/\mathbb{Q}$ and this central torus contains a maximal split torus S. The derived group $G^{(1)}/\mathbb{Q}$ is semi simple and we get an isogeny

$$m : G^{(1)} \times C_1 \times S \to G$$

or briefly $G = G^{(1)} \cdot C_1 \cdot S$.

If for instance $G = R_{L/\mathbb{Q}}(\mathrm{Gl}_n/L)$ then $G^{(1)} = R_{L/\mathbb{Q}}(\mathrm{Sl}_n/L)$ and $C = R_{L/\mathbb{Q}}(\mathbb{G}_m/L)$ and this yields the product decomposition up to isogeny

$$G = G^{(1)} \cdot R^{(1)}_{L/\mathbb{Q}}(\mathbb{G}_m/L) \cdot \mathbb{G}_m. \tag{1.8}$$

For $\mathrm{Gl}_n/\mathbb{Q}$ the central torus is the group $\mathbb{G}_m/\mathbb{Q}$. The center of $\mathrm{Sl}_n/\mathbb{Q}$ is the finite group group scheme μ_n of of n-th roots of unity. The coordinate ring of μ_n is the finite algebra $A(\mu_n) = \mathbb{Q}[t]/(t^n - 1)$. Of course we may replace $\mathbb{Q}$ by any ring commutative ring R.

We can form the quotient group scheme

$$\mathrm{PGl}_n/\mathbb{Q} = (\mathrm{Gl}_n/\mathbb{G}_m)/\mathbb{Q} \xrightarrow{\sim} (\mathrm{Sl}_n/\mathbb{Q})/\mu_n \tag{1.9}$$

this is also the adjoint group of $\mathrm{Gl}_n/\mathbb{Q}$ and $\mathrm{Sl}_n/\mathbb{Q}$, i.e.

$$\mathrm{Ad}(\mathrm{Gl}_n) = \mathrm{PGl}_n = \mathrm{Gl}_n/\mathbb{G}_m = \mathrm{Sl}_n/\mu_n. \tag{1.10}$$

We could certainly drop the assumption that a reductive group should be connected, we could simple say that $G/\mathbb{Q}$ is reductive (semi-simple...) if its connected component of the identity is reductive (semi-simple...).

Another important class of semi simple groups is given by the *quasisplit* groups (see also section 1.1.7. A group $G/\mathbb{Q}$ is called quasisplit if it contains a Borel subgroup $B/\mathbb{Q} \subset G/\mathbb{Q}$. A Borel subgroup $B/\mathbb{Q}$ is a maximal solvable subgroup, it contains a maximal torus $T/\mathbb{Q} \subset B/\mathbb{Q}$, this torus is also a maximal torus in $G/\mathbb{Q}$. Then $B = U \rtimes T$ is the semidirect product of this torus and the *unipotent radical* $U/\mathbb{Q}$. We discuss a special example which is of great relevance for our subject.

Let $L/\mathbb{Q}$ be a quadratic extension, let us denote the non trivial automorphism by $a \mapsto \bar{a}$. Let V/L be a finite dimensional vector space together with a hermitian form $h : V \times_L V \to L$, i.e.

$$h(v,w) = \overline{h(w,v)};\ h(\lambda u + \mu v, w) = \lambda h(u,w) + \mu h(v,w)\ \forall u,v,w \in V, \lambda,\mu \in L.$$

Furthermore we assume that h is non degenerate, i.e. for any $v \in V, v \neq 0$ we find a $w \in V$ such that $h(v,w) \neq 0$. Then we can define the group $\mathrm{SU}(h)/\mathbb{Q}$: For any commutative $\mathbb{Q}$ -algebra R we define

$$\mathrm{SU}(h)(R) = \{g \in \mathrm{Sl}(V \otimes_{\mathbb{Q}} R \mid h(gv,gw) = h(v,w) \text{ and } \det(g) = 1\}. \tag{1.11}$$

Then $\mathrm{SU}(h)/\mathbb{Q}$ is a semi simple group over $\mathbb{Q}$. We can also define the unitary group $\mathrm{U}(h)/\mathbb{Q}$ where we drop the condition that the determinant is one and the group of hermitian similitudes $\mathrm{GU}(h)$ where

$$\mathrm{GU}(h)(R) = \{g \in \mathrm{Gl}(V \otimes_{\mathbb{Q}} R \mid h(gv,gw) = d(g)h(v,w)\ \forall v,w \in V \otimes_{\mathbb{Q}} R\}, \tag{1.12}$$

here $d : \mathrm{GU}(h) \to R_{L/\mathbb{Q}}(\mathbb{G}_m)$ is a homomorphism, the kernel of d is the group $\mathrm{U}(h)$.

We consider the special case where

$$V = Le_1 \oplus \cdots \oplus Le_n \oplus (Le_0) \oplus Lf_n \oplus \cdots \oplus Lf_1$$

the summand Le_0 is left out if $\dim_L V$ is even. The hermitian scalar product is given by

$$h_1(e_i, f_i) = h_1(f_i, e_i) = 1 \ \forall i = 1, \ldots, n, \ \ (h_1(e_0, e_0) = 1)$$

and all other scalar products equal to zero. Then $\mathrm{SU}(h_1)$ is a quasi split semi simple group over $\mathbb{Q}$: The elements $t \in \mathrm{Gl}(V)$ for which

$$t = \{t : e_i \mapsto t_i e_i; t : f_i \mapsto \bar{t_i}^{-1}; \ (t : e_0 \mapsto t_0 e_0 \text{ with } t_0 \bar{t_0} = 1)\}$$

are the $\mathbb{Q}$-valued points of a maximal torus $T_1/\mathbb{Q} \subset \mathrm{SU}(h_1)$. The vector space V/L comes with a natural flag

$$\begin{gathered}\mathcal{F} := \{0\} \subset Le_1 \subset \cdots \subset \oplus Le_1 \oplus \cdots \oplus Le_n \subset (Le_1 \oplus \ldots Le_n + Le_0) \subset \\ (Le_1 \oplus \ldots Le_n \oplus Le_0 \oplus Lf_n) \subset \ldots (Le_1 \oplus \cdots \oplus Le_n \oplus Le_0 \oplus Lf_n \oplus \cdots \oplus Lf_2) \subset V.\end{gathered} \tag{1.13}$$

Now the subgroup $B_1/\mathbb{Q} \subset \mathrm{SU}(h_1)/\mathbb{Q}$ which fixes $\mathcal{F}$ is a maximal solvable subgroup in $\mathrm{SU}(h_1)$.

1.1.4 The Lie-algebra

We need some basic facts about the Lie-algebras of algebraic groups.

For any algebraic group G/k we can consider its group of points with values in $k[\epsilon] = k[X]/(X^2)$. We have the homomorphism $k[\epsilon] \to k$ sending ϵ to zero and hence we get an exact sequence

$$0 \to \mathfrak{g} \to G(k[\epsilon]) \to G(k) \to 1.$$

The kernel $\mathfrak{g}$ is a k-vector space, if the characteristic of k is zero, then its dimension is equal to the dimension of G/k. It is denoted by $\mathfrak{g} = \mathrm{Lie}(G)$.

Let us consider the example of the group $G = SO(f)$, where $f : V \times V \to k$ is a non degenerate symmetric bilinear form. In this case an element in $G(k[\epsilon])$ is of the form $\mathrm{Id} + \epsilon A, A \in \mathrm{End}(V)$ for which

$$f((\mathrm{Id} + \epsilon A)v, (\mathrm{Id} + \epsilon A)w) = f(v, w)$$

for all $v, w \in V$. Taking into account that $\epsilon^2 = 0$ we get

$$\epsilon(f(Av, w) + f(v, Aw)) = 0,$$

i.e. A is skew with respect to the form, and $\mathfrak{g}$ is the k-vector space of skew endomorphisms. If we give V a basis and if $f = \sum x_i^2$ with respect to this basis then this means thematrix of A is skew symmetric.

If we consider $G = \mathrm{Gl}_n/k$ then $\mathfrak{g} = M_n(k)$, the Lie-bracket is given by

$$(A, B) \mapsto AB - BA \tag{1.14}$$

We have some kind of a standard basis for our Lie algebra

$$\mathfrak{g} = \bigoplus_{i=1}^{n} kH_i \oplus \bigoplus_{i,j,i\neq j} kE_{i,j} \tag{1.15}$$

where H_i (resp.$E_{i,j}$) are the matrices

$$H_i = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \ddots & & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ resp. } E_{i,j} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \ddots & & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and the only non zero entries (=1) is at (i, i) on the diagonal (resp. and (i, j) off the diagonal.)

For the group Sl_n/k the Lie-algebra is $\mathfrak{g}^{(0)} = \{A \in M_n(k) | \operatorname{tr}(A) = 0\}$ and again we have a standard basis

$$\mathfrak{g}^{(0)} = \bigoplus_{i=1}^{n-1} k(H_i - H_{i+1}) \oplus \bigoplus_{i,j,i\neq j} kE_{i,j} \tag{1.16}$$

If $\rho : G \to \mathrm{Gl}(V)$ is a rational representation of our group G/k then it is clear from our considerations above that we have a "derivative" of this representation $\boxed{\text{drho}}$

$$d\rho : \mathfrak{g} = \mathrm{Lie}(G/k) \to \mathrm{Lie}(\mathrm{Gl}(V)) = \mathrm{End}(V) \tag{1.17}$$

this is k-linear.

Every group scheme G/k has a very special representation, this is the*Adjoint representation.* We observe that the group acts on itself by conjugation, this is the morphism

$$Inn : G \times_k G \to G$$

which on R valued points is given by

$$Inn(g_1, g_2) \mapsto g_1 g_2 (g_1)^{-1}.$$

This action clearly induces a representation

$$\mathrm{Ad} : G/k \to \mathrm{Gl}(\mathfrak{g})$$

and this is the adjoint representation. This adjoint representation has a derivative and this is a homomorphism of k vector spaces

$$D_{\mathrm{Ad}} = \mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}).$$

We introduce the notation: For $T_1, T_2 \in \mathfrak{g}$ we put

$$[T_1, T_2] := \mathrm{ad}(T_1)(T_2).$$

Now we can state the famous and fundamental result

Theorem 1.1.1. *The map $(T_1, T_2) \mapsto [T_1, T_2]$ is bilinear and antisymmetric. It induces the structure of a Lie-algebra on $\mathfrak{g}$, i.e. we have the Jacobi identity*

$$[T_1, [T_2, T_3]] + [T_2, [T_3, T_1]] + [T_3, [T_1, T_2]] = 0.$$

We do not prove this here. In the case $G/k = \mathrm{Gl}(V)$ and $T_1, T_2 \in \mathrm{Lie}(\mathrm{Gl}(V)) = \mathrm{End}(V)$ we have $[T_1, T_2] = T_1T_2 - T_2T_1$ and in this case the Jacobi Identity is a well known identity.

On any Lie algebra we have a symmetric bilinear form (the *Killing form*)

$$B : \mathfrak{g} \times \mathfrak{g} \to k \tag{1.18}$$

which is defined by the rule

$$B(T_1, T_2) = \mathrm{trace}(\mathrm{ad}(T_1) \circ \mathrm{ad}(T_2))$$

A simple computation shows that for the examples $\mathfrak{g} = \mathrm{Lie}(\mathrm{Gl}_n)$ and $\mathfrak{g}^{(0)} = \mathrm{Lie}(\mathrm{Sl}_n)$ we have

$$B(T_1, T_2) = 2n\ \mathrm{tr}(T_1T_2) - 2\ \mathrm{tr}(T_1)\ \mathrm{tr}(T_2) \tag{1.19}$$

we observe that in case that one of the T_i is central, i.e.$= u\mathrm{Id}$ we have $B(T_1, T_2) = 0$. In the case of $\mathfrak{g}^{(0)}$ the second term is zero.

It is well known that a linear algebraic group is semi-simple if and only if the Killing form B on its Lie algebra is non degenerate. [18]
$\boxed{\text{class}}$

1.1.5 The classical groups and their realisation as split semi-simple group schemes over $\mathrm{Spec}(\mathbb{Z})$

We will not discuss the general notion of a semi-simple group scheme over a base S, instead we will discuss the examples of classical groups and explain the main structure theorems in examples.

The group scheme $\mathrm{Sl}_n/\ \mathrm{Spec}(\mathbb{Z})$

We consider a free module M of rang n over $\mathrm{Spec}(\mathbb{Z})$. We define the group scheme $\mathrm{Sl}(M)/\ \mathrm{Spec}(\mathbb{Z})$: for any $\mathbb{Z}$ algebra R we have $\mathrm{Sl}(M)(R) = \mathrm{Sl}(M \otimes_{\mathbb{Z}} R)$.

This is clearly a semi simple group scheme over $\mathrm{Spec}(\mathbb{Z})$ because :

a) The group scheme is smooth over $\mathrm{Spec}(\mathbb{Z})$
b) For any field k -which is of course a $\mathbb{Z}$-algebra we have

$$\mathrm{Sl}(M) \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(k) = \mathrm{Sl}(M \otimes_{\mathbb{Z}} k)/\ \mathrm{Spec}(k)$$

and for any k this group scheme does not contain a normal subgroup scheme, which is isomorphic to $G_a^r/\operatorname{Spec}(k)$ (hence it is reductive) and its center is a finite group scheme.

If we fix a basis $e_1, e_2, \ldots, e_n$ then we get a split maximal torus $T/\operatorname{Spec}(\mathbb{Z})$ this is the sub group scheme which fixes the lines $\mathbb{Z}e_i$, with respect to this basis we have

$$T(R) = \{\begin{pmatrix} t_1 & 0 & \dots & 0 \\ 0 & t_2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & t_n \end{pmatrix} \quad | \quad t_i \in R^\times, \prod_i t_i = 1 \tag{1.20}$$

With respect to this torus $T/\operatorname{Spec}(\mathbb{Z})$ we define root subgroups. This are smooth subgroup schemes $U \subset G$ which are isomorphic to the additive group scheme $G_a/\operatorname{Spec}(\mathbb{Z})$ and which are normalized by T. It is clear that these root subgroups are given by

$$\tau_{ij} : G_a \to \operatorname{Sl}(M) \tag{1.21}$$

$$\tau_{ij} : x \to \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & x & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.22}$$

where the entry x is placed in the i-th row and j-th collumn. Let us denote the image by $U_{\alpha_{ij}}$.

Then we get the relation

$$t\tau_{ij}(x)t^{-1} = \tau_{ij}((t_i/t_j)x)$$

(If I write such a relation then I always mean that $t, x..$ are elements in $T(R), G_a(R)...$ for some unspecified $\mathbb{Z}-$ algebra R.)

The root system

The characters

$$\alpha_{ij} : T \to G_m$$

$$\alpha_{ij} : \begin{pmatrix} t_1 & 0 & \dots & 0 \\ 0 & t_2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & t_n \end{pmatrix} \to t_i/t_j$$

are form the set Δ of roots in the character module of the torus. We may select a subset of positive roots

$$\Delta^+ = \{\alpha_{ij} \mid i < j\}.$$

Then the torus T and the $U_{\alpha_{ij}}$ with $\alpha_{ij} \subset \Delta^+$ stabilize the flag

$$\mathcal{F} = (0) \subset \mathbb{Z}e_1 \subset \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \subset \cdots \subset M.$$

The stabilizer of the flag is a smooth sub group scheme $B/\operatorname{Spec}(\mathbb{Z})$. It is so-but not entirely obvious- that B is a maximal solvable sub group scheme. These maximal subgroup schemes are called Borel subgroups.

It is clear that the morphism

$$T \times \prod_{\alpha_{ij}, i<j} U_{\alpha_{ij}} \to B,$$

which is induced by the multiplication is an isomorphism of schemes.

The set Δ^+ of positive roots contains the subset $\pi \subset \Delta$ of simple roots $\alpha_i := t_i/t_{i+1}$. Every positive root can be written as a sum of simple roots with positive coefficients.

We consider the normaliser $N(T) \subset \mathrm{Sl}_n$, it acts by permutations on the set of submodules $\mathbb{Z}e_i$. The quotient $N(T)/T = W$ is the Weyl group, in this case it is isomorphic to the symmetric group S_n. It is easy to see that we have a positive definite, symmetric, W invariant bilinear form on $X^*(T)$ which is given by

$$< \alpha_i, \alpha_i >= 2; < \alpha_i, \alpha_{i+1} > -1, \text{ and } < \alpha_i, \alpha_j >= 0 \text{ if } |i-j| > 1 \qquad (1.23)$$

All these data about the set of roots and simple roots are encoded in the Dynkin diagram

$$A_{n-1} := \alpha_1 \quad - \quad \alpha_2 \quad - \cdots - \quad \alpha_{n-1} \qquad (1.24)$$

The flag variety

It is not so difficult to see that the flags form a projective scheme $\mathrm{Gr}/\operatorname{Spec}(\mathbb{Z})$. From this it follows: For any Dedekind ring A and its quotient field K we have

$$\mathrm{Gr}(K) = \mathrm{Gr}(A).$$

If A is even a discrete valuation ring then we can show easily that the group $\mathrm{Sl}_n(A)$ acts transitively on $\mathrm{Gr}(A)$.

The whole point is, that results of this type are true for arbitrary split semi simple groups $\mathcal{G}/\operatorname{Spec}(\mathbb{Z})$. This is not so easy to explain and also much more difficult to prove. But we have the series of so called classical groups and for those these results are again easy to see. (The main problem in the general approach is that we have to start from an abstract definition of a semi simple group and not from a group which is given to us in a rather explicit way like Sl_n or the classical groups)

The group scheme $\mathrm{Sp}_g/$ **Spec**$(\mathbb{Z})$

Now we choose again a free $\mathbb{Z}$ module M but we assume that we have a non degenerate alternating pairing

$$< , >: M \times M \to \mathbb{Z}$$

where non degenerate means: If $x \in M$ and $< x, M > \subset a\mathbb{Z}$ with some integer $a > 1$, then $x = ay$ with $y \in M$. It is well known and also very easy to prove that M is of even rank $2g$ and that we can find a basis

$$\{e_1, \dots, e_g, f_g, \dots, f_1\}$$

such that $< e_i, f_i >= - < f_i, e_i >= 1$ and all other values of the pairing on basis elements are zero.

The automorphism group scheme of $\mathbb{G} = \mathrm{Aut}((M, < \ . \ >))$ is the symplectic group $\mathrm{Sp}_g/\ \mathrm{Spec}(\mathbb{Z})$. Again it is easy to find out how a maximal torus must look like. With respect to our basis we can take

$$T = \{\begin{pmatrix} t_1 & 0 & & & \dots & 0 \\ 0 & \ddots & & & \dots & 0 \\ 0 & 0 & t_g & & & 0 \\ 0 & 0 & 0 & t_g^{-1} & \dots & \\ 0 & & & & \ddots & 0 \\ & 0 & & & & t_1^{-1} \end{pmatrix}\} \tag{1.25}$$

We can say that the torus is the stabilser of the ordered collection of rank 2 submodules $\mathbb{Z}e_i, \mathbb{Z}f_i$. We can define a Borel subgroup $B/\mathbb{Z}$ which is the stabilizer of the flag

$$\mathcal{F} = (0) \subset \mathbb{Z}e_1 \subset \dots \subset \mathbb{Z}e_1 \dots \oplus \dots \mathbb{Z}e_g \subset \mathbb{Z}e_1 \dots \oplus \dots \mathbb{Z}e_g \oplus \mathbb{Z}f_g \subset \dots \subset M$$

(A flag starts with isotropic subspaces until we reach half the rank of the module. But then this lower part of the flag determines the upper half, because it is given by the orthogonal complements of the members in the lower half).

Again we can define the root subgroups (with respect to T)

$$rootsubgroup\tau_\alpha : G_a \xrightarrow{\sim} U_\alpha \subset \mathcal{G} \tag{1.26}$$

which are normalized by T. As before we have the relation

$$\tau(x)t^{-1} = \tau(\alpha(t)x), \tag{1.27}$$

where $\alpha \in \Delta \subset X^*(T)$.

Now it is not quite so easy to write down what these root subgroups are, we write down the simple positive roots in the thecase $g = 2$: We have the maximal torus

$$T = \{\begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix}\}$$

and we want to find one-parameter subgroups $U_\alpha \subset \mathbb{G}$ which stabilize the flag.

The one parameter subgroups corresponding to the simple roots are

$$\tau_{\alpha_1} : x \mapsto \{e_1 \mapsto e_1, e_2 \mapsto e_2 + xe_1, f_2 \mapsto f_2, f_1 \mapsto f_1 - xf_2\}$$

$$\tau_{\alpha_2} : y \mapsto \{e_1 \mapsto e_1, e_2 \mapsto e_2, f_2 \mapsto f_2 + ye_2, f_1 \mapsto f_1\}$$

where $\alpha_1(t) = t_1/t_2, \alpha_2(t) = t_2^2$. The unipotent radical is then

$$\{\begin{pmatrix} 1 & x & v & u \\ 0 & 1 & y & v - xy \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix}\}$$

From here it is not difficult to see that for all values of g the simple roots are $\alpha_i(t) = t_i/t_{i+1}$ with $1 \leq i < g$ and $a_g(t) = t_g^2$. Again we define the Weyl group W as above, We have a W invariant positive definite, symmetric bilinear form on $X^*(T)$ and for this form we have

$$\begin{gathered} < \alpha_i, \alpha_i >= 2 \text{ for } i < g \text{ and } < \alpha_g, \alpha_g >= 4, \\ < \alpha_i, \alpha_{i+1} >= -1 \text{ if } i < g-1 \text{ and } < \alpha_{g-1}, \alpha_g >= -2 \end{gathered} \tag{1.28}$$

and all other values of the pairing between simple roots are zero.

Agin these data are encoded in the Dynkin diagram

$$C_n := \alpha_1 \quad - \quad \alpha_2 \quad - \cdots - <= \quad \alpha_g$$

See [18]. We will see this Dynkin diagram for $g = 3$ at the end of this book.

As before it is not so difficult to show that the flags form a smooth projective scheme $\mathcal{X}/\operatorname{Spec}(\mathbb{Z})$ (see also [book], V.2.4.3). Show that for any discrete valuation ring A the group $\mathbb{G}(A)$ acts transitively on $\mathcal{X}(A) = \mathcal{X}(K)$. It is also easy to verify the statements in 1.1.

The group scheme $\mathrm{SO}(n,n)/$ **Spec**$(\mathbb{Z})$

We can play the same game with symmetric forms. Let M together with its basis as above, we replace g by n. But now we take the quadratic form F

$$F : M \to \mathbb{Z}$$

which is defined by

$$F(x_1e_1 \cdots + x_ne_n + y_nf_n + \cdots + y_1f_1) = \sum x_iy_i$$

and all other values of the pairing on basis elements are zero. We define the group scheme of isomorphisms but in addition we require thedeterminant is one. Hence

$$\mathrm{SO}(n,n)/\operatorname{Spec}(\mathbb{Z}) = \operatorname{Aut}(M, F, \det = 1).$$

The maximal torus and the flags look pretty much the same as in the previous case. But the set of roots looks different. For $n = 2$ the torus and the unipotent radical are given by

$$T = \{\begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix}\},\ U = \{\begin{pmatrix} 1 & x & y & -xy \\ 0 & 1 & 0 & -y \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix}\}.$$

The system of positive roots consists of two roots $\alpha_1(t) = t_1/t_2, \alpha_2(t) = t_1t_2$. This is the Dynkin diagram $A_1 \times A_1$ hence there exists a homomorphism (isogeny) between group schemes over $\operatorname{Spec}(\mathbb{Z})$:

$$\mathrm{Sl}_2 \times \mathrm{Sl}_2 \to \mathrm{SO}(2,2).$$

It is an amusing exercise to write down this isogeny.

Another even more interesting excercise is the computation of the roots for the torus (here $n = 3$)

$$T = \{\begin{pmatrix} t_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_3^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & t_1^{-1} \end{pmatrix}\}. \tag{1.29}$$

In this case we have the root subgroups

$$\tau_{\alpha_1} : x \mapsto \begin{pmatrix} 1 & x & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tau_{\alpha_2} : x \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & x & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -x & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\tau_{\alpha_3} : x \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & x & 0 & 0 \\ 0 & 0 & 1 & 0 & -x & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$\alpha_1(t) = t_1/t_2,\ \alpha_2(t) = t_2/t_3,\ \alpha_3(t) = t_2t_3$$

Use the result of this computation to show that we have an isogeny

$$\mathrm{Sl}_4 \to \mathrm{SO}(3,3).$$

How can we give a linear algebra interpretation of this isogenies?

If we now consider the maximal torus (1.25) and put (1.29) into the middle then we see that the simple roots are

$$\alpha_i(t) = t_i/t_{i+1} \text{ for } i = 1, \dots n-1, \text{ and } \alpha_n(t) = t_{n-1}t_n \tag{1.30}$$

which gives us the Dynkin diagram (wird noch korrigiert!)

$$D_n := \alpha_1 \quad - \quad \alpha_2 \quad - \cdots - \alpha_{n-2} \begin{matrix} \alpha_{n-1} \\ \\ \alpha_n \end{matrix} \tag{1.31}$$

The group scheme $\mathrm{SO}(n+1,n)/$ **Spec**$(\mathbb{Z})$

Of course we can also consider quadratic forms in an odd number of variables. We take a free $\mathbb{Z}$-module of rank $2n+1$ with a basis

$$\{e_1,\dots,e_n,h,f_n,\dots,f_1\}.$$

On this module we consider the quadratic form

$$F : M \to \mathbb{Z}$$

$$F(\sum x_i e_i + zh + \sum y_i f_i) = \sum x_i y_i + z^2.$$

From this quadratic fom we get the bilinear form

$$B(u,v) = F(u+v) - F(u) - F(v).$$

We have the relation

$$F(u) = 2B(u,u),$$

hence we can reconstruct the quadratic form from the bilinear form if we extend $\mathbb{Z}$ to a larger ring where 2 is invertible.

We consider the automorphism scheme

$$\mathcal{G}/\operatorname{Spec}(\mathbb{Z}) = \mathrm{SO}(n+1,n)/\operatorname{Spec}(\mathbb{Z}) = \mathrm{Aut}(M,F,det=1)/\operatorname{Spec}(\mathbb{Z})$$

and I claim that this is indeed a semi simple group scheme over $\operatorname{Spec}(\mathbb{Z})$. To see this I strongly recommend to discuss the case $n=1$.

We have of course the maximal torus

$$T = \{\begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-1} \end{pmatrix}\}.$$

It is also the stabiliser of the collection of three subspaces $\mathbb{Z}e, \mathbb{Z}h, \mathbb{Z}f$, here we use the determinant condition.

Now one has to discuss the root subgroups with respect to this torus.

From this we can derive that we have an isogeny

$$\mathrm{Sl}_2 \to \mathrm{SO}(2,1)$$

It is also interesting to look at the case $n=2$. In this case we can compare the root systems of Sp_2 and $\mathrm{SO}(3,2)$ they are isomorphic. Now it is a general theorem in the theory of split semi simple group schemes that the root system determines the group scheme up to isogeny. Hence we should be able to construct an isogeny between Sp_2 and $\mathrm{SO}(3,2)$. Who ctan do it?

For an arbitrary value of n we get the Dynkin diagram

$$B_n := \alpha_1 \quad - \quad \alpha_2 \quad - \cdots - => \quad \alpha_n \tag{1.32}$$

The element w_0.

Finally we have a short look at the automorphism groups of the Dynkin diagrams.

For the Dynkin diagram of type A the group of automorphisms is trivial if $n = 1$ and for $n > 1$ the group of automorphism is $\mathbb{Z}/2\mathbb{Z}$, the non trivial element ϵ exchanges the roots α_i and α_{n-1-i}.

For the diagrams B_n, C_n the automorphism group is trivial,

For the Dynkin diagram D_n and $n > 4$ the group of automorphisms is $\mathbb{Z}/2\mathbb{Z}$ and the non trivial automorphism ϵ fixes the simple roots α_i with $1 \leq i \leq n-2$ and interchanges α_{n-1}, α_n.

For $n = 4$ the automorphism group is the symmetric group S_3 and it acts by permutations on the three simple roots $\alpha_1, \alpha_3, \alpha_4$.

The Weyl group $W = N(T)/T$ acts simply transitively on the set of Borel subgroups $B' \supset T$. Hence there is a unique element $w_0 \in W$ which sends our Borel subgroup B into its opposite B^- this is the group whose simple roots are the roots $-\alpha_i$.

If the automorphism group of the Dynkin diagram is trivial we have $w_0 = -1$, i.e. it acts by multiplication by -1 on $X^*(T)$.

For the diagram A_n and $n > 1$ the element $w_0 = -\epsilon$ and therefore not equal to -1.

For the diagram D_n the element $w_0 = -1$ if n is even and equal to $-\epsilon$ if n is odd.

The element w_0 will play an important role later in this book.

The fundamental and the dominant weights

The Weyl group $W = N(T)/T$ acts by conjugation on the character module $X^*(T)$ and there a positive definite symmetric bilinear form $< \, , \, >: X^*(T) \times X^*(T) \to \mathbb{Q}$ which is invariant invariant under this action.. The Weyl group is generated by the reflections

$$s_{\alpha_i} : \gamma \mapsto \gamma - \frac{2 < \gamma, \alpha_i >}{< \alpha_i, \alpha_i >} \alpha_i \tag{1.33}$$

and this implies that $\frac{2<\gamma,\alpha_i>}{<\alpha_i,\alpha_i>} \in \mathbb{Z}$. Of course we have $\alpha_i \in X^*(T)$ for all simple roots, the sublattice $\bigoplus \mathbb{Z}\alpha_i$ it is of finite index in $X^*(T)$. To this sublattice belongs a torus T^{ad} and an isogeny $\psi : T \to T^{(\mathrm{ad})}$. The kernel $\ker(\psi) = \mu$ is the centre of our group scheme $G/\mathbb{Z}$ and the quotient $G/\mu = G^{(\mathrm{ad})}$ is the *adjoint* group.

In $X^*(T) \otimes \mathbb{Q}$ we have the elements γ_i which are defined by

$$\frac{2 < \gamma_i, \alpha_i >}{< \alpha_j, \alpha_j >} = \delta_{i,j} \tag{1.34}$$

these elements are the *fundamental weights* . The lattice $\bigoplus \mathbb{Z}\gamma_i$ contains $X^*(T)$ as a sublattice of finite index. It provides a torus $T^{(\mathrm{sc})}$ and an isogeny ψ_1 :

$T^{(\mathrm{sc})} \to T$. This torus is the maximal torus in a semi simple group scheme $G^{((\mathrm{sc}))}/\mathbb{Z}$ which admits an isogeny

$$\psi_1 : G^{(\mathrm{sc})} \to G \tag{1.35}$$

whose kernel is $\ker(\psi_1) \subset T^{(\mathrm{sc})}$. The group $G^{(\mathrm{sc})}/\mathbb{Z}$ is the *simply connected cover* of $G/\operatorname{Spec}(\mathbb{Z})$.

The *dominant weights* are the weights $\lambda = \sum n <_i \gamma_i$ with all $n_i \geq 0$. They parametrise the irreducible representation of G. We come to this in chapter 6.

The abstract group $G^{(\mathrm{sc})}(k)$

We want to show that the abstract group $G^{(\mathrm{sc})}(k)$ is generated by the groups $U_\alpha(k)$.

For any root α we can consider the two root subgroups $U_\alpha, U_{-\alpha}$. It is easy to see -at least in our examples above - that these root subgroups generate a subgroup $H_\alpha \subset G$, this is the smallest subgroup which contains $U_\alpha, U_{-\alpha}$. This subgroup is either PSl_2 or Sl_2. Then $T^{(\alpha)} = H_\alpha \cap T$ is a maximal torus in H_α.

If our group $G = G^{((\mathrm{sc})}$ is simply connected then $H_\alpha = \mathrm{Sl}_2$ and we define define the coroot $\alpha^\vee \in X_*(T^{(\mathrm{sc})})$ by $\alpha^\vee : \mathbb{G}_m \to T^{(\alpha)}$ and $< \alpha^\vee, \alpha >= 2$. We have the relation $< \alpha^\vee, \gamma_j >= \delta_{i,j}$ and this implies that the $\alpha_i^\vee$ form a basis of $X_*(T^{(\mathrm{sc})})$. This in turn implies that the map given by multiplication

$$m : \prod_i \alpha_i^\vee(\mathbb{G}_m) \xrightarrow{\sim} T^{(\mathrm{sc})}. \tag{1.36}$$

is an isomorphism.

Now it is easy to see that for any field k the abstract group $\mathrm{Sl}_2(k)$ is generated by the two root subgroups $U_\alpha(k), U_{-\alpha}(k)$. Combined with the observation above this implies that $T^{(\mathrm{sc})}(k)$ is contained in the subgroup which is generated by the subgroups $U_{\alpha_i}(k), U_{-\alpha_i}(k)$. Now we recall the Bruhat decomposition. The unipotent radical U_+ of B is equal to the product $U_+ = \prod_{\alpha\in\Delta^+} U_\alpha$ and the same holds for $U_- = \prod_{\alpha\in\Delta^-} U_\alpha$. The Bruhat decomposition tells us that the multiplication $m : U_- \times T^{(\mathrm{sc})} \times U_+ \to G$ provides an isomorphism of the left hand side with an Zariski-open $\mathcal{V} \subset G$, (this is the Big Cell). This means that we get a bijection

$$U_-(k) \times T^{(\mathrm{sc})}(k) \times U_+(k) \xrightarrow{\sim} \mathcal{V}(k). \tag{1.37}$$

Our previous arguments imply that $\mathcal{V}(k)$ lies in the subgroup generated by the $U_\alpha(k)$. But then it is clear that $G^{(\mathrm{sc})}(k)$ is generated by the $U_\alpha(k)$.

1.1.6 k-forms of algebraic groups

For the following concepts and results on Galois cohomology we also refer to [70] and [81].

Exercise: 1) *Consider the following two quadratic forms over $\mathbb{Q}$:*

$$f(x, y, z) = x^2 + y^2 - z^2 \ , \ f_1(x, y, z) = x^2 + y^2 - 3z^2.$$

Prove that the first form is isotropic. This means there exists a vector $(a, b, c) \in \mathbb{Q}^3 \setminus \{0\}$ *with*

$$f(a, b, c) = 0.$$

Show that the second form is anisotropic, i.e. it has no such vector.

2) Prove that the two linear algebraic group $G/\mathbb{Q} = \mathrm{SO}(f)/\mathbb{Q}$ *and* $G_1/\mathbb{Q} = \mathrm{SO}(f_1)/\mathbb{Q}$ *cannot be isomorphic. (Hint: This is not so easy since we did not define when two groups are isomorphic.)*

Here is some advice: In general we call an element $e \neq u \in G(\mathbb{Q})$ *unipotent if it is unipotent in* $\mathrm{Gl}_n(\mathbb{Q})$ *where we consider* $G/\mathbb{Q} \hookrightarrow \mathrm{Gl}_n/\mathbb{Q}$*. It turns out that this notion of unipotence does not depend on the embedding.*

Now it is possible to show that our first group $G(\mathbb{Q}) = \mathrm{SO}(f)(\mathbb{Q})$ *has unipotent elements, and* $G_1(\mathbb{Q})$ *does not. Hence these two groups cannot be isomorphic.*

3) Prove that the two algebraic groups $G \times_{\mathbb{Q}} \mathbb{R}$ *and* $G_1 \times_{\mathbb{Q}} \mathbb{R}$ *are isomorphic, and therefore the two groups* $G(\mathbb{R})$ *and* $G_1(\mathbb{R})$ *are isomorphic.*

In this example we see, that we may have two groups $G/k, G_1/k$ which are not isomorphic but which become isomorphic over some extension L/k. Then we say that the groups are *k-forms* of each other. To determine the different forms of a given group G/k is sometimes difficult one has to use the concepts of Galois cohomology. For a separable normal extension L/k we have the almost tautological description

$$G(k) = \{g \in G(L) | \sigma(g) = g \text{ for all elements in the Galois group} \quad \mathrm{Gal}(L/k)\}.$$

Now let we can consider the functor $\mathrm{Aut}(G)$: It attaches to any field extension L/k the group of automorphisms $\mathrm{Aut}(G)(L)$ of the algebraic group $G \times_k L$. We denote this action by $g \mapsto \sigma(g) = g^\sigma$. Note that this notation gives us the rule $g^{(\sigma\tau)} = (g^\tau)^\sigma$. A 1-cocycle of $\mathrm{Gal}(L/k)$ with values in $\mathrm{Aut}(G)$ is a map $c : \sigma \mapsto c_\sigma \in \mathrm{Aut}(G)(L)$ which satisfies the cocycle rule

$$c_{\sigma\tau} = c_\sigma c_\tau^\sigma \tag{1.38}$$

Now we define a new action of $\mathrm{Gal}(L/k)$ on $G(L)$: An element σ acts by

$$g \mapsto c_\sigma g^\sigma c_\sigma^{-1}$$

We define a new algebraic group G_1/k: For any extension E/k we have an action of $\mathrm{Gal}(L/k)$ on $E \otimes_k L$ and we put

$$G_1(E) = \{g \in G(E \otimes_k L) | g = c_\sigma g^\sigma c_\sigma^{-1}\} \tag{1.39}$$

For the trivial cocycle $\sigma \mapsto 1$ this gives us back the original group.

It is plausible and in fact not very difficult to show that $E \to G_1(E)$ is in fact represented by an algebraic group G_1/k. This group is clearly a k-form of G/k.

We can define an equivalence relation on the set of cocycles, we say that

$$\{\sigma \mapsto c_\sigma\} \sim \{\sigma \mapsto c'_\sigma\}$$

if and only if we can find a $a \in G(L)$ such that

$$c'_\sigma = a^{-1} c_\sigma a^\sigma \text{ for all } \sigma \in \mathrm{Gal}(L/k)$$

We define $H^1(L/k, \mathrm{Aut}(G))$ as the set of 1-cocycles modulo this equivalence relation. If we have a larger normal separable extension $L' \supset L \supset k$ then we get an inclusion $H^1(L/k, \mathrm{Aut}(G)) \hookrightarrow H^1(L'/k, \mathrm{Aut}(G))$. If $\bar{k}_s$ is a separable closure of k we can form the limit over all finite extensions $k \subset L \subset \bar{k}_s$ and put

$$H^1(\bar{k}_s/k, \mathrm{Aut}(G)) = \lim_{\rightarrow} H^1(L/k,\ \mathrm{Aut}(G))$$

This set is isomorphic to the set of isomorphism classes of k-forms of G/k.

If L/k is a cyclic extension and if $\sigma \in\ \mathrm{Gal}(/k)$ is a generator, then a cocycle $c:\ \mathrm{Gal}(L/k) \to \mathrm{Aut}(G)(L)$ is determined by its value $g = c(\sigma) \in \mathrm{Aut}(G)(L)$. But we have to satisfy the cocycle relation. We have a useful little

Lemma 1.1.1. *The assigment $\sigma \mapsto c(\sigma) = g$ provides a 1-cocycle if and only iff*

$$\mathrm{Norm}(g) = gg^{\sigma} \dots g^{\sigma^{n-1}} = Id$$

and

$$H^1(\ Gal(L/k, Aut(G)(L)) = \{g \in Aut(G)(L) |\ \mathrm{Norm}(g) = Id\}/hgh^{-\sigma} \sim g\}.$$

Proof. Straightforward calculation □

We may apply the same concepts in a slightly different situation. A $k-$ algebra $\mathcal{D}$ over the field k is called a *central simple algebra*, if it has a unit element $\neq 0$, if it is finite dimensional over k, if its centre is k (embedded via the unit element) and if it has no non trivial two sided ideals. It is a classical theorem, that such an algebra over a separably closed field k_s is isomorphic to a full matrix algebra $M_n(k_s)$. Hence we can say that over an arbitrary field k any central simple algebra of dimension n^2 is a k-forms of $M_n(k)$.

For any algebraic group G/k we may consider the adjoint group $\mathrm{Ad}(G)$, this is the quotient of G/k by its center. It can be shown, that this is again an algebraic group over k. It is clear that we have an embedding

$$\mathrm{Ad}(G) \to\ \mathrm{Aut}(G)$$

which for any $g \in \mathrm{Ad}(G)(L)$ is given by

$$g \mapsto \{x \mapsto g^{-1}xg\}.$$

A k-form G_1/k of a group G/k is called an *inner k-form*, if it is in the image of

$$H^1(\bar{k}_s/k, \mathrm{Ad}(G)) \to H^1(\bar{k}_s/k, \mathrm{Aut}(G)).$$

We call a semi simple group G/k *anisotropic* , if it does not contain a non trivial split torus (See exercise in (1.1.6)) In our example below the group of elements of norm 1 is always semi simple and anisotropic if and only if $D(a, b)$ is a field.

I want to give an example, we consider the algebraic group $\mathrm{Gl}_2/\mathbb{Q}$ we consider two integers $a, b \neq 0$, for simplicity we assume that b is not a square. Then we have the quadratic extension $L = \mathbb{Q}(\sqrt{b})$, let σ be its non trivial automorphism. The element $\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$ defines the inner automorphism

$$\mathrm{Ad}(\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}) : g \mapsto \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix} g \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}^{-1}$$

of the group Gl_2, Then $\sigma \mapsto \mathrm{Ad}(\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix})$ and $\mathrm{Id}_{\mathrm{Gal}(L/k)} \mapsto \mathrm{Id}_{\mathrm{Aut}(\mathrm{Gl}_2)(L)}$ is a 1-cocycle and we get a $\mathbb{Q}$ form of our group.

Hence we get a $\mathbb{Q}$ form $G_1 = G(a,b)/\mathbb{Q}$ of our group Gl_2. It is an inner form.

Now we can see easily that group of rational points of our above group $G(a,b)(\mathbb{Q})$ is the multiplicative group of a central simple algebra $D(a,b)/\mathbb{Q}$. To get this algebra we consider the algebra $M_2(L)$ of (2,2)-matrices over L. We define

$$D(a,b) = \{x \in M_2(L) | x = \mathrm{Ad}(\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}) x^\sigma \mathrm{Ad}(\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix})^{-1}\}. \tag{1.40}$$

We have an embedding of the field L into this algebra, which is given by

$$u \mapsto \begin{pmatrix} u & 0 \\ 0 & u^\sigma \end{pmatrix}$$

Let u_b the image of $\sqrt{b}$ under this map. We also have the element $u_a = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$ in this algebra.

Now I leave it as an exercise to the reader that as a $\mathbb{Q}$ vector space

$$D(a,b) = \mathbb{Q} \oplus \mathbb{Q}u_b \oplus \mathbb{Q}u_a \oplus \mathbb{Q}u_a u_b$$

We have the relation $u_a^2 = a, u_b^2 = b, u_a u_b = -u_b u_a$.

Of course we should ask ourselves: When is $D(a,b)$ split, this means isomorphic to $M_2(\mathbb{Q})$? To answer this question we consider the norm homomorphism, which is defined by

$$x+yu_b+zu_a+wa_a u_b \mapsto (x+yu_b+zu_a+wa_a u_b)(x-yu_b-zu_a-wa_a u_b) = x^2-y^2b-z^2a+w^2ab.$$

It is easy to see that $D(a,b)$ splits if and only if we can find a non zero element whose norm is zero.

If we do this over $\mathbb{R}$ as base field and if we take $a=-1, b=-1$ then we get the Hamiltonian quaternions, which is non split.

We may also look at the p-adic completions $\mathbb{Q}_p$ of our field. Then it is not difficult to see that $D(a,b)$ splits over $\mathbb{Q}_p$ if $p \neq 2$ and $p \nmid ab$. Hence it is clear that there is only a finite number of primes p for which $D(a,b)$ does not split.

If we consider $\mathbb{R}$ as completion at the infinite place, and the $\mathbb{Q}_p$ as the completions at the finite places, then we have

The algebra $D(a,b)$ splits if and only if it splits at all places. The number of places where it does not split is always even.

The first assertion is the so called Hasse-Minkowski principle, the second assertion is essentially equivalent to the quadratic reciprocity law.

Construction of division algebras and anisotropic groups

We give some indication how to construct anisotropic groups over $\mathbb{Q}$ (or even overn any number field). We choose a cyclic extension $L/\mathbb{Q}$ of degree n and we pick a number $a \in \mathbb{Q}^\times$, let $A(a) \in \mathrm{Gl}_n(\mathbb{Q})$ be the following matrix

$$A(a) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \vdots & \ddots & \\ 0 & 0 & \dots & 0 & 1 \\ a & 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.41}$$

Let $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ be a generator then $\sigma^\nu \mapsto A(a)^\nu \mod \mathbb{G}_m$ is a homomorphism from $\mathrm{Gal}(L/\mathbb{Q})$ to $\mathrm{PGl}_n(\mathbb{Q})$ and since $A(a) \in \mathrm{Gl}_n(\mathbb{Q})$ this is also a 1-cocycle $c:\ \mathrm{Gal}(L/K) \to \mathrm{PGl}_n(\mathbb{Q}) := \{\sigma^\nu \mapsto A(a)^\nu\}$. It defines a cohomology class $[A(a)] \in H^1(L/\mathbb{Q}, \mathrm{Ad}(\mathrm{Gl}_n)$ and hence an inner $\mathbb{Q}$-form $G/\mathbb{Q}$ of $\mathrm{Gl}_n/\mathbb{Q}$. In Galois cohomology we have the boundary map

$$\delta : H^1(L/\mathbb{Q}, \mathrm{Ad}(\mathrm{Gl}_n) \to H^2(L/\mathbb{Q}, \mathbb{G}_m) = \mathbb{Q}^\times/N_{L/\mathbb{Q}}(L^\times)$$

and it is clear that

$$\delta([A(a)]) = a \in \mathbb{Q}^\times/N_{L/\mathbb{Q}}(L^\times)$$

Now it is well known that the $\mathbb{Q}$ -form $G/\mathbb{Q}$ of $\mathrm{Gl}_n/\mathbb{Q}$ is anisotropic if and only if the class $a \in \mathbb{Q}^\times/N_{L/\mathbb{Q}}(L^\times)$ is an element of order n. We know from algebraic number theory that there are infinitely many primes p which are inert, i.e. p is unramified in L and the prime ideal (p) stays prime in the ring of integers $\mathcal{O}_L$. Then it easy to see that the order of $p \in \mathbb{Q}^\times/N_{L/\mathbb{Q}}(L^\times)$ is n. Hence we see that the set of isomorphism classes of anisotropic $\mathbb{Q}$ forms of $\mathrm{Gl}_n/\mathbb{Q}$ is abundant.

Obviously the group $M_n(\mathbb{Q})^\times = \mathrm{Gl}_n((\mathbb{Q})$ and we also know that any automorphism of $M_n((\mathbb{Q})^\times$ is inner, hence $\mathrm{Aut}(M_n(\mathbb{Q})) = \mathrm{PGl}_n(\mathbb{Q})$ Therefore the isomorphism classes of ($\mathbb{Q}$-forms of $M_n(\mathbb{Q})$ are equal to the set $H^1(\mathbb{Q}, \mathrm{PGl}_n)$. Such a $\mathbb{Q}$-form $\mathcal{D}/\mathbb{Q}$ is a central simple algebra over $\mathbb{Q}$. The central simple algebra $\mathcal{D}$ defined by the class $[A(a)]$ can be described explicitly:

It contains the field $L/\mathbb{Q}$ as a maximal commutative subalgebra and it is generated by L and another element $a_\sigma \in \mathcal{D}$ which satisfies the following relations

$$\forall x \in L \text{ we have } a_\sigma x \alpha_\sigma^{-1} = \sigma(x)\ ;\ a_\sigma^n = a$$

If we modify a_σ and put $a'_\sigma = a_\sigma y$ with $y \in L^\times$ then the first relation still holds and the second relation becomes $(a'_\sigma)^n = aN_{L/\mathbb{Q}}(y)$. Hence the isomorphism class of $\mathcal{D}$ is determined by the class $a \in \mathbb{Q}^\times/N_{L/\mathbb{Q}}(L^\times)$. It is easy to see that for $a = 1$ the central simple algebra is equal to the endomorphism ring of the $\mathbb{Q}$ vector space $L/\mathbb{Q}$. (This is the linear independence of the elements σ^ν in $\mathrm{End}(L/\mathbb{Q})$.)

1.1.7 Quasisplit $\mathbb{Q}$-forms

We recall that a semi-simple group $G/\mathbb{Q}$ is quasisplit, if contains a Borel subgroup $B/\mathbb{Q}$. This Borel subgroup contains its unipotent radical $U/\mathbb{Q}$ and a maximal torus $T/\mathbb{Q}$. Two such maximal tori $T/\mathbb{Q}, T_1/\mathbb{Q}$ are conjugate by an element $u \in U(\mathbb{Q})$. Let $G_0/\mathbb{Q}$ by a split group which is a $\mathbb{Q}$-form of $G/\mathbb{Q}$. We pick a maximal split torus $T_0/\mathbb{Q}$ and a Borel $B_0/\mathbb{Q} \supset T_0/\mathbb{Q}$. Then we see that the triple $(G, B, T)/\mathbb{Q}$ is a $\mathbb{Q}$-form of $(G_0, B_0, T_0)/\mathbb{Q}$. Hence it can by constructed from a 1-cocycle representing a cohomology class $\xi \in H^1(\mathbb{Q}, \mathrm{Aut}(((G_0, B_0, T_0)))$, where of course $\mathrm{Aut}(((G_0, B_0, T_0))$ is the subgroup of $\mathrm{Aut}(G_0)$ which fixes T_0, B_0. Obviously we have an exact sequence

$$1 \to T_0^{(\mathrm{ad})} \to \mathrm{Aut}(((G_0, B_0, T_0)) \to \mathrm{Autext}((G_0, B_0, T_0)) \to 1, \tag{1.42}$$

here $\mathrm{Autext}((G_0, B_0, T_0))$ is the very "small" group of automorphisms of the Dynkin diagram Φ. This is also the subgroup of $\mathrm{Aut}(X^*(T_0))$ which leaves the set Δ^+ of positive roots invariant. We could say $Autext((G_0, B_0, T_0)) = \mathrm{Aut}(X^*(T_0), \Delta^+))$

It is well known- and easy to see in the examples of classical groups- that this sequence has a section $s_0 : \mathrm{Autext}((G_0, B_0, T_0)) \to \mathrm{Aut}((G_0, B_0, T_0))$ and this gives us a map in Galois cohomology

$$\begin{aligned} s_0^\bullet : H^1(\mathbb{Q}, \mathrm{Autext}((G_0, B_0, T_0)) = \ & \mathrm{Hom}(\ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathrm{Autext}((G_0, B_0, T_0)) \\ & \to H^1(\mathbb{Q}, \mathrm{Autext}((G_0)) \end{aligned} \tag{1.43}$$

Hence we see that the isomorphism classes of quasisplit $\mathbb{Q}$ -forms of $G_0/\mathbb{Q}$ are given homomorphisms ψ : $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Autext}((G_0)$. The maximal torus $T/\mathbb{Q} \subset B/\mathbb{Q}$ is not split (unless $G/\mathbb{Q}$ is split. Hence there is a finite normal extension $F_0/\mathbb{Q}$ such that $T \times_{\mathbb{Q}} F_0$ splits, we assume that $F_0/\mathbb{Q}$ is minimal. i.e. $\mathrm{Gal}(F_0\mathbb{Q}) \subset \mathrm{Aut}(X^*(T \times_{\mathbb{Q}} F_0), \Delta^+)$. We see that a quasisplit form of $G_0/\mathbb{Q}$ is given by a finite normal extension $F_0/\mathbb{Q}$ and a injection $\psi :\ \mathrm{Gal}(F_0/\mathbb{Q}) \hookrightarrow \mathrm{Aut}(X^*(T_0), \Delta^+)$.

In the special case $G_0/\mathbb{Q} = \mathrm{Sl}_n/\mathbb{Q}$ with $T_0/\mathbb{Q}, B_0/\mathbb{Q}$ being the standard diagonal torus and the standard Borel subgroup of upper triangular matrices this looks as follows: We have the element

$$w_0 = \begin{pmatrix} 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 1 & 0 \\ & & \vdots & \ddots & \\ 0 & 1 & \dots & 0 & 0 \\ \pm 1 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{Sl}_n(\mathbb{Q}) \tag{1.44}$$

this element w_0 conjugates B_0 into its opposite B_0^- the group of lower triangular matrices. The standard Cartan involution $\Theta : g \to^t g^{-1}$ does the same and therefore the composition $\mathrm{Ad}(w_0) \circ \Theta$ is an automorphism of $G_0/\mathbb{Q}$ which fixes B_0, T_0. It is an outer automorphism if $n \geq 3$ and gives us the non trivial element of $\mathrm{Autext}(G_0)$. Hence we get a 1-cocycle if choose a quadratic extension $L/\mathbb{Q}$ and send the non trivial element in $\mathrm{Gal}(L/\mathbb{Q})$ to $\mathrm{Ad}(w_0) \circ \Theta$.

We leave it an exercise to the reader to show the$\mathbb{Q}$ form obtained from this cocycle (cohomology class) is isomorphic to the above group $\mathrm{SU}(h_1)/\mathbb{Q}$.

An important class of quasi split groups is given by the groups $G/\mathbb{Q} = R_{F_0/\mathbb{Q}}(G_0)$ where $F_0/\mathbb{Q}$ is a finite extension of $\mathbb{Q}$ and G_0/F_0 is a split group. If $B_0/F_0 \subset G_0$ is a Borel subgroup then $B = R_{F_0/Q}(B_0)$ is a Borel subgroup in $G/\mathbb{Q}$. Let $F \supset F_0$ be a normal closure of F_0 then

$$G \times_{\mathbb{Q}} F = \prod_{\iota : F_0 \to F} G_0 \times_{F_0,\iota} F \tag{1.45}$$

where ι runs over the set Σ of maps from F_0 to F. The Galois group acts on the product via the action on Σ.

1.1.8 Structure of semisimple groups over $\mathbb{R}$ and the symmetric spaces

We need some information concerning the structure of the group $G_\infty = G(\mathbb{R})$ for semisimple groups over $G/\mathbb{R}$. We will provide this information simply by discussing a series of examples.

Of course the group $G(\mathbb{R})$ is a topological group, actually it is even a Lie group. This means it has a natural structure of a $\mathcal{C}_\infty$ -manifold with respect to this structure. Instead of $G(\mathbb{R})$ we will very often write G_∞. Let G_∞^0 be the connected component of the identity in G_∞. It is an open subgroup of finite index. We will discuss the

Theorem of E. Cartan: *The group G_∞^0 always contains a maximal compact subgroup $K_\infty \subset G_\infty^0$ and all maximal compact subgroups are conjugate under G_∞^0. The quotient space $X = G_\infty^0/K_\infty$ is again a $\mathcal{C}_\infty$-manifold. It is diffeomorphic to an $\mathbb{R}^N$ and carries a Riemannian metric which is invariant under the operation of G_∞^0 from the left. It has sectional curvature ≤ 0 and therefore any two points can be joined by a unique geodesic. The maximal compact subgroup $K \subset G_\infty^0$ is connected and equal to its own normalizer. Therefore the space X can be viewed as the space maximal compact subgroups in G_∞^0. See for instance [51].*

For any maximal compact subgroup $K_x \subset G_\infty$ exists an unique automorphism Θ_x with $\Theta_x^2 = e$ such that $K_x = \{g \in G_\infty^0 \, | \Theta(g) = g\}$, this is the Cartan involution corresponding to K_x. The Cartan involutions are in one-to one correspondence with the maximal compact subgroups.

A Cartan involution Θ_x induces an involution also called Θ_x on the Lie algebra $\mathfrak{g}_\mathbb{R}$ of G_∞ and we get a decomposition into $\pm$ eigenspaces

$$\mathfrak{g} = \mathfrak{k}_x \oplus \mathfrak{p}_x;\ \mathfrak{k}_x = \{U \in \mathfrak{g} | \Theta_x(U) = U\}\ ;\mathfrak{p}_x = \{V \in \mathfrak{g} | \Theta_x(V) = -V\}$$

where of course $\mathfrak{k}_x$ is the Lie algebra of K_x. The differential of the action of G_∞ on $G(\mathbb{R})/K_x$ provides an isomorphism $D_x : \mathfrak{p}_x \xrightarrow{\sim} T_x^X$ (then tangent space at x). For $V_1, V_2 \in \mathfrak{p}_x$ we have $[V_1, V_2] \in \mathfrak{k}_x$ the map $R : \mathfrak{p}_x \times \mathfrak{p}_x \to \mathfrak{k}_x$ is the *curvature tensor. The $\mathbb{R}$-vector space $\mathfrak{g}_c := \mathfrak{k}_x + \sqrt{-1}\mathfrak{p}_x \subset \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is a Lie algebra, for $U_1 + \sqrt{-1}V_1, U_2 + \sqrt{-1}V_2 \in \mathfrak{g}_c$ we get for the Lie-bracket*

$$[U_1 + \sqrt{-1}V_1, U_2 + \sqrt{-1}V_2] = [U_1, U_2] - [V_1, V_2] + \sqrt{-1}([U_1, V_2] + [U_2, V_1]) \in \mathfrak{g}_c$$

To this Lie algebra $\mathfrak{g}_c$ corresponds an algebraic group $G_c/\mathbb{R}$ which is a $\mathbb{R}$-form of $G/\mathbb{R}$, the group $G_c(\mathbb{R})$ is compact. The group $G_c/\mathbb{R}$ is called the compact dual of $G/\mathbb{R}$. On $G_c/\mathbb{R}$ we have only one Cartan involution $\Theta = Id$.

This theorem is fundamental. To illustrate this theorem we consider a series of examples:

The groups $\mathbf{Sl}_n(\mathbb{R})$ and $\mathrm{Gl}_n(\mathbb{R})$:

The group $\mathrm{Sl}_d(\mathbb{R})$ is connected. If $K \subset \mathrm{Sl}_d(\mathbb{R})$ is a closed compact subgroup, I claim that we can find a positive definite quadratic form $f : \mathbb{R}^n \to \mathbb{R}$, such that $K \subset SO(f, \mathbb{R})$ Snce the group $SO(f, \mathbb{R})$ itself is compact, it is maximal compact. Two such forms f_1, f_2 define the same maximal compact K_∞ subgroup if there is a $\lambda > 0$ in $\mathbb{R}$ such that $\lambda f_1 = f_2$. We say that f_1 and f_2 are *conformally equivalent.*

This is rather clear, if we believe the first assertion about the existence of f. The existence of f is also easy to see if one believes in the theory of integration on K. This theory provides a positive invariant integral

$$\begin{array}{ccc} \mathcal{C}_c(K) & \longrightarrow & \mathbb{R} \\ \varphi & \longrightarrow & \displaystyle\int_K \varphi(k)dk \end{array}$$

with $\int \varphi > 0$ if $\varphi \geq 0$ and not identically zero (positivity), $\int \varphi(kk_0)dk = \int \varphi(k_0k)dk = \int \varphi(k)dk$ (invariance). To get our form f we start from any positive definite form f_0 on $\mathbb{R}^n$ and put

$$f(\underline{x}) = \int_K f_0(k\underline{x})dk.$$

A positive definite quadratic form on $\mathbb{R}^n$ is the same as a symmetric positive definite bilinear form. Hence the space of positive definite forms is the same as the space of positive definite symmetric matrices

$$\tilde{X} = \left\{A = (a_{ij}) \mid A =^t A, A > 0\right\}.$$

Hence we can say that the space of maximal compact subgroups in $\mathrm{Sl}_n(\mathbb{R})$ is given by

$$X = \tilde{X}/\mathbb{R}^*_{>0}.$$

It is easy to see that a maximal compact subgroup $K_\infty \subset \mathrm{Sl}_d(\mathbb{R})$ is equal to its own normalizer (why?). If we view X as the space of positive definite symmetric matrices with determinant equal to one, then the action of $\mathrm{Sl}_d(\mathbb{R})$ on $X = \mathrm{Sl}_d(\mathbb{R})/K$ is given by

$$(g, A) \longrightarrow g\ A\ {}^tg,$$

and if we view it as the space of maximal compact subgroups, then the action is conjugation.

There is still another interpretation of the points $x \in X$. In our above interpretation a point was a symmetric, positive definite bilinear form $< \, , \, >_x$ on $\mathbb{R}^n$ up to a homothety. From this we get a transposition $g \mapsto {}^{t_x}g$, which is defined by the rule $< gv, u >_x = < v, {}^{t_x}gu >_x$ and from this we get the involution

$$\Theta_x : g \mapsto ({}^{t_x}g)^{-1} \tag{1.46}$$

Then the corresponding maximal compact subgroup is

$$K_x = \{g \in \mathrm{Sl}_n(\mathbb{R}) | \Theta_x(g) = g\} \tag{1.47}$$

This involution Θ_x is a Cartan involution, it also induces an involution also called Θ_x on the Lie-algebra and it has the property that (See 1.18)

$$(u, v) \mapsto B(u, \Theta_x(v)) = B_{\Theta_x}(u, v) \tag{1.48}$$

is negative definite. This bilinear form is K_x invariant. All these Cartan involutions are conjugate.

If we work with $\mathrm{Gl}_n(\mathbb{R})$ instead then we have some freedom to define the symmetric space. In this case we have the non trivial center $\mathbb{R}^\times$ and it is sometimes useful to define

$$X = \mathrm{Gl}_n(\mathbb{R})/\mathrm{SO}(\mathbb{R}) \cdot \mathbb{R}^\times_{>0}, \tag{1.49}$$

then our symmetric space has two components, a point is pair (Θ_x, ϵ) where ϵ is an orientation. If we do not divide by $\mathbb{R}^\times_{>0}$ then we multiply the Riemannian manifold X by a flat space and we get the above space $\tilde{X}$.

A Cartan involution on $\mathrm{Gl}_n(\mathbb{R})$ is an involution which induces a Cartan involution on $\mathrm{Sl}_n(\mathbb{R})$ and which is trivial on the center.

Proposition 1.1.1. *The Cartan involutions on* $\mathrm{Gl}_n(\mathbb{R})$ *are in one to one correspondence to the euclidian metrics on* $\mathbb{R}^n$ *up to conformal equivalence.*

Finally we recall the Iwasawa decomposition. Inside $\mathrm{Gl}_n(\mathbb{R})$ we have the standard Borel- subgroup $B(\mathbb{R})$ of upper triangular matrices and it is well known that

$$\mathrm{Gl}_n(\mathbb{R}) = B(\mathbb{R}) \cdot \mathrm{SO}(\mathbb{R}) \cdot \mathbb{R}^\times_{>0} \tag{1.50}$$

and hence we see that $B(\mathbb{R})$ acts transitively on X.

The compact dual of $\mathrm{Sl}_n(\mathbb{R})$

If $G/\mathbb{R}$ is a semi simple group, then $G_c/\mathbb{R}$ is a $\mathbb{R}$-form of $G/\mathbb{R}$. Therefore we find a cohomology class $\xi_c \in H^1\mathbb{C}/(\mathbb{R}, \mathrm{Aut}(G))$ corresponding to G_c. It is clear from the Theorem of Cartan how we get a cocycle representing this class: We choose a Cartan involution $\Theta \in \mathrm{Aut}(G)$, the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is cyclic of order SO(2 let $\mathbf{c}$ be the generator (the complex conjugation). Then $\mathbf{c} \mapsto \mathbf{c} \circ \Theta$ yields a 1-cocycle in $C^1(\,\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{Aut}(G)(\mathbb{C}))$. (Lemma 1.1.1) and this 1-cocycle represents the class ξ_c.

This means for the group $\mathrm{Sl}_n/\mathbb{R}$ that

$$G_c(\mathbb{R}) = \{g \in \mathrm{Sl}_n(\mathbb{C}) | \mathbf{c}({}^t g^{-1}) = g\}$$

and if we go back to the usual notion and write $\mathbf{c}(g) = \bar{g}$ then we get

$$G_c(\mathbb{R}) = \{g \in \mathrm{Sl}_n(\mathbb{C}) | {}^t\bar{g}g = \mathrm{Id}\} = \mathrm{SU}(n)$$

Here of course $\mathrm{SU}(n) = \mathrm{SU}(h_c)$ where $h_c(z_1, z_2, \ldots, z_n) = \sum_{i=1}^n z_i \bar{z}_i$ is the standard positive definite hermitian form on $\mathbb{C}^n$.

We know that for $G/\mathbb{R} = \mathrm{Sl}_n/\mathbb{R}$ and $n > 2$ the Cartan involution Θ is the generator of $\mathrm{Aut}(G)/\mathrm{Ad}(G)$ and hence it is clear that ξ_c is not in the image of $H^1(\mathbb{C}/\mathbb{R}, \mathrm{Ad}(G)) \to H^1(\mathbb{C}/(\mathbb{R}, \mathrm{Aut}(G))$. This means that in this case $G_c/\mathbb{R} = \mathrm{SU}(n)/\mathbb{R}$ is not an inner $\mathbb{R}$ -form of $\mathrm{Sl}_n/\mathbb{R}$, in turn this also means that $\mathrm{Sl}_n/\mathbb{R}$ is not an inner $\mathbb{R}$ -form of $\mathrm{SU}(n)/\mathbb{R}$.

In this context the following general proposition is of importance

Proposition 1.1.2. *A semi simple group scheme $G/\mathbb{R}$ is an inner $\mathbb{R}$ form of its compact dual $G_c/\mathbb{R}$ if an only if*
a) The Cartan involution Θ of $G/\mathbb{R}$ is an inner automorphism of $G/\mathbb{R}$.
b) The group $G/\mathbb{R}$ has a compact maximal torus $T_c/\mathbb{R} \subset G/\mathbb{R}$.

Give a name to this class of groups ? Examples?

The Arakelow- Chevalley scheme $(\mathrm{Gl}_n/\mathbb{Z}, \Theta_0)$

We start from the free lattice $L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots \oplus \mathbb{Z}e_n$ and we think of $\mathrm{Gl}_n/\mathbb{Z}$ as the scheme of automorphism of this lattice. If we choose an euclidian metric $< \, , \, >$ on $L \otimes \mathbb{R}$, then we call the pair $(L, < \, , \, >)$ an *Arakelow vector bundle.* From the (conformal class of) metric we get a Cartan involution Θ. on $\mathrm{Gl}_n(\mathbb{R})$, and the pair $(\mathrm{Gl}_n/\mathbb{Z}, \Theta)$ is an *Arakelow group scheme*

We may choose the standard euclidian metric with respect to the given basis, i.e. $< e_i, e_j >= \delta_{i,j}$. theresulting Cartan involution is the standard one: $\Theta_0 : g \mapsto ({}^t g)^{-1}$. This pair $(\mathrm{Gl}_n/\mathbb{Z}, \Theta_0)$ is called an Arakelow- Chevalley scheme. (In a certain sense the integral structure of $\mathrm{Gl}_n/\mathbb{Z}$ and the choice of the Cartan involution are "optimally adapted")

In this case we find for our basis elements in (1.15)

$$B_{\Theta_0}(H_i, H_j) = -2n\delta_{i,j} + 2; B_{\Theta_0}(E_{i,j}, E_{k,l}) = -2n\delta_{i,k}\delta_{j,l} \tag{1.51}$$

hence the $E_{i,j}$ are part of an orthonormal basis.

We propose to call a pair $(L, < \, , \, >_x)$ an Arakelow vector bundle over $\mathrm{Spec}(\mathbb{Z}) \cup \{\infty\}$ and $(\mathrm{Gl}_n, \Theta_x)$ an Arakelow group scheme. The Arakelow vector bundles modulo conformal equivalence are in one-to one correspondence with the Arakelow group schemes of type Gl_n.

The group $\mathbf{Sl}_d(\mathbb{C})$

We now consider the group $G/\mathbb{R}$ whose group of real points is $G(\mathbb{R}) = \mathrm{Sl}_d(\mathbb{C})$ (see 1.1 example 4)).

A completely analogous argument as before shows that the maximal compact subgroups are in one to one correspondence to the positive definite hermitian forms on $\mathbb{C}^n$ (up to multiplication by a scalar). Hence we can identify the space of maximal compact subgroups to the space of positive definite hermitian matrices

$$X = \left\{ A \mid A = {}^t\overline{A}\ ,\ A > 0\ ,\ \det A = 1 \right\}.$$

The action of $\mathrm{Sl}_d(\mathbb{C})$ by conjugation on the maximal compact subgroups becomes

$$A \longrightarrow g\, A\, {}^t\overline{g}$$

on the space of matrices.

The orthogonal group:

The next example we want to discuss is the orthogonal group of a non degenerate quadratic form

$$f(x_1, \ldots, x_n) = x_1^2 + \ldots + x_m^2 - x_{m+1}^2 - \ldots - x_n^2,$$

since at this moment we consider only groups over the real numbers, we may assume that our form is of this type. In this case one has the usual notation

$$\mathrm{SO}(f, \mathbb{R}) = \mathrm{SO}(m, n-m).$$

Of course we can use the same argument as before and see that for any maximal compact subgroup $K \subset \mathrm{SO}(f, \mathbb{R})$ we may find a positive definite form ψ

$$\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$$

such that $K = \mathrm{SO}(f, \mathbb{R}) \cap \mathrm{SO}(\psi, \mathbb{R})$. But now we cannot take all forms ψ, i.e. only special forms ψ provide maximal compact subgroup.

We leave it to the reader to verify that any compact subgroup K fixes an orthogonal decomposition $\mathbb{R}^n = V_+ \oplus V_-$ where our original form f is positive definite on V_+ and negative definite on V_-. Then we can take a ψ which is equal to f on V_+ and equal to $-f$ on V_-.

Exercise 3 *a) Let $V/\mathbb{R}$ be a finite dimensional vector space and let f be a symmetric non degenerate form on V. Let $K \subset \mathrm{SO}(f)$ be a compact subgroup. If f is not definite then the action of K on V is not irreducible.*

b) We can find a K invariant decomposition $V = V_- \oplus V_+$ such that f is negative definite on V_- and positive definite on V_+

In this case the structure of the quotient space $G(\mathbb{R})/K$ is not so easy to understand. We consider the special case of the form

$$x_1^2 + \ldots + x_n^2 - x_{n+1}^2 = f(x_1, \ldots, x_{n+1}).$$

We consider in $\mathbb{R}^{n+1}$ the open subset

$$X_- = \{ v = (x_1 \ldots x_{n+1}) \mid f(v) < 0 \}.$$

It is clear that this set has two connected components, one of them is

$$X_-^+ = \{ v \in X_- \mid x_{n+1} > 0 \}$$

Since it is known that $\mathrm{SO}(n,1)$ acts transitively on the vectors of a given length, we find that $\mathrm{SO}(n,1)$ cannot be connected. Let $G^0_\infty \subset \mathrm{SO}(n,1)$ be the subgroup leaving X_-^+ invariant.

Now it is not to difficult to show that for any maximal compact subgroup $K_\infty \subset G^0_\infty$ we can find a ray $\mathbb{R}^*_{>0} \cdot v \subset X_-^{(+)}$ which is fixed by K_∞.

(Start from $v_0 \in X_-^{(+)}$ and show that $R^*_{>0} K_\infty v_0$ is a closed convex cone in $X_-^{(+)}$. It is K_∞ invariant and has a ray which has a "centre of gravity" and this is fixed under K_∞.)

For a vector $v = (x_1, \dots, x_{n+1}) \in X_-^{(+)}$ we may normalise the coordinate x_{n+1} to be equal to one; then the rays $\mathbb{R}^+_{>0} v$ are in one to one correspondence with the points of the ball $\boxed{\text{disc}}$

$$\overset{\circ}{D}_n = \left\{ (x_1, \dots, x_n) \mid x_1^2 + \dots + x_n^2 < 1 \right\} \subset X_-^{(+)}. \tag{1.52}$$

This tells us that we can identify the set of maximal compact subgroups $K_\infty \subset G^0_\infty$ with the points of this ball. The first conclusion is that $G^0_\infty / K_\infty \simeq D^n$ is topologically a cell (diffeomorphic to $\mathbb{R}^n$). Secondly we see that for a $v \in X_-^+$ we have an orthogonal decompositon with respect to f

$$\mathbb{R}^{n+1} = \langle v \rangle + \langle v \rangle^\perp,$$

and the corresponding maximal compact subgroup is the orthogonal group on $\langle v \rangle^\perp$.

The space X_-^+ is often called the n-dimensional hyperbolic space $\mathbb{H}_n$.

Give Cartan Involutions?

1.1.9 Special low dimensional cases

1) We consider the (semi-simple) group $\mathrm{Sl}_2(\mathbb{R})$. It acts on the upper half plane

$$\mathbb{H} = \{ z \mid z \in \mathbb{C}, \Im(z) > 0 \}$$

by

$$(g, z) \longrightarrow \frac{az+b}{cz+b}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}_2(\mathbb{R}).$$

It is clear that the stabiliser of the point $i \in \mathbb{H}$ is the standard maximal compact subgroup

$$K_\infty = \mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \right\}.$$

Hence we have $\mathbb{H} = \mathrm{Sl}_2(\mathbb{R})/K_\infty$. But this quotient has also been realized as the space of symmetric positive definite 2×2-matrices with determinant equal to one

$$z = \left\{ \begin{pmatrix} y_1 & x_1 \\ x_1 & y_2 \end{pmatrix} \mid y_1 y_2 - x_1^2 = 1, y_1 > 0 \right\}.$$

It is clear how to find an isomorphism between these two explicit realizations. The map

$$\begin{pmatrix} y_1 & x_1 \\ x_1 & y_2 \end{pmatrix} \longrightarrow \frac{i + x_1}{y_2}$$

is compatible with the action of $\mathrm{Sl}_2(\mathbb{R})$ on both sides and sends the identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to the point i.

If we start from a point $z \in \mathbb{H}$ the corresponding metric is as follows: We identify the lattices $\langle 1, z\rangle = \{a + bz \mid a, b \in \mathbb{Z}\} = \Omega$ to the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ by sending $1 \to \binom{1}{0}$ and $z \to \binom{0}{1}$. The standard euclidian metric on $\mathbb{C} = \mathbb{R}^2$ induces a metric on $\Omega \subset \mathbb{C}$, and this metric is transported to $\mathbb{R}^2$ by the identification $\Omega \otimes \mathbb{R} \to \mathbb{R}^2$. Hence the symmetric matrix will be $\begin{pmatrix} 1 & x \\ x & z\bar{z} \end{pmatrix}$.

We may also start from the (reductive) group $\mathrm{Gl}_2(\mathbb{R})$, it has the centre $C(\mathbb{R}) = \{\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}\}$. Let $C(\mathbb{R})^{(0)}$ be the connected component of the identity of $C(\mathbb{R})$. In this case we define $K_\infty = \mathrm{SO}(2) \times C(\mathbb{R})^{(0)}$. Then the quotient

$$\mathrm{Gl}_2(\mathbb{R})/K_\infty = \mathbb{H} \cup \mathbb{H}_-$$

where $\mathbb{H}_-$ is the lower half plane.

2) The two groups $\mathrm{Sl}_2(\mathbb{R})$ and $\mathrm{PSl}_2(\mathbb{R})^{(0)} = \mathrm{Sl}_2(\mathbb{R})/\{\pm\mathrm{Id}\}$ give rise to the same symmetric space. The group $P\mathrm{Sl}_2(\mathbb{R})$ acts on the space $M_2(\mathbb{R})$ of 2×2-matrices by conjugation (the group $\mathrm{Gl}_2(\mathbb{R})$ acts by conjugation and the centre acts trivially) and leaves invariant the space

$$\{A \in M_2(\mathbb{R}) \mid \mathrm{trace}(A) = 0\} = M_2^0(\mathbb{R}).$$

On this three-dimensional space we have a symmetric quadratic form

$$\begin{aligned} B &: M_2^0(\mathbb{R}) \longrightarrow \mathbb{R} \\ B &: A \mapsto \frac{1}{2}\,\mathrm{trace}\,(A^2) \end{aligned}$$

and with respect to the basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.53}$$

this form is $x_1^2 + x_2^2 - x_3^2$.

Hence we see that $\mathrm{SO}(M_2^0(\mathbb{R}), B) = \mathrm{SO}(2,1)$, and hence we have an isomorphism between $P\mathrm{Sl}_2(\mathbb{R})$ and the connected component of the identity $G_\infty^0 \subset \mathrm{SO}(2,1)$. Hence we see that our symmetric space $\mathbb{H} = \mathrm{Sl}_2(\mathbb{R})/K_\infty = P\mathrm{Sl}_2(\mathbb{R})/\overline{K}_\infty$ can also be realised as disc

$$D = \{(x_1, x_2) \mid x_1^2 + x_2^2 < 1\}$$

where we normalized $x_3 = 1$ on $X_-^{(+)}$ as in 1.52) .

The group $\mathbf{Sl}_2(\mathbb{C})$.

Recall that in this case the symmetric space is given by the positive definite hermitian matrices

$$A = \left\{ \begin{pmatrix} y_1 & z \\ \overline{z} & y_2 \end{pmatrix} \mid \det(A) = 1, y_1 > 0 \right\}.$$

In this case we have also a realization of the symmetric space as an upper half space. We send

$$\begin{pmatrix} y_1 & w \\ \overline{w} & y_2 \end{pmatrix} \longmapsto \left(\frac{w}{y_2}, \frac{1}{y_2} \right) = (z, \zeta) \in \mathbb{C} \times \mathbb{R}_{>0}$$

The inverse of this isomorphism is given by

$$(z,\zeta) \mapsto \begin{pmatrix} \zeta + z\overline{z}/\zeta & z/\zeta \\ \overline{z}/\zeta & 1/\zeta \end{pmatrix}$$

As explained earlier, the action of $\mathrm{Gl}_2(\mathbb{C})$ on the maximal compact subgroup given by conjugation yields the action

$$G(\mathbb{R}) \times X \longrightarrow X,$$

$$(g, A) \longrightarrow gA^t\overline{g},$$

on the hermitian matrices. Translating this into the realization as an upper half space yield the slightly scaring formula

$$\begin{aligned} G \times (\mathbb{C} \times \mathbb{R}_{>0}) &\longrightarrow \mathbb{C} \times \mathbb{R}_{>0}, \\ (g, (z,\zeta)) &\longrightarrow \left(\frac{(az+b)\,\overline{(cz+d)} + a\overline{c}\,\zeta^2}{(cz+d)\,\overline{(cz+d)} + c\overline{c}\,\zeta^2}\ ,\ \frac{\zeta}{(cz+d)\,\overline{(cz+d)} + c\overline{c}\,\zeta^2} \right) \end{aligned}$$

. Here X is the three dimensional hyperbolic space $\mathbb{H}_3$.

1.3.4. The Riemannian metric: It was already mentioned in the statement of the theorem of Cartan that we always have a G^0_∞ invariant Riemannian metric on X. It is not to difficult to construct such a metric, which in many cases is rather canonical.

In the general case we observe that the maximal compact subgroup is the stabilizer of the point $x_0 = e \cdot K_\infty \in G^0_\infty/K_\infty = X$. Hence it acts on the tangent space of x_0, and we can construct a K_∞-invariant positive definite quadratic form on this tangent sapce. Then we use the action of G^0_∞ on X to transport this metric to an arbitrary point in X: If $x \in X$ we find a g so that $x = gx_0$, it defines an isomorphism between the tangent space at x_0 and the tangent space at x. Hence we get a quadratic form on the tangent space at x, which will not depend on the choice of $g \in G^0_\infty$. In our examples this metric is always unique up to scalars.

a) In the case of the group $\mathrm{Sl}_n(\mathbb{R})$ we may take as a base point $x_0 \in X$ the identity $\mathrm{Id} \in \mathrm{Sl}_n(\mathbb{R})$. The corresponding maximal compact subgroup is the orthogonal group $\mathrm{SO}(n)$. The tangent space at Id is given by the space

$$\mathrm{Sym}^0_n(\mathbb{R}) = T^X_{\mathrm{Id}}$$

of symmetric matrices with trace zero. On this space we have the form

$$Z \longrightarrow \mathrm{trace}(Z^2),$$

which is positive definite (a symmetric matrix has real eigenvalues). It is easy to see that the orthogonal group acts on this tangent space by conjugation, hence the form is invariant.

b) A similar argument applies to the group $G_\infty = \mathrm{Sl}_d(\mathbb{C})$. Again the identity Id is a nice positive definite hermitian matrix. The tangent space consists of the hermitian matrices

$$T_{\mathrm{Id}}^X = \left\{ A \mid A =^t \overline{A} \text{ and } \mathrm{tr}(A) = 0 \right\},$$

and the invariant form is given by

$$A \longrightarrow \mathrm{tr}(A\overline{A}).$$

c) In the case of the group $G_\infty^0 \subset \mathrm{SO}(f)(\mathbb{R})$ where f is the quadratic form

$$f(x_1, \ldots, x_{n+1}) = x_1^2 + \ldots + x_n^2 - x_{n+1}^2.$$

We realized the symmetric space as the open ball

$$\overset{\circ}{D}_n = \{(x_1, \ldots, x_n) \mid x_1^2 + \ldots + x_n^2 < 1\}.$$

The orthogonal group $\mathrm{SO}(n, 1)$ is the stabilizer of $0 \in \overset{\circ}{D}_n$, and hence it is clear that the Riemannian metric has to be of the form

$$h(x_1^2 + \ldots + x_n^2)(dx_1^2 + \ldots dx_n^2)$$

(in the usual notation). A closer look shows that the metrics has to be

$$\frac{dx_1^2 + \ldots + dx_n^2}{\sqrt{1 - x_1^2 - \ldots - x_n^2}}.$$

In our two low dimensional spacial examples the metric is easy to determine. For the action of the group $\mathrm{Sl}_2(\mathbb{R})$ on the upper half plane $\mathbb{H}$ we observe that for any point $z_0 = x + iy \in \mathbb{H}$ the tangent vectors $\frac{\partial}{\partial x}|_{z_0}$, $\frac{\partial}{\partial y}|_{z_0}$ form a basis of the tangent spaces at z_0.

If we take $z_0 = i$ then the stabilizer is the group $\mathrm{SO}(2)$ and for

$$e(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.$$

We have

$$\begin{aligned} e(\varphi) \cdot \left(\frac{\partial}{\partial x} |_i \right) &= \cos 2\varphi \cdot \frac{\partial}{\partial x} |_i + \sin 2\varphi \cdot \frac{\partial}{\partial y} |_i \\ e(\varphi) \left(\frac{\partial}{\partial y} |_i \right) &= \sin 2\varphi \cdot \frac{\partial}{\partial x} |_i + \cos 2\varphi \cdot \frac{\partial}{\partial y} |_i. \end{aligned}$$

Hence we find that $\frac{\partial}{\partial x} |_i$ and $\frac{\partial}{\partial y} |_i$ have to be orthogonal and of the same length.

Now the matrix

$$\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$$

sends i into the point $z = x + iy$. It sends $\frac{\partial}{\partial x} \mid_i$ and $\frac{\partial}{\partial y} \mid_i$ into $y \cdot \frac{\partial}{\partial x} \mid_z$ and $y \cdot \frac{\partial}{\partial y} \mid_z$, and hence we must have for our invariant metric

$$\langle \frac{\partial}{\partial x} \mid_z, \frac{\partial}{\partial y} \mid_z \rangle = 0 \ ; \ \langle \frac{\partial}{\partial x} \mid_z, \frac{\partial}{\partial x} \mid_z \rangle = \frac{1}{y^2} \ ; \ \langle \frac{\partial}{\partial y} \mid_z, \frac{\partial}{\partial y} \mid_z \rangle = \frac{1}{y^2},$$

and this is in the usual notation the metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2). \tag{1.54}$$

A completely analogous argument yields the metric

$$ds^2 = \frac{1}{\zeta^2}\,(d\zeta^2 + dx^2 + dy^2) \tag{1.55}$$

for the space $\mathbb{H}_3$.

1.2 Arithmetic groups

If we have a linear algebraic group $G/\mathbb{Q} \hookrightarrow GL_n$ we may consider the group $\Gamma = G(\mathbb{Q}) \cap GL_n(\mathbb{Z})$. This is the first example of an *arithmetic* group. It has the following fundamental property:

Proposition: *The group* Γ *is a discrete subgroup of the topological group* $G(\mathbb{R})$.

This is rather easily reduced to the fact that $\mathbb{Z}$ is discrete in $\mathbb{R}$. Actually our construction provides a big family of arithmetic groups. For any integer $m > 0$ we have the homomorphism of reduction mod m, namely

$$GL_n(\mathbb{Z}) \longrightarrow GL_n(\mathbb{Z}/m\mathbb{Z}).$$

The kernel $GL_n(\mathbb{Z})(m)$ of this homomorphism has finite index in $GL_n(\mathbb{Z})$ and hence the intersection $\Gamma' = GL_n(\mathbb{Z})(m) \cap \Gamma$ has finite index in Γ.

Definition 2.1.: *A subgroup* Γ'' *of* Γ *is called a congruence subgroup, if we can find an integer* m *such that*

$$GL_n(\mathbb{Z})(m) \cap \Gamma \subset \Gamma'' \subset \Gamma.$$

At this point a remark is in order. We explained already that a linear algebraic group $G/\mathbb{Q}$ may be embedded in different ways into different groups GL_n, i.e.

$$\begin{array}{ccc} & \hookrightarrow & GL_{n_1} \\ G & & \\ & \hookrightarrow & GL_{n_2} \end{array}$$

In this case we may get two different congruence subgroups

$$\Gamma_1 = G(\mathbb{Q}) \cap GL_{n_1}(\mathbb{Z}), \Gamma_2 = G(\mathbb{Q}) \cap GL_{n_2}(\mathbb{Z}).$$

It is not hard to show that in such a case we can find an $m > 0$ such that

$$\begin{array}{l} \Gamma_1 \supset \Gamma_2 \cap GL_{n_2}(\mathbb{Z})(m) \\ \Gamma_2 \supset \Gamma_1 \cap GL_{n_1}(\mathbb{Z})(m) \end{array} \quad .$$

From this we conclude that the notion of congruence subgroup does not depend on the way we realized the group $G/\mathbb{Q}$ as a subgroup in the general linear group.

Now we may also define the notion of an *arithmetic* subgroup. A subgroup $\Gamma' \subset G(\mathbb{Q})$ is called arithmetic if for any congruence subgroup $\Gamma \subset G(\mathbb{Q})$ the group $\Gamma' \cap \Gamma$ is of finite index in Γ' and Γ. (We say that Γ' and Γ are commensurable.) By definition all congruence subgroups are arithmetic subgroups.

The most prominent example of an arithmetic group is the group

$$\Gamma = \mathrm{Sl}_2(\mathbb{Z}).$$

Another example is obtained as follows. We defined for any number field $K/\mathbb{Q}$ the group

$$G/\mathbb{Q} = R_{K/\mathbb{Q}}(\mathrm{Sl}_d)$$

for which $G(\mathbb{Q}) = \mathrm{Sl}_d(K)$. If $\mathcal{O}_K$ is the ring of integers in K, then $\Gamma = \mathrm{Sl}_d(\mathcal{O}_K)$ (and also $\tilde{\Gamma} = GL_n(\mathcal{O}_K)$) is a congruence (and hence arithmetic) subgroup of $G(\mathbb{Q})$.

It is very interesting that the groups $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$ and $\mathrm{Sl}_2(\mathcal{O}_K)$ for imaginary quadratic $K/\mathbb{Q}$ always contain arithmetic subgroups $\Gamma' \subset \Gamma$ which are not congruence subgroups. This means that in general the class of arithmetic subgroups is larger than the class of congruence subgroups. We will prove this assertion in **Non Congruence subgroups**).

If only the group $G(\mathbb{R})$ is given (as the group of real points of a group $G/\mathbb{R}$ or perhaps only as a Lie group, then the notion of arithmetic group $\Gamma \subset G(\mathbb{R})$ is not defined. The notion of an arithmetic subgroup $\Gamma \subset G(\mathbb{R})$ requires the choice of a group scheme $G/\mathbb{Q}$ such that the group $G(\mathbb{R})$ is the group of real points of this group over $\mathbb{Q}$. The exercise in 1.1.2. shows that different $\mathbb{Q}$- forms provide different arithmetic groups.

Exercise 2 *If $\gamma \in Gl_n(\mathbb{Z})$ is a nontrivial torsion element and if $\gamma \equiv \mathrm{Id} \mod m$ then $m = 1$ or $m = 2$. In the latter case the element γ is of order 2.*

This implies that for $m \geq 3$ the congruence subgroup $Gl_n(\mathbb{Z})(m)$ of $Gl_n(\mathbb{Z})$ is torsion free.

This implies of course that any arithmetic group has a subgroup of finite index, which is torsion free.

$\boxed{\text{Affgroup}}$

1.2.1 Affine group schemes over $\mathbb{Z}$

There is a slightly more sophisticated view of arithmetic groups. In our book [40] section 7.5.6 and on p. 50,51 we discuss briefly the general notion of a group scheme over an arbitrary base scheme S. An affine group scheme over $G/\mathbb{Z}$ is a finitely generated $\mathbb{Z}$-algebra $A(G)$ together with a comultiplication $m : A(G) \to A(G) \otimes A(G)$. For any $\mathbb{Z}$ -algebra B (commutative and with identity) the comultiplication m induces a multiplication on the B-valued points

$$^{t}m : \ \mathrm{Hom}_{\mathrm{alg}}(A,B) \times \ \mathrm{Hom}_{\mathrm{alg}}(A,B) \to \ \mathrm{Hom}_{\mathrm{alg}}(A,B)$$

and the requirement is that this multiplication defines a group structure on $G(B) = \ \mathrm{Hom}_{\mathrm{alg}}(A,B)$. In educated language : $G/\mathbb{Z}$ is a functor from the category of affine schemes into the category of groups.

For instance we can define the group scheme $\mathrm{Gl}_n/\mathbb{Z}$. The affine algebra is

$$A(\mathrm{Gl}_n) = \mathbb{Z}[X_{11}, X_{12}, \dots, X_{1n}, X_{21}, \dots, X_{nn}, Y]/(Y \det(X_{ij} - 1)$$

Then the group $\mathrm{Gl}_n(\mathbb{Z})$ of $\mathbb{Z}$-valued points of $\mathrm{Gl}_n/\mathbb{Z}$ is our group $Gl_n(\mathbb{Z})$.

If $G/\mathbb{Q} \subset \mathrm{Gl}_n/\mathbb{Q}$ is a subgroup, then the affine algebra $A(G) = A(\mathrm{Gl}_n)\otimes\mathbb{Q}/I$, where I is an ideal in $A(\mathrm{Gl}_n)\otimes\mathbb{Q}$. Since $G/\mathbb{Q}$ is a subgroup this ideal must satisfy

$$m_{\mathrm{Gl}_n}(I) \subset A(\mathrm{Gl}_n) \otimes \mathbb{Q} \otimes I + I \otimes A(\mathrm{Gl}_n) \otimes \mathbb{Q}.$$

Let $J = A(\mathrm{Gl}_n) \cap I$, then it is easy to check that the comultiplication of $A(\mathrm{Gl}_n)$ satisfies

$$m_{\mathrm{Gl}_n}(J) \subset A(\mathrm{Gl}_n) \otimes J + J \otimes A(\mathrm{Gl}_n)$$

and this tells us that m_{Gl_n} induces a comultiplication

$$m : A(\mathrm{Gl}_n)/J \to A(\mathrm{Gl}_n)/J \otimes A(\mathrm{Gl}_n)/J$$

which provides a group scheme structure. This means that we have extended the group scheme $G/\mathbb{Q}$ to a group scheme over $\mathcal{G}/\mathbb{Z}$. The affine algebra $A(\mathcal{G}) = A(\mathrm{Gl}_n)/J$. This extension depends on the choice of the embedding into $\mathrm{Gl}_n/\mathbb{Q}$ and it is called the *flat extension.* Then the base extension $\mathcal{G} \times_{\mathbb{Z}} \mathbb{Q} = G/\mathbb{Q}$, this base extension is called the *generic fiber* of $\mathcal{G}/\mathbb{Z}$.

We now may understand our arithmetic group $\Gamma = G(\mathbb{Q}) \cap \mathrm{Gl}_n(\mathbb{Z})$ as the group $\mathcal{G}(\mathbb{Z})$ of $\mathbb{Z}$ valued points of a group scheme over $\mathbb{Z}$. Since we know what $\mathcal{G}(\mathbb{Z}/m\mathbb{Z})$ is we can define congruence subgroups Γ_H as inverse images of subgroups $H \subset \mathcal{G}(\mathbb{Z}/m\mathbb{Z})$ under the projection $\mathcal{G}(\mathbb{Z}) \to \mathcal{G}(\mathbb{Z}/m\mathbb{Z})$.

There is the special class of semi-simple or reductive group schemes. Roughly speaking an affine group scheme $G/\mathbb{Z}$ is semi-simple (resp. reductive) , if its generic fiber $G \times_{\mathbb{Z}} \mathbb{Q}$ is semi-simple (resp. reductive) and if for all primes p the group scheme $G \times_{\mathbb{Z}} \mathbb{F}_p$ (the reduction $\mod p$) is a semi-simple ((resp. reductive)) group scheme over $\mathbb{F}_p$.

Of course the simplest example of a semi-simple (resp. reductive) group (scheme) over $\mathbb{Z}$ is the group $\mathrm{Sl}_n/\mathbb{Z}$ (resp. $\mathbb{G}_n/\mathbb{Z}$).

We can also construct semi-simple group-schemes by taking flat extensions of orthogonal (resp. symplectic) groups over $\mathbb{Q}$, (see section1.2.1, example 2) and 3). Here the symmetric (resp. alternating) form has to satisfy certain arithmetic conditions (See chap4.pdf).

$\boxed{\text{lattices}}$

1.2.2 Γ -modules

We consider modules $\mathcal{M}$ (i.e. abelian groups) with an action of Γ, we want to discuss briefly discuss some special classes of such Γ-modules.

The most important classes of Γ-modules are the modules of *arithmetic origin* . To construct such modules we realise our arithmetic group as $\Gamma = G(\mathbb{Q}) \cap \mathrm{Gl}_n(\mathbb{Z})$. Then we take any rational representation $\rho : G/\mathbb{Q} \to \mathrm{Gl}(V)$, where V is a finite dimensional $\mathbb{Q}-$ vector space. Now we look for finitely generated submodules $\mathcal{M} \subset V$ such that $\mathcal{M} \otimes \mathbb{Q} = V$ which are invariant under the action of Γ. Such a module is a Γ-module of arithmetic origin.

It is not to difficult to show that given any finitely generated module $\mathcal{M}'$ which is a full sublattice, i.e. $\mathcal{M}' \otimes \mathbb{Q} = V$, we can find a congruence subgroup $\Gamma_1 \subset \Gamma$ such that $\Gamma_1 \mathcal{M}' = \mathcal{M}'$. Then

$$\mathcal{M} = \bigcap_{\gamma \in \Gamma/\Gamma_1} \gamma \mathcal{M}'$$

is a $\Gamma-$ module of *arithmetic origin.*.

A second class of Γ modules are those of *congruence origin*. To get such a module we simply pick a congruence subgroup $\Gamma(N) \subset \Gamma$ and then we simply look at finitely generated abelian groups V with an action of $\Gamma/\Gamma(N)$ on V.

We get some important examples of Γ modules of congruence origin if we start from a Γ-module $\mathcal{M}$ of arithmetic origin. Then we choose an integer N and consider the Γ module $\mathcal{M} \otimes \mathbb{Z}/N\mathbb{Z}$. On this module $\Gamma(N)$ acts trivially, hence this module is a $\Gamma/\Gamma(N)$ module of congruence origin.

We go back to the more sophisticated point of view above, our arithmetic group is the group $\Gamma = \mathcal{G}(\mathbb{Z})$ of $\mathbb{Z}$ valued points of the flat extension $\mathcal{G}/\mathbb{Z}$.

Now we pick a torsion free finitely generated module $\mathcal{M}$, we know what it means that $\mathcal{M}$ is a $\mathcal{G}/\mathbb{Z}$ module: It simply means that for any commutative ring B with identity we have a B-linear action of $\mathcal{G}(B)$ on the B-module $\mathcal{M} \otimes B$, or in other words we have a homomorphism $\mathcal{G}(B) \to \mathrm{Gl}_B(\mathcal{M} \otimes_{\mathbb{Z}} B)$. Of course we require that this action is functorial in B.

For this book -especially for the first half- the group scheme $\mathrm{Gl}_2/\mathbb{Z}$ plays a dominant role. In this case the irreducible representations of $\mathrm{Gl}_2 \times_{\mathbb{Z}} \mathbb{Q}$ are well known. We consider the $\mathbb{Q}$ vector space of homogenous polynomials in two variables and of degree n

$$\mathcal{M}_{n,\mathbb{Q}} := \{P(X,Y) = \sum_{\nu=0}^{n} a_\nu X^\nu Y^{n-\nu} | a_\nu \in \mathbb{Q}\}. \tag{1.56}$$

We choose an integer m define an action of $\mathrm{Gl}_2(\mathbb{Q})$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} P(X,Y) = P(aX+cY, bX+dY) \det(\begin{pmatrix} a & b \\ c & d \end{pmatrix})^m, \tag{1.57}$$

this gives us the $\mathrm{Gl}_2/\mathbb{Q}$-module $\mathcal{M}_{n,\mathbb{Q}}[m]$.

But now it is easy to get $\mathrm{Gl}_2/\mathbb{Z}$-modules, we simply define

$$\mathcal{M}_n := \{P(X,Y) = \sum_{\nu=0}^{n} a_\nu X^\nu Y^{n-\nu} | a_\nu \in \mathbb{Z}\} \tag{1.58}$$

and then we define the $\mathrm{Gl}_2/\mathbb{Z}$ modules $\mathcal{M}_n[m]$ by the same formula as above. If n is even we will sometimes work with the module $\mathcal{M}[-\frac{n}{2},.$ because in this case the center acts trivially.

At this point a small remark is in order. If look at $\mathcal{M}_n[m]$ only as $\mathrm{Gl}_2(\mathbb{Z})$-module then the module "knows" what n is, clearly $n =$ rank$(\mathcal{M}_n) - 1$. But this $\mathrm{Gl}_2(\mathbb{Z})$- module does not "know" what m is. The only information we get is

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} P = (-1)^m P$$

and from this we only get the value of $m \mod 2$. But if we consider $\mathcal{M}_n[m]$ as module for the group scheme $\mathrm{Gl}_2/\mathbb{Z}$ then the module also knows the value of m because then we know

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} P = \alpha^m P$$

for any $\alpha \in R^\times$ in any commutative ring R with identity. If n is even we may consider the module $\mathcal{M}_n[-\frac{n}{2}]$, this is a module for $\mathrm{PGl}_2/\mathbb{Z} = \mathrm{Gl}_2/\mathbb{G}_m$.

In section 4.1.1 we discuss the corresponding situation for groups $\mathrm{Gl}_2(\mathbb{Z}[\sqrt{-d}])$.

1.2.3 The locally symmetric spaces

We start from a semisimple group $G/\mathbb{Q}$. To this group we attached the group of real points $G(\mathbb{R}) = G_\infty$. In G_∞ we have the connected component G_∞^0 of the identity and in this group we choose a maximal compact subgroup K_∞. The quotient space $X = G_\infty/K_\infty$ is a symmetric space which now may have several connected components. On this space we have the action of an arithmetic group Γ.

We have a fundamental fact:

The action of Γ on X is properly discontinuous, i.e. for any point $x \in X$ there exists an open neighbourhood U_x such that for all $\gamma \in \Gamma$ we have

$$\gamma U_x \cap U_x = \emptyset \quad \text{or} \quad \gamma x = x.$$

Moreover for all $x \in X$ the stabilizer

$$\Gamma_x = \{\gamma \mid \gamma x = x\}$$

is finite.

This is easy to see: If we consider the projection $p : G(\mathbb{R}) \to G(\mathbb{R})/K_\infty = X$, then the inverse image $p^{-1}(U_x)$ of a relatively compact neighbourhood U_x of $x = g_0 K_\infty$ is of the form $V_{g_0} \cdot K_\infty$, where V_{g_0} is a relatively compact neighbourhood of g_0. Hence we look for the solutions of the equation

$$\gamma v k = v' k', \gamma \in \Gamma, v, v' \in V_{g_0}, k, k' \in K_\infty.$$

Since Γ is discrete in $G(\mathbb{R})$ there are only finitely many possibilities for γ and they can be ruled out by shrinking U_x with the exception of those γ for which $\gamma x = x$. If $\gamma x = x$ this means that $\gamma g_0 K_\infty = g_0 K_\infty$ and hence $\gamma \in \Gamma \cap g_0 K_\infty g_0^{-1}$ this intersection is a compact discrete set, hence finite.

If Γ has no torsion then the projection

$$\pi : X \longrightarrow \Gamma\backslash X$$

is locally a $\mathcal{C}_\infty$-diffeomorphism. To any point $x \in \Gamma\backslash X$ and any point $\tilde{x} \in \pi^{-1}(x)$ we find a neighbourhood $U_{\tilde{x}}$ such that

$$\pi : U_{\tilde{x}} \xrightarrow{\sim} U_x.$$

Hence the space $\Gamma\backslash X$ inherits the Riemannian metric and the quotient space is a *locally symmetric space* . In the following we will denote the dimension of $\Gamma\backslash X$ by d, i.e. $d = \dim(\Gamma\backslash X)$.

If our group Γ has torsion, then a point $\tilde{x} \in X$ may have a nontrivial stabilizer $\Gamma_{\tilde{x}}$. Then it is not difficult to prove that $\tilde{x}$ has a neighbourhood $U_{\tilde{x}}$ which is invariant under $\Gamma_{\tilde{x}}$ and that for all $\tilde{y} \in U_{\tilde{x}}$ the stabilizer $\Gamma_{\tilde{y}} \subset \Gamma_{\tilde{x}}$. This gives us a diagram

$$\begin{array}{ccc} U_{\tilde{x}} & \longrightarrow & \Gamma_{\tilde{x}}\backslash U_{\tilde{x}} = U_x \\ \downarrow & & \downarrow \\ X & \xrightarrow{\pi} & \Gamma\backslash X \end{array}$$

i.e. the point $x \in \Gamma\backslash X$ has a neighbourhood which is the quotient of a neighbourhood $U_{\tilde{x}}$ by a finite group.

In this case the quotient space $\Gamma\backslash X$ may have singularities. Such spaces are called orbifolds. They have a natural stratification. Any point x defines a Γ conjugacy class $[\Gamma_{\tilde{x}}]$ of finite subgroups $\Gamma_{\tilde{x}} \subset \Gamma$. On the other hand a conjugacy class $[c]$ of finite subgroups $H \subset \Gamma$ defines the (non empty) subset (stratum) $\Gamma\backslash X([c])$ of those points $x \in \Gamma\backslash X$ for which $\Gamma_{\tilde{x}} \in [c]$.

These strata are easy to describe. We observe that for any finite $H \subset \Gamma$ the fixed point set X^H intersected with a connected component of X is contractible. Let $x_0 \in X^H$ be a point with $\Gamma_{x_0} = H$. Then any other point $x \in X^H$ is of the form $x = gx_0$ with $g \in G(\mathbb{R})$. This implies that $g \in N(H)(\mathbb{R})$, where $N(H)$ is the normaliser of H, it is an algebraic subgroup. Then $N(H)(\mathbb{R}) \cap K_\infty = K_\infty^H$ is compact subgroup, put $\Gamma^H = \Gamma \cap N(H)(\mathbb{R})$, and we get an embedding

$$\Gamma^H\backslash X^H \hookrightarrow \Gamma\backslash X.$$

This space contains the open subset $(\Gamma^H\backslash X^H)^{(0)}$ of those x where $H \in [\Gamma_{\tilde{x}}]$ and this is in fact the stratum attached to the conjugacy class of H.

We have an ordering on the set of conjugacy classes, we have $[c_1] \leq [c_2]$ if for any $H_1 \in [c_1]$ there exists a subgroup $H_2 \in [c_2]$ such that $H_1 \subset H_2$. These strata are not closed, the closure $\overline{\Gamma\backslash X([c])}$ is the union of lower dimensional strata.

If we start investigating the stratification above we immediately hit upon number theoretic problems. Let us pick a prime p and we consider the group $\Gamma = \mathrm{Sl}_{p-1}[\mathbb{Z}]$ and the ring of p-th roots of unity $\mathbb{Z}[\zeta_p]$ as a $\mathbb{Z}$-module is free of rank $p-1$ and hence we get an element

$$\zeta_p \in \mathrm{Sl}(\mathbb{Z}[\zeta_p]) = \mathrm{Sl}_{p-1}(\mathbb{Z})$$

and hence a cyclic subgroup of order p. But clearly we have many conjugacy classes of elements of order p in Γ because any ideal $\mathfrak{a}$ is a free $\mathbb{Z}$-module. If we want to understand the conjugacy classes of elements of order p or the conjugacy classes of cyclic subgroups of order p in $\mathrm{Sl}_{p-1}(\mathbb{Z})$ we need to understand the ideal class group. In the next section we will discuss some simple examples.

These quotient spaces $\Gamma\backslash X$ attract the attention of various different kinds of mathematicians. They provide interesting examples of Riemannian manifolds and they are intensively studied from that point of view. On the other hand number theoretic data enter into their construction. Hence any insight into the structure of these spaces contains number theoretic information. This is the main theme of this book.

It is not difficult to see that any arithmetic group Γ contains a normal congruence subgroup Γ' which does not have torsion. This can be deduced easily from the exercise .... at the end of this section. Hence we see that $\Gamma'\backslash X$ is a Riemannian manifold which is a finite cover of $\Gamma\backslash X$ with covering group Γ/Γ'.

We discuss special examples below.

1.2.4 Low dimensional examples

We consider the action of the group $\Gamma = \mathrm{Sl}_2(\mathbb{Z}) \subset \mathrm{Sl}_2(\mathbb{R})$ on the upper half plane

$$X = \mathbb{H} = \{z \mid \Im(z) = y > 0\} = \mathrm{Sl}_2(\mathbb{R})/\mathrm{SO}(2).$$

We want to describe the quotient $\Gamma\backslash\mathbb{H}$, for this purpose we construct further down the fundamental domain $\mathcal{F}$

As we explained in .section 1.1.9 we may consider the point $z = x + iy$ as a positive definite euclidian metric on $\mathbb{R}^2$ up to a positive scalar. We saw already that this metric can be interpreted as the metric on $\mathbb{C}$ induced on the lattice $\Omega = \langle 1, z\rangle$. The action of $\mathrm{Sl}_2(\mathbb{Z})$ on the upper half plane corresponds to changing the basis $1, z$ of Ω into another basis and then normalising the first vector of the new basis to length equal one. This means that under the action of $\mathrm{Sl}_2(\mathbb{Z})$ we may achieve that the first vector 1 in the lattice is of shortest length. In other words $\Omega = \langle 1, z\rangle$ where now $|z| \geq 1$.

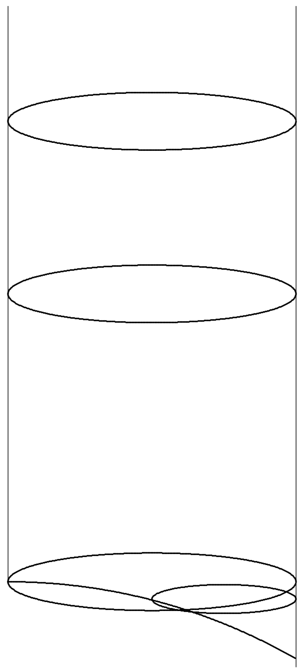
Since we can change the basis by $1 \to 1$ and $z \to z + n$. We still have $|z + n| \geq 1$. Hence see that this condition implies that we can move z by these translation into the strip $-1/2 \leq \Re(z) \leq 1/2$ and since 1 is still the shortest vector we end up in the classical fundamental domain:

$$\mathcal{F} = \{z| - 1/2 \leq \Re(z) \leq 1/2, |z| \geq 1\} \tag{1.59}$$

Two points $z_1, z_2 \in \mathcal{F}$ are inequivalent under the action of $\mathrm{Sl}_2(\mathbb{Z})$ unless they differ by a translation. i.e.

$$z_1 = -\frac{1}{2} + it\ ,\ z_2 = z_1 + 1 = \frac{1}{2} + it, \tag{1.60}$$

or we have $|z_1| = 1$ and $z_2 = -\frac{1}{z_1}$. Hence the quotient $\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}$ is given by the following picture



The circles are actually the images of horizontal lines $iy + x$ where $x \in \mathbb{R}$ or $x \in [9, 1]$ in the quotient. The picture is a little bit misleading because it does not reflect the Riemannian metric: The circumference of the circle at level iy is $\frac{1}{y}$.

It turns out that this quotient is actually a Riemann surface, i.e. the finite stabilisers at i and ρ do not produce singularities. As a Riemann surface the quotient is the complex plane or better the projective line $\mathbb{P}^1(\mathbb{C})$ minus the point at infinity.

It is clear that the points i and $\rho = +\frac{1}{2} + \frac{1}{2}\sqrt{-3}$ in the upper half plane are -up to conjugation by an element $\gamma \in \mathrm{Sl}_2(\mathbb{Z})$- the only points with non-trivial stabiliser . Actually the stabilisers are given by

$$\Gamma_i = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \quad , \quad \Gamma_\rho = \left\{ \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

The second example is given by the group $\Gamma = \mathrm{Sl}_2(\mathbb{Z}[i]) \subset \mathrm{Sl}_2(\mathbb{C}) = G_\infty = R_{\mathbb{C}/\mathbb{R}}(Gl_2/\mathbb{C})(\mathbb{R})$ (See(1.1) . Here we should remember that the choice of G_∞ allows a whole series of arithmetic groups. For any imaginary quadratic extension $K = \mathbb{Q}(\sqrt{-d})$ with $\mathcal{O}_K$ as its ring of integers we may embed K into $\mathbb{C}$ and get

$$\mathrm{Sl}_2(\mathcal{O}_K) = \Gamma \subset G_\infty.$$

If the number d becomes larger then the structure of the group Γ becomes more and more complicated. We only discuss the simplest case $\mathcal{O}_K = \mathbb{Z}[i]$. We will construct a fundamental domain for the action of Γ on the three-dimensional hyperbolic space $\mathbb{H}_3 = \mathbb{C} \times \mathbb{R}_{>0}$.

We identify $\mathbb{H}_3$ with the space of positive definite hermitian matrices

$$X = \{A \in M_2(\mathbb{C}) \mid A =^t \overline{A}, A > 0, \det(A) = 1\}.$$

We consider the lattice

$$\Omega = \mathbb{Z}[i] \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbb{Z}[i] \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

in $\mathbb{C}^2$ and view A as a hermitian metric on $\mathbb{C}^2$ where $\mathbb{C}/\Omega$ has volume 1. Let $e_1' = \binom{\alpha}{\beta}$ be a vector of shortest length. We can find a second vector $e_2' = \binom{\gamma}{\delta}$ so that $\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = 1$. This argument is only valid because $\mathbb{Z}[i]$ is a principal ideal domain. We consider the vectors $e_2' + \nu e_1'$ where $\nu \in \mathbb{Z}[i]$. We have

$$\langle e_2' + \nu e_1', e_2' + \nu e_1' \rangle_A = \langle e_2' + \nu e : 1' \rangle_A + \nu \langle e_1', e_2' \rangle_A + \overline{\nu} \langle e_2', e_1' \rangle_a + \nu\overline{\nu} \langle e_1', e_1' \rangle_A.$$

Since we have the euclidean algorithm in $\mathbb{Z}[i]$ we can choose ν such that

$$-\frac{1}{2}\langle e_1', e_1' \rangle \leq \mathrm{Re}\langle e_1', e_2' \rangle_A, \Im\langle e_1', e_2' \rangle_A \leq \frac{1}{2}\langle e_1', e_1' \rangle_A.$$

If we translate this to the action of $\mathrm{Sl}_2(\mathbb{Z}[i])$ on $\mathbb{H}_3$ then we find that every point $x = (z; \zeta) \in \mathbb{H}_3$ is equivalent to a point in the domain

$$\tilde{F} = \{(z, \zeta) \mid -\frac{1}{2} \leq \mathrm{Re}(z), \Im(z) \leq \frac{1}{2}; z\overline{z} + \zeta^2 \geq 1\}.$$

Since we have still the action of the matrix $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ we even find a smaller fundamental domain

$$F = \{(z, \zeta) \mid -\frac{1}{2} \leq \mathrm{Re}(z), \Im(z) \leq \frac{1}{2}; z\overline{z} + \zeta^2 \geq 1 \text{ and } \mathrm{Re}(z) + \Im(z) \geq 0\}.$$

I also want to discuss the extension of our considerations to the case of the reductive group $\mathrm{Gl}_2(\mathbb{C})$. In such a case we have to enlarge the maximal compact

Figure 1.1: Fundamental Domain

subgroup. In this case the group $\tilde{K} = \mathrm{Sl}_1(2) \cdot \mathbb{C}^* = K \cdot \mathbb{C}^*$ is a good choice where $\mathbb{C}^*$ is the centre of $\mathrm{Gl}_2(\mathbb{C})$. Then we get

$$\mathbb{H}_3 = \mathrm{Sl}_2(\mathbb{C})/K = \mathrm{Gl}_2(\mathbb{C})/\tilde{K}$$

i.e. we have still the same symmetric space. But the group $\tilde{\Gamma} = \mathrm{Gl}_2(\mathbb{Z}[i])$ is still larger. We have an exact sequence

$$1 \to \Gamma \to \tilde{\Gamma} \to \{i^\nu\} \to 1.$$

The centre $Z_{\tilde{\Gamma}}$ of $\tilde{\Gamma}$ is given by the matrices $\left\{\begin{pmatrix} i^v & 0 \\ 0 & i^v \end{pmatrix}\right\}$. The centre Z_Γ has index 2 in $Z_{\tilde{\Gamma}}$. Since the centre acts trivially on the symmetric space, hence the above fundamental domain will be "cut into two halfes" by the action of $\tilde{\Gamma}$. the matrices $\begin{pmatrix} i^v & 0 \\ 0 & 1 \end{pmatrix}$ induce rotation of $\nu \cdot 90^\circ$ around the axis $z = 0$ and therefore it becomes clear that the region

$$F_0 = \{(z, \zeta) \mid 0 \leq \Im(z), \mathrm{Re}(z) \leq \frac{1}{2}, z\overline{z} + \zeta^2 \geq 1\}$$

is a fundamental domain for $\tilde{\Gamma}$.

The translations $z \to z + 1$ and $z \to z + i$ identify the opposite faces of F. This induces an identification on F_0, namely

$$\left(\frac{1}{2} + iy, \zeta\right) \longrightarrow \left(-\frac{1}{2} + iy, \zeta\right) \longrightarrow \left(y + \frac{i}{2}, \zeta\right).$$

On the bottom of the domain F_0, namely

$$F_0(1) = \{(z, \zeta) \in F_0 \mid z\overline{z} + \zeta^2 = 1\}$$

we have the further identification

$$(z, \zeta) \longrightarrow (i\overline{z}, \zeta).$$

Hence we see that the quotient space $\tilde{\Gamma}\backslash\mathbb{H}_3$ is given by the following figure.

I want to discuss the fixed points and the stabilizers of the fixed points of $\tilde{\Gamma}$. Before I can do that, I need some simple facts concerning the structure of Gl_2.

The group $\mathrm{Gl}_2(K)$ acts upon the projective line $\mathbb{P}^1(K) = (K^2 \setminus \{0\})/K^*$. We write

$$\mathbb{P}^1(K) = (K) \cup \{\infty\}\ ;\ K(xe_1 + e_2) = x, Ke_1 = \infty.$$

It is quite clear that the action of $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Gl}_2(K)$ is given by

$$gx = \frac{\alpha x + \beta}{\gamma x + \delta}.$$

The action of $\mathrm{Gl}_2(K)$ on $\mathbb{P}^1(K)$ is transitive. For a point $x \in \mathbb{P}^1(K)$ the stabilizer B_x is clearly a linear subgroup of Gl_2/K. If $x = \infty$, then this stabilizer is the subgroup

$$B_\infty = \left\{\begin{pmatrix} a & u \\ 0 & b \end{pmatrix}\right\},$$

and for $x = 0$ we get

$$B_0 = \left\{ \begin{pmatrix} a & 0 \\ u & b \end{pmatrix} \right\}.$$

It is clear that these subgroups B_x are conjugate under the action of $\mathrm{Gl}_2(K)$. They are in fact maximal solbable subgroups of Gl_2.

If we have two different points $x_1, x_2 \in \mathbb{P}^1(K)$, then this corresponds to a choice of a basis where the basis vectors are only determined up to scalars. Then the intersection of the two groups $B_{x_1} \cap B_{x_2}$ is a so-called maximal torus. If we choose $x_1 = Ke_1$, $x_2 = Ke_2$, then

$$B_{x_1} \cap B_{x_2} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\}.$$

Any other maximal torus of the form B_{x_1}, B_2 is conjugate to T_0 under $\mathrm{Gl}_2(K)$.

Now we assume $K = \mathbb{C}$. We compactify the three dimensional hyperbolic space by adding $\mathbb{P}^1(\mathbb{C})$ at infinity, i.e.

$$\mathbb{H}_3 \hookrightarrow \overline{\mathbb{H}}_3 = \mathbb{H}_3 \cup \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \times \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

(The reader should verify that there is a natural topology on $\overline{\mathbb{H}}_3$ for which the space is compact and for which $\mathrm{Gl}_2(\mathbb{C})$ acts continuously.)

Now let us assume that $a \in \mathrm{Gl}_2(\mathbb{C})$ is an element which has a fixed point on $\mathbb{H}_3$ and which is not central. Since it lies in a maximal compact subgroup times $\mathbb{C}^x$ we see that this element a can be diagonalized

$$a \longrightarrow g_0 \; a \; g_0^{-1} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = a'$$

with $\alpha \neq \beta$ and $|\alpha/\beta| = 1$.

Then it is clear that the fixed point set for a' is the line

$$\mathrm{Fix}\;(a') = \{(0, \zeta) \mid \zeta \in \mathbb{R}_{>0}\},$$

i.e. we do not get an isolated fixed point but a full fixed line.

The element a' has the two fixed points $\infty, 0$ in $\mathbb{P}^1(\mathbb{C})$, and hence ist defines the torus $T_0(\mathbb{C})$. Then it is clear that

$$\mathrm{Fix}(a') = \{(0, \zeta) \mid \zeta > 0\} = T_0(\mathbb{C}) \cdot (0, 1)$$

i.e. the fixed point set is an orbit under the action of $T_0(\mathbb{C})$.

1.2.5 Fixed point sets and stabilizers for $\mathrm{Gl}_2(\mathbb{Z}[i]) = \tilde{\Gamma}$

If we want to describe the stabilizers up to conjugation, we can focus our attention on F_0.

If we have an element $\gamma \in \tilde{\Gamma}$, γ not central and if we assume that γ has fixed points on $\mathbb{H}_3$, then we know that γ defines a torus $T_\gamma = \mathrm{centralizer}_{\mathrm{Gl}_2}(\gamma) =$ stabilizer of $x_\gamma, x_{\gamma'} \in \mathbb{P}^1(\mathbb{C})$. This torus is defined over $\mathbb{Q}(i)$, but it is not necessarily diagonalizable over $\mathbb{Q}(i)$, it may be that the coordinates of $x_\gamma, x_{\gamma'}$ lie in a quadratic extension of $F/\mathbb{Q}(i)$. This is the quadratic extension defined by the eigenvalues of γ.

We look at the edges of the fundamental domain F_0. We saw that they consist of connected pieces of the straight lines

$$G_1 = \{(z,\zeta) \mid z = 0\}, G_2 = \{(z,\zeta) \mid z = \frac{1}{2}\}, G_3 = \{(z,\zeta) \mid z = \frac{1+i}{2}\},$$

and the circles (these circles are euclidean circles and geodesics for the hyperbolic metric)

$$D_1 = \{(z,\zeta) \mid z\overline{z} + \zeta^2 = 1, \Im(z) = \mathrm{Re}(z)\}, D_2 = \{(z,\zeta) \mid z\overline{z} + \zeta^2 = 1, \Im(z) = 0\},$$

$$D_3 = \{(z,\zeta) \mid z\overline{z} + \zeta^2 = 1, \mathrm{Re}(z) = \frac{1}{2}\}.$$

The pair of points $(\infty, (z_0, 0)) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ has as its stabilizer

$$T_{z_0}(\mathbb{C}) = \begin{pmatrix} 1 & z_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & -z_0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & z_0(\beta - \alpha) \\ 0 & \beta \end{pmatrix},$$

the straight line $\{(z_0, \zeta) \mid \zeta > 0\}$ is an orbit u nder $T_{z_0}(\mathbb{C})$ and it consists of fixed points for

$$T_{z_0}(\mathbb{C})(1) = \left\{ \begin{pmatrix} \alpha & z_0(\beta - \alpha) \\ 0 & \beta \end{pmatrix} \,\middle|\, \alpha/\beta \in S^1 \right\}.$$

We can easily compute the pointwise stabilizer of G_1, G_2, G_3 in $\tilde{\Gamma}$. They are

$$\begin{aligned}
\tilde{\Gamma}_{G_1} &= \left\{ \begin{pmatrix} i^\nu & 0 \\ 0 & i^\mu \end{pmatrix} \right\} = \left\{ \begin{pmatrix} i^\nu & 0 \\ 0 & i \end{pmatrix} \right\} \cdot z_{\tilde{\Gamma}} \\
\Gamma_{\tilde{G}_2} &= \left\{ \begin{pmatrix} i^\nu & \frac{1-i^\nu}{2} \\ 0 & 1 \end{pmatrix} \,\middle|\, \frac{1-i^\nu}{2} \in \mathbb{Z}[i] \right\} \cdot Z_{\tilde{\Gamma}} = \left\{ \begin{pmatrix} \pm 1 & \frac{1 \pm 1}{2} \\ 0 & 1 \end{pmatrix} \right\} \cdot Z_{\tilde{\Gamma}} \\
\Gamma_{\tilde{G}_3} &= \left\{ \begin{pmatrix} i^\nu & \frac{(1-i^\nu)(1+i)}{2} \\ 0 & 1 \end{pmatrix} \right\} \cdot Z_{\tilde{\Gamma}},
\end{aligned}$$

where in the last case we have to take into account that $\frac{(1-i^\nu)(1+i)}{2} \in \mathbb{Z}[i]$ for all ν.

Hence modulo the centre $Z_{\tilde{\Gamma}}$ these stabilizers are cyclic groups of order $4, 2, 4$.

The arcs D_i are also pointwise fixed under the action of certain cyclic groups, namely

$$\begin{aligned}
D_1 =& \mathrm{Fix}\left(\begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix} \right) \\
D_2 =& \mathrm{Fix}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \\
D_3 =& \mathrm{Fix}\left(\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right),
\end{aligned}$$

and we check easily that these arcs are geodesics joining the following points in the boundary

D_1 runs from $\sqrt{i}$ to $-\sqrt{i}$

D_2 runs from i to $-i$

D_3 runs from $e = e^{\frac{1\pi i}{6}} = e^{\frac{\pi i}{3}}$ to $\overline{\rho}$.

The corresponding tori are

$$
\begin{aligned}
T_1 =& \mathrm{Stab}(-1,1) = \left\{ \begin{pmatrix} \alpha & i\beta \\ \beta & \alpha \end{pmatrix} \right\} \\
T_2 =& \mathrm{Stab}(-\sqrt{i},\sqrt{i}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \right\} \\
T_3 =& \mathrm{Stab}(\rho,\overline{\rho}) = \left\{ \begin{pmatrix} \delta-\beta & \beta \\ -\beta & \delta \end{pmatrix} \right\}.
\end{aligned}
$$

The torus T_2 splits over $\mathbb{Q}(i)$, the other two tori split over an quadratic extension of $\mathbb{Q}(i)$.

Now it is not difficult anymore to describe the finite stabilizers and the corresponding fixed point sets. If $x \in \mathbb{H}_3$ for which the stabilizer is bigger than $Z_{\tilde{\Gamma}}$, then we can conjugate x into F_0. It is very easy to see that x cannot lie in the interior of F_0 because then we would get an identification of two points nearby x and hence still in F_0 under $\tilde{\Gamma}$.

If x is on one of the lines D_1, D_2, D_3 or on one of the arcs G_1, G_2, G_3 but not on the intersection of two of them, then the stabilizer Γ_x is equal to $Z_{\tilde{\Gamma}}$ times the cyclic group we attached to the line or the arc earlier. Finally we are left with the three special points

$$
\begin{aligned}
x_{12} =& D_1 \cap D_2 \cap G_1 = \{(0,1)\} \\
x_{13} =& D_1 \cap D_3 \cap G_3 = \left\{ \left(\frac{1+i}{2}, \frac{\sqrt{2}}{2} \right) \right\} \\
x_{23} =& D_2 \cap D_3 \cap G_2 = \left\{ \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right) \right\}.
\end{aligned}
$$

In this case it is clear that the stabilizers are given by

$$
\begin{aligned}
\tilde{\Gamma}_{x_{12}} =& \langle \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \rangle = D_4 \\
\tilde{\Gamma}_{x_{13}} =& \langle \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} i & 1 \\ 0 & 1 \end{pmatrix} \rangle = S_4 \\
\tilde{\Gamma}_{x_{23}} =& \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \rangle = S_3.
\end{aligned}
$$

1.2.6 Compactification of $\Gamma\backslash X$

Our two special low dimensional examples show clearly that the quotient spaces $\Gamma\backslash X$ are not compact in general. There exist various constructions to compactify them.

If, for instance, $\Gamma \subset \mathrm{Sl}_2(\mathbb{Z})$ is a subgroup of finite index, then the quotient $\Gamma\backslash\mathbb{H}$ is a Riemann surface. It can be embedded into a compact Riemann surface by adding a finite number of points. this is a special case of a more general theorem of Satake and Baily-Borel: If the symmetric space X is actually hermitian symmetric (this means it has a complex structure) then we have the structure of a quasi-projective variety on $\Gamma\backslash X$. This is the so-called Baily-Borel compactification. It exists only under special circumstances.

I will discuss the process of compactification in some more detail for our special low dimensional examples.

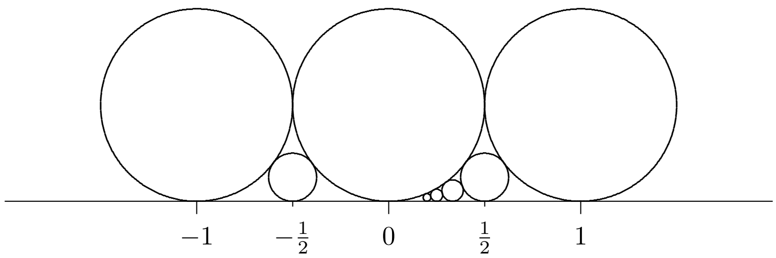
Compactification of $\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}$ by adding points

Let $\Gamma \subset \mathrm{Sl}_2(\mathbb{Z})$ be any subgroup of finite index. The group Γ acts on the rational projective line $\mathbb{P}^1(\mathbb{Q})$. We add it to the upper half plane and form

$$\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}),$$

and we extend the action of Γ to this space. Since the full group $\mathrm{Sl}_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$ we find that Γ has only finitely many orbits on $\mathbb{P}^1(\mathbb{Q})$.

Now we introduce a topology on $\overline{\mathbb{H}}$. We defined a system of neighbourhoods of points $\frac{p}{q} = r \in \mathbb{P}^1(\mathbb{Q})$. We define the Farey circles $S\left(c, \frac{p}{q}\right)$ which touch the real axis in the point $r = p/q$ $(p,q) = 1$ and have the radius $\frac{c}{2q^2}$. For $c = 1$ we get the picture



Let us denote by $D\left(c, \frac{p}{q}\right) = \cup_{c':0<c'\le c} S\left(c', \frac{p}{q}\right)$ the Farey disks. For $c \to 0$ these Farey disks $D\left(c, \frac{p}{q}\right)$ define a system of neighbourhoods of the point $r = p/q$. The Farey disks at $\infty \in \mathbb{P}^1(\mathbb{Q})$ are given by the regions

$$D(T, \infty) = \{z \mid \Im(z) \ge T\}.$$

It is easy to check that an element $\gamma \in \mathrm{Sl}_2(\mathbb{Z})$ which sends $\infty \in \mathbb{P}^1(\mathbb{Q})$ into the point $r = \frac{p}{q}$ sends $D(T,\infty)$ to $D\left(\frac{1}{T}, \frac{p}{q}\right)$. These Farey disks $D(c,r)$ do not meet provided we take $c < 1$. The considerations in 1.6.1 imply that the complement of the union of Farey disks is relatively compact modulo Γ, and since Γ has finitely many orbits on $\mathbb{P}^1(\mathbb{Q})$, we see easily that

$$Y_\Gamma = \Gamma\backslash\overline{\mathbb{H}}$$

is compact (which means of course also Hausdorff).

It is essential that the set of Farey circles $D(c,r)$ and $D\left(\frac{1}{c}, \infty\right)$ is invariant under the action of Γ on the one hand and decomposes into several connected components (which are labeled by the point $r \in \mathbb{P}^1(\mathbb{Q})$) on the other hand. Hence

$$\Gamma\backslash\bigcup_r D(c,r) = \bigcup \Gamma_{r_i}\backslash D(c, r_i)$$

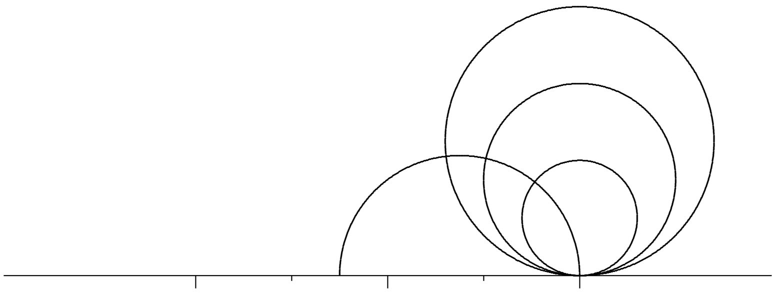
where r_i is a set of representatives for the action of Γ on $\mathbb{P}^1(\mathbb{Q})$ and where Γ_{r_i} is the stabilizer of r_i in Γ.

It is now clear that $\Gamma_{r_i}\backslash D(c, r_i)$ is holomorphically equivalent to a punctured disc and hence the above compactification is obtained by filling the point into this punctured disc and this makes it clear that Y_Γ is a Riemann surface.

$\boxed{\text{BSC}}$

1.2.7 The Borel-Serre compactification of $\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}$

There is another construction of a compactification. We look at the disks $D(c, r)$ and divide them by the action of Γ_r. For any point $y \in S(c', r) - \{r\}$ there exists a unique geodesic joining r and y, passing orthogonally through $S(c', r)$ and hitting the projective line in another point y_∞ ($= -1/4$ in the picture below)



If $r = \infty$, then this system of geodesics is given by the vertical lines $\{y \cdot i + x \mid x \in \mathbb{R}\}$.. This allows us to write the set

$$D(c, r) - \{r\} = X_{\infty,r} \times [c, 0)$$

where $X_{\infty,r} = \mathbb{P}^1(\mathbb{R}) - \{r\}$. The stabilizer Γ_r acts $D(c, r)$ and on the right hand side of the identification it acts on the first factor, the quotient $\Gamma_r\backslash X_{\infty,r}$ is a circle. Hence we can compactify the quotient

$$\Gamma_r\backslash D(c, r) - \{r\} \hookrightarrow \Gamma_r\backslash X_{\infty,r} \times [c, 0] \tag{1.61}$$

This gives us a second way to compactify $\Gamma\backslash\mathbb{H}$, we apply this process to a finite set of representatives of $\mathbb{P}^1(\mathbb{Q}) \mod \Gamma$.

There is a slightly different way of looking at this. We may form the union

$$\mathbb{H} \cup \bigcup_r X_{\infty,r} = \tilde{\mathbb{H}}$$

and topologize it in such a way that

$$D(c, r) = X_{\infty,r} \times [c, 0) \subset X_{\infty,r} \times [c, 0] \tag{1.62}$$

is a local homeomorphism. Then we see that the compactification above is just the quotient $\Gamma\backslash\tilde{\mathbb{H}}$ and the boundary is simply

$$\partial(\Gamma\backslash\bar{\mathbb{H}}) = \Gamma\backslash \bigcup_{r\in\mathbb{P}^1(\mathbb{Q}} X_{\infty,r}. \tag{1.63}$$

This compactification is called the Borel-Serre compactification. Its relation to the Baily-Borel is such that the latter is obtained by the former by collapsing the circles at infinity to a point.

It is quite clear that a similar construction applies to the action of a group $\Gamma \subset \mathrm{Sl}_2(\mathbb{Z}[i])$ on the three-dimensional hyperbolic space. The Farey circles will be substituted by spheres $S(c,\alpha)$ which touch the complex plane $\{(z,0) \mid z \in \mathbb{C}\} \subset \overline{\mathbb{H}}_3$ in the point $(\alpha,0), \alpha \in \mathbb{P}^1(\mathbb{Q}(i))$ and for $\alpha = \infty$ the Farey sphere is the horizontal plane $S(\infty,\zeta_0) = \{(z,\zeta_0) \mid z \in \mathbb{C})$. An element $\gamma \in \Gamma$ which maps $(0,\infty)$ to α maps $S(\infty,\zeta_0)$ to $S(c,\alpha)$, where $c = 1/\zeta_0$. For a given α we may identify the different spheres if we vary c and for any point $\alpha \in \mathbb{P}^1(\mathbb{Q}(i))$ we define $X_{\infty,\alpha} = \mathbb{P}^1(\mathbb{C}) \setminus \{\alpha\}$. Again we can identify

$$D(c,\alpha) \setminus \{\alpha\} = X_{\infty,\alpha} \times (0,c] \subset \overline{D(c,\alpha) \setminus \{\alpha\}} = \partial(\Gamma\backslash\mathbb{H}) = X_{\infty,\alpha} \times [0,c]$$

The stabiliser Γ_α acts on $D(c,\alpha) \setminus \{\alpha\}$ and again this yields an action on the first factor. If we choose $\alpha = \infty$ then

$\boxed{\text{ainfty}}$

$$\Gamma_\infty = \{\begin{pmatrix} \zeta & a \\ 0 & \zeta^{-1} \end{pmatrix} | \zeta \text{ root of unity}, a \in \mathfrak{a}_\infty\} \tag{1.64}$$

where $\mathfrak{a}_\infty$ is a free rank 2 module in $\mathbb{Z}[i]$. If ζ does not assume the value i then $\Gamma_\infty\backslash X_{\infty,\infty}$ is a two-dimensional torus, a product of two circles. If ζ assumes the value i then $\Gamma_\infty\backslash X_{\infty,\infty}$ is a two dimensional sphere. If $\Gamma = \mathrm{Sl}_2(Z[i])$ then $\mathfrak{a}_\infty = \mathbb{Z}[i]$. If course we get the same result for an arbitrary α.

Then we get an action of the group Γ on $\tilde{\mathbb{H}}_3 = \mathbb{H}_3 \cup \bigcup_{\alpha\in\mathbb{P}^1(K)} \overline{D(c,\alpha) \setminus \{\alpha\}}$ and the quotient is compact, the set of orbits of Γ on $\mathbb{P}^1(\mathbb{Q}(i))$ is finite, these orbits are called the cusps.

$\boxed{\text{BSC0}}$

1.2.8 The Borel-Serre compactification, reduction theory of arithmetic groups

This section could be skipped in a first reading. For the particular groups $\mathrm{Sl}_2/\mathbb{Q}$ or $\mathrm{Sl}_2(\mathbb{Z}[\sqrt{-d})$ this compactification has been discussed in detail in the previous sections. A reader who is interested in the specific applications to number theory which will be discussed in the following chapters 2-5 only needs the results from section 1.2.7.

The Borel-Serre compactification works in complete generality for any semi-simple or reductive group $G/\mathbb{Q}$. To explain it, we need the notion of a parabolic subgroup of $G/\mathbb{Q}$.

A subgroup $P/\mathbb{Q} \hookrightarrow G/\mathbb{Q}$ is parabolic if the quotient variety in the sense of algebraic geometry is a projective variety. We mentioned already earlier that for the group $\mathrm{Gl}_2/\mathbb{Q}$ we have an action of Gl_2 on the projective line $\mathbb{P}^1$ and the stabilizers B_x of the points $x \in \mathbb{P}^1(\mathbb{Q})$ are the so-called Borel subgroups of $\mathrm{Gl}_2/\mathbb{Q}$. They are maximal solvable subgroups and

$$\mathrm{Gl}_2/B_x = \mathbb{P}^1,$$

hence they are also parabolic.

More generally we get parabolic subgroups of $\mathrm{Gl}_n/\mathbb{Q}$, if we choose a flag on the vector space $V = \mathbb{Q}^n = \mathbb{Q}e_1 \oplus \cdots \oplus \mathbb{Q}e_n$. This is an increasing sequence of subspaces

$$\mathcal{F} : (0) = \{(0)\} = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_k = V.$$

The stabilizer P of such a flag is always a parabolic subgroup; the quotient space

$$G/P = \text{Variety of all flags of the given type},$$

where the type of the flag is the sequence of the dimensions $n_i = \dim V_i$.

These flag varieties (the Grassmannians) are smooth projective schemes over $\mathrm{Spec}(\mathbb{Z})$ and this implies that any flag $\mathcal{F}$ is induced by a flag

$$\mathcal{F}_{\mathbb{Z}} : (0) = \{(0)\} = L_0 \subset L_1 \subset L_2 \subset \ldots \subset L_k = L = \mathbb{Z}^n \tag{1.65}$$

where $L_i = V_i \cap L$, and of course $L_i \otimes \mathbb{Q} = V_i$. This is the elementary fact which will be used later.

If our group $G/\mathbb{Q}$ is the orthogonal group of a quadratic form

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i^2$$

with $a_i \in \mathbb{Q}^*$. Then we have to replace the flags by sequences of subspaces

$$\mathcal{F} : 0 \subset W_1 \subset W_2 \ldots \qquad\qquad \subset W_2^{\perp} \subset W_1^{\perp} \subset V,$$

where the W_i are isotropic spaces for the form f, i.e. $f \mid W_i \equiv 0$, and where the $W_i^{\perp}$ are the orthogonal complements of the subspaces. Again the stabilizers of these flags are the parabolic subgroups defined over $\mathbb{Q}$.

Especially, if the form f is anisotropic over $\mathbb{Q}$, i.e. there is no non-zero vector $\underline{x} \in K^n$ with $f(\underline{x}) = 0$, then the group $G/\mathbb{Q}$ does not have any parabolic subgroup over $\mathbb{Q}$. This equivalent to the fact that $G(\mathbb{Q})$ does not have unipotent elements.

These parabolic subgroups always have a unipotent radical U_P which is always the subgroup which acts trivially on the successive quotients of the flag. The unipotent radical is a normal subgroup, the quotient $P/U_P = M$ is a reductive group again, it is called the Levi-quotient of P.

We go back to the group $\mathrm{Gl}_n/\mathbb{Q}$. It contains the standard maximal torus whose R valued points are

$$T_0(R) = \{\underline{t} = \begin{pmatrix} t_1 & 0 & \ldots & 0 \\ 0 & t_2 & \ldots & 0 \\ & & \ddots & \\ 0 & 0 & & 0 \\ 0 & 0 & 0 & t_n \end{pmatrix} \quad \mid \quad t_i \in R^{\times}\} \tag{1.66}$$

It is a subgroup of the Borel subgroup (maximal solvable subgroup or minimal parabolic subgroup) whose R-valued points are

$$B_0(R) = \{\underline{b} = \begin{pmatrix} t_1 & u_{1,2} & \dots & u_{1,n} \\ 0 & t_2 & \dots & u_{2,n} \\ 0 & 0 & \ddots & u_{n-1,n} \\ 0 & 0 & 0 & t_n \end{pmatrix} \quad | \quad t_i \in R^\times\} \tag{1.67}$$

and its unipotent radical U_0 consists of those $b \in B_0$ where all the $t_i = 1$. This unipotent radical contains the one dimensional *root subgroups*

$$U_{i,j} = \{\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & x & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\}, x \in R \tag{1.68}$$

where $i < j$, these one dimensional subgroups are isomorphic to the one dimensional additive group $\mathbb{G}_a$. They are normalized by the torus, for an element $t \in T(R)$ and $x_{i,j} \in U_{i,j}(R) = R$ we have

$$\underline{t} x_{i,j} \underline{t}^{-1} = t_i/t_j x_{i,j}. \tag{1.69}$$

For $i = 1, \dots, n, j = 1, \dots, n, i \neq j$ (resp. $i < J$) the characters $\alpha_{i,j}(\underline{t}) = t_i/t_j$ are called the roots (resp. positive roots) of T_0 in Gl_n. We denote these systems of roots by Δ^{Gl_n} (resp)$\Delta_+^{\mathrm{Gl}_n}$. The one dimensional subgroups $U_{i,j}, i \neq j$ are called the root subgroups.

Inside the set of positive roots we have the set of simple roots

$$\pi = \pi^{\mathrm{Gl}_n} = \{\alpha_{1,2}, \dots, \alpha_{i,i+1}, \dots, \alpha_{n-1,n}\} \tag{1.70}$$

If we pass to the semi-simple subgroup $\mathrm{Sl}_n/\mathbb{Q}$ then the torus and the Borel-subgroup has to be replaced by $T_0^{(1)}, B_0^{(1)}$, where we have $\prod_i t_i = 1$. The system of roots does not change, we have $\pi = \pi^{\mathrm{Gl}_n} = \pi^{\mathrm{Sl}_n}$.

We change the notation slightly, for $i = 1, \dots, n-1$ we define $\alpha_i := \alpha_{i,i+1}$ then for $i < j$ we get $\alpha_{i,j} = \alpha_i + \dots \alpha_{j-1}$, and $\pi = \{\alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$

The Borel subgroup B_0 is the stabilizer of the "complete" flag

$$\{0\} \subset \mathbb{Q}e_1 \subset \mathbb{Q}e_1 \oplus \mathbb{Q}e_2 \subset \dots \subset \mathbb{Q}e_1 \oplus \mathbb{Q}e_2 \oplus \dots \oplus \mathbb{Q}e_n, \tag{1.71}$$

the parabolic subgroups $P_0 \supset B_0$ are the stabilizers of "partial" flags

$$\{0\} \subset \mathbb{Q}e_1 \oplus \dots \oplus \mathbb{Q}e_{n_1} \subset \mathbb{Q}e_1 \oplus \dots \oplus \mathbb{Q}e_{n_1} \oplus \mathbb{Q}e_{n_1+1} \oplus \dots \oplus \mathbb{Q}e_{n_1+n_2} \subset \dots \subset \mathbb{Q}^n. \tag{1.72}$$

The parabolic subgroup P_0 also acts on the direct sum of the successive quotients

$$(\mathbb{Q}e_1 \oplus \dots \oplus \mathbb{Q}e_{n_1}) \bigoplus (\mathbb{Q}e_{n_1+1} \oplus \dots \oplus \mathbb{Q}e_{n_1+n_2}) \bigoplus \dots \tag{1.73}$$

and this yields a homomorphism $\boxed{\text{RP}}$

$$r_{P_0} : P_0 \to M_0 = \mathrm{Gl}_{n_1} \times \mathrm{Gl}_{n_2} \times \dots \tag{1.74}$$

hence M_0 is the Levi quotient of P_0. By definition the unipotent radical U_{P_0} of P_0 is the kernel of r_0. The semi-simple component will be $M_0^{(1)} = \mathrm{Sl}_{n_1} \times \mathrm{Sl}_{n_2} \times \dots$

A parabolic subgroups $P_0 \supset B_0$ defines a subset

$$\Delta^{P_0} = \{\alpha_{i,j} \in \Delta^{\mathrm{Gl}_n} \mid U_{i,j} \subset P_0\}$$

and the set decomposes int two sets

$$\Delta^{M_0} = \{\alpha_{i,j} \mid U_{i,j} \text{ and } U_{j,i} \subset \Delta^{P_0}\};\ \Delta^{U_{P_0}} = \Delta^{P_0} \setminus \Delta^{M_0}. \tag{1.75}$$

Intersecting this decomposition with the set π^{Gl_n} yields a disjoint decomposition

$$\pi^{\mathrm{Gl}_n} = \pi^{M_0} \cup \pi^{U} \tag{1.76}$$

where $\pi^U = \{\alpha_{n_1}, \alpha_{n_1+n_2}, \dots, \}$. In turn any such decomposition of π^{Gl_n} yields a well defined parabolic $P_0 \supset B_0$.

We define the *index* of a parabolic subgroup this is the number

$$d(P) = \#\pi^U \tag{1.77}$$

The proper maximal parabolic subgroups are the ones with $d(P) = 1$.

If we choose another maximal split torus T_1 and a Borel subgroup $B_1 \supset T_1$ then this amounts to the choice of a second ordered basis $v_1, v_2, \dots, v_n$ the v_i are given up to a non zero scalar factor. We can find a $g \in \mathrm{Gl}_n(\mathbb{Q})$ which maps $e_1, e_2, \dots, e_n$ to $v_1, v_2, \dots, v_n$, and hence we can conjugate the pair (B_0, T_0) to (B_1, T_1) and hence the parabolic subgroups containing B_0 into the parabolic subgroups containing B_1. The conjugating element g also identifies

$$i_{T_0,B_0,T_1,B_1} : X^*(T_0) \xrightarrow{\sim} X^*(T_1)$$

and this identification does not depend on the choice of the conjugating element g. This allows us to identify the two set of positive simple roots $\pi^{\mathrm{Gl}_n} \subset X^*(T_0)$ and $\pi \subset X^*(T_1)$. Eventually we can speak of the set π of simple roots of Gl_n. Hence we have the fundamental fact

The $\mathrm{Gl}_n(\mathbb{Q})$ *conjugacy classes of parabolic subgroups* $P/\mathbb{Q}$ *are in one to one correspondence with the subsets* $\pi^M \subset \pi$. *Then number of elements in* $\pi \setminus \pi^M = \pi^U$ *is called the rank of* P, *the set* π^U *is called the type of* P.

We will denote the unipotent radical of P by U_P and the reductive quotient of P by U_P will be denoted by $M_P = P/U_P$. Later we will also use a slightly different notation: If we discuss a given P then we put $U (= U_P)$ for the unipotent radical and $M = P/U$ for the reductive quotient. Then we also put $\pi' = \pi^U$.

We formulated this result for $\mathrm{Gl}_n/\mathbb{Q}$ but we can replace $\mathbb{Q}$ by any field k and Gl_n by any reductive group G/k. We have to define the system of relative simple positive roots π^G for any G/k (See [B-T]). We also refer to section 1.1.5.

The group G/k itself is also a parabolic subgroup it corresponds to $\pi' = \pi$. We decide that we do not like it and hence we consider only proper parabolic subgroups $P \neq G$, i.e. $\pi' \neq \emptyset$. We can define the Grassmann variety $\mathrm{Gr}^{[\pi']}$ of parabolic subgroups of type π' This is a smooth projective variety and $\mathrm{Gr}^{[\pi']}(\mathbb{Q})$ is the set of parabolic subgroups of type π'.

There is always a unique minimal conjugacy class it corresponds to $\pi' = \emptyset$. (In our examples this minimal class is given by the maximal flags, i.e. those flags where the dimension of the subspaces increases by one at each step (until we reach a maximal isotropic space in the case of an orthogonal group)). The (proper) maximal parabolic subgroups are those for which $\pi' = \pi \setminus \{\alpha_i\}$, i.e. $\pi^{U_{P_i}} = \{\alpha_i\}$

For any parabolic subgroup $P/\mathbb{Q} \subset G/\mathbb{Q}$ we consider the character module $X^*(P) := \mathrm{Hom}(P/\mathbb{Q}, \mathbb{G}_m)$. Since we do not have any non trivial homomorphisms from the unipotent U_P to $\mathbb{G}_m$ we have $\mathrm{Hom}(P/\mathbb{Q}, \mathbb{G}_m) = \mathrm{Hom}(M_P, \mathbb{G}_m)$.

The reductive quotient $M_P = M_P^{(1)} \cdot C_P$ where C_P is the central torus und $M_P^{(1)}$ the semi-simple part (the derived group). The quotient $M_P/M_P^{(1)} = C'_P$ is a torus and $C_P \to C'_P$ is an isogeny. Hence we have

$$\mathrm{Hom}(P/\mathbb{Q}, \mathbb{G}_m) \otimes \mathbb{Q} = \mathrm{Hom}(M_P, \mathbb{G}_m) \otimes \mathbb{Q} = \mathrm{Hom}(C_P, \mathbb{G}_m) \otimes \mathbb{Q} = \mathrm{Hom}(C'_P, \mathbb{G}_m) \otimes \mathbb{Q} \tag{1.78}$$

For a maximal parabolic subgroup P of type $\pi' = \{\alpha_i\}$ we consider the module $\mathrm{Hom}(P, \mathbb{G}_m) \otimes \mathbb{Q} \subset X^*(T) \otimes \mathbb{Q}$. Of course it always contains the determinant and

$$\mathrm{Hom}(P, \mathbb{G}_m) \otimes \mathbb{Q} = \mathbb{Q}\gamma_i \oplus \mathbb{Q}\det$$

where γ_i is

$$\gamma_i(t) = (\prod_{\nu=1}^{\nu=i} t_\nu) \det(t)^{-i/n}. \tag{1.79}$$

These γ_i are called the dominant fundamental weights.

If our maximal parabolic subgroup is $P/\mathbb{Q}$ is defined as the stabilizer of a flag $0 \subset W \subset V = \mathbb{Q}^n$, then the unipotent radical is $U = \mathrm{Hom}(V/W, W)$. An element $y \in P(\mathbb{Q})$ induces linear maps $y_W, y_{V/W}$ and hence $\mathrm{Ad}(y)$ on $U = \mathrm{Hom}(V/W, W)$. We get a character $\gamma_P(y) = \det(\mathrm{Ad}(y)) \in \mathrm{Hom}(P, \mathbb{G}_m)$ which is called the sum of the positive roots. An easy computation shows that

$$n\gamma_i = \gamma_P \tag{1.80}$$

We add points at infinity to our symmetric space: We consider the disjoint union $\cup_{\pi \neq \pi_G} \mathrm{Gr}^{[\pi']}(\mathbb{Q})$ and form the space

$$\overline{X} = X \cup \bigcup_{\pi' \neq \emptyset} \mathrm{Gr}^{[\pi']}(\mathbb{Q}).$$

This is the analogue of or $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ in our first example, it is now more complicated because we have several Grassmannians, and we also have maps

$$r_{\pi_1,\pi_2} \mathrm{Gr}^{[\pi_1]}(\mathbb{Q}) \to \mathrm{Gr}^{[\pi_2]}(\mathbb{Q}) \text{ if } \pi_2 \subset \pi_1.$$

Our first aim is to put a topology on this space such that $\Gamma \backslash \overline{X}$ becomes a compact Hausdorff space.

In our first example we interpreted the Farey circle $D\left(c, \frac{p}{q}\right)$ with $0 < c < 1$ as an open subset of points in $\mathbb{H}$, which are close to the point $\frac{p}{q}$, but far away from any other point in $\mathbb{P}^1(\mathbb{Q})$.

The point of reduction theory is that for any parabolic $P \in \mathrm{Gr}^{[\pi']}(\mathbb{Q})$ (here we also allow $P = G$) we will define open sets

$$X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'})) \subset X \tag{1.81}$$

which depend on certain parameters $\underline{c}_{\pi'}, r(\underline{c})_{\pi'}$ The points in $X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'}))$ should be viewed as the points, which are "very close" to the parabolic subgroup P (controlled by $\underline{c}_{\pi'}$) but "keep a certain distance" (controlled by $r(\underline{c}_{\pi'})$) to the parabolic subgroups $Q \not\supset P$. They are the analogues of the Farey circles. We will see:

a)This system of open sets is invariant under the action $\mathrm{Gl}_n(\mathbb{Z})$

b) For $P = G$ the set $X^G(\emptyset, r_0)$ is relatively compact modulo the action of $\mathrm{Gl}_n(\mathbb{Z})$.

c) Any subgroup $\Gamma \subset \mathrm{Gl}_n(\mathbb{Z})$ has only finitely many orbits on any $\mathrm{Gr}^{[\pi']}(\mathbb{Q})$

d) For a suitable choice of the parameters $\underline{c}_{\pi'}$, and $r(c_{\pi'})$ we have :

$$X = \bigcup_P X^P(\underline{c}_{\pi'}, r(c_{\pi'})) = X^G(\emptyset, r_0) \cup \bigcup_{P:P\text{proper}} X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'}))$$

and if P and P_1 are conjugate and $P \neq P_1$ then $X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'})) \cap X^{P_1}(\underline{c}_{\pi'}, r(\underline{c}_{\pi'})) = \emptyset$.

Let us assume that we have constructed such a system of open sets, then c) and d) impliy that for a given type π' we have

$$\Gamma \backslash \bigcup_{P:\text{type}(\pi')=\pi} X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'})) = \bigcup \Gamma_{P_i} \backslash X^{P_i}(\underline{c}_{\pi'}, r(\underline{c}_{\pi'}))$$

where $\{\dots, P_i, \dots\} = \Sigma(\pi, \Gamma)$ is a set of representatives of $\mathrm{Gr}^{[\pi']}(\mathbb{Q})$ modulo the action of Γ and $\Gamma_{P_i} = \Gamma \cap P_i(\mathbb{Q})$.

This tells us that we have a covering

$$\Gamma \backslash X = \Gamma \backslash X^G(\emptyset, r_0) \cup \bigcup_{\pi' \neq \emptyset} \bigcup_{P \in \Sigma(\pi', \Gamma)} \Gamma_P \backslash X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'})) \tag{1.82}$$

The philosophy of reduction theory is that $\Gamma \backslash X^G(\emptyset, r_0)$ is relatively compact and that we have an explicit description of the sets $\Gamma_P \backslash X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'}))$ as fiber bundles with nil manifolds as fiber over the locally symmetric spaces $\Gamma_M \backslash X^M$.

We give the definition of the sets $X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'}))$. We stick to the case that $G = \mathrm{Gl}_n/\mathbb{Q}$ and $\Gamma \subset \Gamma_0 = \mathrm{Gl}_n(\mathbb{Z})$ is a (congruence) subgroup of finite index. We defined the positive definite bilinear form (See 1.48)

$$\tilde{B}_{\Theta_x} = -\frac{1}{2n} B_{\Theta_x} : \mathfrak{g}_\mathbb{R} \times \mathfrak{g}_\mathbb{R} \to \mathbb{R}$$

and we have the identification $\mathfrak{g}_{\mathbb{R}} \xrightarrow{\sim} T_e^{\mathbb{G}(\mathbb{R})}$, and hence we get a euclidian metric on the tangent space $T_e^{\mathbb{G}(\mathbb{R})}$ at the identity e. This extends to a left invariant Riemannian metric on $G(\mathbb{R})$, we denote it by $d_{\Theta_x}s^2$. Hence we get a volume form $d^{\Theta_x}_{\mathrm{vol_H}}$ on any closed subgroup $H(\mathbb{R}) \subset G(\mathbb{R})$.

For any point $x \in X$ and any parabolic subgroup $P/\mathbb{Q}$ with unipotent radical $U/\mathbb{Q}$) we define

$$p_P(P,x) = \mathrm{vol_U}^{\Theta_x}(\Gamma_0 \cap U(\mathbb{R}))\backslash U(\mathbb{R})) \tag{1.83}$$

For the Arakelow-Chevalley scheme $(\mathrm{Gl}_n/\mathbb{Z}, \Theta_0)$ See(1.1.8) we have that $\tilde{B}_{\Theta_0}(E_{i,j}) = 1$. We have by construction

$$U_{i,j}(\mathbb{Z})\backslash U_{i,j}(\mathbb{R}) = \mathbb{R}/\mathbb{Z} \tag{1.84}$$

and under this identification $E_{i,j}$ maps to $\frac{\partial}{\partial x}$. Hence we get

$$d^{\Theta_0}_{\mathrm{vol_{U_{i,j}}}}(U_{i,j}(\mathbb{Z})\backslash U_{i,j}(\mathbb{R})) = 1$$

and from this we get immediately

Proposition 1.2.1. *For any parabolic subgroup P_0 containing the torus T_0 we have*

$$p_P(P_0, \Theta_0) = 1.$$

Let $(L, <\ ,\ >_x)$ be an Arakelow vector bundle and $(\mathrm{Gl}_n, \Theta_x)$ the corresponding Arakelow group scheme (of type Gl_n) let

$$\mathcal{F}_{\mathbb{Z}} : (0) = \{(0)\} = L_0 \subset L_1 \subset L_2 \subset \ldots \subset L_k = L = \mathbb{Z}^n$$

be a flag and $P/\mathbb{Z}$ the corresponding parabolic subgroup. Then we have the homomorphism

$$r_P : P/\ \mathrm{Spec}(\mathbb{Z}) \to M/\mathbb{Z} = \prod_{i=1}^{i=k} \mathrm{Gl}(L_i/L_{i-1}) \tag{1.85}$$

with kernel $U_P/\mathbb{Z}$. The metric $<\ ,\ >_x$ on $L \otimes \mathbb{R}$ yields an orthogonal decomposition

$$L \otimes \mathbb{R} = \bigoplus_{i=1}^{i=k} L_i/L_{i-1} \otimes \mathbb{R}$$

and hence an Arakelow bundle structure $(L_i/L_{i-1}, (\Theta_x)_i)$ for all i, and therefore an Arakelow group scheme structure on $M/\mathbb{Z}$.

Hence we get

Proposition 1.2.2. *If (Gl_n, Θ) is an Arakelow group scheme then Θ induces an Arakelow group scheme structure Θ^M on any reductive quotient $M = P/U$.*

Definition : A pair $(\mathrm{Gl}_n/\mathbb{Z}, \Theta)$ is called *stable (resp. semi stable)* if for any proper parabolic subgroup $P/\mathbb{Q} \subset \mathrm{Gl}_n/\mathbb{Q}$ we have

$$p_P(P, \Theta) > 1 \text{ resp.} p_P(P, \Theta) \geq 1 \tag{1.86}$$

In our example in (1.2.6) the stable points are those outside the union of the closed Farey circles.

To get a better understanding of these numbers we have to do some computations with roots and weights. Let us start from an Arakelow vector bundle $(L=\mathbb{Z}^d,<\ ,\ >)$ and let us assume that L is equipped with a complete flag

$$\mathcal{F}_0=\{)\}=L_0\subset L_1\subset\cdots\subset L_{d-1}\subset L_d \tag{1.87}$$

which defines a Borel subgroup $B/\mathbb{Z}$. The quotients $(L_i/L_{i-1},<\ ,\ >_i)$ are Arakelow line bundles over $\mathbb{Z}$ or in a less sophisticated language they are free modules of rank one and the generating vector $\bar{e}_i$ has a length $\sqrt{<\bar{e}_i,\bar{e}_i>_i}$. This length is of course also the volume of $(L_i/L_{i-1}\otimes\mathbb{R})/(L_i/L_{i-1})$.

The unipotent radical $U/\mathbb{Z}\subset B/\mathbb{Z}$ has a filtration $\{(0)\}\subset V_1\subset\ldots,V_{n(n-1)/2-1}\subset V_{n(n-1)/2}=U$ by normal subgroups, the successive quotients are isomorphic to $\mathbb{G}_a$ and the torus $T=B/U$ acts by a positive root $\alpha_{i,j}$ and this is a one to one correspondence between the subquotients and the positive roots. Then it is clear: If ν corresponds to (i,j) then

$$(V_\nu/V_{\nu+1},\Theta_\nu)=(L_i/L_{i-1},<\ ,\ >_i)\otimes(L_j/L_{j-1},<\ ,\ >_j)^{-1}. \tag{1.88}$$

Moreover the quotients $(V_\nu/V_{\nu+1},\Theta_\nu)$ depend only on the conformal class of $<\ ,\ >$ and hence only on the resulting Cartan involution Θ.

The unipotent subgroup $U/\mathbb{Z}$ contains the one parameter subgroup $U_{i,j}/Z$ and this one parameter subgroup maps isomorphically to $(V_\nu/V_{\nu+1})$. Hence our construction defines the Arakelow line bundle $(U_{i,j},\Theta_{i,j})$.

If we now define $n_{\alpha_{i,j}}(B,x)=\ \mathrm{vol}_{\Theta_{i,j}}(U_{i,j}(\mathbb{R})/U_{i,j}(\mathbb{Z}))$ then it is clear that

$$p_B(B,x)=\prod_{i<j}n_{\alpha_{i,j}}(B,x) \tag{1.89}$$

If $P\supset B$ then its unipotent radical $U_P\subset U$ and we defined the set Δ^{U_P} as the set of positive roots for which $U_{i,j}\subset U_P$. Then we have

$$p_P(B,x)=\prod_{(i,j)\in\Delta^{U_P}}n_{\alpha_{i,j}}(B,x) \tag{1.90}$$

Here it is important to notice the right hand side does not depend on the choice of $B\subset P$.

We follow a convention and put $2\rho_P=\sum_{(i,j)\in\Delta^{U_P}}\alpha_{i,j}$ so that ρ_P is the half sum of positive roots in in the unipotent radical. Formula (1.80) tells us that for any maximal parabolic subgroup P_{i_0} $\boxed{\text{rhoP}}$

$$2\rho_{P_{i_0}}=\sum_{i\le i_0,j\ge i_0+1}\alpha_{i,j}=n\gamma_{i_0}. \tag{1.91}$$

For any $\gamma=\sum\alpha_{i,i+1}\otimes z_i\in X^*(T)\otimes\mathbb{C}$ we define the homomorphism

$$|\gamma|:T(\mathbb{R})\to\mathbb{C}^\times:|\gamma|:t\to\prod_i|\alpha_{i,i+1}(t)|^{z_i} \tag{1.92}$$

Since the numbers $n_{\alpha_{i,j}}(B,x)$ are positive real numbers we define for any

$$n_\gamma(B,x) = \prod_{i=1}^{n-1} n_{\alpha_{i,j}}(B,x). \tag{1.93}$$

Here we see that the second argument is a Borel-subgroup B. But if the above character $\gamma : B(\mathbb{R}) \to \mathbb{R}^\times_{>0}$ extends to a character $\gamma : P(\mathbb{R}) \to \mathbb{R}^\times_{>0}$ then we can define

$$n_\gamma(P,x) := n_\gamma(B,x)$$

and this number only depends on P and not on the Borel subgroup $B \subset P$. The characters in $\gamma \in X^*(T)$ for which $|\gamma|$ extend to $P(\mathbb{R})$are exactly the linear combinations (See (1.95) below) $\gamma = \sum_{\alpha_i \in \pi^U} x_i\gamma_i$. The characters $\gamma_P = \sum_{\alpha_i \in \pi^U} r_i\gamma_i$ where the $r_i > 0$ are rational numbers. Let P_i be the maximal parabolic subgroup of type $\pi \setminus \{\alpha_i\}$ containing P then the above formula implies that

$$p_P(P,x) = \prod_{\alpha_i \in \pi^U} n_{\gamma_i}(P_i,x)^{r_i} = \prod_{\alpha_i \in \pi^U} p_{P_i}(P_i,x)^{\frac{r_i}{n}} \tag{1.94}$$

This tells us

The Arakelow scheme $(\mathrm{Gl}_n/\mathbb{Z}, \Theta)$ *is stable if for all maximal parabolic subgroups* $p_{P_i}(P_i,\Theta) = n_{\gamma_i}(P_i,\Theta)^n > 1$.

We need a few more formulas relating roots and weights. For any parabolic subgroup we have the division of the set of simple roots into two parts

$$\pi = \pi^M \cup \pi^{U_P}.$$

This induces a splitting of the character module $\boxed{\text{split}}$

$$X^*(T) \otimes \mathbb{Q} = \bigoplus_{\alpha_i \in \pi^M} \mathbb{Q}\alpha_i \oplus \bigoplus_{\alpha_i \in \pi^{U_P}} \mathbb{Q}\gamma_i \tag{1.95}$$

where γ_i is the dominant fundamental weight attached to α_i (See (1.79)).

If now $\alpha_i \in \pi^{U_P}$ then we can project α_i to the second component, this projection

$$\alpha_i^P = \alpha_i + \sum_{\alpha_\nu \in \pi^M} c_{i,\nu}\alpha_\nu \tag{1.96}$$

Here an elementary - but not completely trivial - computation shows that

$$c_{i,\nu} \geq 0 \tag{1.97}$$

Since $\alpha_i^P \in \oplus \bigoplus_{\alpha_i \in \pi^{U_P}} \mathbb{Q}\gamma_i$ these characters extend to $P(\mathbb{R})$ and hence $n_{\alpha_i^P}(P,x)$ is defined.

We state the two fundamental theorems of reduction theory

Theorem 1.2.1. *For any Arakelow group scheme* $(\mathrm{Gl}_n, \Theta_x)$ *we can find a Borel subgroup* $B \subset \mathrm{Gl}_n$ *for which*

$$n_{\alpha_i}(B, \Theta_x) = n_{\alpha_i}(B, x) < \frac{2}{\sqrt{3}} \text{ for all } i = 1, \dots, n-1$$

Theorem 1.2.2. *For any Arakelow group scheme* (Gl_n, Θ) *we can find a a unique parabolic subgroup* P *such that for all* $\alpha_i \in \pi^{U_P}$ *we have*

$$n_{\alpha_i^P}(P, \Theta) < 1 \text{ and the reductive quotient } (M, \Theta^M) \text{ is semi stable.}$$

The first theorem is due to Minkowski, the second theorem is proved in [Stu], [Gray].

This parabolic subgroup is called the *canonical destabilizing* group. We denote it by $P(x)$, if (G, x) is semi stable then $P(x) = G$. This gives us a dissection of X into the subsets

$$X = \bigcup_{P:\ \text{parabolic subgroups of } G/\mathbb{Q}} X^{[P]} = \{x \in X \mid P(x) = P\} \tag{1.98}$$

Clearly $\gamma X^{[P]} = X^{[\gamma P \gamma^{-1}]}$, if we divide by the group Γ the we get

$$\Gamma \backslash X = \bigcup_{P \in \mathrm{Par}(\Gamma)} \Gamma_P \backslash X^{[P]} \tag{1.99}$$

where Par (Γ) is a set of representatives of Γ conjugacy classes of parabolic subgroups of $\mathrm{Gl}_n/\mathbb{Q}$. This is a decomposition of $\Gamma\backslash X$ into a disjoint union of subsets. The subset $\Gamma\backslash X^{[\mathrm{Gl}_n]}$ is compact, it is the set of semi stable pairs (x, Gl_n), the subsets $\Gamma_P\backslash X^{[P]}$ for $P \neq G$ are in a certain sense "open in some directions" and "closed in some other direction". Therefore this decomposition is not so useful for the study of cohomology groups.

To remedy this we introduce larger subsets. For a real number $r, 0 < r < 1$ we define $\boxed{\text{Gstable}}$

$$X^{\mathrm{Gl}_n}(r) = \{x \in X \mid n_{\gamma_\alpha}(P(x), x) > r, \text{ for all } \alpha \in \pi^{U_{P(x)}}). \tag{1.100}$$

It contains the set of semi-stable (Gl_n, x) If we choose $r < 1$ but close to one then some of the elements in $X^{\mathrm{Gl}_n}(r)$ may be unstable but only a "little bit ".

Together with the first theorem this has a consequence

Proposition 1.2.3. *The quotient* $X^{\mathrm{Gl}_n}(r) = \Gamma\backslash X^{\mathrm{Gl}_n}(r)$ *is relatively compact open subset of* $\Gamma\backslash X$. *It contains the set of semi-stable* (Gl_n, x).

Now we start from a parabolic subgroup P and let $M = P/U_P$ be its Levi-quotient. Our considerations above also apply to $M/\mathbb{Q}$. The group $P(\mathbb{R})$ acts transitively on X and we put (See (1.85))

$$X^M = U_P(\mathbb{R})\backslash X \text{ and let } q_M : X \to X^M \text{ be the projection .}$$

Here $X^M = M(\mathbb{R})/K_\infty^M$ where K_∞^M is the image of $P(\mathbb{R}) \cap K_\infty$ in $M(\mathbb{R})$. Let $Z \subset M$ be the center of M, it is a (split) torus and as usual $M^{(1)}$ the semi simple component. Then we define $X^{M^{(1)}} = M^{(1)}(\mathbb{R})/K_\infty^M$ and if Γ_M is the image of $P(\mathbb{R})\cap\Gamma$ in $M(\mathbb{R})$ then $\boxed{\text{symM}}$

$$\Gamma_M\backslash X^M := \Gamma_M\backslash^{M^{(1)}} \times Z^{(0)}(\mathbb{R}) \tag{1.101}$$

where of course $Z^{(0)}(\mathbb{R})$ is the connected component of the identity of $Z(\mathbb{R})$,

We change the notation and put

$$Z^{(0)}(\mathbb{R}) = A_{\pi'} = \{(\dots, a_\alpha, \dots)_{\alpha\in\pi'=\pi^M\setminus\pi^M} | a_\alpha \in \mathbb{R}^\times_{>0}\} \tag{1.102}$$

For a simple roots $\alpha\in\pi^M$, and a Borel subgroup $\bar{B}\subset M/\mathbb{Q}$ and a point $x_M = q_M(x)$ we can define the numbers $n_\alpha(\bar{B}, x_M)$ essentially in the same way as before and clearly

$$n_\alpha(\bar{B}, x_M) = n_\alpha(B, x)$$

if B is the inverse image of $\bar{B}$.

We have to be a little bit careful with the numbers $p_{\bar{Q}}(\bar{Q}, x^M)$ because the for the inverse image Q the unipotent radical U_Q is larger than $U_{\bar{Q}}$. Therefore we have to look at the dominant fundamental weights $\gamma_\alpha^M \in \bigoplus_{\alpha_i\in\pi^M}\mathbb{Q}\alpha_i$, and formulate the stability condition for x^M in terms of these γ_α^M :

The point x^M is stable, if for all $\alpha_i\in\pi^M$ the inequality $n_{\gamma^M_{\alpha_i}}(\bar{P}_{\alpha_i}, x^M,) > 1$ holds. Again we denote the destabilizing group by $P(x^M)$.

Hence we see that for a number $r_M < 1$ we can define regions

$$X^M(r_M) = \{x^M | n_{\gamma^M_{\alpha_i}}(\bar{P}_{\alpha_i}, x^M) > r_M \text{ whenever } \bar{P}_{\alpha_i} \supset \bar{P}(x^M)\} \tag{1.103}$$

We choose numbers $0 < c_P < 1$, furthermore we choose a number $r(c_P) < 1$ and define

$$^*X^P(c_P, r(c_P)) = \{x|\ n_{\alpha^P}(P,x) < c_P \text{ for all } \alpha\in\pi^{U_P}; x_M \in X^M(r(c_P))\} \tag{1.104}$$

Proposition 1.2.4. *For a given $r(c_P) < 1$ we can find numbers c_P such that for any $x\in {}^*X^P(c_P, r(\underline{c}_P))$ the destabilising parabolic subgroup $P(x)\subset P$. The same is true in the other direction: For a given $0 < c_P < 1$ we can find $r < 1$ such that for $x \in {}^*X^P(c_P, r))$ the destabilising parabolic subgroup $P(x)\subset P$.*

To see this we have to look at the destabilising subgroup $\bar{Q}\subset(M, x_M)$. Its inverse image $Q\subset P$ is a parabolic subgroup of Gl_n. The reductive quotient $(\bar{M}, x_{\bar{M}})$ of Q is semi- stable. We want to show that Q is the destabilising parabolic of (Gl_n, x). We have to show that

$$n_{\alpha^Q}(Q,x) < 1\ \forall\ \alpha\in\pi^{U_Q} = \pi^{U_P}\cup\pi^{U_{\bar{Q}}}.$$

For $\alpha\in\pi^{U_{\bar{Q}}}$ this is true by definition. For $\alpha\in\pi^{U_P}$ we have

$$\alpha^P = \alpha + \sum_{\beta\in\pi^M} a_{\alpha,\beta}\beta \text{ and } \alpha^Q = \alpha + \sum_{\beta'\in\pi^{\bar{M}}} a'_{\alpha,\beta'}\beta,$$

where $a_{\alpha,\beta} \geq 0$. The roots $\beta \in \pi^{U_{\bar{Q}}}$ can be expressed in terms of the $\beta^{\bar{Q}} = \beta^Q$:

$$\beta^Q = \beta + \sum_{\beta' \in \pi^{\bar{M}}} a^*_{\beta,\beta'} \beta' \tag{1.105}$$

and hence

$$\alpha^Q = \alpha^P - \sum_{\beta \in \pi^{U_{\bar{Q}}}} a_{\alpha,\beta} \beta^Q + \sum_{\beta' \in \pi^{\bar{M}}} c_{\alpha\beta'} \beta'. \tag{1.106}$$

The last sum is zero because $\alpha^Q, \alpha^P, \beta^Q$ are orthogonal to the module $\oplus_{\beta'} \mathbb{Z}\beta'$.
We get the relation

$$n_{\alpha^Q}(Q, x) = n_{\alpha^P}(P, x,) \cdot \prod_{\beta \in \pi^{U_{\bar{Q}}}} n_{\beta^Q}(Q, x)^{-a_{\alpha,\beta}}. \tag{1.107}$$

Now it comes down to show that $\boxed{\text{wc}}$

$$\begin{aligned} n_{\alpha^P}(P, x) < c_\alpha, \ \forall\, \alpha \in \pi^{U_P} \text{ and } n_{\beta^Q}(Q, x) > r, \ \forall \beta \ \in \pi^{U_{\bar{Q}}} \\ \Longrightarrow n_{\alpha^Q}(P, x) < 1\,; \forall\, \alpha \in \pi^{U_P} \end{aligned} \tag{1.108}$$

This is certainly true if either the c_α are small enough or if r is sufficiently close to one. In this case we say that (P, c, r) is well chosen
Therefore we define

$$X^P(c_P, r(c_P)) = \{x \in {}^*X^P(c_P, r(c_P)) | P(x) \subset P\} \tag{1.109}$$

we have $X^P(c_P, r(c_P)) = {}^*X^P(c_P, r(c_P))$, if $(c_P, r(c_P))$ is well chosen.

We claim that we can find a family of parameters

$$(\ldots, (c_P, r(c_P)), \ldots)_{P\text{: parabolic over } \mathbb{Q}}$$

where $(c_P, r(c_P))$ only depend on the type of P, such that we get a covering $\boxed{\text{COV}}$

$$X = \bigcup_P X^P(c_P, r(c_P))) \tag{1.110}$$

and hence
$\boxed{\text{UD}}$

$$\Gamma \backslash X = \Gamma \backslash \bigcup_P X^P(c_P, r(c_P)) = \bigcup_{P \in \mathrm{Par}(\Gamma)} \Gamma_P \backslash X^P(c_P, r(c_P)) \tag{1.111}$$

We change the notation slightly, since these numbers only depend on the type $\pi' = \pi^M = t(P)$ we replace c_P by $c_{\pi'}$ and $r(c_P)$ by $r(c_{\pi'})$. We even go one step further and denote a well chosen pair $(c_{\pi'}, r(c_{\pi'}))$ simply by $(\underline{c}_{\pi'})$.

To prove the claim we choose a number $0 < c_\emptyset < 1$. In this case $r_0 = r(c_\emptyset)$ can be any number. Then we choose a number $0 < r_1 < c_\emptyset$. For any $\pi_i = \{\alpha_i\}$

we choose a $c_{\pi_i} < 1$ such that (c_{π_i}, r_1) is well chosen. We continue and chose $0 < r_2 < c_{\pi_i}$ for all i and for any two element subset $J \subset \pi$ we choose numbers $0 < c_J < 1$ such that (c_J, r_2) is well chosen. This goes until we reach top parabolic.

Now we get a covering of X by the open sets $X^P(c_\pi, r(\pi))$. To see this we pick a point $x \in X$, we have to show that it lies in at least one of the sets $X^P(c_P, r(c_P))$. If it is not in $X^{\mathrm{Gl}_n}(r_{n-1})$ then we find a maximal parabolic P_i such that $n_{\alpha_i}(P_i, x) < c_{\pi \setminus \{\alpha_i\}}$. We project x to the point $x^{M_i} \in X^{M_i}$. If this point is in $X^{M_i}(r_{n-2})$ then $x \in X^{P_i}(c_{\pi\setminus\{\alpha_i\}}, r_{n-2})$ and we are done. If not we apply our argument above to x^{M_i} and $\pi' = \pi \setminus \{\alpha_i\}$. We continue the same reasoning and at latest it stops for $\pi' = \emptyset$.

It is clear that we can choose $c_\pi^+, r(c_{\pi'})^+$ a tiny bit larger and these numbers are still well chosen. Then the closure $\overline{\Gamma\backslash \bigcup_P X^P(c_{\pi'}, r(c_{\pi'}))} \subset \Gamma\backslash \bigcup_P X^P(c^+_{\pi'}, r(c^+_{\pi'}))$. We get a second covering by slightly larger open sets. This can be used to produce a partition of unity

Proposition 1.2.5. *We can find a family of $\mathcal{C}_\infty$ functions $h_P \geq 0$ which satisfies*

a) h_P restricted to $\Gamma_P\backslash X^P(c_{\pi'}, r(c_{\pi'})$ is identically equal to one

b) $\sum_P h_P = 1$ and h_P is zero outside of $\Gamma_P\backslash X^P(c^+_{\pi'}, r(c^+_{\pi'}))$

Proof. Well known □

We have a very explicit description of these sets $\Gamma_P\backslash X^P(c_{\pi'}, r(c_{\pi'}))$. We consider the evaluation map

$$\begin{gathered} n^{\pi'}: \Gamma_P\backslash X^P(c_{\pi'}, r(c_{\pi'})) \to \prod_{\alpha\in\pi'}(0, c_\alpha) \\ x \mapsto (\dots, n_{\alpha^P}(P, x), \dots)_{\alpha\in\pi'} \end{gathered} \tag{1.112}$$

Of course we also have the homomorphism

$$|\alpha^{\pi'}| : P(\mathbb{R}) \to \{\dots, |\alpha^P|, \dots\}_{\alpha\in\pi'} \tag{1.113}$$

and the multiplication by an element $y \in P(\mathbb{R})$ induces an isomorphisms of the fibers

$$(n^{\pi'})^{-1}(c_1) \xrightarrow{\sim} (n^{\pi'})^{-1}(c_2) \text{ if } |\alpha^{\pi'}|(y) \cdot c_1 = c_2$$

where the multiplication is taken componentwise. This identification depends on the choice of y.

Delete up to signum

To get a canonical identification we use the geodesic action which is introduced in the paper by Borel and Serre. We define an action of $A = (\prod_{\alpha\in\pi\setminus\pi'} \mathbb{R}^\times_{>0})$ on X. This action depends on P and we denote it by

$$(a, x) \mapsto a \bullet x \tag{1.114}$$

A point $x \in X$ defines a Cartan involution Θ_x and then the parabolic subgroup P^{Θ_x} of $G \times \mathbb{R}$ is opposite to $P \times \mathbb{R}$ and $P \times \mathbb{R} \cap P^{\Theta_x} = M_x$ is a Levi factor, the projection $P \to M$ induces an isomorphism

$$\phi_x : M \times \mathbb{R} \xrightarrow{\sim} M_x. \tag{1.115}$$

The character $\alpha^{\pi'}$ induces an isomorphism

$$s_x : A \xrightarrow{\sim} S_x(\mathbb{R})^{(0)}$$

where S_x is the maximal Hence we $S_x(\mathbb{R})^{(0)}$ is the connected component of the identity of the center $M_x(\mathbb{R}) \cap \mathrm{Sl}_n(\mathbb{R})$ and we put

$$a \bullet x = s_x(a)x$$

We have to verify that this is indeed an action. This is clear because for the Cartan-involution $\Theta_{a\bullet x}$ we obviously have

$$P^{\Theta_x} = P^{\Theta_{a\bullet x}}.$$

It is also clear that this action commutes with the action of $P(\mathbb{R})$ on X because

$$ys_x(a)x = s_{yx}(a)yx \text{ for all } y \in P(\mathbb{R}), x \in X.$$

It follows from the construction that the semigroup $A_- = \{\dots, a_\nu, \dots\}$- where $0 < a_\nu \leq 1$ - acts via the geodesic action on $X^P(c_\pi, r(\underline{c}_{\pi'}))$ and that for $a \in A_-$ we get an isomorphism

$$(n^{\pi^{U_P}})^{-1}(c_1) \xrightarrow{\sim} (n^{\pi'})^{-1}(ac_1).$$

This yields a decomposition

$$X^P(c_{\pi'}, r(\underline{c}_{\pi'})) = (n^{\pi^{U_P}})^{-1}(c_0) \times \prod_{\alpha\in\pi'} (0, c_\alpha]$$

where c_0 is an arbitrary point in the product.

Since we know that $|\alpha^{\pi'}|$ is trivial on Γ_P and since the action of P commutes with the geodesic action we conclude

$$\Gamma_P\backslash X^P(c_{\pi'}, r(\underline{c}_{\pi'})) = \Gamma_P\backslash(n^{\pi'})^{-1}(c_0) \times \prod_{\alpha\in\pi'} (0, c_\alpha] \tag{1.116}$$

signum

The roots $\alpha \in \pi'$ factor over the reductive quotient $M = P/U_P$ and hence we get a surjective homomorphism

$$|\alpha^{\pi'}| : M(\mathbb{R}) \to A_{\pi'} := \prod_{\alpha\in\pi'} \mathbb{R}^\times_{>0}.$$

Let $M_1(\mathbb{R})$ be the kernel of this homomorphism. This homomorphism also yields an isomorphism between $Z(\mathbb{R})^{(0)}$ and $A_{\pi'}$ and therefore we get a canonical identification

$$M(\mathbb{R}) = M_1(\mathbb{R}) \times A_{\pi'} \tag{1.117}$$

We put $K_\infty^M = P(\mathbb{R}) \cap K_\infty$ we identify it with its image in $M_1(\mathbb{R})$ and we get again our symmetric space attached to M

$$X^M = M(\mathbb{R})/K_\infty$$

We have the projection map $p_{P,M} : X \to X^M$ where X^M is the space of Cartan involutions on the reductive quotient M. Hence we get a map

$$p^*_{P,M} = p_{P,M} \times n^{\pi_{U_P}} : X \to X^M \times A_{\pi'} \tag{1.118}$$

The group $U_P(\mathbb{R})$ acts simply transitively on the fibers of this projection,

$$q_{P,M} : \Gamma_P\backslash X^P(c_{\pi'}, r(\underline{c}_{\pi'})) \to \Gamma_M\backslash X^M(r(\underline{c}_P)) \times \prod_{\alpha\in\pi'}(0, c_\alpha] \tag{1.119}$$

is a fiber bundle with fiber isomorphic $\Gamma_U\backslash U(\mathbb{R})$. If we pick a point $x \in \Gamma_M\backslash X^M(r(\underline{c}_P))\times \prod_{\alpha\in\pi'}(0, c_\alpha]$ then the identification of $q^{-1}_{P,M}()$ with $\Gamma_U\backslash U(\mathbb{R})$ depends on the choice of a point $\tilde{x} \in X^P(c_{\pi'}, r(\underline{c}_{\pi'}))$ which maps to x.

This can now be compactified *partially in the A direction* we define the closure

$$\overline{\Gamma_P\backslash X^P(c_{\pi'}), r(\underline{c}_P)} := \Gamma_P\backslash(n^{\pi^{U_P}})^{-1}(c_0) \times \prod_{\alpha\in\pi'}[0, c_{\pi'}], \tag{1.120}$$

and

$$\partial\overline{\Gamma_P\backslash X^P(c_{\pi'}, \Omega_\pi)} = \overline{\Gamma_P\backslash X^P(c_{\pi'}, r(c_{\pi'}))} \setminus \Gamma_P\backslash X^P(c_\pi, \Omega_\pi) \tag{1.121}$$

this is equal to

$$\partial\overline{\Gamma_P\backslash X^P(c_{\pi'}, r(\underline{c}_{\pi'})} = \Gamma_P\backslash(n^{\pi^{U_P}})^{-1}(c_0) \times \partial(\prod_{\nu\in\pi_G\backslash\pi}[0, c_\pi])$$

where of course $\partial(\prod_{\alpha\pi'}[0, c_\pi]) \subset \prod_{\alpha\in\pi'}[0, c_\pi]$ is the subset where at least one of the coordinates x_α is equal to zero.

We form the disjoint union of of these boundaries over the π and set of representatives of Γ conjugacy classes, this is a compact space. Now there is still a minor technical point. If we have two parabolic subgroups $Q \subset P$ then the intersection $X^P(\underline{c}_P, r(\underline{c}_{\pi'}) \cap X^Q(\underline{c}_Q, r(\underline{c}_Q)) \neq \emptyset$. If we now have points

$$x \in \partial\overline{\Gamma_P\backslash X^P(c_\pi, r(\underline{c}_{\pi'})}, y \in \partial\overline{\Gamma_Q\backslash X^Q(c_{\pi'}, r(\underline{c}_{P'})}$$

then we identify these two points if we have a sequence of points $\{x_n\}_{n\in\mathbb{N}}$ which lies in the intersection $X^P(c_\pi, r(\underline{c}_{\pi'}))\cap X^Q(c_{\pi'}, r(\underline{c}_{P'}))$ and which converges to x in $\overline{\Gamma_P\backslash X^P(c_\pi, r(\underline{c}_{\pi'})}$ and to y in $\overline{\Gamma_Q\backslash X^Q(c_{\pi'}, r(\underline{c}_{P'})}$. A careful inspection shows that this provides an equivalence relation $\sim$, and we define

$$\partial(\Gamma\backslash X) = \bigcup_{\pi', P\in\mathrm{Par}(\Gamma)} \partial\overline{\Gamma_P\backslash X^P(c_\pi, r(\underline{c}_{\pi'})}/\sim \tag{1.122}$$

and the Borel-Serre compactification will be the manifold with corners

$$\overline{\Gamma\backslash X} = \Gamma\backslash(X \cup \bigcup_{P:P\mathrm{proper}} \overline{X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'}))}. \tag{1.123}$$

We define a "tubular" neighbourhood of the boundary we put

$$\mathcal{N}(\Gamma\backslash X)(\mathbf{c}) = \Gamma\backslash \bigcup_{P:P\text{proper}} \overline{X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'}))} \tag{1.124}$$

where $\mathbf{c}$ stands for the collection of parameters $c_{\pi'}, r(c_{\pi'})$. Then we define the *punctured tubular neighbourhood* as

$\boxed{\text{ptub}}$

$$\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X)(\mathbf{c}) = \Gamma\backslash \bigcup_{P:P\text{proper}} X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'})) = \Gamma\backslash X \cap \mathcal{N}(\Gamma\backslash X) \tag{1.125}$$

We also define the Borel-Serre stratification of the Borel-Serre boundary

$$\partial(\Gamma\backslash\bar{X}) = \bigcup_P \partial_P(\Gamma\backslash\bar{X}) \tag{1.126}$$

where $\partial_P(\Gamma\backslash\bar{X})(r(c_{\pi'}))$ is the the subset of $\overline{\Gamma_P\backslash X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'})}$ where $a_\alpha = 0$ for all $\alpha \in \pi'$ and where we take the union over all l $r(c_{\pi'}) > 0$.

The projection $q_{P,M}$ extends to a fibration on the closure and restriction to the boundary yields the fibration with fibre $\Gamma_U\backslash U(\mathbb{R})$.

$\boxed{\text{fibPM}}$

$$q_{P,M}(0) : \partial_P(\Gamma\backslash\bar{X}) \to \Gamma_M\ X^M \times \{(0)\} \tag{1.127}$$

Eventually we want to use the above covering as a tool to understand cohomology (See section **??**)) For this it is also necessary to understand the intersections

$$X^{P_1}(c_{\pi_1}, r(\underline{c}_{\pi_1})) \cap \cdots \cap X^{P_\nu}(c_{\pi_k}, r(\underline{c}_{\pi_k})) \tag{1.128}$$

Our proposition 1.2.4 implies that for any point x in the intersection the destabilizing parabolic subgroup $P(x) \subset P_1 \cap \cdots \cap P_k$. Hence we see that the above intersection can only be non empty if $Q = P_1 \cap \cdots \cap P_k$ is a parabolic subgroup. Then $\pi^{U_Q} = \cup_{\nu=1}^k \pi^{U_{P_\nu}}$. Let M be the reductive quotient of Q.

Now we look at the product $\prod_{\alpha\in\pi^{U_Q}} \mathbb{R}^\times_{>0}$, here it seems to be helpful to identify it - using the logarithm - with $\mathbb{R}^{d_Q}$:

$$\log : \prod_{\alpha\in\pi^{U_Q}} \mathbb{R}^\times_{>0} \xrightarrow{\sim} \mathbb{R}^{d_Q} \tag{1.129}$$

We consider the map

$$\begin{aligned} N^Q &: X^{P_1}(c_{\pi_1}, r(\underline{c}_{\pi_1})) \cap \cdots \cap X^{P_k}(c_{\pi_k}, r(\underline{c}_{\pi_k})) \to \mathbb{R}^{d_Q} \\ N^Q &: x \mapsto (\ldots, -\log(n_{\alpha^Q}(Q,x)), \ldots)_{\alpha^Q\in\pi^{U_Q}} \end{aligned} \tag{1.130}$$

Consider a point $x \in X^{P_\nu}(c_{\pi_\nu}, r(\underline{c}_{\pi_\nu}))$, for $\alpha \in \pi^{U_{P_\nu}}$ we have

$$-\log(n_{\alpha^{P_\nu}}(P_\nu, x)) \geq -\log(c_{\pi_\nu})$$

We can express $-\log(n_{\alpha^{P_\nu}}(P_\nu,x))$ as a linear combination of the $-\log(n_{\alpha^Q}(Q,x)$, with $\alpha\in\pi^{U_Q}$. This means that the root $\alpha\in\pi^{U_{P_\nu}}$ defines a half space $H^+_\nu(\alpha)$ in $\mathbb{R}^{d_Q}$ and $N^Q(x)\subset H^+_\nu(\alpha)$ in $\mathbb{R}^{d_Q}$.

Now we assume that x is in the intersection (1.128). For the roots $\alpha\in\pi\setminus\pi^{U_{P_\nu}}$ we have the condition (1.103). For the roots $\alpha\in\pi^{U_Q}\setminus\pi^{U_{P_\nu}}$ this yields

$$-\log(n_{\gamma_\alpha^{M_\nu}}(P_\nu,x))\le-\log(r(\pi_\nu)).$$

Therefore we see that the image of N^Q is contained in the intersection of a finite number of half spaces, which are obtained from a finite family of hyperplanes. These hyperplanes depend on the parameters $c_{\pi_\nu},r(\pi_\nu)$, let us call this intersection $C(\underline{c},\underline{r})$, it is a convex -possibly empty- subset of $\mathbb{R}^{d_Q}$.

We investigate the restriction

$$N^Q:X^{P_1}(c_{\pi_1},r(\underline{c}_{\pi_1}))\cap\cdots\cap X^{P_\nu}(c_{\pi_k},r(\underline{c}_{\pi_k}))\to C(\underline{c},\underline{r})$$

We observe that the unipotent radical $U_Q(\mathbb{R})$ acts by left translations on the intersection, we get a diagram

$$\begin{array}{ccc} X^{P_1}(c_{\pi_1},r(\underline{c}_{\pi_1}))\cap\cdots\cap X^{P_\nu}(c_{\pi_k},r(\underline{c}_{\pi_k})) & \to & C(\underline{c},\underline{r}) \\ \downarrow p_M & & \\ X^M\times\mathbb{R}^{d_Q} & \to & \mathbb{R}^{d_Q} \end{array} \tag{1.131}$$

Now it is clear from the definitions that the image of p_M is a set

$$\mathrm{Im}(p_M)=\Omega^M(\underline{c},\underline{r})\times C(\underline{c},\underline{r})$$

where $\Omega^M(\underline{c},\underline{r})\subset X^M$ is a subset containing the set $X^{M,st}$ of semi stable points and is described by certain inequalities as in (1.100). This subset is Γ_M invariant and $\Gamma_M\backslash\Omega^M(\underline{c},\underline{r})$ is relatively compact.

Hence we see that we have essentially the same situation as in (1.119). The map

$$q_M:X^{P_1}(c_{\pi_1},r(\underline{c}_{\pi_1}))\cap\cdots\cap X^{P_\nu}(c_{\pi_k},r(\underline{c}_{\pi_k}))\to\Gamma_M\backslash\Omega^M(\underline{c},\underline{r})\times C(\underline{c},\underline{r}) \tag{1.132}$$

is a fiber bundle with fiber isomorphic to $\Gamma_{U_Q}\backslash U_Q(\mathbb{R})$.

In the following we refer to the book of S. Helgason [51].

We mention an important property of the sets $X^P(\underline{c}_{\pi'},r(c_P))$. We assume that our symmetric space X is connected, then it is well known that it is convex, any two points $p,q\in X$ can be joined by a unique geodesic $[p,q]$. We say that a subset $U\subset X$ is convex if for any two points $p,q\in U$ also the geodesic $[p,q]\subset U$.

Proposition 1.2.6. *Let $\Omega\subset\Omega^M(\underline{c},\underline{r})$ be a convex subset. Then the inverse image $p_M^{-1}(\Omega\times C(\underline{c},\underline{r}))$ is a convex subset of $X^{P_1}(c_{\pi_1},r(\underline{c}_{\pi_1}))\cap\cdots\cap X^{P_\nu}(c_{\pi_k},r(\underline{c}_{\pi_k}))$*

Proof. The assertion is easily reduced to the following:

Let P be a maximal parabolic subgroup, let M be its reductive quotient, let α be the simple root not in π^M and $\Omega\subset X^{M^{(1)}}$. Then the set for any choice of We choose a $c_\alpha > 0$ and claim that $X^P(c_\alpha,\Omega) = \{x \in X \mid n_{\alpha^P}(P,x) \leq c_\alpha\, ; q_M(x) \in \Omega\}$ is convex .

To see this we pick a point $x \in X^P(c_\alpha,\Omega)$, let T_x^X be the tangent space at x. The action of $G(\mathbb{R})$ on X gives us a surjective map $D_x : \mathfrak{g}_\mathbb{R} \to T_{x_0}^X$ and this induces an isomorphism $D_x : \mathfrak{g}_\mathbb{R}/\mathfrak{k}_x \xrightarrow{\sim} T_x^X$, here of course $\mathfrak{k}_x$ is the Lie-algebra of K_x. We get the well known Cartan decomposition of the Lie-algebra

$$\mathfrak{g}_\mathbb{R} = \mathfrak{k}_x \oplus \mathfrak{p}_x \text{ where } \mathfrak{p}_x = \{V \in \mathfrak{g}_\mathbb{R} \mid \Theta_x(V) = -V\} \tag{1.133}$$

and we get the isomorphism $D_x : \mathfrak{p}_x \xrightarrow{\sim} T_x^X$. Starting from our parabolic subgroup P we get a finer decomposition of $\mathfrak{p}_x$.

Let $\mathfrak{P}_\mathbb{R}$ be theLie algebra of $P\times\mathbb{R}$. The intersection $P\times\mathbb{R} \cap \Theta_x(P\times\mathbb{R}) = M_x$ and we get for the Lie algebras $\mathfrak{m}_x = \mathfrak{m}^{(0)} \oplus \mathfrak{a}$ and this gives the finer decoposition $\mathfrak{m}_x = \mathfrak{k}_{x_M} \oplus \mathfrak{p}^{(M_x} \oplus \mathfrak{a}$ and then

$$\mathfrak{p}_x = \mathfrak{p}^{(M_x)} \oplus \mathfrak{a} \oplus \{V - \Theta_x(V)\}_{V\in\mathfrak{u}} \tag{1.134}$$

where $V \in \mathfrak{u}_\mathbb{R}$ and $\mathfrak{a} = \mathbb{R}Y_A$. We normalise Y_A such that $d\alpha^P(Y_A) = 1$.Then we can write a tangent vector T_x^X as image of

$$Y = Y_M + aY_A + (V - \theta(V));$$

We know that there is a unique geodesic $c : \mathbb{R} \to X$ starting at x with $c'(t) = Y$ The theorem 3.3 in Chapter IV in [51] says that this geodesic is $c(t) = \exp(tY) \cdot x$. A tedious computation using the Iwasawa decomposition and the Campell-Hausdorff formula shows that

$$-\log(n_{\alpha^P}(\exp(tY)\cdot x) = -\log(n_{\alpha^P}(x)) + at - a^2 q(Y_A, V)t^2 \tag{1.135}$$

where $q(Y_A,,V)$ is a positive definite form in V.

If now $x_1 \in X^P(c_\alpha,\Omega)$ is a second point, We find a tangent vector $Y = Y_M + aY_A + (V - \theta(V))$ such that $t \mapsto \exp(tY)\cdot x$ is the geodesic joining x and $x_1 = \exp(Y)\cdot x$. If we project these two points to $X^{M^{(1)}}$ then the images $\bar{x}, \bar{x_1} \in \Omega$ and $\exp(t(Y_M)\bar{x}$ is the geodesic in $X^{M^{(1)}}$. and hence for $t \in [0,1]$ we have $\exp(t(Y_M)\bar{x}$. But now

$$-\log(n_{\alpha^P}(x)) \geq -\log(c_\alpha);\ -\log(n_{\alpha^P}(\exp(Y)\cdot x) = -\log((n_{\alpha^P}(x_1) \geq -\log(c_\alpha).$$

Since the second derivative is always > 0 (see(1.135) it follows that $-\log(n_{\alpha^P}(\exp(tY)\cdot x) \geq -\log(c_\alpha)\ \forall\, t \in [0,1]$.

□

We formulated the main theorems of reduction theory only for $\mathrm{Gl}_n/\mathbb{Z}$ because we did not want to use to much from the theory of reductive groups (for instance [15]). But for any ring of algebraic integers $\mathcal{O}$ and for any Chevalley scheme $\mathcal{G}/\mathcal{O}$ we can define the notion of Arakelow group scheme $(\mathcal{G},\Theta_x)$ and we can define the numbers $n_{\alpha_i}(B,\Theta_x)$. Then we can prove the two theorems 1.2.1,1.2.2. For the first theorem we just copy the classical approach. For the second one we refer to the paper [2] of Kai Behrends.

For any semi simple group G/K over a number field $K/\mathbb{Q}$ we can find a normal extension F/K such that $G\times_K F$ splits. Then we can extend $G\times_K F$ to a split i simple group scheme (i.e. a Chevalley scheme) $\mathcal{G}_F/\mathcal{O}_F$. Then we also can a group scheme $\mathcal{G}_K/\mathcal{O}_K$ such that $\mathcal{G}_K(\mathcal{O}_K) = G(K) \cap \mathcal{G}(\mathcal{O}_F)$. Now any arithmetic group $\Gamma \subset G(K)$ is commensurable to $\mathcal{G}(\mathcal{O}_F)$. With these preparations it is easy to show

Theorem 1.2.3. *(Borel-Harish-Chandra): If $G/\mathbb{K}$ is a reductive group and $\Gamma \subset G(\mathbb{Q})$ is an arithmetic subgroup then*

$$\Gamma\backslash X = \Gamma\backslash G(\mathbb{R})/K_\infty$$

is compact if and only if G/K is anisotropic.

Let $\tilde{\Gamma}$ be a $\mathrm{Gal}(F/K)$ invariant arithmetic subgroup of $\mathcal{G}(\mathcal{O}_F)$ and let us assume that $\Gamma = \tilde{\Gamma} \cap G(K)$ is torsion free. Then an easy argument shows that

$$j : \Gamma\backslash X \to \tilde{\Gamma}\backslash \tilde{X}$$

is injective. Then we have for $x \in X$ the numbers and for any $\sigma \in \mathrm{Gal}(F/K)$

$$n_{\alpha_i}(j(x), B, \mathcal{G}(\mathcal{O}_F) \sim n_{\alpha_i}(j(x), B, \mathcal{G}(\mathcal{O}_F)^\sigma)$$

i.e. the ratio is bounded away from zero. And this implies that the destabilising parabolic for $\mathcal{G}(\mathcal{O}_F)$ equals the destabilising parabolic is invariant under the Galois group, i.e. defined over K, provided it is very unstable. Hence it is clear: If the image of j is not compact, then G/K is not anisotropic, and the theorem follows.

Chapter 2

The Cohomology groups

SHCOH

2.1 Cohomology of arithmetic groups as cohomology of sheaves on $\Gamma\backslash X$.

We are now in the position to unify — at least for the special case of arithmetic groups — the two cohomology theories from our chapter II and chapter IV in [39].

We start from a semi simple group $G/\mathbb{Q}$ and we choose an arithmetic congruence subgroup $\Gamma \subset G(\mathbb{Q})$. Let $X = G(\mathbb{R})/K_\infty$ as before. A second datum will be a Γ- module $\mathcal{M}$, in principle this can be any Γ- module.

To such a $\Gamma-$ module we attach a sheaf $\tilde{\mathcal{M}}$ on $\Gamma\backslash X$. This sheaf has values in the category of abelian groups. For any open subset $U \subset X$ we have to define the group of sections $\tilde{\mathcal{M}}(U)$. We start from the projection

$$\pi : X \longrightarrow \Gamma\backslash X \tag{2.1}$$

and define sheaf

$$\tilde{\mathcal{M}}(U) = \{f : \pi^{-1}(U) \to \mathcal{M} \mid f \text{ is locally constant } f(\gamma u) = \gamma f(u)\}. \tag{2.2}$$

This is clearly a sheaf. For any point $x \in \Gamma\backslash X$ we can find a neighbourhood V_x with the following property: We choose a point $\tilde{x} \in \pi^{-1}(x)$, then $\tilde{x}$ has a convex $\Gamma_{\tilde{x}}$-invariant neighbourhood $U_{\tilde{x}}$, for which $\gamma U_{\tilde{x}} \cap U_{\tilde{x}} \neq \emptyset \iff \gamma \notin \Gamma_{\tilde{x}}$. Then we put $V_x = \Gamma_{\tilde{x}}\backslash U_{\tilde{x}}$. We call such a neighbourhood V_x an *orbiconvex* neighbourhood. It is clear that

$$\tilde{\mathcal{M}}(V_x) = \mathcal{M}^{\Gamma_{\tilde{x}}}.$$

Since x has a cofinal system of neighbourhoods of this kind, we see that we get an isomorphism

$$j_{\tilde{x}} : \tilde{\mathcal{M}}(V_x) = \tilde{\mathcal{M}}_x \xrightarrow{\sim} \mathcal{M}^{\Gamma_{\tilde{x}}}.$$

If we are in the special case that Γ has no fixed points then we can cover $\Gamma\backslash X$ by open sets U so that $\tilde{\mathcal{M}}/U$ is isomorphic to a constant sheaf $\underline{\mathcal{M}_U}$. These sheaves are called *local systems* If we have fixed points we call them *orbilocal systems*

Sometimes we will denote the functor, which sends $\mathcal{M}$ to $\tilde{\mathcal{M}}$ by

$$\mathrm{sh}_\Gamma : \mathbf{Mod}_\Gamma \to \mathcal{S}_{\Gamma\backslash X},$$

this may be useful if we are dealing with varying subgroups Γ.

The motivations for these constructions are

1) The spaces $\Gamma\backslash X$ are interesting examples of so-called locally symmetric spaces (provided Γ has no torsion). Hence they are of interest for differential geometers and analysts.

2) If we have some understanding of the geometry of the quotient space $\Gamma\backslash X$ we gain some insight into the structure of Γ. This will become clear when we discuss the examples in (2.1.4)

3) The cohomology groups $H^\bullet(\Gamma, \mathcal{M})$ are closely related - and in many cases even isomorphic - to the sheaf cohomology groups $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})$. Again the geometry provides tools to compute these cohomology groups in some cases (again 2.1.4).

4) If the Γ-module $\mathcal{M}$ is a $\mathbb{C}$-vector space and obtained from a rational representation of $G/\mathbb{Q}$, then we can apply analytic tools to get insight (de Rham cohomology, Hodge theory See Chapter 8).

2.1.1 The relation between $H^\bullet(\Gamma, \mathcal{M})$ and $H^\bullet(\Gamma\backslash X, \mathcal{M})$

For the following we refer to [39] Chapter 2. In this section we assume that X is connected. The functor

$$\mathcal{M} \to H^0(\Gamma\backslash X, \tilde{\mathcal{M}}) = \mathcal{M}^\Gamma.$$

is a functor from the category of $\Gamma-$ modules to the category $\mathbf{Ab}$ of abelian groups. We can write our functor $\mathcal{M} \to \mathcal{M}^\Gamma$ as a composition of

$$\mathrm{sh}_\Gamma : \mathcal{M} \longrightarrow \tilde{\mathcal{M}} \text{ and } H^0 : \tilde{\mathcal{M}} \to H^0(\Gamma\backslash X, \tilde{\mathcal{M}}).$$

We want to apply the composition rule from [39] 4.6.4.

In a first step we have to convince ourselves that sh_Γ *sends injective* Γ-*modules to acyclic sheaves.*

In [39], 2.2.4. we constructed the induced Γ -module $\mathrm{Ind}_{\{1\}}^\Gamma \mathcal{M}$, for any Γ module $\mathcal{M}$. This is the module of all functions $f : \Gamma \to \mathcal{M}$ and $\gamma_1 \in \Gamma$ acts on this module by $(\gamma_1 f)(\gamma) = f(\gamma\gamma_1)$. The map

$$m \mapsto f_m = \{\gamma \mapsto \gamma m\} \tag{2.3}$$

is an injective $\Gamma-$ module homomorphism.

In a first step we prove that for any such induced module the sheaf $\mathrm{sh}_\Gamma(\mathrm{Ind}_{\{1\}}^\Gamma \mathcal{M})$. is acyclic. We have a little

Lemma 2.1.1. *Let us consider the projection $\pi : X \to \Gamma\backslash X$ and the constant sheaf $\underline{\mathcal{M}}_X$ on X. Then we have a canonical isomorphism of sheaves*

$$\pi_*(\underline{\mathcal{M}}_X) \xrightarrow{\sim} \widetilde{Ind_{\{1\}}^{\Gamma}}\mathcal{M}.$$

Proof. This is rather obvious. Let us consider a small neighbourhood U_x of a point x, such that $\pi^{-1}(U_x)$ is the disjoint union of small contractible neighbourhoods $U_{\tilde{x}}$ for $\tilde{x} \in \pi^{-1}(x)$. Then for all points $\tilde{x}$ we have $\underline{\mathcal{M}}_X(U_{\tilde{x}}) = \mathcal{M}$ and

$$\pi_*(\underline{\mathcal{M}}_X)(U_x) = \prod_{\tilde{x}\in\pi^{-1}(x)} \mathcal{M}.$$

On the other hand

$$\widetilde{\mathrm{Ind}_{\{1\}}^{\Gamma}}\mathcal{M}(U_x) = \left\{h : \pi^{-1}(U_x) \to \ \mathrm{Ind}_{\{1\}}^{\Gamma}\mathcal{M} \mid h \text{ is locally constant } h(\gamma u) = \gamma h(u)\right\}$$

For $u \in \pi^{-1}(U_x)$ the element $h(u)$ itself is a map

$$h(u) : \Gamma \longrightarrow \mathcal{M},$$

and $(\gamma h(u))(\gamma_1) = h(u)(\gamma_1\gamma)$ (here $\gamma_1 \in \Gamma$ is the variable.) Hence we know the function $u \to h(u)$ from $\pi^{-1}(U_x)$ to $\mathrm{Ind}_{\{1\}}^{\Gamma}\mathcal{M}$ if we know its value $h(u)(1)$ and this value can be prescribed on the connected components of $\pi^{-1}(U_x)$. On these connected components it is constant, we may take its value at $\tilde{x}$ and hence

$$h \longrightarrow (\dots, h(\tilde{x})(1), \dots)_{\tilde{x}\in\pi^{-1}(x)}$$

yields the desired isomorphism.

Now acyclicity is clear.. We apply example d) in [39], 4.6.3 to this situation. The fibre of π is a discrete space and hence

$$\pi_*(\underline{\mathcal{M}}_X) = \widetilde{\mathrm{Ind}_{\{1\}}^{\Gamma}}\mathcal{M}$$

and $R^q(\pi_*)(\underline{\mathcal{M}}_X) = 0$ for $q > 0$. Therefore the spectral sequence yields

$$H^q(X, \underline{\mathcal{M}}_X) = H^q(\Gamma\backslash X, \pi_*(\underline{\mathcal{M}}_X)) = H^q\left(\Gamma\backslash X,\ \widetilde{\mathrm{Ind}_{\{1\}}^{\Gamma}}\mathcal{M}\right),$$

and since X is a cell, we see that this is zero for $q \geq 1$. □

We apply this to the case that $\mathcal{M} = \mathcal{I}$ is an injective Γ-module. Clearly we can always embed $\mathcal{I} \longrightarrow \mathrm{Ind}_{\{1\}}^{\Gamma}\mathcal{I}$. But this is now a direct summand; hence it follows from the acyclicity of $\widetilde{\mathrm{Ind}_{\{1\}}^{\Gamma}}\mathcal{I}$ that also $\tilde{\mathcal{I}}$ must be acyclic.

Hence we can apply the composition rule and get spectral sequence with E_2 term

$$H^p(\Gamma\backslash X, R^q(\mathrm{sh}_\Gamma)(\mathcal{M})) \Rightarrow H^n(\Gamma, \mathcal{M}).$$

The edge homomorphism yields a homomorphism

$$H^n(\Gamma\backslash X, \mathrm{sh}_\Gamma(\mathcal{M})) \to H^n(\Gamma, \mathcal{M}) \tag{2.4}$$

which in general is neither injective nor surjective.

We have seen in section (1.2.2) that -under our assumption that $G/\mathbb{Q}$ is semisimple- the stabilisers Γ_x are finite. This implies hat the stalks $R^q(\mathrm{sh}_\Gamma)(\mathcal{M})_x = H^q(\Gamma_{\tilde{x}}, \mathcal{M})$ for $q > 0$ are torsion groups actually they are annihilated by $\#\Gamma_x$. This implies that the edge homomorphism has finite kernel and kokernel.

In this book we are mainly interested in the cohomology groups $H^n(\Gamma\backslash X, \mathrm{sh}_\Gamma(\mathcal{M}))$ and not so much in the group cohomology $H^\bullet(\Gamma, \mathcal{M})$.

2.1.2 Functorial properties of cohomology

We investigate the functorial properties of the cohomology with respect to the change of Γ. If $\Gamma' \subset \Gamma$ is a subgroup of finite index, then we have, the functor

$$\mathbf{Mod}_\Gamma \longrightarrow \mathbf{Mod}_{\Gamma'},$$

which is obtained by restricting the Γ-module structure to Γ'. Since for any Γ-module $\mathcal{M}$ we have $\mathcal{M}^\Gamma \longrightarrow \mathcal{M}^{\Gamma'}$, we obtain a homomorphism

$$\mathrm{res}\ : H^i(\Gamma, \mathcal{M}) \longrightarrow H^i(\Gamma', \mathcal{M}).$$

We give an interpretation of this homomorphism in terms of sheaf cohomology. We have the diagram

$$\begin{array}{ccc} & X & \\ \pi_{\Gamma'} \swarrow & & \searrow \pi_\Gamma \\ \pi = \pi_{\Gamma,\Gamma'} : \Gamma'\backslash X & \longrightarrow & \Gamma\backslash X \end{array}$$

and a Γ-module $\mathcal{M}$ produces sheaves $\mathrm{sh}_\Gamma(\mathcal{M}) = \tilde{\mathcal{M}}$ and $\mathrm{sh}_{\Gamma'}(\mathcal{M})\tilde{=}\mathcal{M}'$ on $\Gamma'\backslash X$ and $\Gamma\backslash X$ respectively. It is clear that we have a homomorphism

$$\pi^*(\tilde{\mathcal{M}}) \longrightarrow \tilde{\mathcal{M}}'.$$

To get this homomorphism we observe that for $y_1 \in \Gamma'\backslash X$ we have $\pi_1^*(\tilde{\mathcal{M}})_{y_1} = \tilde{\mathcal{M}}_{\pi_1(y_1)}$, and this is

$$\{f : \pi^{-1}(\pi_1(y)) \to \mathcal{M} \mid f(\gamma\tilde{y}) = \gamma f(\tilde{y}) \text{ for all } \gamma \in \Gamma, \tilde{y} \in \pi^{-1}(\pi(y_1))\}$$

and

$$\tilde{\mathcal{M}}'_{y_1} = \{fg : (\pi')^{-1}(y_1) \to \mathcal{M} \mid f(\gamma'\tilde{y}) = \gamma' f(\tilde{y}) \text{ for all } \gamma \in \Gamma', \tilde{y} \in (\pi')^{-1}(y_1)\},$$

and if we pick a point $\tilde{y} \in (\pi')^{-1}(y_1) \subset \pi^{-1}(\pi_1(y_1))$ then

$$\pi^*(\mathcal{M})_{y_1} \simeq \mathcal{M}^{\Gamma_{\tilde{y}_1}} \subset \tilde{\mathcal{M}}'_{y_1} = \mathcal{M}^{\Gamma'_{\tilde{y}_1}}.$$

Hence we get (or define) our restriction homomorphism as

$$H^i(\Gamma\backslash X, \mathrm{sh}_\Gamma(\mathcal{M})) \longrightarrow H^i(\Gamma'\backslash X, \pi_1^*(\mathrm{sh}_\Gamma(\mathcal{M})) \longrightarrow H^i(\Gamma'\backslash X, \mathrm{sh}_{\Gamma'}(\mathcal{M})). \quad (2.5)$$

There is also a map in the opposite direction.

Since the fibres of π are discrete we have

$$H^i(\Gamma'\backslash X, \tilde{\mathcal{M}}) \xrightarrow{\sim} H^i(\Gamma\backslash X, \pi_*(\tilde{\mathcal{M}})).$$

But the same reasoning as in the previous section yields an isomorphism

$$\pi_{1,*}(\tilde{\mathcal{M}}) \xrightarrow{\sim} \widetilde{\mathrm{Ind}_{\Gamma'}^{\Gamma}}\mathcal{M}.$$

Hence we get an isomorphism

$$H^i(\Gamma'\backslash X, \tilde{\mathcal{M}}) \xrightarrow{\sim} H^i(\Gamma\backslash X, \widetilde{\mathrm{Ind}_{\Gamma'}^{\Gamma}}\mathcal{M}) \tag{2.6}$$

which is well known as Shapiro's lemma. But we have a Γ-module homomorphism

$$e: \mathrm{Ind}_{\Gamma'}^{\Gamma}\mathcal{M} \longrightarrow \mathcal{M}$$

which sends an $f : \Gamma \to \mathcal{M}$, in $f \in \mathrm{Ind}_{\Gamma'}^{\Gamma}\mathcal{M}$ to the sum

$$\mathrm{tr}(f) = \sum \gamma_i^{-1} f(\gamma_i)$$

where the γ_i are representatives for the classes of $\Gamma'\backslash\Gamma$. This homomorphism induces a map in the cohomology. We get a composition

$$\pi_{1,\bullet} : H^i(\Gamma'\backslash X, \tilde{\mathcal{M}}) \longrightarrow H^i(\Gamma\backslash X, \tilde{\mathcal{M}}). \tag{2.7}$$

It is not difficult to check that

$$\pi_\bullet \circ \pi^\bullet = [\Gamma : \Gamma']\mathrm{Id} \tag{2.8}$$

2.1.3 How to compute the cohomology groups $H^i(\Gamma\backslash X, \tilde{\mathcal{M}})$?

The Čech complex of an orbiconvex Covering

We consider a point $\tilde{x} \in X$ and an open neighbourhood $\tilde{U}_{\tilde{x}} \subset X$. We say that $\tilde{U}_{\tilde{x}}$ is an *orbiconvex* neighbourhood of $\tilde{x}$ if

a) The set $\tilde{U}_{\tilde{x}}$ is convex, i.e. for any two points in $\tilde{x}_1, \tilde{x}_2 \in \tilde{U}_{\tilde{x}}$ the geodesic joining $\tilde{x}_1$ and $\tilde{x}_2$ lies in $\tilde{U}_{\tilde{x}}$.

irgendwo früher was zu Geodäten sagen,)

b) We have $\gamma\tilde{U}_{\tilde{x}} \cap \tilde{U}_{\tilde{x}} = \emptyset$ unless $\gamma\tilde{x} = \tilde{x}$ and in this case we even have $\gamma\tilde{U}_{\tilde{x}} = \tilde{U}_{\tilde{x}}$.

A family of orbiconvex neighbourhoods $\{\tilde{U}_{\tilde{x}_i}\}_{i=1,\dots,r}$ ($\tilde{x}_1, \dots, \tilde{x}_r$ set of points) will be called an *orbiconvex covering,* if

$$\bigcup_{i=1}^{r} \bigcup_{\gamma\in\Gamma} \gamma\tilde{U}_{\tilde{x}_i} = X. \tag{2.9}$$

We will show later that we can always find a finite orbiconvex covering of X.

If now $\{\tilde{U}_{\tilde{x}_i}\}_{i=1,\dots,r}$ is an orbiconvex covering we put $U_{x_i} = \pi(\tilde{U}_{\tilde{x}_i})$, and then we get finite covering by open sets

$$\bigcup_{x_i} U_{x_i} = \Gamma\backslash X.$$

We abbreviate and use the usual notation $\mathfrak{U} = \{U_{x_i}\}$ for this orbiconvex covering of $\Gamma\backslash X$.

We will see further down that the intersections $U_{\underline{i}} = U_{x_{i_1}} \cap U_{x_{i_2}} \cap \dots \cap U_{x_{i_q}}$ are acyclic, more precisely

$$H^q(U_{\underline{i}}) = \begin{cases} \mathcal{M}^{\Gamma_{\underline{i}}} & q = 0 \\ 0 & q > 0 \end{cases} \tag{2.10}$$

This implies that the Čzech complex (See [39], Chap. 4)

$$C^\bullet(\mathfrak{U}, \tilde{\mathcal{M}}) \quad := 0 \to \bigoplus_{i\in I} \tilde{\mathcal{M}}(U_{x_i}) \xrightarrow{d_0} \bigoplus_{i<j} \tilde{\mathcal{M}}(U_{x_i} \cap U_{x_j}) \to \tag{2.11}$$

computes the cohomology, i.e. the cohomology groups $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})$. are the cohomology groups of the Čzech complex.

For the implementation on a computer we need to resolve the definition of the spaces of sections and the definition of the boundary maps. (By this I mean that we have to write explicitly

$$\tilde{\mathcal{M}}(U_{\underline{i}}) = \bigoplus_\eta \mathcal{M}_\eta$$

where η runs through an index set and the $\mathcal{M}_\eta$ are explicit subspaces of $\mathcal{M}$ and then we have to write down certain explicit linear maps $\mathcal{M}_\eta \to \mathcal{M}_{\eta'}$.)

We have to be aware that the intersections $U_{\underline{i}}$ are not necessarily connected. We write $U_{\underline{i}} = \cup U_\eta$ as the union of its connected components, we have to choose a connected component $\tilde{U}_\eta$ in $\pi^{-1}(U_\eta)$ for each value of η. Then the evaluation of a section $m \in \tilde{\mathcal{M}}(U_{\underline{i}})$ on these $\tilde{U}_\eta$ yields an isomorphism

$$\bigoplus_\eta ev_{\tilde{U}_\eta} : \bigoplus_\eta \tilde{\mathcal{M}}(U_\eta)(= \mathcal{M}(U_{\underline{i}}) \xrightarrow{\sim} \bigoplus_\eta \mathcal{M}^{\Gamma_\eta}.$$

If we replace $\tilde{U}_\eta$ by $\gamma\tilde{U}_\eta$ then we get for $m \in \tilde{\mathcal{M}}(\pi(\tilde{U}_\eta))$ the equality

$$\gamma ev_{\tilde{U}_\eta}(m) = ev_{\gamma\tilde{U}_\eta} \tag{2.12}$$

In degree zero the U_{x_i} are connected and this gives for the first term of the complex

$$ev_{U_{x_i}} : \tilde{\mathcal{M}}(U_{x_i}) \xrightarrow{\sim} \mathcal{M}^{\Gamma_{\tilde{x}_i}}. \tag{2.13}$$

The computation the second term is a little bit more delicate. We have to understand the connected components of $U_{x_i} \cap U_{x_j}$. To get these connected components we have to find the elements $\gamma \in \Gamma$ for which

$$\tilde{U}_{\tilde{x}_i} \cap \gamma(\tilde{U}_{\tilde{x}_j}) \neq \emptyset \tag{2.14}$$

It is clear that this gives us a finite set $\mathcal{G}_{i,j}$ of elements $\gamma \in \Gamma/\Gamma_{x_j}$. We have a little lemma

Lemma 2.1.2. *The images $\pi(\tilde{U}_{\tilde{x_i}} \cap \gamma(\tilde{U}_{\tilde{x_j}}))$ are the connected components of $U_{x_i} \cap U_{x_j}$, two elements γ, γ_1 give the same connected component if and only if $\gamma_1 \in \Gamma_{x_i}\gamma\Gamma_{x_j}$.*

Let $\mathcal{F}_{i,j} \subset \mathcal{G}_{i,j}$ be a set of representatives for the action of Γ_{x_1} on $G_{i,j}$ this set can be identified to the set of connected components. Of course the set $\tilde{U}_{\tilde{x_i}} \cap \gamma(\tilde{U}_{\tilde{x_j}})$ may have a non trivial stabilizer $\Gamma_{i,j,\gamma}$ and then we get an identification

$$\oplus_{\gamma \in F_{i,j}} ev_{\tilde{U}_{x_i} \cap \gamma \tilde{U}_{x_j}} : \tilde{\mathcal{M}}(U_{x_i} \cap U_{x_j}) \xrightarrow{\sim} \bigoplus_{\gamma \in \mathcal{F}_{i,j}} \mathcal{M}^{\Gamma_{i,j,\gamma}} \tag{2.15}$$

This is now an explicit (i.e. digestible for a computer) description of the second term in our complex above. We still need to give the explicit formula for d_0 in the complex

$$0 \to \bigoplus_{i \in I} \mathcal{M}^{\Gamma_{\tilde{x}_i}} \xrightarrow{d_0} \bigoplus_{i<j} \bigoplus_{\gamma \in \mathcal{F}_{i,j}} \mathcal{M}^{\Gamma_{i,j,\gamma}} \tag{2.16}$$

Looking at the definition it is clear that this map is given by

$$(\dots, m_i, \dots, m_j, \dots) \mapsto (\dots, m_i - \gamma m_j, \dots) \tag{2.17}$$

Here we have to observe that $\gamma \in \Gamma/\Gamma_{x_j}$ but this does not matter since $m_j \in \mathcal{M}^{\Gamma_{\tilde{x}_j}}$. So we have an explicit description of the beginning of the Čech complex. A little reasoning shows of course that a different choice $\mathcal{F}'_{i,j}$ of the representatives provides an isomorphic complex.

Now it is clear, how to proceed. At first we have to understand the combinatorics of the covering $\mathfrak{U} = \{U_{x_i}\}_{i \in I}$. We consider sets

$$G_{\underline{i}} = \{\underline{\gamma} = (e, \gamma_1, \dots, \gamma_q) | \gamma_i \in \Gamma/\Gamma_{x_i}; \tilde{U}_{\tilde{x}_0} \cap \dots \cap \gamma_i \tilde{U}_{\tilde{x_i}} \cap \gamma_q \tilde{U}_{\tilde{x}_q} \neq \emptyset\}$$

on these sets we have an action of Γ_{x_0} by multiplication from the left. Again let $F_{\underline{i}}$ be a system of representatives modulo the action of Γ_{x_0}.

We abbreviate

$$\tilde{U}_{\underline{i},\underline{\gamma}} = \tilde{U}_{\tilde{x}_0} \cap \dots \cap \gamma_i \tilde{U}_{\tilde{x_i}} \cap \gamma_q \tilde{U}_{\tilde{x}_q},$$

let $\Gamma_{\underline{i},\underline{\gamma}}$ be the stabiliser of $\tilde{U}_{\underline{i},\underline{\gamma}}$.

The images $\pi(\tilde{U}_{\underline{i},\underline{\gamma}})$ under the projection map π are the connected components $\pi(\tilde{U}_{\underline{i},\underline{\gamma}}) = U_{\underline{i},\underline{\gamma}} \subset U_{\underline{i}} = U_{x_{i_0}} \cap \dots \cap U_{x_{i_\nu}} \cap \dots U_{x_{i_q}}$. On the other hand each set $\tilde{U}_{\underline{i},\underline{\gamma}}$ is a connected component in $\pi^{-1}(U_{\underline{i},\underline{\gamma}})$. We get an isomorphism

$$\bigoplus_{\underline{\gamma} \in \mathcal{F}_{\underline{i}}} ev_{\tilde{U}_{\underline{i},\gamma}} : \tilde{\mathcal{M}}(U_{\underline{i}}) = \tilde{\mathcal{M}}(U_{x_{i_0}} \cap \dots \cap U_{x_{i_\nu}} \cap \dots U_{x_{i_q}}) \xrightarrow{\sim} \bigoplus_{\underline{\gamma} \in \mathcal{F}_{\underline{i}}} \mathcal{M}^{\Gamma_{\underline{i},\underline{\gamma}}}. \tag{2.18}$$

We need to give explicit formulas for the boundary maps

$$\bigoplus_{\underline{i} \in I^q} \tilde{\mathcal{M}}(U_{\underline{i}}) \xrightarrow{d_q} \bigoplus_{\underline{i} \in I^{q+1}} \tilde{\mathcal{M}}(U_{\underline{i}}).$$

Abstractly this boundary operator is defined as follows: We look at pairs $\underline{i} \in I^{q+1}, \underline{i}^{(\nu)} \in I^q$ where $\underline{i}^{(\nu)}$ is obtained from $\underline{i}$ by deleting the ν-th entry. Then we have $U_{\underline{i}} \subset U_{\underline{i}^{(\nu)}}$ and from this we get the resulting restriction homomorphism $R_{\underline{i}^{(\nu)},\underline{i}} : \tilde{\mathcal{M}}(U_{\underline{i}^{(\nu)}}) \to \tilde{\mathcal{M}}(U_{\underline{i}})$. Then

$$d_q = \sum_{\underline{i}} \sum_{\nu=0}^{q} (-1)^\nu R_{\underline{i}^{(\nu)},\underline{i}} \tag{2.19}$$

and hence we have to give an explicit description of $R_{\underline{i}^{(\nu)},\underline{i}}$ with respect to the isomorphism in the diagram (2.18).

We pick two connected components $\pi(\tilde{U}_{\underline{i},\underline{\gamma}}) \subset U_{\underline{i}})$ and $\pi(\tilde{U}_{\underline{i}^{(\nu)},\underline{\gamma}'} \subset U_{\underline{i}^{(\nu)}})$, then we know that

$$\tilde{U}_{\underline{i},\underline{\gamma}} \subset \tilde{U}_{\underline{i}^{(\nu)},\underline{\gamma}'} \iff \exists\, \eta_{\gamma,\gamma'} \in \Gamma \text{ such that } \eta_{\gamma,\gamma'}\gamma'_\mu = \gamma_\mu \text{ for all } \mu \neq \nu$$

and then the restriction of $R_{\underline{i}^{(\nu)},\underline{i}}$ to these two components is given by

$$\begin{array}{ccc} \tilde{\mathcal{M}}(\pi(\tilde{U}_{\underline{i}^{(\nu)},\underline{\gamma}'})) & \xrightarrow{ev_{\tilde{U}_{\underline{i}^{(\nu)},\underline{\gamma}'}}} & \mathcal{M}^{\Gamma_{\underline{i}^{(\nu)},\underline{\gamma}'}} \\ \downarrow R_{\underline{i}^{(\nu)},\underline{i}} & & \downarrow\ \eta_{\gamma,\gamma'} \\ \tilde{\mathcal{M}}(\pi(\tilde{U}_{\underline{i},\underline{\gamma}})) & \xrightarrow{ev_{\tilde{U}_{\underline{i},\underline{\gamma}}}} & \mathcal{M}^{\Gamma_{\underline{i},\underline{\gamma}}} \end{array} \tag{2.20}$$

Here the two horizontal maps are isomorphisms, we observe that $\eta_{\gamma,\gamma'}$ is unique up to an element in $\Gamma_{\underline{i}^{(\nu)},\underline{\gamma}'}$ and hence the vertical arrow $\eta_{\gamma,\gamma'}$ is well defined.

Hence we conclude:

Once we have found a a finite orbiconvex covering of $\Gamma\backslash X$, *we can write down an explicit complex, which computes the cohomology groups* $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})$.

We may also look at this situation from a different point of view: If $x \in X$ is any point and $\Gamma_x \subset \Gamma$ its stabilizer, then we define the induced Γ module

$$\mathrm{Ind}_{\Gamma_x}^{\Gamma} \mathbb{Z} := \{f : \Gamma \to \mathbb{Z} \mid f \text{ has finite support and } f(a\gamma) = f(\gamma),\ \forall a \in \Gamma_x, \gamma \in \Gamma\} \tag{2.21}$$

If V_x is an open neighbourhood of x which satisfies b) an c) then we have $\pi^{-1}(\pi(V_x)) = \bigcup_{\gamma\in\Gamma/\Gamma_x} \gamma V_x$ and

$$\pi^*(\tilde{\mathcal{M}})(\bigcup_{\gamma\in\Gamma/\Gamma_x} \gamma V_x) = \mathrm{Hom}(\,\mathrm{Ind}_{\Gamma_x}^{\Gamma}\mathbb{Z}, \mathcal{M}).$$

We have the covering

$$\tilde{\mathfrak{U}} = \bigcup_{i,\gamma\in\Gamma/\Gamma_{\tilde{x}_i}} \gamma \tilde{U}_{\tilde{x}_i} = X$$

of the symmetric space. The Čzech-complex $C^\bullet(\tilde{\mathfrak{U}}, \pi^*(\tilde{\mathcal{M}}))$ computes the cohomology groups $H^q(X, \pi^*(\tilde{\mathcal{M}}))$ which are trivial for $q > 0$. Our considerations above yield

$$C^\bullet(\tilde{\mathfrak{U}}, \pi^*(\tilde{\mathcal{M}})) = 0 \to \bigoplus_{i=1}^{r} \mathrm{Hom}(\,\mathrm{Ind}_{\Gamma_{x_i}}^{\Gamma}\mathbb{Z}, \mathcal{M}) \xrightarrow{d^1} \bigoplus_{i<j,\tilde{x}_{i,j}} \mathrm{Hom}(\,\mathrm{Ind}_{\Gamma_{\tilde{x}_{i,j}}}^{\Gamma}\mathbb{Z}, \mathcal{M}) \xrightarrow{d^2} \dots$$

Now it is easy see that the boundary maps are induced by maps between the induced modules

$$\xrightarrow{\delta^2} \bigoplus_{i<j,\tilde{x}_{i,j}} \mathrm{Ind}_{\Gamma_{\tilde{x}_{i,j}}}^{\Gamma} \mathbb{Z} \xrightarrow{\delta^1} \bigoplus_{i=1}^{r} \mathrm{Ind}_{\Gamma_{\tilde{x}_i}}^{\Gamma} \mathbb{Z} \to 0,$$

where for $f \in \bigoplus \mathrm{Ind}_{\Gamma_{\tilde{x}_J}}^{\Gamma} \mathbb{Z}$, in degree ν and $\omega \in C^{\nu-1}(\tilde{\mathfrak{U}}, \pi^*(\tilde{\mathcal{M}}))$ the relation $\omega(\delta^\nu(f)) = d^{\nu-1}(\omega)(f)$ defines δ^ν. We get an augmented complex

$$P^\bullet :=\to \bigoplus_{\tilde{x}_J} \mathrm{Ind}_{\Gamma_{\tilde{x}_J}}^{\Gamma} \mathbb{Z} \to \cdots \to \bigoplus_{\tilde{x}_{i,j}} \mathrm{Ind}_{\Gamma_{\tilde{x}_{i,j}}}^{\Gamma} \mathbb{Z} \to \bigoplus_{i=1}^{r} \mathrm{Ind}_{\Gamma_{\tilde{x}_i}}^{\Gamma} \mathbb{Z} \to \mathbb{Z} \to 0 \tag{2.22}$$

and since $C^\bullet(\tilde{\mathfrak{U}}, \pi^*(\tilde{\mathcal{M}}))$ is acyclic in degree > 0, we get that $P^\bullet$ is an acyclic resolution of the trivial module $\mathbb{Z}$.

Let $N = \prod_i \#\Gamma_{\tilde{x}_i}$ and $R := \mathbb{Z}[\frac{1}{N}]$ then the $R[\Gamma]$ module $\mathrm{Ind}_{\Gamma_x}^{\Gamma} \otimes R$ is a direct summand in $R[\Gamma]$ and hence a projective $R[\Gamma]$ module. This implies of course that

$$P^\bullet \otimes R =\to \bigoplus_{\tilde{x}_J} \mathrm{Ind}_{\Gamma_{\tilde{x}_J}}^{\Gamma} R \to \cdots \to \bigoplus_{\tilde{x}_{i,j}} \mathrm{Ind}_{\Gamma_{\tilde{x}_{i,j}}}^{\Gamma} R \to \bigoplus_{i=1}^{r} \mathrm{Ind}_{\Gamma_{\tilde{x}_i}}^{\Gamma} R \to R \to 0 \tag{2.23}$$

is indeed a projective resolution of the trivial Γ -module R. Therefore we know that

$$H^\bullet(\Gamma, \mathcal{M}_R) = H^\bullet(0 \to \ \mathrm{Hom}_\Gamma(\bigoplus_{i=1}^{r} \mathrm{Ind}_{\Gamma_{\tilde{x}_i}}^{\Gamma} R, \mathcal{M}_R) \to \bigoplus_{i<j,\tilde{x}_{i,j}} \mathrm{Hom}_\Gamma(\ \mathrm{Ind}_{\Gamma_{\tilde{x}_{i,j}}}^{\Gamma} R, \mathcal{M}_R) \to) \tag{2.24}$$

where now on the left hand side we have the group cohomology.

If we do not tensor by R then the Čzech-complex

$$0 \to \bigoplus_{i=1}^{r} \mathrm{Hom}_\Gamma(\ \mathrm{Ind}_{\Gamma_{x_i}}^{\Gamma} \mathbb{Z}, \mathcal{M}) \to \bigoplus_{i<j,_{i,j}} \mathrm{Hom}_\Gamma(\ \mathrm{Ind}_{\Gamma_{\tilde{x}_{i,j}}}^{\Gamma} \mathbb{Z}, \mathcal{M}) \to \dots \tag{2.25}$$

is isomorphic to the Čzech complex (2.11) and it computes the sheaf cohomology $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})$.

It follows from reduction theory that

Theorem 2.1.1. *We can construct a finite covering $\Gamma\backslash X = \bigcup_{i\in E} U_{x_i} = \mathfrak{U}$ by orbiconvex sets.*

Proof. This is rather clear. We start from the covering by the sets $X^P(c_{\pi'}, r(c_{\pi'}))$.) The set of "almost stable" points $X^G(r) \subset X$ is relatively compact modulo Γ. For any point $\tilde{x} \in X$ we look at the minimum distance

$$d(\tilde{x}) := \min_{\gamma\in\Gamma\backslash\Gamma_{\tilde{x}}} d(\tilde{x}, \gamma\tilde{x}).$$

since the action of Γ is properly discontinuous this minimum distance $d(\tilde{x}) > 0$. Let $D(\tilde{x}, d(\tilde{x})/2) := \{\tilde{y} | d(\tilde{y}, \tilde{x}) < d(\tilde{x})/2\}$, (-the Dirichlet-ball around $\tilde{x}$-) then $D(\tilde{x}, d(\tilde{x})/2)$ is an orbiconvex neighbourhood of $\tilde{x}$. Then we can find finitely many points $\tilde{x}_1, \dots, \tilde{x}_r$ such that

$$\bigcup_{i=1}^{r} \bigcup_{\gamma \in \Gamma} \gamma D(\tilde{x}_i, d(\tilde{x}_i)/2)) \supset X^G(r).$$

We have to find a covering for the $X^P(c_P, r(c_P)))$. We recall the fibration (See (1.118))

$$p^*_{P,M} : X^P(c_{\pi'}, r(\underline{c}_{\pi'})) \to X^M(r(\underline{c}_{\pi'})) \times \prod_{\alpha \in \pi'} (0, c_\alpha].$$

We apply our previous argument and find a finite covering

$$\bigcup_{i=1}^{s} \bigcup_{\gamma \in \Gamma_M} \gamma D(\tilde{y}_i, d(\tilde{y}_i)/2)) \supset X^M(r(\underline{c}_{\pi'})).$$

We pick a point $\underline{c}_0 \in \prod_{\alpha \in \pi'} (0, c_\alpha]$ then the inverse image $(p^*_{P,M})^{-1}(D(\tilde{y}_i, d(\tilde{y}_i)/2)) \times \underline{c}_0$ is relatively compact and we can find an orbiconvex covering $\{\mathfrak{V}\{V_{\tilde{x}_\mu}\}$ of this set. Then the products $V_{\tilde{x}_\mu} \times \prod_{\alpha \in \pi'} (0, c_\alpha]$ provide an orbiconvex covering of $X^P(c_{\pi'}, r(\underline{c}_{\pi'}))$. Of course these sets are not (relatively) compact anymore.

$\square$

I think that it is a very important problem to write algorithms which compute the cohomology effectively. The main goal would be to collect experimental data which may suggest conjectures or give support for conjectures which come from different sources. We come back to this aspect in the following chapter 3 and also in the final chapter 9.

I do not claim that my proposal using orbiconvex coverings provides an acceptable solution to this problem. It solves the problem in principle but it not clear how far it reaches in practice. We see that the fixed points create some problems if we want to write down the explicit complexes. But it is certainly no solution to avoid these problems by passing to a congruence subgroup. Then the number of members in the covering growth rapidly and the complexes become much bigger. For the groups $\mathrm{Sl}_n(\mathbb{Z})$ (for some small values of n and for some congruence subgroups) various authors have computed the cohomology (Ash, Stevens, Gunnels) using the Voronoi decomposition of the cone of positive definite symmetric matrices.

A first step would be to find effectively an optimal orbiconvex covering $\{U_{\tilde{x}_\nu}\}$ of the set $X^G(r)$ of almost stable points. The covering sets must not necessarily be Dirichlet balls. We could proceed and apply this also to the different $X^M(r(\underline{c}_{\pi'}))$ and find orbiconvex covers $\{V^M_{\tilde{y}_\mu}\}$ for them. Then we may consider the inverse images $(p^*_{P,M})^{-1}(V^M_{\tilde{y}_\mu} \times \prod_{\alpha \in \pi'} (0, c_\alpha]) = \tilde{V}^M_{\tilde{y}_\mu}$. This family of sets $\{\{\gamma U_{\tilde{x}_\nu}\}, \dots, \gamma_1 \tilde{V}^M_{\tilde{y}_\mu}, \dots\}$ provide a covering of X by open sets, hence the images under the projection provide a covering

$$\mathfrak{W} = \{W_i\}_{i \in I} = \{\{U_{x_\nu}\}, \dots, \{\tilde{V}^M_{y_\mu}\}, \dots\}$$

of $\Gamma\backslash X$, here the index set I is the union of the $x_]\tilde{x}_\nu, \dots, ^M y_{\tilde{y}_\mu}$.

Of course we have a problem: The sets $\tilde{V}^M_{\tilde{y}_\mu}$ are not acyclic anymore, so we can not use the Čzech complex of this covering for the computation of the cohomology. But we know that

$$\tilde{V}^M_{\tilde{y}_\mu} \to V^M_{\tilde{y}_\mu} \times \prod_{\alpha\in\pi'} (0, c_\alpha]$$

is a fiber bundle with fiber $U(\mathbb{Z})\backslash U(\mathbb{R})$, Since the base $V^M_{\tilde{y}_\mu} \times \prod_{\alpha\in\pi'}(0, c_\alpha]$ is acyclic we know that

$$H^\bullet(\tilde{V}^M_{\tilde{y}_\mu}) \xrightarrow{\sim} \mathbb{H}^\bullet(U(\mathbb{Z})\backslash U(\mathbb{R}), \tilde{\mathcal{M}}) \tag{2.26}$$

and we have a good understanding of the cohomology on the right. If for instance we tensor by the rationals the Theorem of Kostant (See section **??**) gives us a complete description of the cohomology $H^\bullet(U(\mathbb{Z})\backslash U(\mathbb{R}), \tilde{\mathcal{M}} \otimes \mathbb{Q})$.

For $\underline{i} \in I^{p+1}$ we put $\mathfrak{W}_{\underline{i}} = W_{i_0} \cap W_{i_1} \cap \dots \cap \ W_{i_p}$ Now we follow [39], 4.6.6, for any $q \geq 0$ write the Čzech complexe

$$C^\bullet(\mathfrak{W}, \mathcal{H}^q) :=\to \prod_{\underline{i}\in I^{p+1}} H^q(W_{\underline{i}}) \to \prod_{\underline{i}\in I^{p+2}} H^q W_{\underline{i}}) \tag{2.27}$$

and then we know that we get a spectral sequence

$$H^p(C^\bullet(\mathfrak{W}, \mathcal{H}^q)) = E_1^{p,q} \implies H^{p+q}(\Gamma\backslash X, \tilde{\mathcal{M}}) \tag{2.28}$$

$\boxed{\text{Lowdim}}$

2.1.4 Special examples in low dimensions.

We consider the group $\Gamma = \mathrm{Sl}_2(\mathbb{Z})/\{\pm\mathrm{Id}\}$ and its action on the upper half plane $\mathbb{H}$. We want to investigate the cohomology groups $H^i(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$ for any module Γ-module $\mathcal{M}$. Let $p : \mathbb{H} \to \Gamma\backslash\mathbb{H}$ be the projection. We have the two special points i and ρ in $\mathbb{H}$. They are - up to conjugation by Γ - the only points which have a non trivial stabilizer. We construct two nice orbiconvex neighbourhoods of these two points. The stabilizers $\Gamma_i,$ resp. Γ_ρ are cyclic and generated by the two elements

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, R = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

respectively.

We begin with i. We consider the strip $V_i = \{z | -1/2 < \Re(z) < 1/2\}$, the element S maps the two vertical boundary lines $\Re(z) = \pm\frac{1}{2}$ into geodesic circles starting from 0 and ending in ± 2. Then the intersection $\tilde{U}_i = V_i \cap S(V_i)$ is an orbiconvex neighbourhood of i.

Let us look at ρ. We consider the strip $V_\rho = \{z \mid -0 < \Re(z) < 1\}$ and now we define $\tilde{U}_\rho = V_\rho \cap R(V_\rho) \cap R^2(V_\rho)$. This is a nice orbiconvex neighbourhood of ρ.

Now it is clear that these two sets provide an orbiconvex covering of $\mathbb{H}$, if $U_i = p(\tilde{U}_i), U_\rho = p(\tilde{U}_\rho)$ then

$$\Gamma\backslash\mathbb{H} = U_i \cup U_\rho. \tag{2.29}$$

We have $\tilde{\mathcal{M}}(U_i) = \mathrm{sh}_\Gamma(\mathcal{M})(U_i) = \mathcal{M}^{\Gamma_i}, \tilde{\mathcal{M}}(U_\rho) = \mathcal{M}^{\Gamma_\rho}$ and hence the cohomology groups are given by the cohomology of the complex

$$0 \to \mathcal{M}^{\Gamma_i} \oplus \mathcal{M}^{\Gamma_\rho} \to \mathcal{M} \to 0 \tag{2.30}$$

Then $H^0(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) = \mathcal{M}^\Gamma = \mathcal{M}^{\Gamma_i} \cap \mathcal{M}^{\Gamma_\rho}$. Since this is true for any Γ module we easily conclude that Γ is generated by Γ_i, Γ_ρ. And we get
$\boxed{\text{H1}}$

$$H^1(\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}, \tilde{\mathcal{M}}_\mathbb{Z}) = \mathcal{M}/(\mathcal{M}^{(<S>} \oplus \mathcal{M}^{<R>}), \tag{2.31}$$

and the cohomology vanishes in higher degrees.

Exercise 1: *Let $\Gamma' \subset \Gamma = \mathrm{Sl}_2(\mathbb{Z})/\pm\mathrm{Id}$ be a subgroup of finite index. Prove*
ii) We have (Shapiros lemma)

$$H^1(\Gamma'\backslash\mathbb{H}, \mathbb{Z}) = H^1(\Gamma\backslash\mathbb{H}, \widetilde{\mathrm{Ind}_{\Gamma'}^{\Gamma}\mathbb{Z}}).$$

These cohomology groups are free of rank

$$[\Gamma : \Gamma'] - n_i - n_\rho + 1$$

where n_i (resp. n_ρ) is the number of orbits of Γ_i (resp. Γ_ρ) on $\Gamma'\backslash\Gamma$. If Γ' is torsion free then

$$\mathrm{rank}(H^1\Gamma\backslash\mathbb{H}, \widetilde{\mathrm{Ind}_{\Gamma'}^{\Gamma}\mathbb{Z}}) = \frac{1}{6}[\Gamma : \Gamma'] + 1$$

The Euler-characteristic of $\Gamma'\backslash\mathbb{H}$ is $\frac{1}{6}[\Gamma : \Gamma']$.

Exercise 2: *Let $\mathcal{M}_n$ be the $\mathrm{Sl}_2(\mathbb{Z})$-module of homogenous polynomials in the two variables X, Y and coefficients in $\mathbb{Z}$. (See 1.2.2). We have the usual action of $\mathrm{Sl}_2(\mathbb{Z})$ on this module by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} P(X, Y) = P(aX + cY, bX + dY).$$

these modules define a sheaf $\tilde{\mathcal{M}}_n$ on $\Gamma\backslash\mathbb{H}$., We compute the cohomology groups $H^\bullet(.(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n)$.

Prove:
i) If n is odd, then $\tilde{\mathcal{M}}_n = 0$.
Hence we assume $n \geq 2$ and n even from now on.

ii) For $n > 0$ we have $H^0(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n) = 0$.

iii) If we tensorize by $\mathbb{Q}$ *, then* $H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n\otimes\mathbb{Q})$ *is a vector space of rank* $n-1-2\left[\frac{n}{4}\right]-2\left[\frac{n}{6}\right]$.

Hint: *Diagonalise the action of* Γ_i *and* Γ_ρ *on* $\mathcal{M}_n\otimes\overline{\mathbb{Q}}$ *separately and look at the eigenspaces. To say it differently: Over* $\bar{\mathbb{Q}}$ *we can conjugate the matrices* $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$, $\begin{pmatrix}1&1\\-1&0\end{pmatrix}$ *into the diagonal maximal torus* $\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}$ *, and then look at the decomposition of* $\mathcal{M}_n$ *into eigenspaces.*

iv) Investigate the torsion in $H^1(\Gamma\backslash\mathbb{H},\mathcal{M}_n)$. *(Start from the sequence* $0\to\mathcal{M}_n\to\mathcal{M}_n\to\mathcal{M}_n/\ell\mathcal{M}_n\to 0$.*)*

v) Now we consider $\Gamma=\mathrm{Sl}_2(\mathbb{Z})$. *The two matrices* $S=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ *and* $R=\begin{pmatrix}-1&1\\-1&0\end{pmatrix}$ *are generators of the stabilisers of* i *and* ρ *respectively.*

We take for our module $\mathcal{M}$ *the cyclic group* $\mathbb{Z}/12\mathbb{Z}$, *consider the spectral sequence*

$$H^p(\Gamma\backslash\mathbb{H},R^q(\mathrm{sh}_\Gamma)(\mathbb{Z}/12\mathbb{Z})).$$

Show that $H^0(\Gamma\backslash\mathbb{H},R^1(\mathrm{sh}_\Gamma)(\mathbb{Z}/12\mathbb{Z})=\mathbb{Z}/12\mathbb{Z}$. *Show that the differential*

$$H^0(\Gamma\backslash\mathbb{H},R^1(\mathrm{sh}_\Gamma)(\mathbb{Z}/12\mathbb{Z}))\to H^2(\Gamma\backslash\mathbb{H},\mathrm{sh}_\Gamma(\mathbb{Z}/12\mathbb{Z}))$$

vanishes and conclude

$$H^1(\Gamma,\mathbb{Z}/12\mathbb{Z})=\mathbb{Z}/12\mathbb{Z}.$$

The group $\Gamma=\mathrm{Gl}_2(\mathbb{Z}[i])$

A similar computation can be made up to compute the cohomology in the case of $\tilde{\Gamma}=\mathrm{Gl}_2(\mathbb{Z}[i]))$. We have the three special points x_{12},x_{13} and x_{23} (See(1.2.5), and we choose closed sets A_{ij} containing these points which just leave out a small open strip containing the opposite face. If $\tilde{A}_{ij}$ is a connected component of the inverse image of A_{ij} in $\mathbb{H}_3$, then

$$A_{ij}=\Gamma_{ij}\backslash\tilde{A}_{ij}.$$

The intersections $A_{ij}\cap A_{i'j'}=A_\nu$ are closed sets. They are of the form

$$A_\nu=\Gamma_\nu\backslash\tilde{A}_\nu$$

where Γ_ν is the stabilizer of the arc joining x_{ij} and $x_{i'j'}$. The restrictions of our sheaves $\tilde{\mathcal{M}}$ to the A_{ij} and A_ν and to $A=A_{12}\cap A_{23}\cap A_{13}$ are acyclic and hence we get a complex

$$0\longrightarrow\tilde{\mathcal{M}}\longrightarrow\bigoplus_{(i,j)}\tilde{\mathcal{M}}_{A_{ij}}\longrightarrow\bigoplus\tilde{\mathcal{M}}_{A_\nu}\longrightarrow\tilde{\mathcal{M}}_A\longrightarrow 0 \tag{2.32}$$

where the $\tilde{\mathcal{M}}_?$ are the restrictions of $\tilde{\mathcal{M}}$ to ? and then extended to the space again.

Hence we find that our cohomology groups are equal to the cohomology groups of the complex

$$0 \longrightarrow \bigoplus_{(i,j)} \mathcal{M}^{\Gamma_{ij}} \xrightarrow{d^1} \bigoplus_{\nu} \mathcal{M}^{\Gamma_\nu} \xrightarrow{d^2} \mathcal{M} \longrightarrow 0 \tag{2.33}$$

with boundary maps

$$\begin{aligned} d^1 :& (m_{12}, m_{13}, m_{23}) \longmapsto (m_{12} - m_{13}, m_{23} - m_{12}, m_{13} - m_{23}) \\ d^2 :& (m_1, m_2, m_3) \longmapsto m_1 + m_2 + m_3. \end{aligned}$$

If we take for instance $\tilde{\mathcal{M}} = \mathbb{Z}$ then we get $H^0(\tilde{\Gamma}\backslash\mathbb{H}_3, \mathbb{Z}) = \mathbb{Z}$ and $H^i(\tilde{\Gamma}\backslash\mathbb{H}_3, \mathbb{Z}) = 0$ for $i > 0$ as it should be.

We do not get a satisfying answer to our question. We consider the special case $\Gamma = \mathrm{Sl}_2(\mathbb{Z}[i])$ and the coefficient system $\mathcal{M}_{n_1} \otimes_{\mathbb{Z}[i]} \mathcal{M}_{n_2}$ where

$$\mathcal{M}_{n_1} := \{P(X,Y) = \sum_{\nu=0}^{n_1} a_\nu X^\nu Y^{n_1-\nu} | a_\nu \in \mathbb{Z}[i]\}, \tag{2.34}$$

$$\mathcal{M}_{n_2} := \{P(X,Y) = \sum_{\nu=0}^{n_2} a_\nu \bar{X}^\nu \bar{Y}^{n_2-\nu} | a_\nu \in \mathbb{Z}[i]\}, \tag{2.35}$$

and where a matrix $\gamma \in \Gamma$ acts by

$$\gamma P(X,Y) \otimes Q(\bar{X}, \bar{Y}) = P(aX + cY, bX + dY) \otimes Q(\bar{a}\bar{X} + \bar{c}\bar{Y}, \bar{b}\bar{X} + \bar{d}\bar{Y}) \tag{2.36}$$

If we now choose for our module $\mathcal{M}$ in (2.33) the module $\mathcal{M} = \mathcal{M}_{n_1} \otimes_{\mathbb{Z}[i]} \mathcal{M}_{n_2}$ then it i clear that for any given (not too large) values n_1, n_2 we can compute the cohomology explicitly. But we do not get any theoretical insight, for instance we do not get a formula for the dimensions of the cohomology groups (tensorised by $\mathbb{Q}$),

$\boxed{\text{Poindual}}$

2.1.5 Homology, Cohomology with compact support and Poincaré duality.

Here we have to use the theory of compactifications. For any locally symmetric space we can embed $\Gamma\backslash X$ into its Borel-Serre compactification

$$i : \Gamma\backslash X \longrightarrow \Gamma\backslash\overline{X}_{BS},$$

and this process was explained in detail for our low dimensional examples. Especially we give an explicit description of a neighbourhood of a point $x \in \partial(\Gamma\backslash\overline{X}_{BS})$. If we have a sheaf $\tilde{\mathcal{M}}$ on $\Gamma\backslash X$, we can extend it to the compactification by using the functor i_*. We get a sheaf

$$i_*(\tilde{\mathcal{M}}) \quad \text{on} \quad \Gamma\backslash\overline{X}_{BS},$$

it is clear from the description of a neighbourhood of a point in the boundary, that i_* is exact. (This is not true for the Baily-Borel compactification.)

Our construction $\mathcal{M} \to \tilde{\mathcal{M}}$ can be extended to the action of Γ on $\overline{X}_{BS}$ and clearly

$$i_*(\tilde{\mathcal{M}}) = \text{ result of the construction } \mathcal{M} \to \tilde{\mathcal{M}} \text{ on } \Gamma\backslash\overline{X}_{BS}.$$

Hence we get from our general results in Chapter I, ... that

$$H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) = H^\bullet(\Gamma\backslash\overline{X}_{BS}, i_*(\tilde{\mathcal{M}})).$$

But we have another construction of extending the sheaf $\tilde{\mathcal{M}}$ from $\Gamma\backslash X$ to $\Gamma\backslash\overline{X}_{BS}$. This is the so called extension by zero. We define the sheaf $i_!(\tilde{\mathcal{M}})$ on $\Gamma\backslash\overline{X}_{BS}$ by giving the stalks. For $x \in \Gamma\backslash\overline{X}_{BS}$ we put

$$i_!(\tilde{\mathcal{M}})_x = \begin{cases} \tilde{\mathcal{M}}_x & \text{if} \quad x \in \Gamma\backslash X \\ 0 & \text{if} \quad x \notin \Gamma\backslash X \end{cases}.$$

It is clear that $i_!$ is an exact functor sending sheaves on $\Gamma\backslash X$ to sheaves on $\Gamma\backslash\overline{X}_{BS}$, and we have for an arbitrary sheaf

$$H^0(\Gamma\backslash\overline{X}_{BS}, i_!(\mathcal{F})) = H^0_c(\Gamma\backslash X, \mathcal{F})$$

where $H^0_c(\Gamma\backslash X, \mathcal{F})$ is the abelian group of those sections $s \in H^0(\Gamma\backslash X, \mathcal{F})$ for which the support

$$\text{supp } (s) = \{x \mid s_x \neq 0\}$$

is compact.

Hence we define the cohomology with compact supports as

$$H^q_c(\Gamma\backslash X, \mathcal{F}) = H^q(\Gamma\backslash\overline{X}_{BS}, i_!\mathcal{F})).$$

If $\tilde{\mathcal{M}}$ is a sheaf on $\Gamma\backslash X$ which is obtained from a Γ-module $\mathcal{M}$, then it is quite clear that

$$H^0_c(\Gamma\backslash X, \tilde{\mathcal{M}}) = 0,$$

provided our quotient $\Gamma\backslash X$ is not compact.

The cohomology with compact supports is actually related to the homology of the group: I want to indicate that we have a natural isomorphism

$$H_i(\Gamma, \mathcal{M}) \simeq H^{d-i}_c(\Gamma\backslash X, \tilde{\mathcal{M}})$$

under the assumption that X is connected and the orders of the stabilizers are invertible in R.

This is the analogous statement to the theorem .... which we discussed when we introduced cohomology.

Our starting point is the fact that the projective Γ-modules have analogous vanishing properties as the induced modules.

Lemma 2.1.3. *Let us assume that Γ acts on the connected symmetric space X. If P if a projective module then*

$$H^i_c(\Gamma\backslash X, \tilde{P}) = \begin{cases} 0 & \text{if} \quad i \neq \dim X \\ \\ P_\Gamma & \text{if} \quad i = \dim X. \end{cases}$$

Let us believe this lemma. Then it is quite clear that

$$H_i(\Gamma, \mathcal{M}) \simeq H_c^{d-i}(\Gamma\backslash X, \tilde{P}),$$

because both sides can be computed from a projective resolution.

2.1.6 The homology as singular homology

We have still another description of the homology. We form the singular chain complex

$$\to C_i(X) \to C_{i-1}(X) \to \ldots \to C_0(X) \to 0.$$

This is a complex of Γ-modules, and we can form the tensor product with $\mathcal{M}$. We get a complex of Γ-modules

$$\xrightarrow{d_{i+1}} C_i(X) \otimes \mathcal{M} \xrightarrow{d_i} C_{i-1}(X) \otimes \mathcal{M} \longrightarrow \ldots.$$

We define the chain complex

$$C_\bullet(\Gamma\backslash X, \underline{\mathcal{M}}),$$

simply as the resulting complex of Γ-coinvariants. The homology groups are defined as

$$H_i(\Gamma\backslash X, \underline{\mathcal{M}}) = \ker(d_i)/\mathrm{Im}(d_{i+1}) \tag{2.37}$$

The cosheaves

The symbol $\underline{\mathcal{M}}$ should be interpreted as the cosheaf attached to our Γ-module, this is an object which is dual to the sheaf $\tilde{\mathcal{M}}$. For a point $\bar{x} \in \Gamma\backslash X$ costalk $\underline{\mathcal{M}}_{\bar{x}}$ is given as follows: As in (2.2) we consider the projection $\pi_\Gamma : X \to \Gamma\backslash X$ and maps with finite support

$$\mathcal{C}(\bar{x}, \mathcal{M}) := \{f : \pi_\Gamma^{-1}(\bar{x}) \to \mathcal{M}\}. \tag{2.38}$$

On this module we have an action of Γ which is given by $\boxed{\text{act}}$

$$(\gamma f)(x) = \gamma(f(\gamma^{-1}x). \tag{2.39}$$

Then our costalk is given by the coinvariants

$$\underline{\mathcal{M}}_{\bar{x}} = \mathcal{C}(\bar{x}, \mathcal{M})_\Gamma = \mathcal{C}(\bar{x}, \mathcal{M})/\{f - \gamma f, \gamma \in \Gamma, f \in \mathcal{C}(\bar{x}, \mathcal{M})\} \tag{2.40}$$

We have the homomorphism $\int : \mathcal{M}_{\bar{x}} \to \mathcal{M}$ which is given by summation $f \mapsto \sum_{x\in\pi_\Gamma^{-1}(\bar{x})} f(x)$ and this induces an isomorphism $\boxed{\text{invint}}$

$$\int : \mathcal{C}(\bar{x},\mathcal{M})_\Gamma \xrightarrow{\sim} \underline{\mathcal{M}}_{\bar{x}} \tag{2.41}$$

We pick a point $x \in \pi_\Gamma^{-1}(\bar{x})$ and an open neighbourhood U_x of x such that $\gamma U_x \cap U_x \neq \emptyset$ implies $\gamma \in \Gamma_x$. We consider the space $\mathcal{C}(\bar{x}, x, \mathcal{M})$ of those maps, which are supported in in the point x. This space is of course equal to $\mathcal{M}$ and the composition

$$\delta_x : \mathcal{C}(\bar{x}, x, \mathcal{M}) \to \mathcal{C}(\bar{x},\mathcal{M}) \to \underline{\mathcal{M}}_{\bar{x}}$$

induces an isomorphism

$$\delta_x : \mathcal{M}_{\Gamma_x} \xrightarrow{\sim} \underline{\mathcal{M}}_{\bar{x}} \tag{2.42}$$

If we pick a second point $\bar{y} \in \pi_\Gamma(U_x)$ and a $y \in \pi_\Gamma^{-1}(\bar{y}) \cap U_x$ then clearly $\Gamma_y \subset \Gamma_x$ and therefore we get a specialization map

$$r_{\bar{y},\bar{x}} : \underline{\mathcal{M}}_{\bar{y}} \to \underline{\mathcal{M}}_{\bar{x}}. \tag{2.43}$$

Now it becomes clear why these objects are called cosheaves. For the sheaf $\tilde{\mathcal{M}}$ we get in the corresponding situation a map in the opposite direction

$$\tilde{\mathcal{M}}_{\bar{x}} \to \tilde{\mathcal{M}}_{\bar{y}} \tag{2.44}$$

as a specialization map between the stalks of $\tilde{\mathcal{M}}$. An element $f^* \in \underline{\mathcal{M}}_{\bar{x}}$ can be represented as an array $\boxed{\text{refcos}}$

$$f^* = \{\dots, f(x), \dots\}_{x\in\pi^{-1}(\bar{x})} \tag{2.45}$$

where $f(x) \in (\underline{\mathcal{M}}_{\bar{x}})_{\Gamma_x}$ and $f(\gamma x) = \gamma f(x)$.

Now we can give a different description of the group of i-chains $C_i(\Gamma\backslash X, \underline{\mathcal{M}})$: An i-chain with values in the cosheaf $\underline{\mathcal{M}}$ is of the form $\sigma\otimes f$ where $\sigma : \Delta^i \to \Gamma\backslash X$ is a continuous (differentiable) map from the i dimensional simplex Δ^i to $\Gamma\backslash X$ and where f is a section in the cosheaf, i.e. $f_x \in \underline{\mathcal{M}}_{\sigma(x)}$ and where f_x varies continuously. (This means: If $\sigma(y)$ specializes to $\sigma(x)$ then $r_{\sigma(y),\sigma(x)}(f_y) = f_x$.)

Then $C_i(\Gamma\backslash X, \underline{\mathcal{M}})$ is the free abelian group generated by these i chains with values in $\underline{\mathcal{M}}$). Then the boundary maps d_i are defined in the usual way and we get a slightly different description of the homology groups $H_i(\Gamma\backslash X, \underline{\mathcal{M}})$.

But we may choose for our module $\mathcal{M}$ simply the group ring. Then

$$(C_\bullet(X) \otimes \mathbb{Z}[\Gamma])_\Gamma \simeq C_\bullet(X),$$

and hence we have, since X is a cell, that

$$H_i(\Gamma\backslash X, \underline{\mathbb{Z}[\Gamma]}) = 0 \qquad \text{for} \qquad i > 0.$$

On the other hand we have

$$H_0(\Gamma\backslash X, \underline{\mathcal{M}}) = \mathcal{M}_\Gamma.$$

This follows directly from looking at the complex

$$(C_1(X)\otimes \mathcal{M})_\Gamma \longrightarrow (C_0(X)\otimes \mathcal{M})_\Gamma.$$

First of all we observe that 0-cycles

$$x_1\otimes m - x_0\otimes m$$

are boundaries since X is pathwise connected. On the other hand we have that

$$x_0\otimes m - \gamma x_0\otimes \gamma m\in C_0(X)\otimes \mathcal{M}$$

becomes zero if we go to the coinvatiants and this implies the assertion.

If we have in addition that the orders of the stabilizers are invertible in R than it is clear that a short exact sequence of R-Γ-modules

$$0\longrightarrow \mathcal{M}'\longrightarrow \mathcal{M}\longrightarrow \mathcal{M}''\longrightarrow 0$$

leads to an exact sequence of complexes

$$0\longrightarrow C_\bullet(\Gamma\backslash X,\underline{\mathcal{M}'})\longrightarrow C_\bullet(\Gamma\backslash X,\underline{\mathcal{M}})\longrightarrow C_\bullet(\Gamma\backslash X,\underline{\mathcal{M}''})\longrightarrow 0,$$

and hence to a long exact cohomology sequence

$$H_i(\Gamma\backslash X,\underline{\mathcal{M}'})\longrightarrow H_i(\Gamma\backslash X,\underline{\mathcal{M}})\longrightarrow H_i(\Gamma\backslash X,\underline{\mathcal{M}''})\longrightarrow H_{i-1}(\Gamma\backslash X,\underline{\mathcal{M}'}).$$

Now it is clear that

$$H_i(\Gamma,\mathcal{M})\simeq H_i(\Gamma\backslash X,\underline{\mathcal{M}})\simeq H_c^{d-i}(\Gamma\backslash X,\tilde{\mathcal{M}}).$$

$\boxed{\text{fundex}}$

2.1.7 The fundamental exact sequence

By construction we have the exact sequence

$$0\to i_!(\tilde{\mathcal{M}})\to i_*(\tilde{\mathcal{M}})\to i_*(\tilde{\mathcal{M}})/i_!(\tilde{\mathcal{M}})\to 0$$

of sheaves and clearly $i_*(\mathcal{M})/i_!(\mathcal{M})$ is simply the restriction of $i_*(\tilde{\mathcal{M}})$ to the boundary extended by zero to the entire space. This yields the *fundamental exact sequence*

$$\to H^{q-1}(\partial(\Gamma\backslash X),\tilde{\mathcal{M}})\to H^q_c(\Gamma\backslash X,\tilde{\mathcal{M}})\to H^q(\Gamma\backslash \bar{X},\tilde{\mathcal{M}})\xrightarrow{r} H^q(\partial(\Gamma\backslash X),\tilde{\mathcal{M}})\to\dots \tag{2.46}$$

We define the "inner cohomology" $\boxed{\text{inncoh}}$

$$H^q_!(\Gamma\backslash X,\tilde{\mathcal{M}}):=\mathrm{Im}(H^q_c(\Gamma\backslash X,\tilde{\mathcal{M}})\to H^q(\Gamma\backslash \bar{X},\tilde{\mathcal{M}}))=\ker H^q(\Gamma\backslash \bar{X},\tilde{\mathcal{M}})\xrightarrow{r} H^q(\partial(\Gamma\backslash X),\tilde{\mathcal{M}}) \tag{2.47}$$

(This a little bit misleading because these groups are not honest cohomology groups, they are not the cohomology groups of a space with coefficients in a sheaf. An exact sequence of sheaves $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ does not provide an exact sequence for these $H_!$ groups.)

In the special case that the underlying group $G/\mathbb{Q}$ is anisotropic the fundamental exact sequence becomes trivial, in this case the quotient $\Gamma\backslash X$ is compact and we have

$$H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) = H_c^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) = H_!^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}).$$

Many authors prefer to consider the case of a compact quotient $\Gamma\backslash X$, but I think we loose some very interesting phenomena if we concentrate on this case. On the other hand we do not need to read the next subsection. Also readers who are more interested in the low dimensional cases and the more specific results in these cases may well skip reading the next subsection.

The cohomology of the boundary

We want to have a slightly different look at this sequence. We recall the covering (See 1.124,1.125)

$$\Gamma\backslash X = \Gamma\backslash X(r) \cup \overset{\bullet}{\mathcal{N}}(\Gamma\backslash X) = \Gamma\backslash X(r) \cup \bigcup_{P:P\text{proper}} \Gamma_P\backslash X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'})) \qquad (2.48)$$

where the union runs over Γ conjugacy classes of parabolic subgroups over $\mathbb{Q}$ and $\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X)$ is a punctured tubular neighbourhood of ∞, i.e. the boundary of the Borel-Serre compactification.

It is well known (See for instance [book] vol I , 4.5) that from a covering $\Gamma\backslash X = \bigcup_i V_i$ we get a Čzech complex and a spectral sequence with $E_1^{p,q}$- term

$$\prod_{\underline{i}=\{i_0,i_1\dots,i_p\}} H^q(V_{\underline{i}}, \tilde{\mathcal{M}}) \qquad (2.49)$$

where $V_{\underline{i}} = V_{i_0} \cap \cdots \cap V_{i_p}$. The boundary in the Čzech complex gives us the differential

$$d_1^{p,q} : \prod_{\underline{i}=\{i_0,i_1\dots,i_p\}} H^q(V_{\underline{i}}, \tilde{\mathcal{M}}) \to \prod_{\underline{j}=\{j_0,j_1\dots,j_{p+1}\}} H^q(V_{\underline{j}}, \tilde{\mathcal{M}}) \qquad (2.50)$$

Here we work with the alternating Čzech complex, we also assume that we have an ordering on the set of simple positive roots. If such a $V_{\underline{i}}$ is non empty then it of the form $\Gamma_Q\backslash X^Q(C(\tilde{\underline{c}}))$.

We return to the diagram (1.132), on the left hand side we can divide by Γ_Q. We have the map which maps a Cartan involution on X to a Cartan-involution on M. Then we get a diagram

$$\begin{array}{ccc} f^\dagger : X^Q(C(\tilde{\underline{c}})) & \to & X^M(r) \times C_{U_Q}(\tilde{\underline{c}}) \\ \downarrow p_Q & & \downarrow p_M \\ f : \Gamma_Q\backslash X^Q(C(\tilde{\underline{c}})) & \to & \Gamma_M\backslash X^M(r) \times C_{U_Q}(\tilde{\underline{c}})) \end{array} \qquad (2.51)$$

where the bottom line is a fibration. To describe the fiber in a point $\tilde{x}$ we pick a point $x \in (p_m \circ f^\dagger)^{-1}$. Then $U_Q(\mathbb{R})$ acts simply transitively on the fiber $(f^\dagger)^{-1}(f^\dagger(x))$ hence $U_Q(\mathbb{R}) = (f^\dagger)^{-1}(f^\dagger(x))$. Then $p_Q : U_Q(\mathbb{R}) \to \Gamma_{U_Q}\backslash U_Q(\mathbb{R})$ yields the identification $i_x : \Gamma_{U_Q}\backslash U_Q(\mathbb{R}) \xrightarrow{\sim} f^{-1}(\tilde{x})$. If we replace x by $\gamma x = x_1$ with $\gamma \in \Gamma_{U_Q}$ then we get $i_{x_1} = \mathrm{Ad}(\gamma)\circ i_x$ where for $u \in U_{U_Q}$ $\mathrm{Ad}(\gamma)(u) = \gamma u \gamma^{-1}$ where for $u \in U_Q(\mathbb{R})$, under this action of Γ_Q.

We have the spectral sequence

$$H^p(\Gamma_M\backslash X^M(r), R^q f_*(\tilde{\mathcal{M}})) \Rightarrow H^{p+q}(\Gamma_Q\backslash X^Q(C(\underline{c}_{\pi_1}, \dots, c_{\pi_\nu})), \tilde{\mathcal{M}})$$

and clearly $R^q f_*(\tilde{\mathcal{M}})$ is a locally constant sheaf. This sheaf is easy to determine. Under the above identification we get an isomorphism

$$i_x^\bullet : H^\bullet(\Gamma_{U_Q}\backslash U_Q(\mathbb{R}), \tilde{\mathcal{M}})) \xrightarrow{\sim} R^\bullet(\tilde{\mathcal{M}})_{\tilde{x}}.$$

The adjoint action $\mathrm{Ad} : \Gamma_Q \to \mathrm{Aut}(\Gamma_{U_Q}\backslash U_Q(\mathbb{R}))$ induces an action of Γ_Q on the cohomology $H^\bullet((\Gamma_{U_Q}\backslash U_Q(\mathbb{R})), \tilde{\mathcal{M}})$. Since the functor cohomology is the derived functor of taking Γ_{U_Q} invariants it follows that the restriction of Ad to Γ_{U_Q} acts trivially on $H^\bullet(\Gamma_{U_Q}\backslash U_Q(\mathbb{R}), \tilde{\mathcal{M}})$. Consequently $H^\bullet((\Gamma_{U_Q}\backslash U_Q(\mathbb{R})), \tilde{\mathcal{M}})$ is a Γ_M- module. We get

$$R^\bullet f_*(\tilde{\mathcal{M}}) \xrightarrow{\sim} H^\bullet(\widetilde{\Gamma_{U_Q}\backslash U_Q}(\mathbb{R}), \tilde{\mathcal{M}})$$

and hence our spectral sequence becomes

$\boxed{\text{vEst}}$

$$H^p(\Gamma_M\backslash X^M(r), H^\bullet(\widetilde{\Gamma_{U_Q}\backslash U_Q}(\mathbb{R}), \tilde{\mathcal{M}})) \Rightarrow H^{p+q}(\Gamma_Q\backslash X^Q(C(\underline{\tilde{c}})), \tilde{\mathcal{M}}) \tag{2.52}$$

We can take the composition $r_Q \circ f$. Then it is obvious that for any point $c_0 \in C_{U_Q}(\underline{\tilde{c}}))$ the restriction map

$$H^\bullet(X^Q(C(\underline{\tilde{c}})), \tilde{\mathcal{M}}) \to H^\bullet(X^Q((r_Q \circ f)^{-1}(c_0), \tilde{\mathcal{M}}) \tag{2.53}$$

is an isomorphism. On the other hand it is clear that we may vary our parameter $\underline{\tilde{c}}$ we may assume that the $C_{U_Q}(\underline{\tilde{c}})$ go to infinity. Then we may enlarge the parameter r without violating the assumptions in proposition 1.2.3. Hence we get that the inclusion $\Gamma_Q\backslash X^Q(C(\underline{\tilde{c}})) \subset \Gamma_Q\backslash X^Q$ induces an isomorphism in cohomology

$$H^\bullet(\Gamma_Q\backslash X^Q(C(\underline{\tilde{c}}), \tilde{\mathcal{M}}) \xrightarrow{\sim} H^\bullet(\Gamma_Q\backslash X, \tilde{\mathcal{M}}) \tag{2.54}$$

We choose a total ordering on the set of Γ conjugacy classes of parabolic subgroups, i.e. we enumerate them by a finite interval of integers $[1, N]$. We also enumerate the set of simple roots $\{\alpha_1, \dots, \alpha_d)$ in our special case $\alpha_i = \alpha_{i,i+1}$. For any conjugacy class $[P]$ we define the type of P to be $t(P) = \pi^{U_P}$ the subset of unipotent simple roots and $d(P) = \#\pi^{U_P}$ the cardinality of this set. If $P_{i_1}, \dots, P_{i_r}$ are maximal, $i_1 < i_2 \cdots < i_r$ and if $P_{i_1}\cap, \dots \cap P_{i_r} = Q$ is a parabolic subgroup then we require that $t(P_{i_1}) < \dots < t(P_{i_r})$.

The indexing set $\mathrm{Par}(\Gamma)$ of our covering is the Γ conjugacy classes of parabolic subgroups over $\mathbb{Q}$. If we have a finite set $[P_{i_0}], [P_{i_1}], \dots, [P_{i_p}]$ of conjugacy classes then we say $[Q] \in [P_{i_0}], [P_{i_1}], \dots, [P_{i_p}]$ if we can find representatives $P'_{i_\nu} \in [P_{i_\nu}]$ and $Q' \in [Q]$ such that $Q' = P'_{i_0} \cap \dots \cap P'_{i_p}$.

Hence we see that the $E_1^{\bullet,q}$ complex in our spectral sequence (2.50) is given by

$$\prod_i H^q(\Gamma_{Q_i}\backslash X^{Q_i}(C(\underline{\tilde{c}})), \tilde{\mathcal{M}}) \to \prod_{i<j} \prod_{[R]\in[Q_i]\cap[Q_j]} H^q(\Gamma_R\backslash X^R(C(\underline{\tilde{c}})), \tilde{\mathcal{M}}) \to \tag{2.55}$$

this obtained from our covering (1.125). Now we replace our covering by a simplicial space, i.e. we consider the diagram of maps between spaces

$$\mathfrak{Par} := \prod_i \Gamma_{Q_i}\backslash X \underset{p_2}{\overset{p_1}{\leftleftarrows}} \prod_{i<j} \prod_{[R]\in[Q_i]\cap Q_j]} \Gamma_R\backslash X \begin{matrix}\longleftarrow\\ \longleftarrow\\ \longleftarrow\end{matrix} \tag{2.56}$$

this yields a spectral sequence with $E_1^{\bullet,q}$ term

$$\prod_i H^q(\Gamma_{Q_i}\backslash X, \tilde{\mathcal{M}}) \xrightarrow{d^{(0)}} \prod_{i<j} \prod_{[R]\in[P_i]\cap[P_j]} H^q(\Gamma_R\backslash X^R, \tilde{\mathcal{M}}) \xrightarrow{d^{(1)}} \tag{2.57}$$

Our covering also yields a simplicial space which is a subspace of (2.56) we get a map from (2.50) to (2.57) and this map is an isomorphism of complexes.

We replace $\mathfrak{Par}$ by another simplicial complex

$$\mathfrak{Parmax} := \prod_{[P]:d(P)=1} \Gamma_P\backslash X \underset{p_2}{\overset{p_1}{\leftleftarrows}} \prod_{[Q]:d(Q)=2} \Gamma_Q\backslash X \begin{matrix}\longleftarrow\\ \longleftarrow\\ \longleftarrow\end{matrix} \tag{2.58}$$

We have an obvious projection $\Pi : \mathfrak{Par} \to \mathfrak{Parmax}$ which induces a homomorphism

$$\begin{array}{ccccc} \prod_i H^q(\Gamma_{Q_i}\backslash X, \tilde{\mathcal{M}}) & \xrightarrow{d^{(0)}} & \prod_{i<j} \prod_{[R]\in[P_i]\cap[P_j]} H^q(\Gamma_R\backslash X^R, \tilde{\mathcal{M}}) & \xrightarrow{d^{(1)}} \\ \uparrow & & \uparrow & \\ \prod_{[P]:d(P)=1} H^q(\Gamma_P\backslash X, \tilde{\mathcal{M}}) & \xrightarrow{d^{(0)}} & \prod_{[R]:d(R)=2} H^q(\Gamma_R\backslash X^R, \tilde{\mathcal{M}}) & \xrightarrow{d^{(1)}} \end{array} \tag{2.59}$$

and an easy argument in homological algebra shows that this induces an isomorphism in cohomology or in other words an isomorphism of the $E_2^{p,q}$ terms of the two spectral sequences.

We had the covering

$$\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X) = \bigcup_{P:P\text{proper}} \Gamma_P\backslash X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'})) \tag{2.60}$$

which gives us the spectral sequence converging to $H^\bullet(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X), \tilde{\mathcal{M}})$ with

$$E_1^{p,q} = \bigoplus_{i_0<i_1<\dots<i_p} \bigoplus_{[Q]\in[P_{i_0}]\cap[P_{i_1}]\cap\dots\cap[P_{i_p}]} H^q(\Gamma_Q\backslash X^Q(\underline{c}_{\pi'}, r(\underline{c}_{\pi'}), \tilde{\mathcal{M}})) \tag{2.61}$$

Our covering of $\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X)$ gives us a simplicial space $\mathfrak{Cov}(\overset{\bullet}{\mathcal{N}})\Gamma\backslash X)$ and we have maps

$$\mathfrak{Cov}(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X)) \hookrightarrow \mathfrak{Par} \to \mathfrak{Parmax}. \tag{2.62}$$

We saw that the resulting maps induced an isomorphism in the $E_2^{p,q}$ terms of the spectral sequences. Hence we see that $\mathfrak{Parmax}$ yields a spectral sequence

$$E_1^{p,q} = \bigoplus_{[P]:d(P)=p+1} H^q(\Gamma_P\backslash X, \tilde{\mathcal{M}}) \Rightarrow H^{p+q}(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X), \tilde{\mathcal{M}})) \tag{2.63}$$

the differentials $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ are simply obtained from the restriction maps.

This is of course the "same" spectral sequence as the one above (2.57) and we may also restrict to the cohomology of the Borel-Serre boundary and get

$$E_1^{p,q} = \bigoplus_{[P]:d(P)=p+1} H^q(\partial_P(\Gamma\backslash\bar{X}), \tilde{\mathcal{M}}) \Rightarrow H^{p+q}(\partial((\Gamma\backslash\bar{X}, \tilde{\mathcal{M}})) \tag{2.64}$$

At this point we want to raise an interesting question

Does this spectral sequence degenerate at $E_2^{p,q}$ level?

The author of this book is hoping that the answer to this question is no! And this is so for interesting reasons! We come back to this question when we discuss the Eisenstein cohomology.

The complement of $\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X)$ is a relatively compact open set $V \subset \Gamma\backslash X$, this set contains the stable points. We define $\tilde{\mathcal{M}}_V^! = i_{V,!}(\tilde{\mathcal{M}})$ then we get an exact sequence

$$0 \to \tilde{\mathcal{M}}_V^! \to \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}/\tilde{\mathcal{M}}_V^! \to 0 \tag{2.65}$$

and $\tilde{\mathcal{M}}/\tilde{\mathcal{M}}_V^!$ is obviously the extension of the restriction of $\tilde{\mathcal{M}}$ to $\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X)$ and the extended by zero to $\Gamma\backslash X$. We claim (easy proof later) that

$$H_c^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) = H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_V^!) \tag{2.66}$$

and this gives us again the fundamental exact sequence $\boxed{\text{fux}}$

$$H^{q-1}(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X), \tilde{\mathcal{M}}) \to H^q(\Gamma\backslash X, \tilde{\mathcal{M}}_V^!) \to H^q(\Gamma\backslash X, \tilde{\mathcal{M}}) \to H^q(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X), \tilde{\mathcal{M}}) \to \tag{2.67}$$

2.1.8 How to compute the cohomology groups $H_c^q(\Gamma\backslash X, \tilde{\mathcal{M}})$

We apply the considerations in 4.8 from [39] Again we cover $\Gamma\backslash X$ by orbiconvex open neighbourhoods U_{x_i}, and now we define

$$\tilde{\mathcal{M}}_{\underline{x}}^! = (i_{\underline{x}})_! i_{\underline{x}}^*(\tilde{\mathcal{M}}).$$

These sheaves have properties, which are dual to those of the sheaves $\tilde{\mathcal{M}}_{\underline{x}}$. If $\underline{x} = (x_1, \dots, x_s)$ and if we add another point $\underline{x}' = (x_1, \dots, x_s, x_{s+1})$ then we have the restriction $\tilde{\mathcal{M}}_{\underline{x}} \to \tilde{\mathcal{M}}_{\underline{x}'}$, which were used to construct the Čech resolution.

Now let $d = \dim(X)$. For the ! sheaves we get (See [39] , loc. cit.) get a morphism $\tilde{\mathcal{M}}^!_{\underline{x}'} \to \tilde{\mathcal{M}}^!_{\underline{x}}$. For $\underline{x} = (x_1, \dots, x_s)$ we define the degree $d(\underline{x}) = d+1-s$. Then we construct the Čech-coresolution (See [39] , 4.8.3)

$$\to \prod_{\underline{x}:d(\underline{x})=q} \tilde{\mathcal{M}}^!_{\underline{x}} \to \cdots \to \prod_{(x_i,x_j)} \tilde{\mathcal{M}}^!_{x_i,x_j} \to \prod_{x_i} \tilde{\mathcal{M}}^!_{x_i} \to i_!(\tilde{\mathcal{M}}) \to 0. \tag{2.68}$$

Now we have a dual statement to (2.10)

Proposition: *If* $d = \dim(X)$ *then*

$$H^q(U_{\underline{\tilde{x}}}, \tilde{\mathcal{M}}^!_{\underline{x}}) = \begin{cases} \mathcal{M}_{\Gamma_{\underline{x}}} & q = d \\ 0 & q \neq d \end{cases} \tag{2.69}$$

Hence the above complex of sheaves provides a complex of modules

$$C^\bullet_!(\mathfrak{U}, \tilde{\mathcal{M}}) :=$$

$$\to \prod_{\underline{x}:d(\underline{x})=q} H^d(U_{\underline{x}}, \tilde{\mathcal{M}}^!_{\underline{x}}) \to \cdots \to \prod_{(x_i,x_j)} H^d(U_{x_i,x_j}, \tilde{\mathcal{M}}^!_{x_i,x_j}) \to \prod_{x_i} \tilde{H}^d(U_{x_i}, \tilde{\mathcal{M}}^!_{x_i}) \to 0. \tag{2.70}$$

and then

$$H^q(\Gamma\backslash X, i_!(\tilde{\mathcal{M}})) = H^q_c(\Gamma\backslash X, \tilde{\mathcal{M}}) = H^q(C^\bullet_!(\mathfrak{U}, \tilde{\mathcal{M}})). \tag{2.71}$$

Now let us assume that $\mathcal{M}$ is a finitely generated module over some commutative noetherian ring R with identity. Then clearly all our cohomology groups will be R-modules.

Our Theorem A above implies

Theorem (Raghunathan) *Under our general assumptions all the cohomology groups* $H^q_c(\Gamma\backslash X, \tilde{\mathcal{M}}), H^q(\Gamma\backslash X, \tilde{\mathcal{M}}), H^q_!(\Gamma\backslash X, \tilde{\mathcal{M}}), H^q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}})$ *are finitely generated R modules.*

$\boxed{\text{MC}}$

2.1.9 Modified cohomology groups

Most of the time our module $\mathcal{M}$ will be a finitely generated $\mathbb{Z}$ module and the theorem of Raghunathan says that the cohomology groups are also finitely generated $\mathbb{Z}$ modules. Sometimes we replace $\mathbb{Z}$ ring of integers $\mathcal{O}_F$ of a finite extension $F/\mathbb{Q}$ and then we will even invert some finite numbers of primes. Hence we our coefficient modules will be finitely generated R-modules where $\mathcal{O}_F \subset R \subset F$. In any case these rings R will be Dedekind rings.

Starting from the fundamental exact sequence we have introduced the modified cohomology groups $H^q_!(\ )$. There is a second process of modification: If $H^\bullet(\ )$ is any of these cohomology groups then

$\boxed{\text{Hint}}$

$$H^\bullet(\ )_{\text{ int}} := H^\bullet(\ )/\text{Tors} = \text{Im}(H^\bullet(\ ) \to H^\bullet(\ ) \otimes \mathbb{Q}) \tag{2.72}$$

We have to discuss a minor problem: These two processes of modification do not quite commute. This is due to the fact that the resulting sequence

$$\to H^{q-1}(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}_R)_{\text{ int}} \to H^q_c(\Gamma\backslash X, \tilde{\mathcal{M}}_R)_{\text{ int}} \xrightarrow{j} H^q(\Gamma\backslash \bar{X}, \tilde{\mathcal{M}}_R)_{\text{ int}} \xrightarrow{r} H^q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}_R)_{\text{ int}}$$

is not necessarily exact anymore. Clearly we have $H^q_!(\Gamma\backslash X, \tilde{\mathcal{M}}_R)_{\text{ int}} = \text{Im}(j)$ and if we now define $H^q\Gamma\backslash X, \tilde{\mathcal{M}}_R)_{\text{ int},!} :=\ \ker(r)$ then we have

$$H^q_!(\Gamma\backslash X, \tilde{\mathcal{M}}_R)_{\text{ int}} \subset H^q(\Gamma\backslash X, \tilde{\mathcal{M}}_R)_{\text{ int},!} \tag{2.73}$$

but this inclusion may be proper. The following proposition is an elementary exercise in homological algebra. $\boxed{\text{supqbd}}$

Proposition 2.1.1. *The quotient* $H^q(\Gamma\backslash X, \tilde{\mathcal{M}}_R)_{\text{ int},!}/H^q_!(\Gamma\backslash X, \tilde{\mathcal{M}}_R)_{\text{ int}}$ *is finite and isomorphic to a subquotient of* $H^q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}_R)$.

We will discuss an example in section 3.3.1

This may be a good place to introduce some terminology. If V is a torsion free, finitely generated R -module and we have a direct sum of submodules $V \supset \oplus_\nu V_\nu$ then we say that this direct sum is *decomposition up to isogeny* if the quotient $V/\oplus_\nu V_\nu$ is a torsion module and if for any ν the quotient V/V_ν is torsion free. Sometimes we also call this a *saturated decomposition* (see section 2.1.9).

$\boxed{\text{PDgen}}$

2.1.10 Poincare duality in general

Let us assume that the symmetric space $X = G(\mathbb{R})/K_\infty$ is connected and let us also assume that Γ operates orientation preserving, this means that any finite stabiliser Γ_x acts trivially on $\Lambda^d(T_{X,x})$. Then we know that $H^d_c(\Gamma\backslash X, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$, the isomorphism depends on the choice of an orientation.

Assume that $\mathcal{M}$ is a finitely generated torsion free $\mathbb{Z}$ module with an action of Γ on it and let $\mathcal{M}^\vee =\ \text{Hom}(\mathcal{M}, \mathbb{Z})$ be the dual module. Then we have an obvious pairing between the two complexes $C^\bullet_!(\mathfrak{U}, \tilde{\mathcal{M}})$: and $C^\bullet(\mathfrak{U}, \tilde{\mathcal{M}})$ (See (2.70),(2.11) which induced by the obvious pairing $H^0(U_{\underline{x}}, \tilde{\mathcal{M}}) \times H^d_c(U_{\underline{x}}, \tilde{\mathcal{M}}^\vee)$ and summation over the components. This pairing induces a pairing on the cohomology groups which are computed by these complexes (See [39],4.8.4.)

$$H^q_c(\Gamma\backslash X, \tilde{\mathcal{M}}) \otimes H^{d-q}(\Gamma\backslash X, \tilde{\mathcal{M}}) \xrightarrow{<,>_{PD}} H^q_c(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes \tilde{\mathcal{M}}^\vee) \to H^d_c(\Gamma\backslash X, \mathbb{Z}) = \mathbb{Z} \tag{2.74}$$

Of course $< x, y >_{PD} = 0$ if one of the entries is a torsion element and hence we get a pairing between the modified cohomology groups

PDZcp

$$H^q_c(\Gamma\backslash X,\tilde{\mathcal{M}})_{\text{ int}}\otimes H^{d-q}(\Gamma\backslash X,\tilde{\mathcal{M}}^\vee)_{\text{ int}}\xrightarrow{<,>}_{PD}\mathbb{Z} \tag{2.75}$$

If Γ is torsion free this is a non degenerate pairing (see for instance ([39], Thm. 4.8.4). If Γ has non trivial torsion the same is true if we replace $\mathbb{Z}$ by $R=\mathbb{Z}[1/N]$ where N is the product of the orders of the finite stabilisers.

If we analyse the fundamental exact sequence we also find a non degenerate pairing

PDaus

$$H^q_!(\Gamma\backslash X,\tilde{\mathcal{M}})_{\text{ int}}\otimes H^{d-q}(\Gamma\backslash X,\tilde{\mathcal{M}}^\vee)_{\text{ int},!}\xrightarrow{<,>_{PD}}\mathbb{Z} \tag{2.76}$$

Eventually we can pass to rational coefficient systems we get a non degenerate pairing

PDQ

$$H^q_!(\Gamma\backslash X,\tilde{\mathcal{M}}_\mathbb{Q})\otimes H^{d-q}_!(\Gamma\backslash X,\tilde{\mathcal{M}}^\vee_\mathbb{Q})\xrightarrow{<,>_{PD}}\to\mathbb{Q} \tag{2.77}$$

If we look again at the fundamental exact sequence we see the two homomorphisms

$$\delta H^{q-1}(\partial(\Gamma\backslash X),\tilde{\mathcal{M}}_\mathbb{Q})\to H^q_c(\Gamma\backslash X,\tilde{\mathcal{M}}_\mathbb{Q})\text{ and }r: H^{d-q}(\Gamma\backslash X,\tilde{\mathcal{M}}^\vee_\mathbb{Q})\to H^{d-q}(\partial(\Gamma\backslash X),\tilde{\mathcal{M}}^\vee_\mathbb{Q}) \tag{2.78}$$

The Borel-Serre boundary has the homotopy type of a $d-1$ dimensional manifold hence we we also have a non degenerate pairing PDbound

$$<\ ,\ >_\partial: H^{q-1}(\partial(\Gamma\backslash X),\tilde{\mathcal{M}}_\mathbb{Q})\times H^{d-q}(\Gamma\backslash X,\tilde{\mathcal{M}}^\vee_\mathbb{Q})\to\mathbb{Q}. \tag{2.79}$$

Now an easy computation shows that for $x\in H^{q-1}(\partial(\Gamma\backslash X),\tilde{\mathcal{M}}_\mathbb{Q}), y\in H^{d-q}(\Gamma\backslash X,\tilde{\mathcal{M}}^\vee_\mathbb{Q})$ we have the formula

$$<x,r(y)>_\partial=<\delta(x),y>_! \tag{2.80}$$

and this has the following important consequence

Proposition 2.1.2. *The spaces* $\mathrm{Im}(r(H^{q-1}(\Gamma\backslash X,\tilde{\mathcal{M}}_\mathbb{Q}))$ *and* $\mathrm{Im}(r(H^{d-q}(\Gamma\backslash X,\tilde{\mathcal{M}}^\vee_\mathbb{Q}))$ *are mutual orhogonal complements of each other, i.e.*

$$\mathrm{Im}(r(H^{d-q}(\Gamma\backslash X,\tilde{\mathcal{M}}^\vee_\mathbb{Q}))=\{y\mid\ <y,\mathrm{Im}(r(H^{q-1}(\Gamma\backslash X,\tilde{\mathcal{M}}_\mathbb{Q}))>_\partial=0$$

2.1.11 The case $\Gamma=\mathrm{Sl}_2(\mathbb{Z})$

We return to the case $\Gamma=\mathrm{Sl}_2(\mathbb{Z})$. (See section 2.1.4). We will see that for this seemingly very easy case we can formulate and prove some deep results, for instance we understand the denominators of the Eisenstein classes (Theorem 3.89).

In the following $\mathcal{M}$ can be any Γ-module. We investigate the fundamental exact sequence for this special group. We computed already the cohomology groups $H^\bullet(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})$ in (2.1.4).

We compute the cohomology with compact supports. Of course we start again from our covering $\Gamma\backslash\mathbb{H} = U_i \cup U_\rho$. The cohomology with compact supports is the cohomology of the complex (see 2.70)

$$0 \to H^2(U_i \cap U_\rho, \tilde{\mathcal{M}}^!_{i,\rho}) \to H^2(U_i\ , \tilde{\mathcal{M}}^!_i) \oplus H^2(U_\rho, \tilde{\mathcal{M}}^!_\rho) \to 0.$$

Now we have $H^2(U_i \cap U_\rho, \tilde{\tilde{\mathcal{M}}}^!_{i,\rho}) = \mathcal{M}, H^2(U_i\ , \tilde{\tilde{\mathcal{M}}}^!_i) = \mathcal{M}_{\Gamma_i} = \mathcal{M}/(\mathrm{Id} - S)\mathcal{M}, H^2(U_\rho\ , \tilde{\tilde{\mathcal{M}}}^!_\rho) = \mathcal{M}_{\Gamma_\rho} = \mathcal{M}/(\mathrm{Id} - R)\tilde{\mathcal{M}}$ and hence we get the complex

$$0 \to \mathcal{M} \to \mathcal{M}_{\Gamma_i} \oplus \mathcal{M}_{\Gamma_\rho} \to 0$$

and from this we obtain

$$H^1(\Gamma\backslash\mathbb{H}, i_!(\mathcal{M})) = \ \ker(\mathcal{M} \to (M/(\mathrm{Id} - S)\mathcal{M} \oplus \mathcal{M}/(\mathrm{Id} - R)\mathcal{M}))$$

and

$$H^0(\Gamma\backslash\mathbb{H}, i_!(\mathcal{M})) = 0, H^2(\Gamma\backslash\mathbb{H}, i_!(\mathcal{M})) = \mathcal{M}_\Gamma.$$

Now we consider the cohomology of the boundary $H^\bullet(\partial(\Gamma\backslash\mathbb{H}), \tilde{M})$. We discussed the Borel-Serre compactification and saw that in this case we get this compactification if we add a circle at infinity to our picture of the quotient. But we may as well cut the cylinder at any level $c > 1$, i.e. we consider the level line $\mathbb{H}(c) = \{z = x + ic | z \in \mathbb{H}\}$ and divide this level line by the action of the translation group

$$\Gamma_U = \{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} | n \in \mathbb{Z}\} = \{\begin{pmatrix} \epsilon & n \\ 0 & \epsilon \end{pmatrix} | n \in \mathbb{Z}, \epsilon = \pm 1\}/\{\pm \mathrm{Id}\}.$$

But this quotient is homotopy equivalent to the cylinder

$$\Gamma_U\backslash\mathbb{H} \simeq \Gamma_U\backslash\mathbb{H}(c).$$

We apply our general consideration on cohomology of arithmetic groups to this situation and find

$$H^\bullet(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}) = H^\bullet(\Gamma_U\backslash\mathbb{H}, \mathrm{sh}_{\Gamma_U}(\mathcal{M})) = H^\bullet(\Gamma_U\backslash\mathbb{H}(c), \mathrm{sh}_{\Gamma_U}(\mathcal{M})).$$

This cohomology is easy to compute. The group Γ_U is generated by the element $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. It is rather clear that

$$H^0(\Gamma_U\backslash\mathbb{H}, \mathrm{sh}_{\Gamma_U}(\mathcal{M})) = \mathcal{M}^{\Gamma_U}, H^1(\Gamma_U\backslash\mathbb{H}, \mathrm{sh}_{\Gamma_U}(\mathcal{M})) = \mathcal{M}_{\Gamma_U} = \mathcal{M}/(\mathrm{Id} - T)\mathcal{M}.$$

Then our fundamental exact sequence becomes (See(2.31))
$\boxed{\text{fundexsq}}$

$$\begin{gathered} 0 \to \mathcal{M}^\Gamma \to \mathcal{M}^{\Gamma_U} \to \ \ker(\mathcal{M} \to (\mathcal{M}/(\mathrm{Id} - S)\mathcal{M} \oplus \mathcal{M}/(\mathrm{Id} - R)\mathcal{M})) \xrightarrow{j} \\ \mathcal{M}/(\mathcal{M}^{\Gamma_i} \oplus \mathcal{M}^{\Gamma_\rho}) \xrightarrow{r} \mathcal{M}/(\mathrm{Id} - T)\mathcal{M} \to \mathcal{M}_\Gamma \to 0 \end{gathered} \tag{2.81}$$

Now it may come as a little surprise to the readers, that we can formulate a little exercise which is not entirely trivial

Exercise: *Write down explicitly all the arrows in the above fundamental sequence*

We give the answer without proof. I change notation slightly and work with the matrices
$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, R = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$
and we have the relation
$$RS = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
Then $\Gamma_i =< S >, \Gamma_\rho =< R >$. The map
$$\mathcal{M}/(\mathcal{M}^{<S>} \oplus \mathcal{M}^{<R>}) \to \mathcal{M}/(\mathrm{Id} - T)\mathcal{M}$$
is given by
$$m \mapsto m - Sm$$
We have to show that this map is well defined: If $m \in \mathcal{M}^{<S>}$ then $m \mapsto 0$. If $m \in \mathcal{M}^{<R>}$ then
$$m - Sm = m - SR^{-1}m = m - Tm$$
and this is zero in $\mathcal{M}/(\mathrm{Id} - T)\mathcal{M}$.

The map
$$\ker(\mathcal{M} \to (\mathcal{M}/(\mathrm{Id} - S)\mathcal{M} \oplus \mathcal{M}/(\mathrm{Id} - R)\mathcal{M})) \to \mathcal{M}/(\mathcal{M}^{<S>} \oplus \mathcal{M}^{<R>})$$
is a little bit delicate. We pick an element m in the kernel, hence we can write it as
$$m = m_1 - Sm_1 = m_2 - R^{-1}m_2$$
and send $m \mapsto m_1 - m_2$ (Here we have to use the orientation). If we modify m_1, m_2 to $m_1' = m_1 + n_1, m_2' = m_2 + n_2$ then $m_1' - m_2'$ gives the same element in $\mathcal{M}/(\mathcal{M}^{<S>} \oplus \mathcal{M}^{<R>})$.

This answer can only be right if $m_1 - m_2$ goes to zero under the map r, i.e. we have to show that
$$m_1 - m_2 - S(m_1 - m_2) \in (\mathrm{Id} - T)\mathcal{M}$$
We compute
$$m_1 - m_2 - S(m_1 - m_2) = m - m_2 + Sm_2 = m - m_2 + R^{-1}m_2 - R^{-1}m_2 + Sm_2 =$$
$$-R^{-1}m_2 + Sm_2 = -T^{-1}Sm_2 + Sm_2 \in (\mathrm{Id} - T)\mathcal{M}.$$
Finally we claim that the map $\mathcal{M}^{<T>} \to \ker(\mathcal{M} \to (\mathcal{M}/(\mathrm{Id} - S)\mathcal{M} \oplus \mathcal{M}/(\mathrm{Id} - R)\mathcal{M}))$ is given by $m \mapsto m - Sm = m - R^{-1}T^{-1}m = m - R^{-1}m$.

There is still another element of structure. The map $c : z \mapsto -\bar{z}$ induces an (differentiable) involution of $\mathbb{H}$. We put $S_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ then $\gamma c z = cS_1\gamma S_1^{-1}z$

and therefore c induces an involution on $\Gamma\backslash\mathbb{H}$. We get an isomorphism of cohomology groups

$$c^{(1)} : H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \xrightarrow{\sim} H^1(\Gamma\backslash\mathbb{H}, c_*(\tilde{\mathcal{M}})) \tag{2.82}$$

The direct image sheaf $c_*(\tilde{\mathcal{M}})$ is by definition the sheaf attached to the Γ module $\mathcal{M}^{[S_1]}$: This is the module which equal to $\mathcal{M}$ as an abstract module, but the action is twisted by a conjugation by the above matrix S_1, i.e.

$$\gamma * m = S_1 \gamma S_1^{-1} m \tag{2.83}$$

Now we assume that $\mathcal{M}$ is actually a $\mathrm{Gl}_2(\mathbb{Z})$ module. Then the map $m \to S_1 m$ provides an isomorphism $\mathcal{M}^{(S_1)} \xrightarrow{\sim} \mathcal{M}$ and hence we get in involution on the cohomology groups

$\boxed{\text{cc}}$

$$\mathbf{c}^\bullet : H^\bullet(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \to H^\bullet(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \tag{2.84}$$

We have the explicit description of the cohomology groups $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$ and we can compute this involution in terms of this description.The map $m \mapsto S_1 m$ induces an isomorphism

$$S_1^\bullet \mathcal{M}/(\mathcal{M}^{\Gamma_i} + \mathcal{M}^{\Gamma_\rho}) \to \mathcal{M}/(\mathcal{M}^{\Gamma_i} + \mathcal{M}^{\Gamma_{\rho^2}}), \tag{2.85}$$

both sides are equal to the cohomology $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$. Hence $\mathbf{c}^\bullet$ is the map induced by $S_1^{|}\bullet$

The cohomology has a + and a − eigen submodule under this involution, and

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \supset H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})_+ \oplus H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})_-, \tag{2.86}$$

the sum of the two eigen modules has finite index which is a power of 2.

Poincare' duality

We assume that our Γ module $\mathcal{M}$ is a finitely generated and locally free module over R , where R is a Dedekind ring or a field. We assume $\frac{1}{2} \in R$. In section 2.1.10 we discuss Poincare duality in greater generality, here we consider the pairing (see 2.76)

$$H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})_{\text{ int}} \times H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\vee)_{\text{ int},!} \to H^2_!(\Gamma\backslash\mathbb{H}, R) = R \tag{2.87}$$

It is clear that the involution $\mathbf{c}$ induces multiplication by -1 on $H^2_!(\Gamma\backslash\mathbb{H}, R)$. On the other hand we have the decompositions of the above cohomology groups into $\pm$ eigen modules. The pairings of the $+,+$ parts and the $-,-$ give zero and then we get pairings

$$\begin{aligned} &H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})_{\text{ int},+} \times H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\vee)_{\text{ int},!,-} \to R \\ &H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})_{\text{ int},!,+} \times H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\vee)_{\text{ int},-} \to R \end{aligned} \tag{2.88}$$

both of them are partially non degenerate.

If we have $\mathcal{M} = \mathcal{M}^\vee$ then we get $\boxed{\text{eqrank}}$

$$rank(H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})_{\text{ int},+}) = rank(H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})_{\text{ int},-}) \tag{2.89}$$

Final remark: The reader may get the impression that - at least in the case $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$-it is easy to compute the cohomology, but the contrary is true. In the case $\Gamma = \mathrm{Sl}_2(\mathbb{Z})/\pm\mathrm{Id}$ we found formulae for the rank of the cohomology groups, this seems to be a satisfactory answer, but it is not. The point is that there is an additional structure. In the next section we will introduce the Hecke operators, these Hecke operators form an algebra of endomorphisms of the cohomology groups.

It is a fundamental question (see further down) to understand the cohomology as a module under the action of this Hecke algebra.

It is not so easy to write down the effect of a Hecke operator on a module like $\mathcal{M}/(\mathcal{M}^{\Gamma_i} + \mathcal{M}^{\Gamma_\rho})$. We will discuss an explicit example in section 3.3.

We mentioned already that the situation is even less satisfying if we consider the case $\Gamma = \mathrm{Sl}_2(\mathbb{Z}[\mathbf{i}])$. In this case we considered the coefficient system $\mathcal{M} = \mathcal{M}_{n_1} \otimes \mathcal{M}_{n_2}$. (see 2.36). Then it turns out that the complex (**??**) does not tell us very much about the cohomology.

We may for instance employ the complex (2.33). and the resulting complex for the cohomology with compact supports, then we can compute -for a given pair n_1, n_2- the modules ($\mathbb{Q}[\mathbf{i}]$ -vector spaces) in the exact sequence

$$0 \to H^1_!)\Gamma\backslash\mathbb{H}_3, \tilde{\mathcal{M}}\otimes\mathbb{Q}) \to H^1(\Gamma\backslash\mathbb{H}_3,\tilde{\mathcal{M}}\otimes\mathbb{Q}) \xrightarrow{r} H^1(\partial(\Gamma\backslash\mathbb{H}_3),\tilde{\mathcal{M}}\otimes\mathbb{Q}).$$

An easy computation (see later) shows that

$$H^1(\partial(\Gamma\backslash\mathbb{H}_3),\tilde{\mathcal{M}}\otimes\mathbb{Q}) = \mathbb{Q}[\mathbf{i}]e_{0,1} \oplus \mathbb{Q}[\mathbf{i}]e_{1,0}$$

where the $e_{ii,j}$ are some naturally given generators. Another simple argument using Poincare' duality shows that the image under the restriction map r will be a one dimensional subspace

$$\mathrm{Im}(r)(H^1(\Gamma\backslash\mathbb{H}_3,\tilde{\mathcal{M}}\otimes\mathbb{Q})) = \mathbb{Q}[\mathbf{i}](\mathcal{L}(n_1,n_2)e_{0,1} + \mathcal{L}(n_2,n_1)e_{1,0})$$

This far we get with our topological methods.

In chapter 4 we will use tools from analysis and prove a vanishing theorem

$$H^\bullet_!(\Gamma\backslash\mathbb{H}_3,\tilde{\mathcal{M}}\otimes\mathbb{Q}) = 0 \text{ if } n_1 \neq n_2 \tag{2.90}$$

(See 4.1.7) I do not see how such a result can not be obtained from studying the complex (2.33).

At this point we ask a natural question: Can we compute the position of the one dimensional subspace $\mathrm{Im}(r)$ the cohomology of the boundary, i.e. can we compute the point $(\mathcal{L}(n_1,n_2),\mathcal{L}(n_2,n_1)) \in \mathbb{P}^1(\mathbb{Z}[\mathbf{i}])$?

Here the answer is yes, but we have to use transcendental tools The ratio $\frac{\mathcal{L}(n_1,n_2)}{\mathcal{L}(n_2,n_1)}$ (or its inverse) is the quotient of a Hecke L function function evaluated at two consecutive critical integer arguments arguments (depending on n_1, n_2 we come back to this later) So the complex can not give an elementary expression for it.

On the other hand our purely combinatorial considerations imply that the ratios $\frac{\mathcal{L}(n_1,n_2)}{\mathcal{L}(n_2,n_1)} \in \mathbb{Q}[\mathbf{i}]$ (or $=\infty$) this implies a non trivial rationality relation between special values of the Hecke L function..

The rationality relation are easy consequences the results of Hurwitz on the special values of his Zeta-function.

The complex computes the $\mathbb{Z}[\mathbf{i}]-$ modules $H^i_!(\Gamma\backslash\mathbb{H}_3, \tilde{\mathcal{M}}_\lambda)$ and therefore it also computes the torsion. Again it seems to be difficult to derive general theorems, except the obvious finiteness assertions.

Chapter 3

Hecke Operators

3.1 The construction of Hecke operators

We mentioned already that the cohomology and homology groups of an arithmetic group have an additional structure: They are modules for the Hecke algebra. The following description of the Hecke algebra is somewhat provisorial, we get a richer Hecke algebra, if we work in the adelic context (See Chapter 6). The description here is more intuitive.

We start from the arithmetic group $\Gamma \subset G(\mathbb{Q})$ and an arbitrary Γ-module $\mathcal{M}$. The module $\mathcal{M}$ is also a module over a ring R which in the beginning may be simply $\mathbb{Z}$. More generally is R may the ring of integers in an algebraic number field, where we also inverted a finite number of primes. At this point it is better to have a notation for this action

$$\Gamma \times \mathcal{M} \to \mathcal{M}, (\gamma, m) \mapsto r(\gamma)(m)$$

where now $r : \Gamma \to \mathrm{Aut}_R(\mathcal{M})$. We abbreviate $r(\gamma)m = \gamma m$.

If we have a subgroup $\Gamma' \subset \Gamma$ of finite index, then we constructed maps

$$\begin{aligned} \pi^{\bullet}_{\Gamma',\Gamma} :& H^{\bullet}(\Gamma\backslash X, \tilde{\mathcal{M}}) \longrightarrow H^{\bullet}(\Gamma'\backslash X, \tilde{\mathcal{M}}) \\ \pi_{\Gamma',\Gamma,\bullet} :& H^{\bullet}(\Gamma'\backslash X, \tilde{\mathcal{M}}) \longrightarrow H^{\bullet}(\Gamma\backslash X, \tilde{\mathcal{M}}) \end{aligned}$$

(see section 2.1.2).

We pick an element $\alpha \in G(\mathbb{Q})$. The group

$$\Gamma(\alpha^{-1}) = \alpha^{-1}\Gamma\alpha \cap \Gamma$$

is a subgroup of finite index in Γ and the conjugation by α induces an isomorphism

$$\mathrm{inn}(\alpha) : \Gamma(\alpha^{-1}) \longrightarrow \Gamma(\alpha).$$

We get an isomorphism

$$j(\alpha) : \Gamma(\alpha^{-1})\backslash X \longrightarrow \Gamma(\alpha)\backslash X$$

which is induced by the map $x \longrightarrow \alpha x$ on the space X. This yields an isomorphism of cohomology groups

$$j(\alpha)^{\bullet} : H^{\bullet}(\Gamma(\alpha^{-1})\backslash X, \tilde{\mathcal{M}}) \longrightarrow H^{\bullet}(\Gamma(\alpha)\backslash X, j(\alpha)_*(\tilde{\mathcal{M}})).$$

We compute the sheaf $j(\alpha)_*(\tilde{\mathcal{M}})$. For a point $x \in \Gamma(\alpha)\backslash X$ we have $j(\alpha)_*(\tilde{\mathcal{M}})_x = \tilde{\mathcal{M}}_{x'}$ where $j(\alpha)(x') = x$. We have the projection $\pi_{\Gamma(\alpha^{-1})} : X \to \Gamma(\alpha^{-1})\backslash X$, and the definition yields

$$\tilde{\mathcal{M}}_{x'} = \left\{ s : \pi^{-1}_{\Gamma(\alpha^{-1})}(x') \to \mathcal{M} \mid s(\gamma m) = \gamma s(m) \text{ for all } \gamma \in \Gamma(\alpha^{-1}) \right\} \tag{3.1}$$

The map $z \longrightarrow \alpha z$ provides an identification $\pi^{-1}_{\Gamma(\alpha^{-1})}(x') \xrightarrow{\sim} \pi^{-1}_{\Gamma(\alpha)}(x)$ in terms of this fibre we can describe the stalk at x as

$$(\alpha)_*(\tilde{\mathcal{M}})_x = \left\{ s : \pi^{-1}_{\Gamma(\alpha)}(x) \to \mathcal{M} \mid s(\gamma v) = \alpha^{-1}\gamma\alpha s(v) \text{ for all } \gamma \in \Gamma(\alpha) \right\}. \tag{3.2}$$

Hence we see: We may use α to define a new $\Gamma(\alpha)$-module $\mathcal{M}^{(\alpha)}$: The underlying abelian group of $\mathcal{M}^{(\alpha)}$ is $\mathcal{M}$ but the operation of $\Gamma(\alpha)$ is given by

$$(\gamma, m) \longrightarrow (\alpha^{-1}\gamma\alpha)m = \gamma *_\alpha m.$$

Then the sheaf $j(\alpha)_*(\tilde{\mathcal{M}})$ is equal to $\tilde{\mathcal{M}}^{(\alpha)}$. Now every element

$$u_\alpha \in \operatorname{Hom}_{\Gamma(\alpha)}(\mathcal{M}^{(\alpha)}, \mathcal{M})$$

defines a map $\tilde{u}_\alpha : j(\alpha)_*(\tilde{\mathcal{M}}) \to \tilde{\mathcal{M}}$, and we get a commuting diagram

$$\begin{tikzcd}
H^\bullet(\Gamma(\alpha^{-1})\backslash X, \tilde{\mathcal{M}}) \arrow[r, "j(\alpha)^\bullet"] & H^\bullet(\Gamma(\alpha)\backslash X, j(\alpha)_*(\tilde{\mathcal{M}})) \arrow[r, "\tilde{u}_\alpha^\bullet"] & H^\bullet(\Gamma(\alpha)\backslash X, \mathcal{M}) \arrow[d, "\pi_\bullet"] \\
H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) \arrow[u, "\pi^\bullet"] \arrow[rr, "T(\alpha, u_\alpha)"] & & H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})
\end{tikzcd}, \tag{3.3}$$

and now the operator the bottom line is the Hecke operator.

The Hecke operator depends on two data:

a) the element $\alpha \in G(\mathbb{Q})$,

b) the choice of $u_\alpha \in \operatorname{Hom}_{\Gamma(\alpha)}(\mathcal{M}^{(\alpha)}, \mathcal{M})$.

It is not difficult to show that the operator $T(\alpha, u_\alpha)$ only depends on the double coset $\Gamma\ \alpha\ \Gamma$, provided we adapt the choice of u_α. To be more precise if

$$\alpha_1 = \gamma_1 \alpha \gamma_2 \qquad \gamma_1, \gamma_2 \in \Gamma,$$

then we have an obvious bijection

$$\Phi_{\gamma_1,\gamma_2} : \operatorname{Hom}_{\Gamma(\alpha)}(\mathcal{M}^{(\alpha)}, \mathcal{M}) \longrightarrow \operatorname{Hom}_{\Gamma(\alpha_1)}(\mathcal{M}^{\alpha_1}, \mathcal{M})$$

which is given by

$$\Phi_{\gamma_1,\gamma_2}(u_\alpha) = u_{\alpha_1} = \gamma_1 u_\alpha \gamma_2.$$

The reader will verify without difficulties that

$$T(\alpha, u_\alpha) = T(\alpha_1, u_{\alpha_1}).$$

(Verify this for H^0 and then use some kind of resolution (See next section))

The choice of u_α may be delicate in some situations. There are cases where we also have a canonical choice of u_α. The first case is that our Γ-module $\mathcal{M}$ is of arithmetic origin. In this case $G(\mathbb{Q})$ acts upon $\mathcal{M}_{\mathbb{Q}} = \mathcal{M} \otimes \mathbb{Q}$. Then the canonical choice of an

$$u_{\alpha,\mathbb{Q}} : \mathcal{M}_{\mathbb{Q}}^{(\alpha)} \longrightarrow \mathcal{M}_{\mathbb{Q}},$$

is given by $u_\alpha : m \mapsto \alpha m$. Hence we can speak of the Hecke-opertor $T(\alpha) : H^\bullet(\Gamma\backslash X, \mathcal{M}_{\mathbb{Q}}) \to H^\bullet(\Gamma\backslash X, \mathcal{M}_{\mathbb{Q}})$.

But if we return to the R-module sheaf $\mathcal{M}$ this morphism $u_{\alpha,\mathbb{Q}}$ will not necessarily map the lattice $\mathcal{M}^{(\alpha)}$ into $\mathcal{M}$. Clearly we can find a rational number $d(\alpha) > 0$ for which

$$d(\alpha) \cdot u_{\alpha,\mathbb{Q}} : \mathcal{M}^{(\alpha)} \longrightarrow \mathcal{M} \text{ and } d(\alpha) \cdot u_{\alpha,\mathbb{Q}}(\mathcal{M}^{(\alpha)}) \not\subset b\mathcal{M} \text{ for any integer } b > 1.$$

Then $u_\alpha = d(\alpha) \cdot u_{\alpha,\mathbb{Q}}$ is called the *normalised choice*, and then $T(\alpha, u_\alpha)$ will be the *normalised Hecke operator.*

The canonical choice defines endomorphisms on the rational cohomology, i.e. the cohomology with coefficients in $\tilde{\mathcal{M}}_{\mathbb{Q}}$ whereas the normalised Hecke operators induce endomorphism of the integral cohomology. The normalised choice and the canonical choice differ only by a scalar factor.

In the second case we assume that $\Gamma_0 = G(\mathbb{Z})$, let $\Gamma(N) \subset \Gamma_0$ be the full congruence subgroup $\mod N$. Then $\Gamma_0/\Gamma(N) \subset G(\mathbb{Z}/N\mathbb{Z})$ Now we assume that $\mathcal{V}$ is a $G(\mathbb{Z}/N\mathbb{Z})$ module, in section 1.2.2 we called this a module of congruence origin. Then we have some constraints on the choice of elements α. We introduce the semi local ring $\mathbb{Z}_{(N)}$ where we invert all primes not dividing N. Now we pick our elements $\alpha \in G(\mathbb{Z}_{(N)})$ Since we have the homomorphism $\mathbb{Z}[_{(N)} \to \mathbb{Z}/N\mathbb{Z}$ our module $\mathcal{V}$ is also a $G(\mathbb{Z}_{(N)}))$ module. Therefore we can simply choose $u_\alpha := m \mapsto \alpha m$.

We see that we can construct many endomorphisms $T(\alpha, u_\alpha) : H^\bullet(\Gamma\backslash X, \tilde{\mathcal{V}}) \to H^\bullet(\Gamma\backslash X, \tilde{\mathcal{V}})$. These endomorphisms will generate an algebra

$$\mathcal{H}_{N,\tilde{\mathcal{V}}} \subset \operatorname{End}(H^\bullet(\Gamma\backslash X, \tilde{\mathcal{V}})). \tag{3.4}$$

This is now the so-called *Hecke algebra.*.

We can also define endomorphisms $T(\alpha, u_\alpha)$ on the cohomology with compact supports, on the inner cohomology and the cohomology of the boundary. Since the operators are compatible with all the arrows in the fundamental exact sequence we denote them by the same symbol.

The Hecke algebra also acts on the inner cohomology $H^q_!(\Gamma\backslash X, \tilde{\mathcal{M}})$. Of course we may tensorize our coefficient system with any number field $L \supset \mathbb{Q}$, then we write $\mathcal{M}_L = \mathcal{M} \otimes L$..

We state without proof the following fundamental theorem :

$\boxed{\text{He-ss}}$

Theorem 3.1.1. *Let $\mathcal{M}$ be a module of arithmetic origin. For any extension $L/\mathbb{Q}$ the $\mathcal{H}_\Gamma \otimes L$ module $H^q_!(\Gamma\backslash X, \tilde{\mathcal{M}}_L)$ is semi simple, i.e. a direct sum of irreducible $\mathcal{H}_\Gamma$ modules.*

The proof of this theorem will be discussed in Chapter 6 (section 6.1.8) it requires some input from analysis. We give a brief sketch. We tensorize our coefficient system by $\mathbb{C}$, i.e. we consider $\mathcal{M}_L \otimes_L \mathbb{C} = \mathcal{M}_\mathbb{C}$. Let us assume that Γ is torsion free. First of all start from the well known fact, that the cohomology $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_\mathbb{C})$ can be computed from the de-Rham-complex

$$H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_\mathbb{C}) = H^\bullet(\Omega^\bullet(\Gamma\backslash X) \otimes \tilde{\mathcal{M}}_\mathbb{C}. \tag{3.5}$$

We introduces some specific positive definite hermitian form on $\mathcal{M}_\mathbb{C}$ and this allows us to define a hermitian scalar product between two $\tilde{\mathcal{M}}_\mathbb{C}$ -valued p-forms

$$< \omega_1, \omega_2 >= \int_{\Gamma\backslash X} \omega_1 \wedge *\omega_2,$$

provided one of the forms is compactly supported.

This will allow us a positive definite scalar product on $H^p_!(\Gamma\backslash \mathbb{H}, \tilde{\mathcal{M}}_{n,\mathbb{C}})$, We apply theorem 6.1.1 , this theorem tells us that we can find representatives ω_1^h, ω_2^h which are harmonic (they satisfy certain differential equations) and then

$$< [\omega_1], [\omega_2] >:= \int_{\Gamma\backslash X} \omega_1^h \wedge *\omega_2^h, \tag{3.6}$$

defines a positive definite hermitian scalar product on $H^q_!(\Gamma\backslash X, \tilde{\mathcal{M}}_\mathbb{C})$. Finally we show that $\mathcal{H}_\Gamma$ is self adjoint with respect to this scalar product,(see 6.1 and then semi-simplicity follows from the standard argument.

For the groups $\Gamma \subset \mathrm{Sl}_2(\mathbb{Z})$ and the cohomology groups $H^1_!(\Gamma\backslash \mathbb{H}, \mathcal{M}_n \otimes \mathbb{C})$ these harmonic representatives are given linear combinations of holomorphic and antiholomorphic cusp forms of weight $n+2$ (See 4.1.7). The scalar product on this space of modular forms is given by the by the Peterson scalar product (see section 4.1.8.)

3.1.1 Commuting relations

We want to say some words concerning the structure of the Hecke algebra.

To begin we discuss the action of the Hecke-algebra on $H^0(\Gamma\backslash X, \tilde{\mathcal{M}})$. We do this since we defined the cohomology in terms of injective (or acyclic) resolutions and therefore the general results concerning the structure of the Hecke algebra can be reduced to this special case.

If we have a Γ-module $\mathcal{M}$ and if we look at the diagram defining the Hecke operators, then we see that we get in degree 0

$$\begin{array}{ccc} \mathcal{M}^{\Gamma(\alpha^{-1})} & \longrightarrow (\mathcal{M}^{(\alpha)})^{\Gamma(\alpha)} \xrightarrow{u_\alpha} & \mathcal{M}^{\Gamma(\alpha)} \\ \uparrow & & \downarrow \\ \mathcal{M}^\Gamma & \xrightarrow{\quad T(\alpha,u_\alpha) \quad} & \mathcal{M}^\Gamma \end{array}$$

where the first arrow on the top line is induced by the identity map $\mathcal{M} \to \mathcal{M}^{(\alpha)} = \mathcal{M}$ and the second by a map $u_\alpha \in \mathrm{Hom}_{\mathbf{Ab}}(\mathcal{M}, \mathcal{M})$ which satisfies $u_\alpha((\alpha\gamma\alpha^{-1})m) = \gamma u_\alpha(m)$. Recalling the definition of the vertical arrow on the right, we find

$$T(\alpha, u_\alpha)(v) = \sum_{\gamma \in \Gamma/\Gamma(\alpha)} \gamma \cdot u_\alpha(v).$$

We are interested to get formulae for the product of Hecke operators, for instance, we would like to show that under certian assumptions on α, β and certain adjustment of u_α, u_β and $u_{\alpha\beta}$ we can show

$$T(\alpha, u_\alpha) \cdot T(\beta, u_\beta) = T(\beta, u_\beta) \cdot T(\alpha, u_\alpha) = T(\alpha\beta, u_{\alpha\beta}).$$

It is easy to see what the conditions are if we want such a formula to be true. We look at what happens in H^0. For $v \in \mathcal{M}^\Gamma$ we get

$$T(\alpha, u_\alpha) \cdot T(\beta, u_\beta)(v) = \sum_{\gamma \in \Gamma/\Gamma(\alpha)} \gamma u_\alpha(\sum_{\eta \in \Gamma/\Gamma(\beta)} \eta u_\beta(v))$$

We assume that the following three conditions hold

(i) for each η we can find an $\eta' \in \Gamma$ such that

$$\eta' \circ u_\alpha = u_\alpha \circ \eta,$$

(ii) The elements $\gamma\eta'$ form a system of representatives for $\Gamma/\Gamma(\alpha\beta)$

(iii) $u_\alpha u_\beta(v) = u_\beta u_\alpha(v) = u_{\alpha\beta}(v)$.

Then we get

$$T(\alpha, u_\alpha) \cdot T(\beta, u_\beta)(v) = \sum_{\gamma \in \Gamma/\Gamma(\alpha)} \sum_{\eta' \in \Gamma/\Gamma(\beta)} \gamma\eta' u_\alpha u_\beta(v) = \sum_{\xi \in \Gamma/\Gamma(\alpha\beta)} \xi u_{\alpha\beta}(v) =$$

$$T(\alpha\beta, u_{\alpha\beta})(v)$$

We want to explain in a special case that we may have relations like the one above.

Let S be a finite set of primes, let $|S|$ be the product of these primes. Then we define $\Gamma_S = G(\mathbb{Z}[\frac{1}{|S|}])$. We say that $\alpha \in G(\mathbb{Q})$ has support in S if $\alpha \in G(\mathbb{Z}[\frac{1}{|S|}])$.

We take the group $\Gamma = \mathrm{Sl}_d(\mathbb{Z})$, and we take two disjoint sets of primes S_1, S_2. For the group Γ one can prove the so-called *strong approximation* theorem (see [57]) which asserts that for any natural number m the map

$$\mathrm{Sl}_d(\mathbb{Z}) \longrightarrow \mathrm{Sl}_d(\mathbb{Z}/m\mathbb{Z})$$

is surjective. (This special case is actually not so difficult. The theorem holds for many other arithmetic groups, for instance for simply connected Chevalley schemes over $\mathrm{Spec}(\mathbb{Z})$.)

We consider the case

$$\alpha = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_d \end{pmatrix} \in \Gamma_{S_1}, \beta = \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_d \end{pmatrix} \in \Gamma_{S_2},$$

where $a_d|a_{d-1}\dots|a_1$ and $b_d|b_{d-1}|\dots|b_1$. It is clear that we can find integers n_1 and n_2 which are only divisible by the primes in S_1 and S_2 respectively, so that

$$\Gamma(n_i) \subset \Gamma(\alpha^{-1}), \Gamma(n_2) \subset \Gamma(\beta^{-1}),$$

where the $\Gamma(n_i)$ are the full congruence subgroups $\mod n_i$. Since we have

$$\mathrm{Sl}_d(\mathbb{Z}/n\mathbb{Z}) = \mathrm{Sl}_d(\mathbb{Z}/n_1\mathbb{Z}) \times \mathrm{Sl}_d(\mathbb{Z}/n_2\mathbb{Z})$$

we get

$$\Gamma/\Gamma(\alpha^{-1}\beta^{-1}) \xrightarrow{\sim} \Gamma/\Gamma(\alpha^{-1}) \times \Gamma/\Gamma(\beta^{-1}).$$

On the right hand side we can chose representatives γ for $\Gamma/\Gamma(\alpha^{-1})$ which satisfy $\gamma \equiv \mathrm{Id} \mod n_2$ and η for $\Gamma/\Gamma(\beta^{-1})$ which satisfy $\eta \equiv \mathrm{Id} \mod n_1$. Then the products $\gamma\eta$ will form a system of representatives for $\Gamma/\Gamma(\alpha^{-1}\beta^{-1})$. But then we clearly have $u_\alpha\eta = \eta u_\alpha$ and we see that (i) and (ii) above are true. Then we can put $u_{\alpha\beta} = u_\alpha u_\beta$.

We consider the case that our module $\mathcal{M}$ is a R-lattice in $\mathcal{M}_\mathbb{Q}$, where $\mathcal{M}_\mathbb{Q}$ is a rational $G(\mathbb{Q})$-module. Then we saw that we can write

$$u_\alpha = d(\alpha) \cdot \alpha$$

where $d(\alpha)$ will be a product of powers of the primes p dividing n_1 and an analogous statement can be obtained for β and n_2.

Since we have $\alpha\beta = \beta\alpha$ and since clearly $d(\alpha)d(\beta) = d(\alpha\beta)$ we also get the commutation relation.

So far we only proved this relation only for the action on $H^0(\Gamma\backslash X, \tilde{\mathcal{M}})$. If we want to prove it for cohomology in higher degrees, we have to choose an acyclic resolution

$$0 \longrightarrow \mathcal{M} \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \dots = 0 \longrightarrow \mathcal{M} \longrightarrow A^\bullet$$

and compute the cohomology from this resolution. We have to extend the maps u_α, u_β to this complex

$$\begin{array}{ccccc} 0 \longrightarrow & \mathcal{M}^{(\alpha)} & \longrightarrow & (A^\bullet)^{(\alpha)} \\ & \Big\downarrow u_\alpha & & \Big\downarrow u_\alpha^{(\bullet)} \\ 0 \longrightarrow & \mathcal{M} & \longrightarrow & A^\bullet, \end{array}$$

and we have to prove that the relation

$$u_\alpha \eta u_\beta = \eta' u_\alpha u_\beta = \eta' u_{\alpha\beta}$$

also holds on the complex. Once we can prove this, it becomes clear that the commutation rule also holds in higher degrees.

We choose the special resolution

$$\begin{gathered} 0 \to \mathcal{M} \to \mathrm{Ind}^\bullet(\mathcal{M}) = \\ 0 \longrightarrow \mathcal{M} \longrightarrow \mathrm{Ind}_{\{1\}}^\Gamma \mathcal{M} \longrightarrow \mathrm{Ind}_{\{1\}}^\Gamma(\mathrm{Ind}_{\{1\}}^\Gamma \mathcal{M}/\mathcal{M}) \longrightarrow \end{gathered} \tag{3.7}$$

It is clear that if suffices to show: If we selected the u_α, u_β in such a way that we have the condition (i), (ii) and (iii) above satisfied, then we can choose extensions $u_\alpha, u_\beta, u_{\alpha\beta}$ to $\mathrm{Ind}_{\{1\}}^\Gamma \mathcal{M}$ so that (i), (ii) and (iii) are also satisfied. Once we have done this we can proceed by induction.

In other words we have the diagram of $\Gamma(\alpha)$-modules

$$\begin{array}{ccccc} 0 \longrightarrow & \mathcal{M}^{(\alpha)} & \longrightarrow & (\mathrm{Ind}_{\{1\}}^\Gamma \mathcal{M})^{(\alpha)} \\ & \big\downarrow u_\alpha & & \big\downarrow ? \\ 0 \longrightarrow & \mathcal{M} & \longrightarrow & \mathrm{Ind}_{\{1\}}^\Gamma \mathcal{M}, \end{array}$$

and we are searching for a suitable vertical arrow ?. The horizontal arrows are given by (as before see (2.3)) by $i : m \longrightarrow f_m : \{\gamma \longrightarrow \gamma m\}$.

We make another assumption concerning our α, β. We assume that there exists an automorphism Θ of $G/\mathbb{Q}$ such that $\Theta(\alpha) = \alpha^{-1}, \Theta(\beta) = \beta^{-1}$ and $\Theta\Gamma = \Gamma$. This assumption is certainly fulfilled in the case above, we simply take $\Theta(g) =^t g^{-1}$, i.e. transpose inverse.

We choose representatives $\xi_1, \ldots, \xi_r$ for $\Gamma/\Gamma(\alpha^{-1})$, then $\Theta\xi_1, \ldots, \Theta\xi_r$ is a system of representatives for $\Gamma/\Gamma(\alpha)$. To define the vertical arrow $? = u_\alpha^{(0)}$ we require

$$u_\alpha^{(0)}(f)(\Theta\xi_\nu) = u_\alpha(f(\xi_\nu)) \;\; \forall\, \nu = 1, \ldots, r$$

and this yields a unique $\Gamma(\alpha)$- module isomorphism, for all $\gamma \in \Gamma(\alpha)$ we must have

$$u_\alpha^{(0)}(f)(\Theta\xi_\nu\gamma) = u_\alpha(f(\xi_\nu\alpha^{-1}\gamma\alpha) \;\; \forall\, \nu = 1, \ldots, r.$$

Iterating this construction gives us the $u_\alpha^{(\bullet)}$, by construction these morphisms satisfy (i), (ii), (iii). Since the complex $H^0(\Gamma\backslash X, \tilde{\mathrm{Ind}(\mathcal{M})})$ computes the cohomology groups $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})$ the commutation rules hold in all degrees.

$\boxed{\text{HSO}}$

3.1.2 More relations between Hecke operators

We look at the algebra of Hecke operators in the special case that $G/\mathbb{Z} = \mathrm{Gl}_2/\mathbb{Z}$, we consider the action on $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$ where $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$, we assume n even and $\mathcal{M} = \mathcal{M}[-\frac{n}{2}]$. This has the effect that the centre of $G/\mathbb{Z}$ acts trivially on $\mathcal{M}$ and this makes life simpler.

We attach a Hecke operator to any coset $\Gamma\alpha\Gamma$ where $\alpha \in \mathrm{Gl}_2^+(\mathbb{Q})$ (i.e. $\det(\alpha) > 0$, we want α to act on the upper half plane). Then α and $\lambda\alpha$ with $\lambda \in \mathbb{Q}^*$ define the same operator. Hence we may assume that the matrix entries of α are integers. The theorem of elementary divisors asserts that the double cosets

$$\Gamma \cdot M_n(\mathbb{Z})_{\det\neq 0} \cdot \Gamma \subset \mathrm{Gl}_2^+(\mathbb{Q})$$

are represented by matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

where $b \mid a$. But here we can divide by b, and we are left with the matrix

$$\alpha = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad a \in \mathbb{N}.$$

We can attach a Hecke operator to this matrix provided we choose u_α. We see that α induces on the basis vectors of our module $\mathcal{M}$

$$X^\nu Y^{n-\nu} \longrightarrow a^{\nu-n/2} \cdot X^\nu Y^{n-\nu}.$$

Hence we see that we have the following natural choice for u_α

$$u_\alpha : P(X.Y) \longrightarrow a^{n/2}\alpha \cdot P(X,Y).$$

(See the general discussion of the Hecke operators)

Hence we get a family of endomorphisms

$$T\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, u_{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}}\right) = T(a) \tag{3.8}$$

of the cohomology $H^i(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$ We have seen already that we have $T_aT_b = T_{ab}$ if a, b are coprime.

Hence we have to investigate the local algebra $\mathcal{H}_p$ which is generated by the

$$T_{p^r} = T\left(\begin{pmatrix} p^r & 0 \\ 0 & 1 \end{pmatrix}, u_{\begin{pmatrix} p^r & 0 \\ 0 & 1 \end{pmatrix}}\right) \tag{3.9}$$

for the special case of the group $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$ and the coefficient system $\mathcal{M} = \mathcal{M}_n[-\frac{n}{2}]$. To do this we compute the product

$$T_{p^r} \cdot T_p = T\left(\begin{pmatrix} p^r & 0 \\ 0 & 1 \end{pmatrix}, u_{\alpha_p^r}\right) \cdot T\left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, u_{\alpha_p}\right) \tag{3.10}$$

where the u'_{α^r} are the canonical choices.

Again we investigate first what happens in degree zero, i.e. on $H^0(\Gamma\backslash\mathbb{H}, \tilde{I})$ here I is any Γ-module. Let $\alpha = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \xi \in H^0(\Gamma\backslash X, \tilde{I})$ then

$$T(\alpha^r, u_{\alpha^r})T(\alpha, u_\alpha)\xi = (\sum_{\gamma\in\Gamma/\Gamma(\alpha^r)} \gamma u_{\alpha^r})(\sum_{\eta\in\Gamma/\Gamma(\alpha)} \eta u_\alpha)(\xi)$$

We have the classical system of representatives

$$\Gamma/\Gamma(\alpha^r) = \bigcup_{j \mod p^r} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \Gamma(\alpha^r) \quad \bigcup \quad \bigcup_{j' \mod p^{r-1}} \begin{pmatrix} 1 & 0 \\ j'p & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Gamma(\alpha^r),$$

and our product of Hecke operators becomes

$$\Big(\sum_{j \mod p^r} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} + \sum_{j' \mod p^{r-1}} \begin{pmatrix} 1 & 0 \\ j'p & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u_{\alpha^r} \Big) \Big(\Big(\sum_{j_1 \mod p} \begin{pmatrix} 1 & j_1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Big) u_\alpha \Big)(\xi) =$$

$$\Big[\sum_{j \mod p^r, j_1 \mod p} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} u_{\alpha^r} \begin{pmatrix} 1 & j_1 \\ 0 & 1 \end{pmatrix} u_\alpha)(\xi)$$

$$+ \sum_{j' \mod p^{r-1}, j_1 \mod p} \begin{pmatrix} 1 & 0 \\ j'p & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u_{\alpha^r} \begin{pmatrix} 1 & j_1 \\ 0 & 1 \end{pmatrix} u_\alpha(\xi) \Big] +$$

$$+ \Big[\sum_{j \mod p^r} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} u_{\alpha^r} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) u_\alpha(\xi) +$$

$$(\sum_{j' \mod p^{r-1}} \begin{pmatrix} 1 & 0 \\ j'p & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u_{\alpha^r} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u_\alpha(\xi) \Big]$$

Now we have to assume t u_{α^ν} satisfy commutation rules

$$\begin{aligned} u_{\alpha^r} u_\alpha &= u_{\alpha^{r+1}} \\ u_{\alpha^r} \begin{pmatrix} 1 & j_1 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & j_1 p^r \\ 0 & 1 \end{pmatrix} u_{\alpha^r} \\ u_{\alpha^r} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= c_I(p) u_{\alpha^{r-1}} \end{aligned} \tag{3.11}$$

where $c_I(p)$ is a non zero integer. If we exploit the first two commutation relation then we get as the sum in the first $[\dots]$

$$\begin{aligned} &[\textstyle\sum_{j \mod p^r, j_1 \mod p} \begin{pmatrix} 1 & j + p^r j_1 \\ 0 & 1 \end{pmatrix} \\ &\textstyle\sum_{j' \mod p^{r-1}, j_1 \mod p} \begin{pmatrix} 1 & 0 \\ (j' + p^{r-1} j_1)p & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}] u_{\alpha^{r+1}}(\xi)] \\ &= T(p^{r+1}, u_{\alpha^{r+1}})(\xi). \end{aligned} \tag{3.12}$$

To compute the contribution of the second $[\dots]$ we observe that $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma$ and hence we have $w\xi = \xi$.Then the second commutation relation yields for the sum of the terms in the second $[\dots]$

$$c_I(p)(\sum_{j \mod p^r} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} + \sum_{j' \mod p^{r-1}} \begin{pmatrix} 1 & 0 \\ j'p & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) u_{\alpha^{r-1}}(\xi). \quad (3.13)$$

We observe that for $j \equiv 0 \mod p^{r-1}$ we get

$$\begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \alpha^{r-1}(\xi) = u_{\alpha^{r-1}} \begin{pmatrix} 1 & \frac{j}{p^{r-1}} \\ 0 & 1 \end{pmatrix} (\xi) = u_{\alpha^{r-1}}(\xi)$$

and in case $r > 1$ for $j' \equiv 0 \mod p^{r-2}$

$$\begin{pmatrix} 1 & 0 \\ j'p & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) u_{\alpha^{r-1}}(\xi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) u_{\alpha^{r-1}} \begin{pmatrix} 1 & \frac{pj'}{p^{r-1}} \\ 0 & 1 \end{pmatrix} (\xi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) u_{\alpha^{r-1}}(\xi). \quad (3.14)$$

Here we used again (3.11) and $\xi \in H^0(\Gamma\backslash X, \tilde{I})$. In other words in the summation (3.13) the first term only depends on $j \mod p^{r-1}$ and the second only on $j' \mod p^{r-2}$. For $r > 1$ this yields for the second term (3.13)

$$pc_I(p)(\sum_{j \mod p^{r-1}} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} + \sum_{j' \mod p^{r-2}} \begin{pmatrix} 1 & 0 \\ j'p & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) u_{\alpha^{r-1}}(\xi) = pc_I(p)T(p^{r-1})\xi$$

If $r = 1$ the value for (3.13) is $c_I(p)(p+1)u_{\alpha^0}$ and hence we get the general formula

$$T_{p^r} \cdot T_p = T_{p^{r+1}} + (p + \epsilon(p))c_I(p)T_{p^{r-1}} \quad (3.15)$$

where $\epsilon(r) = 0$ if $r > 1$ and $\epsilon(r) = 1$ for $r = 1$.

This formula is valid for all values of $r \geq 0$ if we put $T_{p^{-1}} = 0$.

We want to know what this means for the action on $H^1(\Gamma\backslash\mathbb{H}, \mathcal{M})$, we start again from our special resolution. (3.7). A simple calculation gives that the u_{α^r} satisfy the relations (3.11) with $c_{\mathcal{M}}(p) = p^n$. Hence we get for the action on $H^1(\Gamma\backslash\mathbb{H}, \mathcal{M})$

$$T_{p^r} \cdot T_p = T_{p^{r+1}} + p^{n+1}T_{p^{r-1}} + \epsilon(r))p^n T_{p^{r-1}} \quad (3.16)$$

where $\epsilon(r) = 0$ if $r > 1$ and $\epsilon(r) = 1$ for $r = 1$.

Interlude

We assume that a majority of the readers has seen Hecke operators in the context of modular forms and also has seen formulas for these Hecke operators acting on spaces of modular forms, which look very similar to the formulas above. (See [80]), [50]) This is of course not accidental, in the following chapter we will discuss the Eichler-Shimura isomorphism, which provides an injection of the space of modular forms of weight k into the cohomology $H^1(\Gamma\backslash\mathbb{H}, \mathcal{M}_{k-2} \otimes \mathbb{C})$). (See Thm. 4.1.3). This is a Hecke-module isomorphism and this explains the relation between the classical Hecke operators and the "cohomological" Hecke operators.

There is a slight difference between the formulas here and in [80], the reason is that our T_{p^r} differ slightly from the classical Hecke operators. But we always have T_p defined as above is equal to T_p in (3.1.2).

Here we want to stress that in this text so far -except in the introduction- there is no mentioning of modular forms, this is intentional.

End Interlude

This can be generalised. We choose an integer $N > 1$ and we take as our arithmetic group the full congruence group $\Gamma = \Gamma(N)$. For any prime $p \nmid N$ the $T(\alpha, u_\alpha)$ with $\alpha \in \mathrm{Gl}_2^+(\mathbb{Z}[1/p])$ form a commutative subalgebra $\mathcal{H}_p$ which is generated by T_p. It has an identity element $e_p = T_{p^0}$ This is the so called *unramified Hecke algebra.*

For $p|N$ we can also consider the $T(\alpha, u_\alpha)$ with $\alpha \in \mathrm{Gl}_2^+(\mathbb{Z}[1/p])$. They will also generate a local algebra $\mathcal{H}_p$ of endomorphisms in any of our cohomology groups, but this algebra will not necessarily be commutative. But if we have two different primes p, p_1 then we saw that the $\mathcal{H}_p, \mathcal{H}_{p_1}$ commute with each other. All these algebras $\mathcal{H}_p$ have an identity element e_p, we form the algebra

$$\mathcal{H}_\Gamma = \bigotimes_p{}' \mathcal{H}_p$$

where the superscript indicates that a tensor $h_f = \bigotimes_p h_p, h_p \in \mathcal{H}_p$ has a a factor e_p for almost all p. (In part II we will give a better construction of the Hecke-algebra which uses the language of adeles). This algebra acts on all our cohomology groups. We recall that the action of $\mathcal{H}_\Gamma$ on the inner cohomology groups is semi-simple (See Thm. 3.1.1). This has important consequences, which we discuss after a brief recapitulation of the theory of semi simple modules.

3.2 Some results on semi-simple $\mathcal{A}$- modules

We fix a field L and its algebraic closure $\bar{L}$, for simplicity we assume that the characteristic of L is zero, or that L is perfect. We consider an L-algebras $\mathcal{A}$, not necessarily commutative, but with identity. We need a few results and concepts from the theory on finite dimensional vector spaces V/L with an action of $\mathcal{A}$, i.e equipped with a homomorphism $\mathcal{A} \to \mathrm{End}_L(V)$.

Such an $\mathcal{A}$ module V is called *irreducible* if it does not contain an $\mathcal{A}$ invariant proper submodule $W \subset V$, i.e $\{0\} \neq W \neq V$. It is called *absolutely irreducible* if $\mathcal{A} \otimes \bar{L}$ module $V \otimes \bar{L}$ is irreducible. We say that V is *indecomposable* if it can not be written as the direct sum of two non zero submodules. An irreducible module is also indecomposable.

We say that the action of $\mathcal{A}$ on V is semi-simple, if the action of $\mathcal{A} \otimes \bar{L}$ on $V \otimes \bar{L}$ is semi simple and this means that any $\mathcal{A}$ submodule $W \subset V \otimes \bar{L}$ has a complement, i.e. we can find an $\mathcal{A}$-submodule $W^\perp \subset V \otimes \bar{L}$ such that $V \otimes \bar{L} = W \oplus W^\perp$.

Then it is clear that we get a decomposition indexed by a finite set E

$$V \otimes \bar{L} = \bigoplus_{i \in E} W_i$$

where the W_i are (absolutely) irreducible submodules. In general this decomposition will not be unique. For any two W_i, W_j of these submodules we have (Schur's lemma)

$$\mathrm{Hom}_{\mathcal{A}}(W_i, W_j) = \begin{cases} \bar{L} & \text{if they are isomorphic as } \mathcal{A} \text{ -modules} \\ 0 & \text{else} \end{cases}$$

We decompose the indexing set $E = E_1 \cup E_2 \cup .. \cup E_k$ according to isomorphism types. For any E_ν we choose an $\mathcal{A}$ module $W_{[\nu]}$ of this given isomorphism type. Then by definition

$$\mathrm{Hom}_{\mathcal{A}}(W_{[\nu]}, W_j) = \begin{cases} \bar{L} & \text{if } j \in E_\nu \\ 0 & \text{else} \end{cases}.$$

Now we define $H_{[\nu]} = \mathrm{Hom}_{\mathcal{A}}(W_{[\nu]}, V \otimes \bar{L})$ we get an inclusion $H_{[\nu]} \otimes W_{[\nu]} \hookrightarrow V \otimes \bar{L}$. The image X_ν will be an $\mathcal{A}$ submodule, which is a direct sum of copies of $W_{[\nu]}$, it is the unique such submodule.

We get a direct sum decomposition

$$V \otimes \bar{L} = \bigoplus_{\nu} \bigoplus_{i \in E_\nu} W_i = \bigoplus_{\nu} X_\nu$$

then this last decomposition is easily seen to be unique, it is called the *isotypical decomposition*.

If V is a semi simple $\mathcal{A}$ module then any submodule $W \subset V$ also has a complement (this is not entirely obvious because by definition only $W \otimes_L \bar{L}$ has a complement in $V \otimes_L \bar{L}$. But a small moment of meditation gives us that finding such a complement is the same as solving an inhomogeneous system of linear equations over L. If this system has a solution over $\bar{L}$ it also has a solution over L.) Therefore we also can decompose the $\mathcal{A}$ module V into irreducibles. Again we can group the irreducibles according to isomorphism types and we get the *isotypical* decomposition

$$V = \bigoplus_{i \in E} U_i = \bigoplus_{\nu} \bigoplus_{i \in E_\nu} U_i = \bigoplus_{\nu} Y_\nu. \tag{3.17}$$

But of course a summand U_i may become reducible if we extend the scalars to $\bar{L}$ (See examplple below). Since it is clear that for any two $\mathcal{A}$- modules V_1, V_2 we have

$$\mathrm{Hom}_{\mathcal{A}}(V_1, V_2) \otimes \bar{L} = \mathrm{Hom}_{\mathcal{A} \otimes \bar{L}}(V_1 \otimes \bar{L}, V_2 \otimes \bar{L})$$

we know that we get the isotypical decomposition of $V \otimes \bar{L}$ by taking the isotypical decomposition of the $Y_\nu \otimes \bar{L}$ and then taking the direct sum over ν.

Example: Let L_1/L be a finite extension of degree > 1, then we put $\mathcal{A} = L_1$ and $V = L_1$, the action is given by multiplication. Clearly V is irreducible, but $V \otimes \bar{L}$ is not. If L_1/L is separable then the module is semisimple, otherwise it is not.

We have a classical result:

Proposition 3.2.1. *Let V be a semi simple $\mathcal{A}$ module. Then the following assertions are equivalent*
i) The $\mathcal{A}$ module V is absolutely irreducible
ii) The image of $\mathcal{A}$ in the ring of endomorphisms is $End(V)$
iii) The vector space of $\mathcal{A}$ endomorphisms $End_{\mathcal{A}}(V) = L$.

This can be an exercise for an algebra class. Where do we need the assumption that V is semi simple?

We return to our algebra $\mathcal{A}$ over L. Let V be an irreducible semi-simple $\mathcal{A}$ -module, which is not necessarily absolutely irreducible. Let I_V be the two sided ideal which annihilates V, i.e. the kernel of $\mathcal{A} \to \mathrm{End}_L(V)$. Let $\mathcal{C}_L$ be the centre of $\mathcal{A}/I_V$. This centre is a field, because any $c \in \mathcal{C}_L$ is either zero or an isomorphism, in other words V is a $\mathcal{C}_L$ vector space. The $\mathcal{C}_L$-algebra $\mathcal{A}/I_V$ is a central simple algebra. There is a central division algebra $\mathcal{D}/\mathcal{C}_L$ such that $\mathcal{A}/I_V \xrightarrow{\sim} M_r(\mathcal{D})$, this is the algebra of (r,r) matrices with coefficients in $\mathcal{D}$. This algebra has exactly one -up to isomorphism- non zero irreducible module, this is the module of column vectors $\mathcal{D}^r$, the algebra acts by multiplication from the left. Let us denote this module by $\mathcal{X}[\mathcal{A}/I_V]$

Theorem 3.2.1. *The extension $\mathcal{C}_L/L$ is separable. Let L_1/L be a normal closure of $\mathcal{C}_L$. Then we have the isotypical decomposition*

$$V \otimes_L L_1 = \bigoplus_{\sigma:\mathcal{C}_L \to L_1} V \otimes_{\mathcal{C}_L,\sigma} L_1 \tag{3.18}$$

The Galois group $Gal(L_1/L)$ permutes the summands simply transitively. The $\mathcal{A}/I_V \otimes_{\mathcal{C}_L,\sigma} L_1$ module $V \otimes_{\mathcal{C}_L,\sigma} L_1$ is isomorphic to the standard module $\mathcal{X}[\mathcal{A}/I_V \otimes_{\mathcal{C}_L,\sigma} L_1]$.

Here $M_r(\mathcal{D})$ is the L_1 algebra of (r,r) matrices with coefficients in $\mathcal{D}$. This is essentially the classical Wedderburn theorem.

Proposition 3.2.2. *For any semi -simple $\mathcal{A}$ module V we can find a finite extension L_2/L such that the irreducible sub modules in the decomposition into irreducibles are absolutely irreducible.*

Clear, we have to take an extension which splits $\mathcal{D}$.

If V is any $\mathcal{A}$ module- not necessarily semi simple but finite dimensional over L-then there is a finite extension L_2/L and a filtration

$$\{0)\} \subset V_1 \subset V_2 \subset \cdots \subset V_{r-1} \subset V \otimes_L L_2$$

such that the successive quotients V_i/V_{i-1} are absolutely irreducible. A very elementary argument shows that the set of isomorphism types occurring in this filtrations does not depend on the filtration, let us denote this set of isomorphism types by $\mathrm{Spec}_V(\mathcal{A} \otimes L_2)$.

We say that an $\mathcal{A}$- sub module $W \subset V$ is *complete in* V if the two sets $\mathrm{Spec}_W(\mathcal{A} \otimes L_2)$ and $\mathrm{Spec}_{V/W}(\mathcal{A} \otimes L_2)$ are disjoint. We have the simple

Proposition 3.2.3. *a) If V is a semi simple $\mathcal{A}$-module and if $W \subset V$ is complete in V then we have a canonical splitting $V = W \oplus W'$.*

b) If V is not necessarily semi simple but if $\mathcal{A}$ is commutative instead then any $W \subset V$ which is complete in V also has a canonical complement W', i.e. $V = W \oplus W'$.

Proof. For the second assertion we observe that an absolutely irreducible $\mathcal{A}$ module U is simply one dimensional over L_2 and given by a homomorphism $\pi : \mathcal{A} \to L_2$, i.e. it is an eigenspace for $\mathcal{A}$. □

Let us call such a decomposition a *isotypical decomposition*into complete summands.i.

Let us now assume that we have two algebras $\mathcal{A}, \mathcal{B}$ acting on V, let us assume that these two operations commute i.e. for $A \in \mathcal{A}, B \in \mathcal{B}, v \in V$ we have $A(Bv) = B(Av)$. This structure is the same as having a $\mathcal{A} \otimes_L \mathcal{B}$ structure on V. Let us assume that $\mathcal{A}$ acts semi simply on V and let us assume that the irreducible $\mathcal{A}$ submodules of V are absolutely irreducible. Then it is clear that the isotypical summands $Y_\nu = \bigoplus W_i$ are invariant under the action $\mathcal{B}$. Now we pick an index i_0 then the evaluation maps gives us a homomorpism

$$W_{i_0} \otimes \ \mathrm{Hom}_{\mathcal{A}}(W_{i_0}, Y_\nu) \to Y_\nu.$$

Under our assumptions this is an isomorphism. Then we see that we get

$$V = \bigoplus_\nu W_{i_\nu} \otimes \ \mathrm{Hom}_{\mathcal{A}}(W_{i_0}, Y_\nu)$$

where i_ν is any element in E_ν and where $\mathcal{A}$ acts upon the first factor and $\mathcal{B}$ acts upon the second factor via the action of $\mathcal{B}$ on Y_ν.

We apply this to our Hecke algebra $\mathcal{H}_\Gamma = \bigotimes_p{}' \mathcal{H}_p$ and consider its action on $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$. We anticipate the theorem that this action is semi-simple. Hence we can find a finite normal extension $F/\mathbb{Q}$ such that we get an isotypical decomposition

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes F) = \bigoplus_{\pi_f} H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes F)(\pi_f), \tag{3.19}$$

here π_f is the isomorphism type of an absolutely irreducible $\mathcal{H}$ module over F. We can realise this module by a vector space H_{π_f}/F with an absolutely irreducible action of $\mathcal{H}_\Gamma$ on it. Then $H_{\pi_f} = \bigotimes' H_{\pi_p}$ where H_{π_p} is an absolutely $\mathcal{H}_p$ module. For almost all primes H_{π_p} is one dimensional and π_p is simply simply a homomorphism

$$\pi_p : \mathcal{H}_p \to F \text{ which is determined by its value } \pi_f(T_p) \in F \tag{3.20}$$

The Galois group $\mathrm{Gal}(F/\mathbb{Q})$ acts on $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes F)$ and hence it permutes the π_f which occur ih decomposition. Then for any π_f the Hecke module

$$\bigoplus_{\sigma \in \ \mathrm{Gal}(F/\mathbb{Q})} H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes F)({}^\sigma\pi_f), \tag{3.21}$$

is invariant under the action of the Galois group, hence defined over $\mathbb{Q}$ Theefore we get an isotypical decomposition

$$H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})=\bigoplus_{\Pi_f}H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})(\Pi_f) \tag{3.22}$$

$\boxed{\text{HEOP}}$

3.2.1 Explicit formulas for the Hecke operators, a general strategy.

In the following section we discuss the Hecke operators and for numerical experiments it is useful to have an explicit procedure to compute them in a given case. The main obstruction to get such an explicit procedure is to find an explicit way to compute the arrow $j^\bullet(\alpha)$ in the top line of the diagram (3.3). (we change notation $j(\alpha)$ to $m(\alpha)$).

Let us assume that we have computed the cohomology groups on both sides by means of orbiconvex coverings $\mathfrak{V}:\cup_{i\in I}V_{y_i}=\Gamma(\alpha^{-1})\backslash X$ and $\mathfrak{U}:\cup_{j\in J}U_{y_j}=\Gamma(\alpha)\backslash X$.

The map $m(\alpha)$ is an isomorphism between spaces and hence $m(\alpha)(\mathfrak{V})$ is an acyclic covering of $\Gamma(\alpha)\backslash X$. This induces an identification

$$C^\bullet(\mathfrak{V},\tilde{\mathcal{M}})=C^\bullet(m(\alpha)(\mathfrak{V}),\tilde{\mathcal{M}}^{(\alpha)})$$

and the complex on the right hand side computes $H^\bullet(\Gamma(\alpha)\backslash X,\tilde{\mathcal{M}}^{(\alpha)})$. But this cohomology is also computable from the complex $C^\bullet(\mathfrak{U},\tilde{\mathcal{M}}^{(\alpha)})$. We take the disjoint union of the two indexing sets $I\cup J$ and look at the covering $m_\alpha(\mathfrak{V})\cup\mathfrak{U}$. (To be precise: We consider the disjoint union $\tilde{I}=I\cup J$ and define a covering $\mathfrak{W}_\mathfrak{i}$ indexed by $\tilde{I}$. If $i\in\tilde{I}$ then $W_i=m(\alpha)(V_{y_i})$ and if $i\in J$ then we put $W_i=U_{x_i}$. We get a diagram of Čzech complexes Čzech

$$\begin{array}{ccccc}
\to & \bigoplus_{\underline{i}\in I^q}\tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}}) & \to & \bigoplus_{\underline{i}\in I^{q+1}}\tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}}) & \to\\
& \uparrow & & \uparrow & \\
\to & \bigoplus_{\underline{i}\in \tilde{I}^q}\tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}}) & \to & \bigoplus_{\underline{i}\in \tilde{I}^{q+1}}\tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}}) & \to\\
& \downarrow & & \downarrow & \\
\to & \bigoplus_{\underline{i}\in J^q}\tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}}) & \to & \bigoplus_{\underline{i}\in J^{q+1}}\tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}}) & \to
\end{array} \tag{3.23}$$

The sets $I^\bullet,J^\bullet$ are subsets of $\tilde{I}^\bullet$ and the up- and down-arrows are the resulting projection maps. We know that these up- and down-arrows induce isomorphisms in cohomology.

Hence we can start from a cohomology class $\xi\in H^q(\Gamma(\alpha)\backslash X,\tilde{\mathcal{M}}^{(\alpha)})$, we represent it by a cocycle

$$c_\xi\in\bigoplus_{\underline{i}\in I^q}\tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}}).$$

Then we can find a cocycle $\tilde{c}_\xi\in\bigoplus_{\underline{i}\in\tilde{I}^q}\tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}})$ which maps to c_ξ under the uparrow. To get this cocycle we have to do the following: our cocycle c_ξ is an array with components $c_\xi(\underline{i})$ for $\underline{i}\in I^q$. We have $d_q(c_\xi)=0$. To get $\tilde{c}_\xi$ we have to give the values $\tilde{c}_\xi(\underline{i})$ for all $\underline{i}\in\tilde{e}I^q\setminus I^q$. We must have

$$d_q\tilde{c}_\xi=0.$$

this yields a system of linear equations for the remaining entries. We know that this system of equations has a solution -this is then our $\tilde{c}_\xi$ - and this solution is unique up to a boundary $d_{q-1}(\xi')$. Then we apply the downarrow to $\tilde{c}_\xi$ and get a cocycle $c_\xi^\dagger$, which represents the same class ξ but this class is now represented by a cocycle with respect to the covering $\mathfrak{U}$. We apply the map $\tilde{u}^\alpha : \tilde{\mathcal{M}}^{(\alpha)} \to \tilde{\mathcal{M}}$ to this cocycle and then we get a cocycle which represents the image of our class ξ under T_α.

The adjunction formula

Let $\mathcal{M}^\vee$ be the dual module to $\mathcal{M}$, We will define Hecke operators

$$T_c(\alpha, u_{u_{\alpha^\vee}}) : H_c^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}^\vee) \to H_c^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}^\vee) \tag{3.24}$$

such that for $x \in H^q(\Gamma\backslash X, \tilde{\mathcal{M}}), y \in H_c^{d-q}(\Gamma\backslash X, \tilde{\mathcal{M}}^\vee)$ we have

$$< T(\alpha, u_\alpha)x, y >_{PD} = < x, T_c(\alpha^{-1}, u_{\alpha^\vee})y >_{PD} \tag{3.25}$$

We proceed as in section 3.1. If $\Gamma' \subset \Gamma$ is of finite index we have again the two maps

$$\pi_c^\bullet : H_c^\bullet(\Gamma'\backslash X, \tilde{\mathcal{M}}) \to H_c^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) \text{ and } \pi_{\bullet,,c} : H_c^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) \to H_c^\bullet(\Gamma'\backslash X, \tilde{\mathcal{M}})$$

Then we can define

$$\begin{tikzcd}
H_c^\bullet(\Gamma(\alpha^{-1})\backslash X, \tilde{\mathcal{M}}^\vee) \arrow[d, "\pi^\bullet"] & H_c^\bullet(\Gamma(\alpha)\backslash X, j(\alpha^{-1})_*(\tilde{\mathcal{M}}^\vee)) \arrow[l, "j(\alpha^{-1})^\bullet"'] & H_c^\bullet(\Gamma(\alpha)\backslash X, \mathcal{M}^\vee) \arrow[l, "\tilde{u}_\alpha^\bullet"'] \\
H_c^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}^\vee) & & H_c^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}^\vee) \arrow[ll, "T_c(\alpha^{-1}, u_{\alpha^{-1}})"'] \arrow[u, "\pi_\bullet"']
\end{tikzcd} , \tag{3.26}$$

Now it is easy to see that $\pi^\bullet$ and $\pi_c^\bullet$ and as well $pi_\bullet$ and $\pi_{\bullet,c}$ are adjoint to each other with respect to Poincare duality and then it becomes clear that (6.1) holds.

In the following section we discuss the explicit computation of a Hecke operator in a very specific situation. We start from our computation in section (2.1.4) and write down some $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})$ explicitly. On these modules we give explicit procedures to compute a Hecke operator. We get some supply of data and we look for some interesting laws or we try to verify some conjectures (see (3.89)).

3.3 Hecke operators for Gl_2:

For the rest of this chapter we discuss a very specific case. The algebraic group scheme will be $\mathrm{Gl}_2/\mathbb{Z}$. The symmetric space will be

$$X = \mathrm{Gl}_2(\mathbb{R})/K_\infty \text{ where } K_\infty = \mathrm{SO}(2) \times \{\begin{pmatrix} t & 0 \\ 0 & t) \end{pmatrix} \,|t \in \mathbb{R}^\times, t > 0\}.$$

Then the space X is the union of an upper and a lower half plane. We choose $\tilde{\Gamma} = \mathrm{Gl}_2(\mathbb{Z})$, then

$$\tilde{\Gamma}\backslash G_\infty/K_\infty = \Gamma\backslash\mathbb{H},$$

where $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$ and $\mathbb{H}$ is the upper half plane.

Earlier we defined the Γ-modules $\mathcal{M}_n[m]$ (See1.2.2), in the following we put $\mathcal{M} = \mathcal{M}_n[0]$.

We refer to Chapter 2 2.1.3. We have the two open sets $\tilde{U}_{\mathbf{i}}$, resp. $\tilde{U}_\rho \subset \mathbb{H}$, they are fixed under

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } R = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$

respectively. We also will use the elements

$$\begin{aligned} T_+ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; S_1^+ = T_- S T_-^{-1} = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix} \in \Gamma_0^+(2) \\ T_- &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \; S_1^- = T_+ S T_+^{-1} = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \in \Gamma_0^-(2) \end{aligned}.$$

The elements S_1^+ and S_1^- are elements of order four, i.e. $(S_1^+)^2 = (S_1^-)^2 = -\mathrm{Id}$, the corresponding fixed points are $\frac{\mathbf{i}+1}{2}$ and $\mathbf{i}+1$ respectively. Hence S_1^- fixes the sets $\alpha\tilde{U}_{\frac{\mathbf{i}+1}{2}}$ and $\tilde{U}_{\mathbf{i}+1}$, this is the only occurrence of a non trivial stabilizer.

3.3.1 The boundary cohomology

It is easier to compute the action of the Hecke operator T_p on the cohomology of the boundary, i. e. to compute the endomorphism

$$T_p : H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}) \to H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}).$$

We know (see 2.81) that $H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}) = \mathcal{M}/(1-T_+)\mathcal{M}$, we collect some easy facts concerning this module. For $n \geq k \geq 0$ we define the submodules

$$\mathcal{M}^{(k)} = \mathbb{Z}\, X^k Y^{n-k} \oplus \mathbb{Z}\, X^{k+1} Y^{n-k-1} \oplus \cdots \oplus \mathbb{Z}\, X^n$$

for $k = 0$(resp. $k = n$) we have $\mathcal{M}^{(0)} = \mathcal{M}$(resp. $\mathcal{M}^{(n)} = \mathbb{Z}\; X^n$). These modules are invariant under the action of T_+ we have $(1-T_+)\mathcal{M}^{(k)} \subset \mathcal{M}^{(k+1)}$, and $\mathcal{M}^{(0)}/\mathcal{M}^{(1)} \xrightarrow{\sim} \mathbb{Z}$. The map $(1-T_+)$ induces a map

$$\partial_k : \mathcal{M}^{(k)}/\mathcal{M}^{(k+1)} \to \mathcal{M}^{(k+1)}/\mathcal{M}^{(k+2)}$$

which is given by multiplication with $n-k$. Hence it is clear that

$$\mathcal{M}/(1-T_+)\mathcal{M} = \mathbb{Z}[Y^n] \oplus \mathcal{M}^{(1)}/(1-T_+)\mathcal{M}$$

and the second summand is a finite module. The filtration of $\mathcal{M}$ by the $\mathcal{M}^{(k)}$ induces a filtration on $H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}})$, we put

$$H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}})^{(k)} := \mathrm{Im}(H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}^{(k)}) \to H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}) \qquad (3.27)$$

Then $\boxed{\text{pn1}}$

Proposition 3.3.1. *For $k>0$ the quotient*

$$H^1(\partial(\Gamma\backslash\mathbb{H}),\tilde{\mathcal{M}})^{(k)}/H^1(\partial(\Gamma\backslash\mathbb{H}),\tilde{\mathcal{M}})^{(k+1)} \xrightarrow{\sim} \mathbb{Z}/(n-k+1)\mathbb{Z}$$

The Hecke operator T_p acts on $H^1(\partial(\Gamma\backslash\mathbb{H}),\tilde{\mathcal{M}})^{(k)}/H^1(\partial(\Gamma\backslash\mathbb{H}),\tilde{\mathcal{M}})^{(k+1)}$ by multiplication with p^k+p^{n-k+1}. Especially we have

$$T_p[Y^n]=(p^{n+1}+1)[Y^n]$$

Proof. We introduce the polynomials

$$\epsilon_k(X,Y):=X^n\tfrac{Y}{X}(\tfrac{Y}{X}-1)\dots(\tfrac{Y}{X}-k+1)=X^n\textstyle\prod_{\nu=0}^{k-1}(\tfrac{Y}{X}-\nu)=k!X^n\begin{pmatrix}\frac{Y}{X}\\k\end{pmatrix}=$$

$$X^{n-k}(Y-X)\dots(Y-(k-1)X)=X^{n-k}Y^k+\dots+(-1)^k k!X^n$$

Obviously these $\epsilon_k(X,Y)$ form a basis of $\mathcal{M}$. Pascal's rule for binomial coefficient says $\binom{\frac{Y}{X}+1}{k}=\binom{\frac{Y}{X}}{k}+\binom{\frac{Y}{X}}{k-1}$ and this yields

$$T_+\epsilon_k(X,Y)=\epsilon_k(X,X+Y)=\epsilon_k(X,Y)+k\epsilon_{k-1}(X,Y)$$

and from this we get

$$\mathcal{M}/(1-T_+)\mathcal{M}=\mathbb{Z}\epsilon_n(X,Y)\oplus\bigoplus_{k=n-1}^{0}(\mathbb{Z}/(k+1)\mathbb{Z})\epsilon_k(X,Y) \tag{3.28}$$

this is the first assertion.

We pick a prime p and investigate the action of T_p on $H^1(\partial(\Gamma\backslash\mathbb{H}),\tilde{\mathcal{M}})$. We recall the definition of the Hecke operator: We start from the matrix $\alpha=\begin{pmatrix}p&0\\0&1\end{pmatrix}$ and consider the diagram (3.3) adapted to our situation

$$\begin{array}{ccccc} H^1(\partial(\Gamma(\alpha^{-1})\backslash\mathbb{H}),\tilde{\mathcal{M}}) & \xrightarrow{j(\alpha)^{(1)}} & H^1(\partial(\Gamma(\alpha)\backslash\mathbb{H}),j(\alpha)_*(\tilde{\mathcal{M}})) & \xrightarrow{\tilde{u}_\alpha^{(1)}} & H^1(\partial(\Gamma(\alpha)\backslash\mathbb{H}),\mathcal{M}) \\ \uparrow{\scriptstyle\pi^{(1)}} & & & & \downarrow{\scriptstyle\pi_{(1)}} \\ H^1(\Gamma\backslash X,\tilde{\mathcal{M}}) & & \xrightarrow{T(\alpha,u_\alpha)} & & H^1(\partial(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}) \end{array} \tag{3.29}$$

The group $\Gamma(\alpha^{-1})=\{\begin{pmatrix}a&b\\c&d\end{pmatrix}|c\equiv 0 \mod p\}$, it acts on $\mathbb{P}^1(\mathbb{Q})$ and has two orbits which can be represented by ∞ and 0. The stabilisers of these two cusps are $\Gamma_\infty=\{\pm\mathrm{Id}\ T_+^\nu\}$ and $\Gamma_0=\{\pm\mathrm{Id}\ T_-^{p\nu}\}$ respectively. Hence we get

$$H^1(\partial(\Gamma(\alpha^{-1})\backslash\mathbb{H}),\tilde{\mathcal{M}})=\mathcal{M}/(\mathrm{Id}-T_+)\mathcal{M}\oplus\mathcal{M}/(\mathrm{Id}-T_-^p)\mathcal{M} \tag{3.30}$$

We identify $H^1(\partial(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})=\mathcal{M}/(\mathrm{Id}-T_+)\mathcal{M}\xrightarrow{w_0}\mathcal{M}/(\mathrm{Id}-T_-)\mathcal{M}$ where the last arrow is induced by the map $m\mapsto w_0m$ with $w_0=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$. Then

$$\pi^{(1)}(m)=(m,\sum_{j=0}^{p-1}\begin{pmatrix}1&0\\j&1\end{pmatrix}w_0m). \tag{3.31}$$

Therefore the composition

$$u_\alpha^{(1)}\circ j(\alpha)^{(1)}: \mathcal{M}/(\mathrm{Id}-T_+)\mathcal{M}\oplus\mathcal{M}/(\mathrm{Id}-T_-^p)\mathcal{M}\to\mathcal{M}/(\mathrm{Id}-T_+^p)\mathcal{M}\oplus\mathcal{M}/(\mathrm{Id}-T_-)\mathcal{M}$$

is given by $u_\alpha^{(1)}\circ j(\alpha)^{(1)}(m_\infty,m_0)\mapsto(\alpha m_\infty,\alpha m_0)$. and $\pi_{(1)}((n_\infty,n_0))=n_\infty+w_0n_0$. This yields

$$T_p(m)=\alpha m+w_0\alpha w_0^{-1}\sum_{j=0}^{p-1}\begin{pmatrix}1 & j\\ 0 & 1\end{pmatrix}m.$$

On $\mathcal{M}^{(k)}/\mathcal{M}^{(k+1)}$ the element $\begin{pmatrix}1 & j\\ 0 & 1\end{pmatrix}$ acts as identity, α is multiplication by p^k and $w_0\alpha w_0^{-1}$ is multiplication p^{n-k}. □

Here we encounter a situation where the quotient $H^1(\Gamma\backslash\mathbb{H},\mathcal{M})_{\text{ int},!}/H^1_!(\Gamma\backslash\mathbb{H},\mathcal{M})_{\text{ int}}$ may become non trivial and somewhat interesting (see(2.72)). We have to consider the exact sequence

$$0\to H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})\to H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})\xrightarrow{r}H^1(\partial(\Gamma\backslash\mathbb{H}),\tilde{\mathcal{M}}).\tag{3.32}$$

Our cohomology groups may have some torsion $\mathcal{T}_1\subset H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}),\mathcal{T}_2\subset H^1(\partial(\Gamma\backslash\mathbb{H}),\tilde{\mathcal{M}})$ and the map r maps the torsion $\mathcal{T}_1$ to a submodule $r(\mathcal{T}_1)\subset\mathcal{T}_2$. But it will happen that $r(r^{-1}(\mathcal{T}_2))$ is strictly larger than $r(\mathcal{T}_1)$ this means that some non torsion elements are mapped to torsion elements under r. By definition $H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})_{\text{ int},!}=r^{-1}(\mathcal{T}_2)$ and therefore

$$H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})_{\text{ int},!}/H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})_{\text{ int}}=r(r^{-1}(\mathcal{T}_2)/\mathcal{T}_1)\tag{3.33}$$

This has been investigated extensively by Taiwang Deng in [27].

Let $\pi_1:\mathbb{H}\to\Gamma\backslash\mathbb{H}$ be the projection. We get a covering $\Gamma\backslash\mathbb{H}=\pi_1(\tilde{U}_{\mathbf{i}})\cup\pi_1(\tilde{U}_\rho)=U_i\cap U_\rho$. From this covering we get the Čzech complex

$$\begin{array}{ccccc}
0\ \to & \tilde{\mathcal{M}}(U_{\mathbf{i}})\oplus\tilde{\mathcal{M}}(U_\rho) & \to & \tilde{\mathcal{M}}(U_{\mathbf{i}}\cap U_\rho) & \to 0\\
 & \downarrow ev_{\tilde{U}_{\mathbf{i}}}\oplus ev_{\tilde{U}_\rho} & & \downarrow ev_{\tilde{U}_{\mathbf{i}}\cap\tilde{U}_\rho} & \\
 & \mathcal{M}^{<S>}\oplus\mathcal{M}^{<R>} & \to & \mathcal{M} & \to 0
\end{array}\tag{3.34}$$

and this gives us our formula for the first cohomology

$$H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})=\mathcal{M}/(\mathcal{M}^{<S>}\oplus\mathcal{M}^{<R>})\tag{3.35}$$

We want to discuss the Hecke operator T_2. To do this we pass to the subgroups

$$\begin{aligned}\Gamma_0^+(2)&=\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\mid c\equiv 0\mod 2\}\\ \Gamma_0^-(2)&=\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\mid b\equiv 0\mod 2\}\end{aligned}\tag{3.36}$$

we form the two quotients and introduce the projection maps $\pi_2^\pm : \mathbb{H} \to \Gamma_0^\pm(2)\backslash\mathbb{H}$. We have an isomorphism between the spaces

$$\Gamma_0^+(2)\backslash\mathbb{H} \xrightarrow{\alpha_2} \Gamma_0^-(2)\backslash\mathbb{H}$$

which is induced by the map $m_2 : z \mapsto \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} z = 2z$. This map induces an isomorphism

$$\alpha_2^\bullet : H^1(\Gamma_0^+(2)\backslash\mathbb{H}, \tilde{\mathcal{M}}) \xrightarrow{\sim} H^1(\Gamma_0^-(2)\backslash\mathbb{H}, \tilde{\mathcal{M}}^{(\alpha)}). \tag{3.37}$$

We also have the map between sheaves $u_2 : m \mapsto \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} m$ and the composition with this map induces a homomorphism in cohomology

$$H^1(\Gamma_0^+(2)\backslash\mathbb{H}, \tilde{\mathcal{M}}) \xrightarrow{u_2^\bullet \circ \alpha_2^\bullet} H^1(\Gamma_0^-(2)\backslash\mathbb{H}, \tilde{\mathcal{M}}). \tag{3.38}$$

This is the homomorphism we need for the computation of the Hecke operator; it is easy to define but it may be difficult in practice to compute it.

Each of the spaces $\Gamma_0^+(2)\backslash\mathbb{H}, \Gamma_0^-(2)\backslash\mathbb{H}$ has two cusps which can be represented by the points $\infty, 0 \in \mathbb{P}^1(\mathbb{Q})$. The stabilizers of these two cusps in $\Gamma_0^+(2)$ resp. $\Gamma_0^-(2)$ are

$$< T_+ > \times\{\pm\mathrm{Id}\} \text{ and } < T_-^2 > \times\{\pm\mathrm{Id}\} \subset \Gamma_0^+(2)$$

resp.

$$< T_+^2 > \times\{\pm\mathrm{Id}\} \text{ and } < T_- > \times\{\pm\mathrm{Id}\} \subset \Gamma_0^-(2)$$

the factor $\{\pm\mathrm{Id}\}$ can be ignored. Then we get

$$H^1(\partial(\Gamma_0^+(2)\backslash\mathbb{H}), \tilde{\mathcal{M}}) \xrightarrow{\sim} \mathcal{M}/(\mathrm{Id} - T_+)\mathcal{M} \oplus \mathcal{M}/(\mathrm{Id} - T_-^2)\mathcal{M}$$
$$H^1(\partial(\Gamma_0^-(2)\backslash\mathbb{H}), \tilde{\mathcal{M}}) \xrightarrow{\sim} \mathcal{M}/(\mathrm{Id} - T_+^2)\mathcal{M} \oplus \mathcal{M}/(\mathrm{Id} - T_-)\mathcal{M}.$$

But now it is obvious that α maps the cusp ∞ to ∞ and 0 to 0 and then it is also clear that for the boundary cohomology the map

$$\alpha_2^\bullet : \mathcal{M}/(\mathrm{Id} - T_+)\mathcal{M} \oplus \mathcal{M}/(\mathrm{Id} - T_-^2)\mathcal{M} \to \mathcal{M}/(\mathrm{Id} - T_+^2)\mathcal{M} \oplus \mathcal{M}/(\mathrm{Id} - T_-)\mathcal{M}$$

is simply the map which is induced by $u_2 : \mathcal{M} \to \mathcal{M}$. If we ignore torsion then the individual summands are infinite cyclic.

Our module $\mathcal{M}$ is the module of homogenous polynomials of degree n in 2 variables X, Y with integer coefficients. Then the classes $[Y^n], [X^n]$ of the polynomials Y^n (resp.) X^n are generators of $(\mathcal{M}/(\mathrm{Id}-T_+^\nu)\mathcal{M})/\mathrm{tors}($ resp. $(\mathcal{M}/(\mathrm{Id}-T_+^\nu)\mathcal{M})/\mathrm{tors})$ where $\nu = 1($ resp. 2.) Then we get for the homomorphism $\alpha_2^\bullet$

$$\alpha_2^\bullet : [Y^n] \mapsto [Y^n], \alpha_2^\bullet : [X^n] \mapsto 2^n[X^n]. \tag{3.39}$$

Nochmal ansehen

3.3.2 The explicit description of the cohomology

We give the explicit description of the cohomology $H^1(\Gamma_0^+(2)\backslash\mathbb{H}, \tilde{\mathcal{M}})$. We introduce the projections

$$\mathbb{H} \xrightarrow{\pi_2^+} \Gamma_0^+(2)\backslash\mathbb{H};\ \mathbb{H} \xrightarrow{\pi_2^-} \Gamma_0^-(2)\backslash\mathbb{H}$$

and get the covering $\mathfrak{U}_2$

$$\Gamma_0^+(2)\backslash\mathbb{H} = \pi_2^+(\tilde{U}_{\mathbf{i}}) \cup \pi_2^+(T_-\tilde{U}_{\mathbf{i}}) \cup \pi_2^+(\tilde{U}_\rho) = \pi_2^+(\tilde{U}_{\mathbf{i}}) \cup \pi_2^+(\tilde{U}_{\frac{\mathbf{i}+1}{2}}) \cup \pi_2^+(\tilde{U}_\rho)$$

where we put $T_-\tilde{U}_{\mathbf{i}} = \tilde{U}_{\frac{\mathbf{i}+1}{2}}$. Our set $\{x_\nu\}$ of indexing points is $\mathbf{i}$, $\frac{\mathbf{i}+1}{2}$, ρ, we put $U_{x_i}^+ = \pi_2^+(\tilde{U}_{x_i})$. Note $T_- \notin \Gamma_0^+(2), T_+ \in \Gamma_0^+(2)$.

Again the cohomology is computed by the complex

$$0 \to \tilde{\mathcal{M}}(U_{\mathbf{i}}^+) \oplus \tilde{\mathcal{M}}(T_-\tilde{U}_{\mathbf{i}}^+) \oplus \tilde{\mathcal{M}}(U_\rho^+) \to \tilde{\mathcal{M}}(U_{\mathbf{i}}^+ \cap U_\rho^+) \oplus \tilde{\mathcal{M}}(T_-\tilde{U}_{\mathbf{i}}^+ \cap U_\rho^+) \to 0$$

we have to identify the terms as submodules of some $\bigoplus \mathcal{M}$ and write down the boundary map explicitly. We have

$$\begin{array}{ccc}
\tilde{\mathcal{M}}(U_{\mathbf{i}}^+) \oplus \tilde{\mathcal{M}}(U_{\frac{\mathbf{i}+1}{2}}^+) \oplus \tilde{\mathcal{M}}(U_\rho^+) & \xrightarrow{d_0} & \tilde{\mathcal{M}}(U_i^+ \cap U_\rho^+) \oplus \tilde{\mathcal{M}}(U_{\frac{\mathbf{i}+1}{2}}^+ \cap U_\rho^+) \\
\downarrow ev_{\tilde{U}_{\mathbf{i}}} \oplus ev_{T_-\tilde{U}_{\mathbf{i}}} \oplus ev_{\tilde{U}_\rho} & & \downarrow ev_{\tilde{U}_{\mathbf{i}}\cap\tilde{U}_\rho} \oplus ev_{\tilde{U}_{\mathbf{i}}\cap T_+^{-1}\tilde{U}_\rho} \oplus ev_{T_-\tilde{U}_{\mathbf{i}}\cap\tilde{U}_\rho} \\
\mathcal{M} \oplus \mathcal{M}^{<S_1^+>} \oplus \mathcal{M} & \xrightarrow{\bar{d}_0} & \mathcal{M} \oplus \mathcal{M} \oplus \mathcal{M}
\end{array} \tag{3.40}$$

where the vertical arrows are isomorphisms. The boundary map $\bar{d}_0$ in the bottom row is given by

$$(m_1, m_2, m_3) \mapsto (m_1 - m_3, m_1 - T_+^{-1}m_3, m_1 - m_2) = (x, y, z)$$

We may look at the (isomorphic) sub complex where $x = z = 0$ and $m_1 = m_2 = m_3$ then we obtain the complex

$$0 \to \mathcal{M}^{<S_1^+>} \to \mathcal{M} \to 0;\ m_2 \mapsto m_2 - T_+^{-1}m_2$$

which provides an isomorphism

$$H^1(\Gamma_0^+(2)\backslash\mathbb{H}, \tilde{\mathcal{M}}) \xrightarrow{\sim} \mathcal{M}/(\mathrm{Id} - T_+^{-1})\mathcal{M}^{<S_1^+>}. \tag{3.41}$$

A simple computation shows that the cohomology class represented by the class (x, y, z) is equal to the class represented by $(0, y-x+T_+^{-1}z-z, 0)$ we write

$$[(x, y, z)] = [(0, y - x + T_+^{-1}z - z, 0)] \tag{3.42}$$

3.3.3 The map to the boundary cohomology

We have the restriction map for the cohomology of the boundary

$$\begin{array}{ccc} H^1(\Gamma_0^+(2)\backslash\mathbb{H},\tilde{\mathcal{M}}) & \xrightarrow{\sim} & \mathcal{M}/(\mathrm{Id}-T_+^{-1})\mathcal{M}^{<S_1^+>} \\ \downarrow & & r^+\oplus r^- \downarrow \\ H^1(\partial(\Gamma_0^+(2)\backslash\mathbb{H}),\tilde{\mathcal{M}}) & \xrightarrow{\sim} & \mathcal{M}/(\mathrm{Id}-T_+)\mathcal{M}\oplus\mathcal{M}/(\mathrm{Id}-T_-^2)\mathcal{M} \end{array} \tag{3.43}$$

we give a formula for the second vertical arrow. We represent a class $[m]$ by an element $m\in\mathcal{M}$ and send m to its class in each the two summands, respectively. This is well defined, for r^+ it is obvious, while for r^- we observe that if $m = x - T_+^{-1}x$ and $S_1^+x = x$ then $m = x - T_+^{-1}S_1^+x = x - T_-^2x$.

Restriction and Corestriction

Now we have to give explicit formulas for the two maps π^*,π_* in the big diagram ((3.3).Here we should change notation: The map π will now be denoted by :

$$\varpi_2^+ : \Gamma_0^+(2)\backslash\mathbb{H}\to\Gamma\backslash\mathbb{H} \tag{3.44}$$

We have the two complexes which compute the cohomology $H^1(\Gamma_0^+(2)\backslash\mathbb{H},\tilde{\mathcal{M}})$ and $H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})$, and we have defined arrows between them. We realized these two complexes explicitly in (3.40) resp. (3.34) and we have

$$\begin{array}{ccc} \tilde{\mathcal{M}}(U_i^+)\oplus\tilde{\mathcal{M}}(U^+_{\frac{i+1}{2}})\oplus\tilde{\mathcal{M}}(U_\rho^+) & \xrightarrow{d_0} & \tilde{\mathcal{M}}(U_i^+\cap U_\rho^+)\oplus\tilde{\mathcal{M}}(U^+_{\frac{i+1}{2}}\cap U_\rho^+) \\ (\varpi_2^+)^{(0)}\uparrow\downarrow(\varpi_2^+)_{(0)} & & (\varpi_2^+)^{(1)}\uparrow\downarrow(\varpi_2^+)_{(1)} \\ \tilde{\mathcal{M}}(U_i)\oplus\tilde{\mathcal{M}}(U_\rho) & \xrightarrow{d_0} & \tilde{\mathcal{M}}(U_i\cap U_\rho) \end{array} \tag{3.45}$$

and in terms of our explicit realization in diagram (3.40) this gives

$$\begin{array}{ccc} \mathcal{M}\oplus\mathcal{M}^{<S_1>}\oplus\mathcal{M} & \xrightarrow{d_0} & \mathcal{M}\oplus\mathcal{M}\oplus\mathcal{M} \\ (\varpi_2^+)^{(0)}\uparrow\downarrow(\varpi_2^+)_{(0)} & & (\varpi_2^+)^{(1)}\uparrow\downarrow(\varpi_2^+)_{(1)} \\ \mathcal{M}^{<S>}\oplus\mathcal{M}^{<R>} & \xrightarrow{d_0} & \mathcal{M} \end{array} \tag{3.46}$$

Looking at the definitions we find

$$\begin{gathered} (\varpi_2^+)^{(0)} : (m_1,m_2)\mapsto(m_1,T_-m_1,m_2) \\ (\varpi_2^+)_{(0)} : (m_1,m_2,m_3)\mapsto(m_1+Sm_1+T_-^{-1}m_2,(1+R+R^2)m_3) \end{gathered} \tag{3.47}$$

and we check easily that the composition $(\varpi_2^+)_{(0)}\circ(\varpi_2^+)^{(0)}$ is the multiplication by 3 as it should be, since this is the index of $\Gamma_0(2)^+$ in Γ.

For the two arrows in degree one we find

$$\begin{aligned}(\varpi_2^+)^{(1)} &: m \mapsto (m, Sm, T_- m)\\ (\varpi_2^+)_{(1)} &: (m_1, m_2, m_3) \mapsto (m_1 + Sm_2 + T_-^{-1} m_3)\end{aligned} \tag{3.48}$$

We apply equation (3.42) and we see that $(\varpi_2^+)^{(1)}(m)$ is represented by

$$[(\varpi_2^+)^{(1)}(m)] = [0, Sm + T_+^{-1}T_- m - m - T_- m, 0] \tag{3.49}$$

We do the same calculation for $\Gamma_0^-(2)$. As before we start from a covering

$$\Gamma_0^-(2)\backslash\mathbb{H} = \pi_2^-(\tilde{U}_{\mathbf{i}}) \cup \pi_2^-(T_+\tilde{U}_{\mathbf{i}}) \cup \pi_2^-(\tilde{U}_\rho) = \pi_2^-(\tilde{U}_{\mathbf{i}}) \cup \pi_2^-(\tilde{U}_{i+1}) \cup \pi_2^-(\tilde{U}_\rho)$$

and as before we put $U_{y_\nu}^- = \pi_2^-(\tilde{U}_{y_\nu})$. In this case $\tilde{U}_{i+1} = T_+\tilde{U}_i$ is fixed by $S_1^- = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \in \Gamma_0^-(2)$ and we get a diagram for the Čzech complex

$$\begin{array}{ccc} \tilde{\mathcal{M}}(U_{\mathbf{i}}^-) \oplus \tilde{\mathcal{M}}(U_{\mathbf{i+1}}^-) \oplus \tilde{\mathcal{M}}(U_\rho^-) & \xrightarrow{d_0} & \tilde{\mathcal{M}}(U_{\mathbf{i}}^- \cap U_\rho^-) \oplus \tilde{\mathcal{M}}(U_{\mathbf{i+1}}^- \cap U_\rho^-) \\ ev_{\tilde{U}_{\mathbf{i}}} \oplus ev_{\tilde{U}_{\mathbf{i+1}}} \downarrow \oplus ev_{\tilde{U}_\rho} & & ev_{\tilde{U}_{\mathbf{i}} \cap \tilde{U}_\rho} \oplus ev_{\tilde{U}_{\mathbf{i}} \cap T_-^{-1}\tilde{U}_\rho} \downarrow \oplus ev_{\tilde{U}_{\mathbf{i+1}} \cap \tilde{U}_\rho} \\ \mathcal{M} \oplus \mathcal{M}^{<S_1^->} \oplus \mathcal{M} & \xrightarrow{\bar{d}_0} & \mathcal{M} \oplus \mathcal{M} \oplus \mathcal{M} \end{array} \tag{3.50}$$

Again we can modify this complex and get

$$H^1(\Gamma_0^-(2)\backslash\mathbb{H}, \tilde{\mathcal{M}}) \xrightarrow{\sim} \mathcal{M}/(\mathrm{Id} - T_-^{-1})\mathcal{M}^{<S_1^->}. \tag{3.51}$$

We compute the arrows $(\varpi_2^-)^*, (\varpi_2^-)_*$ in degree one

$$\begin{aligned}(\varpi_2^-)^{(1)} &: m \mapsto (m, Sm, T_+ m),\\ (\varpi_2^-)_{(1)} &: (m_1, m_2, m_3) \mapsto (m_1 + Sm_2 + T_+^{-1} m_3).\end{aligned} \tag{3.52}$$

The computation of $\alpha_2^\bullet$.

We recall our isomorphism α between the spaces and the resulting isomorphism (3.37). The identity map of the module $\mathcal{M}$ and the isomorphism α on the space identifies the two complexes

$$\begin{array}{ccc} \tilde{\mathcal{M}}(U_{\mathbf{i}}^+) \oplus \tilde{\mathcal{M}}(U_{\frac{\mathbf{i+1}}{2}}^+) \oplus \tilde{\mathcal{M}}(U_\rho^+) & \xrightarrow{d_0} & \tilde{\mathcal{M}}(U_{\mathbf{i}}^+ \cap U_\rho^+) \oplus \tilde{\mathcal{M}}(U_{\frac{\mathbf{i+1}}{2}}^+ \cap U_\rho^+) \\ \tilde{\mathcal{M}}^{(\alpha)}(\alpha(U_{\mathbf{i}}^+)) \oplus \tilde{\mathcal{M}}^{(\alpha)}(\alpha(U_{\frac{\mathbf{i+1}}{2}}^+)) \oplus \tilde{\mathcal{M}}^{(\alpha)}(\alpha(U_\rho^+)) & \xrightarrow{d_0} & \tilde{\mathcal{M}}^{(\alpha)}(\alpha(U_{\mathbf{i}}^+ \cap U_\rho^+)) \oplus \tilde{\mathcal{M}}^{(\alpha)}(\alpha(U_{\frac{\mathbf{i+1}}{2}}^+ \cap U_\rho^+)) \end{array} \tag{3.53}$$

and if we consider their explicit realization then this identification is given by the equality of $\mathbb{Z}$ modules $\mathcal{M} = \mathcal{M}^{(\alpha)}$. This equality of complexes expresses the identification (3.37). We can compute the cohomology $H^1(\Gamma_0^-(2)\backslash\mathbb{H},\tilde{\mathcal{M}}^{(\alpha)})$ from any of the two coverings

$$\begin{aligned}&\Gamma_0^-(2)\backslash\mathbb{H}=\alpha(U_{\mathbf{i}}^+)\cup\alpha(U_{\frac{\mathbf{i}+1}{2}}^+)\cup\alpha(U_\rho^+)=U_{x_1}\cup U_{x_2}\cup U_{x_3}\\ &\text{and}\\ &\quad\Gamma_0^-(2)\backslash\mathbb{H}=U_{\mathbf{i}}^-\cup U_{\mathbf{i}+1}^-\cup U_\rho^-=U_{x_4}\cup U_{x_5}\cup U_{x_6}.\end{aligned}\tag{3.54}$$

We have to pick a class $\xi\in H^1(\Gamma_0^-(2)\backslash\mathbb{H},\tilde{\mathcal{M}}^{(\alpha)})$ and represent it by a cocycle

$$c_\xi\in\bigoplus_{1\le i<j\le 3}\tilde{\mathcal{M}}^{(\alpha)}(U_{x_i}\cap U_{x_j})$$

(The cocycle condition is empty since $U_{x_1}\cap U_{x_2}\cap U_{x_3}=\emptyset$.)

Then we have to produce a cocycle

$$c_\xi^{(\alpha)}\in\bigoplus_{4\le i<j\le 6}\tilde{\mathcal{M}}^{(\alpha)}(U_{x_i}\cap U_{x_j})$$

which represents the same class.

To get this cocycle we write down the three complexes

$$\begin{array}{ccc}\bigoplus_{1\le i<j\le 3}\tilde{\mathcal{M}}^{(\alpha)}(U_{x_i}\cap U_{x_j}) & \to & 0\\ \uparrow & & \\ \bigoplus_{1\le i<j\le 6}\tilde{\mathcal{M}}^{(\alpha)}(U_{x_i}\cap U_{x_j}) & \to & \bigoplus_{1\le i<j<k\le 6}\tilde{\mathcal{M}}^{(\alpha)}(U_{x_i}\cap U_{x_j}\cap U_{x_k})\\ \downarrow & & \\ \bigoplus_{4\le i<j\le 6}\tilde{\mathcal{M}}^{(\alpha)}(U_{x_i}\cap U_{x_j}) & \to & 0\end{array}\tag{3.55}$$

for our cocycle c_ξ we find a cocycle $c_\xi^\dagger$ in the complex in the middle which maps to c_ξ under the upwards arrow and this cocycle is unique up to a coboundary. Then we project it down by the downwards arrow, i.e. we only take its $4\le i<j\le 6$ components, and this is our cocycle $c_\xi^{(\alpha)}$.

We apply the principles from section (2.1.3) and we write down these complexes explicitly. For any pair $\underline{i}=(i,j),i<j$ of indices we have to compute the set $\mathcal{F}_{\underline{i}}$. We drew some pictures and from these pictures we get (modulo errors) the following list (of lists):

$$\begin{array}{cccc}\mathcal{F}_{1,2}=\emptyset & \mathcal{F}_{1,3}=\{\mathrm{Id},T_+^{-2}\} & \mathcal{F}_{1,4}=\{\mathrm{Id}\} & \mathcal{F}_{1,5}=\{\mathrm{Id},T_+^{-2}\}\\ \mathcal{F}_{1,6}:=\{\mathrm{Id},T_-^{-1}\} & \mathcal{F}_{2,3}=\{\mathrm{Id}\} & \mathcal{F}_{2,4}=\{\mathrm{Id},T_-\} & \mathcal{F}_{2,5}=\{\mathrm{Id}\}\\ \mathcal{F}_{2,6}=\{\mathrm{Id}\} & \mathcal{F}_{3,4}=\{\mathrm{Id},T_+^2\} & \mathcal{F}_{3,5}=\{\mathrm{Id}\} & \mathcal{F}_{3,6}=\{\mathrm{Id},S_1^-\}\\ \mathcal{F}_{4,5}=\emptyset & \mathcal{F}_{4,6}=\{\mathrm{Id},T_-^{-1}\} & \mathcal{F}_{5,6}=\{\mathrm{Id}\} & \end{array}\tag{3.56}$$

Now we have to follow the rules in the first section and we can write down an explicit version of the diagram (3.55) . We refer to section 2.1.3 and get

$$
\begin{array}{ccc}
\bigoplus_{1\le i<j\le 3}\bigoplus_{\gamma\in\mathcal{F}_{i,j}}(\mathcal{M}^{(\alpha)})^{\Gamma_{i,j,\gamma}} & \to & 0\\
\uparrow & & \\
\bigoplus_{1\le i<j\le 6}\bigoplus_{\gamma\in\mathcal{F}_{i,j}}(\mathcal{M}^{(\alpha)})^{\Gamma_{i,j,\gamma}} & \to & \bigoplus_{1\le i<j<k\le 6}\bigoplus_{\gamma\in\mathcal{F}_{i,j,k}}(\mathcal{M}^{(\alpha)})^{\Gamma_{i,j,k,\gamma}}\\
\downarrow & & \\
\bigoplus_{4\le i<j\le 6}\bigoplus_{\gamma\in\mathcal{F}_{i,j}}(\mathcal{M}^{(\alpha)})^{\Gamma_{i,j,\gamma}} & \to & 0
\end{array}
\tag{3.57}
$$

Here we have to interpret this diagram. The module $\mathcal{M}^{(\alpha)}$ is equal to $\mathcal{M}$ as an abstract module, but an element $\gamma\in\Gamma_0^-(2)$ acts by the twisted action (See ChapII, 2.2)

$$m\mapsto \gamma *_\alpha m=\alpha^{-1}\gamma\alpha * m$$

here the $*$ denotes the original action. Hence we have to take the invariants $(\mathcal{M}^{(\alpha)})^{\Gamma_{i,j,\gamma}}$ with respect to this twisted action. In our special situation this has very little effect since almost all the $\Gamma_{i,j,\gamma}$ are trivial, except for the intersection $\alpha(\tilde{U}_{\frac{\mathbf{i}+1}{2}})\cap\tilde{U}_{\mathbf{i}}$ in which case $\Gamma_{i,j,\gamma}=<S_1^->$. Hence

$$(\mathcal{M}^{(\alpha)})^{<S_1^->}=\mathcal{M}^{<S_1^+>}.$$

Each of the complexes in (3.57) compute the cohomology group $H^1(\Gamma_0^-(2)\backslash\mathbb{H},\tilde{\mathcal{M}})$ and the diagram gives us a formula for the isomorphism in (3.37). To get $u_\alpha^\bullet$ in (3.37) we apply the multiplication m_2:$m\mapsto\alpha m$ to the complex in the middle and the bottom. Then the cocycle c_ξ^α is now an element in $\bigoplus\tilde{\mathcal{M}}^{(\alpha)}$ and αc_ξ^α represents the cohomology class $u_\alpha^\bullet(\xi)\in H^1(\Gamma_0^-(2)\backslash\mathbb{H},\tilde{\mathcal{M}})$.

Now it is clear how we can compute the Hecke operator

$$T_2=T_{\begin{pmatrix}2&0\\0&1\end{pmatrix}}:\mathcal{M}/(\mathcal{M}^{<S>}\oplus\mathcal{M}^{<R>})\to\mathcal{M}/(\mathcal{M}^{<S>}\oplus\mathcal{M}^{<R>})$$

We pick a representative $m\in\mathcal{M}$ of the cohomology class. We apply $(\varpi_2^+)^{(1)}$ in the diagram (3.46) to it and this gives the element $(Sm,m,T_-m)=c_\xi$. We apply the above process to compute $c_\xi^{(\alpha)}$. Then $\alpha c_\xi^{(\alpha)}=(m_1,m_2,m_3)$ is an element in $\tilde{\mathcal{M}}(U_{\mathbf{i}}^-\cap U_\rho^-)\oplus\tilde{\mathcal{M}}(U_{\mathbf{i}+1}^-\cap U_\rho^-)$ and this module is identified with $\mathcal{M}\oplus\mathcal{M}\oplus\mathcal{M}$ by the vertical arrow in (3.50). To this element we apply the trace

$$(\varpi_2^-)_{(1)}(m_1,m_2,m_3)=m_1+m_2+T_+^{-1}m_3$$

and the latter element in $\mathcal{M}$ represents the class $T_2([m])$.

We have written a computer program which for a given $\mathcal{M}=\mathcal{M}_n$, i.e. for a given even positive integer n, computes the module $H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})$ and the endomorphism T_2 on it.

Looking our data we discovered the following (surprising?) fact: We consider the isomorphism in equation (3.37). We have the explicit description of the cohomology in (3.41)

$$H^1(\Gamma_0^+(2)\backslash\mathbb{H},\tilde{\mathcal{M}}) \xrightarrow{\sim} \mathcal{M}/(\mathrm{Id}-T_+^{-1})\mathcal{M}^{<S_1^+>}$$

and

$$H^1(\Gamma_0^-(2)\backslash\mathbb{H},\tilde{\mathcal{M}}^{(\alpha)}) \xrightarrow{\sim} \mathcal{M}/(\mathrm{Id}-T_-^{-1})(\mathcal{M}^{(\alpha)})^{<S_1^->}$$

We know that we may represent any cohomology class by a cocycle

$$c_\xi = (0,c_\xi,0) \in \mathcal{M}^{(\alpha)}(\pi_2^-(\alpha(U_\mathbf{i})\cap\alpha(U_\rho))\oplus\mathcal{M}^{(\alpha)}(\pi_2^-(\alpha(U_\mathbf{i})\cap\alpha(T_+^{-1}U_\rho))\oplus\mathcal{M}^{(\alpha)}(\pi_2^-(\alpha(U_{\frac{\mathbf{i}+1}{2}})\cap\alpha(T_+^{-1}U_\rho))$$

so it is non zero only in the middle component and then it is simply an element in $\mathcal{M}$. If we now look at our data, then it seems to by so that $c_\xi^{(\alpha)}$ is also non zero only in the middle, hence

$$c_\xi^{(\alpha)} \in (0,c'_\xi,0) \in 0\oplus\mathcal{M}^{(\alpha)}(\pi_2^-(U_\mathbf{i}\cap T_-^{-1}U_\rho))\oplus 0$$

hence it is also in $\mathcal{M}^{(\alpha)}$ and then our data seem to suggest that

$$c'_\xi = c_\xi$$

Hence we see that the homomorphism in equation (3.38) is simply given by

$$X^\nu Y^{n-\nu} \mapsto 2^\nu X^\nu Y^{n-\nu}.$$

Is there a kind of homotopy argument (- 2 moves continuously to 1?-)-, which explains this?

We get an explicit formula for the Hecke operator T_2 : We pick an element $m\in\mathcal{M}$ representing the class $[m]$. We send it by $(\varpi_2^+)^{(1)}$ to $H^1(\Gamma_0^+(2)\backslash\mathbb{H},\tilde{\mathcal{M}})$, i.e.

$$(\varpi_2^+)^{(1)} : m \mapsto (m, Sm, T_-m) \tag{3.58}$$

We modify it so that the first and the third entry become zero see(3.42)

$$[(m,Sm,T_-m)] = [(0, Sm-m+T_+^{-1}T_-m-T_-m, 0)] \tag{3.59}$$

To the entry in the middle we apply $M_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and then apply $(\varpi_2^-)_{(1)}$ and get

$$T_2([m]) = [S\cdot M_2(Sm-m+T_+^{-1}T_-m-T_-m)] \tag{3.60}$$

$\boxed{\text{Eisn}}$

3.3.4 The first interesting example

We give an explicit formula for the cohomology in the case of $\mathcal{M} = \mathcal{M}_{10}$. We define the sub-module

$$\mathcal{M}^{\mathrm{tr}} = \bigoplus_{\nu=0}^{5} \mathbb{Z} Y^{10-\nu} X^{\nu}$$

and we have the truncation operator trunc : $\mathcal{M} \to \mathcal{M}^{\mathrm{tr}}$

$$\mathrm{trunc} : Y^{10-\nu}X^{\nu} \mapsto \begin{cases} Y^{10-\nu}X^{\nu} & \text{if } \nu \le 5, \\ (-1)^{\nu+1} Y^{\nu} X^{10-\nu} & \text{else,} \end{cases},$$

it identifies the quotient module $\mathcal{M}/\mathcal{M}^{<S>}$ to $\mathcal{M}^{\mathrm{tr}}$. To get the cohomology we have to divide by the relations coming from $\mathcal{M}^{<R>}$, i.e. we have to divide by the submodule $\mathrm{trunc}(\mathcal{M}^{<R>})$ The module of these relations is generated by

$$\begin{gathered} \mathrm{R}_1 = 10Y^9X + 20Y^7X^3 + Y^5X^5 \\ \mathrm{R}_2 = 9Y^8X^2 - 36Y^7X^3 + 14Y^6X^4 - 45Y^5X^5 \\ \mathrm{R}_3 = 8Y^7X^3 + 10Y^5X^5 \end{gathered}$$

and then

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) = \bigoplus_{\nu=0}^{5} \mathbb{Z} Y^{10-\nu}X^{\nu} / \{\mathrm{R}_1, \mathrm{R}_2, \mathrm{R}_3\} \tag{3.61}$$

We simplify the notation and put $e_\nu = Y^\nu X^{n-\nu}$. Using R_1 we can eliminate $e_5 = -10e_9 - 20e_7$ and then

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) = \bigoplus_{\nu=10}^{\nu=6} \mathbb{Z} e_\nu / \{-50e_9 + 9e_8 - 96e_7 + 14e_6, -100e_9 - 192e_7\} \tag{3.62}$$

introduce a new basis $\{f_{10}, f_9, f_8, f_7, f_6, f_5\}$ of the $\mathbb{Z}$ module $\mathcal{M}^{\mathrm{tr}}$:

$$\begin{gathered} f_{10} = e_{10}; f_8 = -2e_8 - 3e_6; f_6 = 9e_8 + 14e_6 \\ f_9 = -12e_9 - 23e_7; f_7 = 25e_9 + 48e_7; f_5 = 10e_9 + 20e_7 + e_5 \end{gathered} \tag{3.63}$$

and hence in the quotient we get $\bar{f}_5 = 0$ and $2\bar{f}_7 = \bar{f}_6$ and therefore

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) = \mathbb{Z}\bar{f}_{10} \oplus \mathbb{Z}\bar{f}_9 \oplus \mathbb{Z}\bar{f}_8 \oplus \mathbb{Z}/(4)\bar{f}_7 \tag{3.64}$$

x

We defined the action of complex conjugation (see 2.84) on $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$ and we leave it as an exercise to the reader to show that

$$\mathbf{c}^{(1)}(\bar{f}_{10}) = -\bar{f}_{10}, \mathbf{c}^{(1)}(\bar{f}_9) = -\bar{f}_9, \mathbf{c}^{(1)}(\bar{f}_8) = \bar{f}_8 \tag{3.65}$$

If we can apply the above procedure to compute the action of T_2 on cohomology. It is turns out to be reasonable to compute the matrix for T_2 with respect to the basis $\bar{f}_{10}, \bar{f}_8, \bar{f}_9$ then our program with Gangl yields

$$T_2 = \begin{pmatrix} 2049 & 0 & -68040 & 0 \\ 0 & -24 & 0 & 0 \\ 0 & 0 & -24 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}. \tag{3.66}$$

Hence we see that T_2 is non trivial on the torsion subgroup. If we divide by the torsion then the matrix reduces to a (3,3)-matrix and this matrix gives us the endomorphism on the "integral" cohomology which is defined in general by

$$H^\bullet_{\text{int}}(\Gamma\backslash X, \tilde{\mathcal{M}}) = H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})/\text{tors} \subset H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_{\mathbb{Q}}). \tag{3.67}$$

Here we should be careful: the functor $H^\bullet \to H^\bullet_{\text{int}}$ is not exact. In our case we get (perhaps up to a little piece of 2-torsion) exact sequences of Hecke modules

$$\begin{array}{ccccccc} 0 & \to & \mathbb{Z}\bar{f}_9 \oplus \mathbb{Z}\bar{f}_8 & \to \mathbb{Z}\bar{f}_{10} \oplus \mathbb{Z}\bar{f}_9 \oplus \mathbb{Z}\bar{f}_8 \xrightarrow{r} & \mathbb{Z}\bar{f}_{10} \to & & 0 \\ & & \| & \| & \| & & \\ 0 & \to & H^1_{\text{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \to & H^1_{\text{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \xrightarrow{r} & H^1_{\text{int}}(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}) \to & & 0 \end{array} \tag{3.68}$$

where $T_2(\bar{f}_{10}) = (2^{11}+1)\bar{f}_{10}$. If we tensor by $\mathbb{Q}$ then we can find an unique element - *the Eisenstein class*- $f^\dagger_{10} \in H^1_{\text{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Q})$ which maps to $\bar{f}_{10}$ and which satisfies $T_2(f^\dagger_{10}) = (2^{11}+1)f^\dagger_{10}$. This element is not necessarily integral, in our case an easy computation shows that $f^\dagger \notin H^1_{\text{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$, but $691 f^\dagger \in H^1_{\text{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$. This means that 691 is the denominator of $f^\dagger_{10}$, i.e. 691 is the *denominator of the Eisenstein class*

Hence we see that

$$H^1_{\text{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \supset H^1_{\text{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \oplus \mathbb{Z}(691 f^\dagger) \tag{3.69}$$

the quotient of these modules is isomorphic to $\mathbb{Z}/691\mathbb{Z}$.

The exact sequence $\mathcal{X}_{10}$ in (3.68) is an exact sequence of modules for the Hecke algebra $\mathcal{H} \supset \mathbb{Z}[T_2]$ and hence it yields an element

$$[\mathcal{X}_{10}] \in \text{Ext}^1_{\mathcal{H}}(\mathbb{Z}f_{10}, H^1_{\text{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})), \tag{3.70}$$

and an easy calculation shows that this Ext^1 group is cyclic of order 691 and that it is generated by $\mathcal{X}_{10}$.

We look at the action of the full Hecke algebra $\mathcal{H}$ on these cohomology groups. It turns out that for any prime p the Hecke operator T_p acts by the eigenvalue $p^{11}+1$ on f_{10}(see proposition 3.3.1). We will also see that a simple argument using Poincare duality and the self adjointness of the Hecke operators shows that

T_p acts by multiplication by a scalar $\tau(p)$ on the inner cohomology .

Then we can conclude

For all primes p we have

$$\tau(p) \equiv p^{11} + 1 \mod 691-$$

f

3.3.5 Interlude: Ramanujan's $\Delta(z)$

We want to stress that the previous considerations are of purely algebraic and combinatorial nature, no analysis is involved. In the next chapter we will use analytic methods -especially we will use the results from the theory modular forms- to obtain some further insight into the structure of the cohomology groups. In our special case here it comes down to the following.

In his paper [72] Ramanujan introduced the function

$$\Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^{24} \tag{3.71}$$

is a function on the upper half plane $\mathbb{H} = \{z | \Im(z) > 0\}$ and it satisfies

$$\Delta(\frac{az+b}{cz+d}) = (cz+d)^{12}\Delta(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}_2(\mathbb{Z})$. This means that it is a modular form of weight 12. Since it goes to zero if $z = iy \to \infty$ it is even a modular cusp form. It is the unique (up to a non zero scalar) cusp form of weight 12 for $\mathrm{Sl}_2(\mathbb{Z})$,

(See [80]). We can expand

$$\Delta(z) = e^{2\pi iz} - 24e^{4\pi iz} + 252e^{6\pi iz} + \cdots + a_n e^{2n\pi iz} + \ldots$$

@ The coefficients satisfy (conjectured by Ramanujan) the following recursions

$$\begin{aligned} a_{n_1 n_2} &= a_{n_1} a_{n_2} \text{ if } n_1, n_2 \text{ are coprime;} \\ a_{p^r} &= a_p a_{p^{r-1}} + p^{11} a_{p^{r-2}} \text{ if } p \text{ is a prime and } r \geq 2 \end{aligned} \tag{3.72}$$

These recursion formulas for the coefficients of the expansion were proved by Mordell [66] (essentially by using Hecke operators) and later by Hecke in a more general framework.

In the next section we discuss the Eichler-Shimura isomorphism (see 4.1.7), in this special case it implies that for any prime p we have $a_p = \tau(p)$. Therefore we define the Ramanujan τ function by $\tau(n) = a_n$. With this definition of $\tau(n)$ Ramanujan proved the famous congruence $\tau(p) \equiv p^{11} + 1 \mod 691$.

Ramanujan also made the famous conjecture saying that for all primes p we have the inequality

$$\tau(p) \leq 2\, p^{\frac{11}{2}}$$

This inequality implies of course that for all primes p (and especially for $p = 2$) $\tau(p) \neq p^{11} + 1$ and this implies that any Hecke operator T_p provides a canonical splitting into eigenspaces $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Q}) = H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Q}) \oplus \mathbb{Q}f_{10}$. This is the simplest instance where the Manin-Drinfeld principle works.

Other congruences

It is easy to check that $H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})$ and $H^2_c(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})$ do not have 5 or 7 torsion. Therefore we have (see Prop. 3.3.1,)

$$\mathbb{Z}/10\mathbb{Z}\epsilon_9(X,Y)\oplus\mathbb{Z}/5\mathbb{Z}\epsilon_4(X,Y)\oplus\mathbb{Z}/7\mathbb{Z}\epsilon_6(X,Y)\subset r(r^{-1}(\mathcal{T}_2))/\mathcal{T}_1 \tag{3.73}$$

and this implies the well known congruences

$$\tau(p)\equiv p^{10}+p\equiv p^6+p^5 \mod 5;\ \tau(p)\equiv p^7+p^5 \mod 7 \tag{3.74}$$

[86] [27] These congruences are called congruences of *local origin* whereas the congruence mod 691 is a congruence of *global origin.*

End of interlude

Reduction mod 691

Of course our program also runs if we reduce mod 691, and in principle it runs much faster. Since there is at most 2 torsion we get an exact sequence of Hecke-modules

$$0\to\ H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes\mathbb{F}_{691})\to H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes\mathbb{F}_{691})\xrightarrow{r}\ H^1_{\mathrm{int}}(\partial(\Gamma\backslash\mathbb{H}),\tilde{\mathcal{M}}\otimes\mathbb{F}_{691})\to 0. \tag{3.75}$$

The matrix giving the Hecke operator mod 691 becomes

$$T_2=\begin{pmatrix}667&0&369\\0&667&0\\0&0&667\end{pmatrix}. \tag{3.76}$$

Now we see that our computation mod 691 yields the extension class $[\mathcal{X}_{10}\otimes\mathbb{F}_{691}]$ is an element of order 691, or in other words the sequence (3.75)does not split under the action of T_2. Therefore we get from the computation mod 691 that 691 divides the order of $[\mathcal{X}_{10}]$ and hence divides the order of the denominator of the Eisenstein class.

Of course we may also consider the other Hecke operators T_p acting on $H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})$ then the corresponding matrix will be

$$T_p=\begin{pmatrix}p^{11}+1&0&t^{(p)}\\0&\tau(p)&0\\0&0&\tau(p)\end{pmatrix}. \tag{3.77}$$

But there is no reason that for the other Hecke operators T_p the sequence (3.75) is always non split. Here we have a little proposition

Proposition 3.3.2. *The sequence (3.75) splits for the action of T_p if and only if*

$$p^{11}+1-\tau(p)\equiv 0 \mod 691^2 \tag{3.78}$$

Proof. The sequence mod 691 splits for T_p if and only if $t^{(p)} \equiv 0 \mod 691$. But we have seen that the equation $x(p^{11}+1-\tau(p)) = t^{(p)}$ has no solution in the local ring $\mathbb{Z}_{(691)}$, and this implies the above congruence. □

For the curious reader we mention that this happens for $p = 3559$ and for the first ten thousand primes it happens 13 times and 13 is roughly equal to 10000/691.

At the end of Chapter 5 we presume the result of Deligne which says that we have an action of the Galois group on $H^1_{\text{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{F}_{691})$. We will see that the structure of this cohomology as a module for the Hecke algebra has interesting consequences for this action (See Theorem 5.1.5).

$\boxed{\text{Eisng}}$

3.3.6 The general case

Now we describe the general case $\mathcal{M} = \mathcal{M}_n$ where $n > 0$ is an even integer > 0. We define $\mathcal{M}^{\text{tr}}$ as above, if $n/2$ is even, then we leave out the summand $X^{n/2}Y^{n/2}$, we get

$$\mathcal{M}^{\text{tr}} = \mathcal{M}/\mathcal{M}^{<S>}.$$

This gives us for the cohomology and the restriction to the boundary cohomology

$$\begin{array}{ccc} H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) & \xrightarrow{\sim} & \mathcal{M}^{\text{tr}}/\text{Rel} \\ \downarrow & & \downarrow \\ H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}) & \xrightarrow{\sim} & \mathcal{M}/(\text{Id}-T)\mathcal{M}. \end{array} \tag{3.79}$$

We have the basis

$$e_n = \text{trunc}(Y^n), e_{n-1} = \text{trunc}(Y^{n-1}X), \dots, \begin{cases} Y^{n/2}X^{n/2} & n/2 \text{ odd} \\ 0 & \text{else} \end{cases}$$

for $\mathcal{M}^{\text{tr}}$. Let us put $n_2 = n/2$ or $n/2-1$. Then the algorithm *Smithnormalform* provides a second basis $f_n = e_n, f_{n-1}, \dots, f_{n_2}$ such that the module of relations becomes

$$d_n f_n = 0, d_{n-1}f_{n-1} = 0, \dots, d_t f_t = 0, \dots, d_{n_2} f_{n_2} = 0$$

where $d_{n_2}|d_{n_2+1}|\dots|d_n$. We have $d_n = d_{n-1} = \dots = d_{n-2s} = 0$ where $2s+1 = \dim H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \otimes \mathbb{Q}$ and $d_{n-2s-1} \neq 0$.

Now H. Gangl and I we have written a computer program which for a given n gives us an explicit matrix for T_2, it is of the form

$$T_2(f_i) = \sum_{j=n}^{j=n_2} t^{(2)}_{i,j} f_j \tag{3.80}$$

where we have (the numeration of the rows and columns is downwards from n to n_2)

$$\begin{array}{c} t^{(2)}_{\nu,n} = 0 \text{ for } \nu < n \text{ and } t^{(2)}_{i,j} \in \text{ Hom}(\mathbb{Z}/(d_i), \mathbb{Z}/(d_j)) \\ \text{and } t^{(2)}_{i,j} = 0 \text{ for } i \geq n-2s, j < n-2s \end{array} \tag{3.81}$$

If we divide by the torsion we get for the restriction map to the boundary cohomology

$$H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})_{\mathrm{int}} = \bigoplus_{\nu=n}^{n-2s}\mathbb{Z}f_\nu \xrightarrow{r} H^1(\partial(\Gamma\backslash\mathbb{H}),\tilde{\mathcal{M}})_{\mathrm{int}} = \mathbb{Z}Y^n \tag{3.82}$$

where $f_n \mapsto Y^n$ and $T_2(Y^n) = (2^{n+1}+1)Y^n$. Now we will find that the endomorphism $T_2 - (2^{n+1}+1)\mathrm{Id}$ of $H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})_{\mathrm{int}}$ is injective.

We mentioned the exercise (see3.65) to determine the action of $\mathbf{c}^{(1)}$, in the general case we find $\mathbf{c}^{(1)}(f_n) = -(f_n)$ and for $\nu < n$ we find $\mathbf{c}^{(1)}(f_\nu) = (-1)^\nu(f_\nu)$. This has the consequence that $t^{(2)}_{n,j} = 0$ for $j \equiv 0 \mod 2$. It turns out that it may be wise to reorder the basis and take as a basis the list $\{f_n, f_{n-2}, f_{n-1}, f_{n-4}, \dots\}$ with respect to this basis the matrix for T_2 looks like this

$$\begin{pmatrix} 2^{n+1}+1 & 0 & t^{(2)}_{n,n-2} & 0 & t^{(2)}_{n,n-4} & \dots \\ 0 & t^{(2)}_{n-1,n-1} & 0 & t^{(2)}_{n-1,n-3} & \dots & \\ 0 & 0 & t^{(2)}_{n-2,n-2} & 0 & t^{(2)}_{n-2,n-4} & \dots \\ & & \vdots & & & \end{pmatrix} \tag{3.83}$$

Comment : We can verify this of course for any given n experimentally. But this assertion follows from the *Manin-Drinfeld principle*. This principle exploits the fact that we have estimates for the eigenvalues of T_2 (or more generally for T_p on $H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes\mathbb{C})$.) These estimates say that for all primes p we always have the Ramanujan inequality

$$|\pi_f(T_p)| \le 2p^{\frac{n+1}{2}} : \tag{3.84}$$

(This is a very deep theorem which has been proved by Deligne)

This implies that $2^{n+1}+1$ can not be an eigenvalue of T_2 on $H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes\mathbb{C})$ and this proves the injectivity. This implies that we can find a vector

$$\mathrm{Eis}_n = f_n + \sum_{\nu=n-1}^{\nu=n-2s} x_\nu f_\nu,\ \ x_\nu \in \mathbb{Q} \tag{3.85}$$

which is an eigenvector for T_2 i.e.

$$T_2(\ \mathrm{Eis}_n) = (2^{n+1}+1)\ \mathrm{Eis}_n. \tag{3.86}$$

The least common multiple $\Delta(n)$ of the denominators of the x_ν is the denominator of the Eisenstein class, it is the smallest positive integer for which

$$\Delta(n)\mathrm{Eis}_n \in H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})_{\mathrm{int}}. \tag{3.87}$$

This denominator is of great interest and our computer program allows us to compute it for any given not to large n. We simply have to compute the x_ν.

We know that $T_2(f_n) = (2^{n+1}+1)f_n + \sum_{\mu=n-1}^{\mu=n-2s} t^{(2)}_{n,\mu} f_\mu$ and then the x_ν are the unique solution of

$$\sum_{\nu=n-1}^{\nu=n-2s} ((2^{n+1}+1)\delta_{\nu,\mu} - t^{(2)}_{\nu,\mu})x_\nu = t^{(2)}_{n,\mu}; \{\mu = n-1, \dots, n-2s\} \tag{3.88}$$

With the help of H. Gangl we carried the computation of the x_ν and hence the $\Delta(n)$ and we found for some not too large values of n (roughly $n \leq 150$) that

$$\Delta(n) = \mathrm{numerator}(\zeta(-1-n)). \tag{3.89}$$

Here of course $\zeta(s)$ is the Riemann ζ function, it is well known that for any even positive integer n the value $\zeta(-1-n)$ is a rational number, hence it makes sense to speak of the numerator. A prime number p is called *irregular* if it divides the numerator of such a value of the Riemann ζ function. The most famous irregular prime is $p = 691$ we have

$$\zeta(-11) = \frac{691}{32760}.$$

Actually 3.89 is a theorem, we will give a proof in Chapter 5 (Theorem 5.1.2).

The reader might argue, why do you make such efforts to find out some experimental evidence for something you know to be true?

There are several reasons for doing this, but the main motivation is the following. The Theorem 5.1.2 is hopefully a special case of a much more general ensemble of assertions. The problem to determine (estimate) denominators of Eisenstein classes is ubiquitous in the cohomology of arithmetic groups. And we have many cases where we have conjectures relating these denominators to special values of L-functions. In our case above this are the values of the numerator $\zeta(-1-n)$. Some further examples will be discussed in Chapter 9 (See also [42]) But in many of these cases the methods to prove theorems like Theorem 5.1.2 fail.

On the other hand we explained in section 3.2.1 that in any given case we can compute the denominator -in principle- . Therefore it seems to be of interest to develop algorithms which compute the cohomology and the action of Hecke operators explicitly in given cases and verify or falsify these conjectures. A general strategy for such an algorithm has been outlined in section 3.2.1 and H. Gangl and I wrote a toy model program in the above case. See also [44].

We are aware that these algorithms may become very slow for more general reductive groups, and it is very likely that we need clever new ideas to achieve this task. Finally I want to say that in many cases the resulting congruences have been checked for certain finite sets of primes (see also Chapter 9).

3.3.7 Localisation at a prime ℓ

We will see later the we should not consider the denominator of the Eisenstein class as a number but rather as an ideal. Hence we are only interested in the decomposition into prime ideals, i.e. for a prime ℓ we want to know the exact power of ℓ which divides $\Delta(n)$. To achieve this we replace in the considerations above the coefficient system $\tilde{\mathcal{M}}$ by $\tilde{\mathcal{M}}_{(\ell)} := \tilde{\mathcal{M}} \otimes \mathbb{Z}_{(\ell)}$, here $\mathbb{Z}_{(\ell)} \subset \mathbb{Q}$ is the local ring at ℓ. Then our cohomology modules will be finitely generated $\mathbb{Z}_{(\ell)}$-modules $H^\bullet(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{(\ell)})$.

ℓ -ordinary endomorphisms

In this subsection we fix a prime ℓ we consider finitely generated modules M over the local ring $\mathbb{Z}_{(\ell)} \subset \mathbb{Q}$. We consider such a module together with an endomorphism $\Phi : M \to M$. Then

Proposition 3.3.3. *We have a canonical decomposition into Φ submodules $M = M_{\,ord} \oplus M_{nilpt}$ where M_{nilpt} is the maximal submodule such that*

$$\bigcap_k \Phi^k(M_{nilpt}) = \{0\}$$

Proof. This is rather obvious if M is a finite, i.e. a torsion module. If M is a free $\mathbb{Z}_{(\ell)}$ module then we find a finite, normal extension $F/\mathbb{Q}$ such that $M \otimes F$ can be decomposed into generalised eigenspaces

$$M \otimes F = \bigoplus_\mu M[\mu] \; ; \; M[\mu] \neq 0$$

where $\mu \in \mathcal{O}_F \otimes \mathbb{Z}_{(\ell)}$, $M[\mu] := \{m \in M \otimes F \mid (\Phi - \mu \cdot \mathrm{Id})^k m = 0 \text{ for some } k > 0\}$ The Galois group acts on the set of eigenvalues μ. We consider the set of primes $\mathfrak{l}_1, \dots, \mathfrak{l}_\mathfrak{f} \subset \mathcal{O}_F$ which lie above ℓ, the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ acts transitively on this set. We say that μ is ordinary if there is a prime $\mathfrak{l}_\mu$ such that $\mu \notin \mathfrak{l}_\mu$, the set of ordinary eigenvalues is invariant under the action of the Galois group. We get a decomposition into

$$M \otimes F = \bigoplus_{\mu \text{ ordinary}} M[\mu] \oplus \bigoplus_{\mu \text{ not ordinary}} M[\mu]$$

the two summands are invariant under the action of the Galois group. We put

$$M_{\,\mathrm{ord}} = \bigoplus_{\mu \text{ is a unit}} M[\mu] \cap M \text{ and } M_{\mathrm{nilpt}} = \bigoplus_{\mu \text{ is not a unit}} M[\mu] \cap M.$$

Because of the Galois invariance it is clear that $M \otimes \mathbb{Q} = M_{\,\mathrm{ord}} \otimes \mathbb{Q} \oplus M_{\mathrm{nilpt}} \otimes \mathbb{Q}$. But a little bit of semi-local algebra shows that actually $M = M_{\,\mathrm{ord}} \oplus M_{\mathrm{nilpt}}$ and this decomposition has the desired properties. $\square$

We call $M_{\,\mathrm{ord}}$ the ordinary part with respect to Φ and ℓ and we call $M_{\,\mathrm{ord}}$ an ℓ-ordinary Φ module. Of course the functor $M \to M_{\,\mathrm{ord}}$ is exact.

This has some nice consequences for our considerations above. Since the functor $X \to X_{\,\mathrm{int}}$ is not exact the surjectivity in (3.82) is problematic, because $H^2_c(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \neq 0$. But if we localise our fundamental exact sequence

$$H^1_c(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{(\ell)}) \to H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{(\ell)}) \to H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}_{(\ell)}) \to H^2_c(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}_{(\ell)}) \tag{3.90}$$

and choose for Φ the Hecke operator T_ℓ then it follows from our computations in section 3.3.1 that T_ℓ acts nilpotently on $H^2_c(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}_{(\ell)})$, and therefore $H^2_{c,\,\mathrm{ord}}(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}_{(\ell)}) = 0$. We get the exact sequence

$$H^1_{c,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{(\ell)}) \to H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{(\ell)}) \to H^1_{\mathrm{ord}}(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}_{(\ell)}) \to 0. \quad (3.91)$$

It follows from our earlier computations (prop. 3.3.1) that $H^1_{\mathrm{ord}}(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}_{(\ell)}) = \mathbb{Z}_{(\ell)}[Y^n]$. Then we get for all Hecke operators $T_p[Y^n] = (p^{n+1}+1)[Y^n]$, we denote this Hecke-module by $\mathbb{Z}_{(\ell)}[n]$.

Now we can replace the sequence (3.82) by the above sequence if we want to study the power $\ell^{\delta_\ell(n)} = \Delta_\ell(n)$ in $\Delta(n)$.

The $\mathrm{Gl}_2/\mathbb{Z}$ module $\mathcal{M}_n$ contains the submodule

$$\mathcal{M}^\flat_n = \{\sum a_\nu \binom{n}{\nu} X^\nu Y^{n-\nu} \mid a_\nu \in \mathbb{Z}\} \quad (3.92)$$

(see 4.1.1) , this is actually the smallest submodule of $\mathcal{M}_n$ which contains X^n. Then we consider the cohomology $H^\bullet(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_n)$ and again we can ask for the denominator of the Eisenstein class. Here the method of localising at ℓ provides a simple answer. We consider the exact sequence of coefficients

$$0 \to \tilde{\mathcal{M}}^\flat_n \otimes \mathbb{Z}_{(\ell)} \to \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)} \to \tilde{\mathcal{M}}_n/\tilde{\mathcal{M}}^\flat_n \otimes \mathbb{Z}_{(\ell)} \to 0.$$

Now it follows easily from the definition that the Hecke operator T_ℓ acts nilpotently on the cohomology modules $H^\bullet(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n/\tilde{\mathcal{M}}^\flat_n \otimes \mathbb{Z}_{(\ell)})$ and hence we see that

$$H^1_{\mathrm{ord},?}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_n \otimes \mathbb{Z}_{(\ell)}) \xrightarrow{\sim} H^1_{\mathrm{ord},?}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)}) \quad (3.93)$$

is an isomorphism. This implies that the denominator of the Eisenstein class does not depend on the choice of the coefficient system.

At this point it seems to be appropriate to use some homological algebra. We consider the exact sequence of modules for the Hecke algebra $\mathcal{H}$

$$\mathcal{X}_n := 0 \to H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)}) \to H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)}) \quad \to \mathbb{Z}_{(\ell)}[n] \to 0. \quad (3.94)$$

We consider the sequence $\mathrm{Hom}_{\mathcal{H}}(\mathbb{Z}_{(\ell)}[n], \mathcal{X}_n)$ which is not exact anymore, this sequence yields a long exact sequence, we are interested in the boundary map $\boxed{\text{Ext}}$

$$\to \mathrm{Hom}_{\mathcal{H}}(\mathbb{Z}_{(\ell)}[n], H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)}) \to$$

$$\to \mathrm{Hom}_{\mathcal{H}}(\mathbb{Z}_{(\ell)}[n], \mathbb{Z}_{(\ell)}[n]) \xrightarrow{\delta} \mathrm{Ext}^1_{\mathcal{H}}(\mathbb{Z}_{(\ell)}[n], H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)})) \to \quad (3.95)$$

It is clear that the boundary map δ maps the identity element $\mathbf{1} \in \mathrm{Hom}_{\mathcal{H}}(\mathbb{Z}_{(\ell)}[n], \mathbb{Z}_{(\ell)}[n])$ to an element of order $\Delta_\ell(n)$, in other words ∂_1 maps $\mathrm{Hom}_{\mathcal{H}}(\mathbb{Z}_{(\ell)}[n], \mathbb{Z}_{(\ell)}[n])$ to a cyclic subgroup of the Ext^1 of order $\Delta_\ell(n)$.

We introduce the Eisenstein ideal $\mathcal{IE} \subset \mathcal{H}$, this is the ideal which is generated the elements $(p^{n+1}+1)\mathrm{Id}-T_p$, where p runs through all primes. It is not difficult to see that

$$\begin{gathered}\mathrm{Ext}^1_{\mathcal{H}}(\mathbb{Z}_{(\ell)}[n], H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)}) = \\ H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)})/\mathcal{IE}H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)}).\end{gathered} \tag{3.96}$$

Now we choose a prime p and look at the sub algebra $\mathbb{Z}[T_p] \subset \mathcal{H}$ which is generated by the Hecke operator T_p. We consider the exact sequence (3.95) but we change the subscript $\mathcal{H}$ to $Z[T_p]$. As before the map

$$\mathrm{Hom}_{\mathbb{Z}[T_p]}(\mathbb{Z}_{(\ell)}[n], \mathbb{Z}_{(\ell)}[n]) \xrightarrow{\delta} \mathrm{Ext}^1_{\mathbb{Z}[T_p]]}(\mathbb{Z}_{(\ell)}[n], H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)})) \tag{3.97}$$

maps the identity element $\mathbf{1}$ to an element to an element of order $\Delta_\ell(n)$. But now it follows from the definition of δ_1 that

$$\delta(\mathbf{1}) = \mathbf{t}_n^{(p)} = \{\dots, t^{(p)}_{n,\nu}, \dots\}_\nu \mod (\mathrm{Id}(p^{n+1}+1) - T_p)H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)})) \tag{3.98}$$

Hence we see that we simply have to compute the order of $\mathbf{t}_n^{(p)}$ in $\mathrm{Ext}^1_{\mathbb{Z}[T_p]]}\mathbb{Z}_{(\ell)}[n], H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n\otimes \mathbb{Z}_{(\ell)}))$ for just one prime p.

3.3.8 Computing $\mod \ell$

Of course the coefficients $t^{(p)}_{\nu,\mu}$ will become very large if n or p becomes larger, hence we can verify (3.89) only in a very small range of degrees n. But if we only want to verify that $\ell|\Delta_\ell(n)$ then it is sufficient to compute the coefficients $t^{(p)}_{\nu,\mu}$ modulo ℓ and to check whether $\mathbf{t}_n^{(p)}$ represents a non zero class in $\mathrm{Ext}^1_{\mathbb{Z}[T_p]]}(\mathbb{F}_\ell[n], H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{F}_\ell)$. Hence we see: We have $\ell|\Delta_\ell(n)$ if for some choice of p the equation

$$\sum_{\nu=n-1}^{\nu\equiv n-2s} ((p^{n+1}+1)\delta_{\nu,\mu} - t^{(p)}_{\nu,\mu})x_\nu \equiv t^{(p)}_{n,\mu} \mod \ell \tag{3.99}$$

has no solution. But now the coefficients are elements in $\mathbb{F}_\ell$ and this reduces the computational complexity considerably.

We have to be careful, it may and will happen that $t^{(p)}_{n,\mu} \mod \ell$ is zero (for some values of p) but still $\ell|\Delta_\ell(n)$.

$\boxed{\text{higher}}$

Higher powers of ℓ

This reasoning can also be applied if we look at higher powers of ℓ dividing a numerator of a $\zeta(-1-n))$. Let us assume that $\ell^{\delta_\ell(n)}||\mathrm{numerator}(\zeta(-1-n))$. We have to show that $\ell^{\delta_\ell(n)}$ divides the lcm of the denominators of the x_ν in equation (3.88). If we assume that $\mathbf{t}_n^{(p)}$ is not zero modulo ℓ then this follows if we show that the equation

$$\sum_{\nu=n-1}^{\nu\equiv n-2s}((p^{n+1}+1)\delta_{\nu,\mu}-t^{(p)}_{\nu,\mu})x_\nu \equiv \ell^{\delta_\ell(n)-1}t^{(2)}_{n,\mu} \mod \ell^{\delta_\ell(n)} \tag{3.100}$$

has no solution in $\mathbb{Z}/(\ell^{\delta_\ell(n)})$. Then the class

$$[\mathcal{X}_n\otimes\mathbb{Z}/\ell^{\delta_\ell(n)}\mathbb{Z}]\in \mathrm{Ext}^1_{\mathcal{H}}((\mathbb{Z}/\ell^{\delta_\ell(n)}\mathbb{Z})(-1-n), H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes(\mathbb{Z}/\ell^{\delta_\ell(n)}\mathbb{Z}))$$

has exact order $\ell^{\delta_\ell(n)}$.

If ℓ is an irregular prime, then there is always an even positive integer n_0 such that $n_0<p-1$ such that $\ell|\zeta(-1-n_0)$. One does not know any pair(ℓ,n) with $n_0<\ell-1$ such that we even have $\ell^2|\zeta(-1-n_0)$. But if we drop the assumption $n<\ell-1$ then we may find arbitrary high powers of ℓ dividing $\zeta(-1-n)$ (See also section 3.3.11) We have some examples

$$\zeta(-31)\equiv 0 \mod 37;\ \zeta(-283)\equiv 0 \mod 37^2;$$

$$\zeta(-37579)\equiv 0 \mod 37^3;\ \zeta(-1072543)\equiv 0 \mod 37^4;\dots$$

$$\zeta(-43)\equiv 0 \mod 59;\ \zeta(-913)\equiv 0 \mod 59^2$$

$$\zeta(-23)\equiv 0 \mod 103\ ;\zeta(-227)\equiv 0 \mod 103^2$$

We verified (3.89) in the cases $(37,282),(59,912),(103,226)$ using our program with Gangl. The case $(59,912)$ used roughly 18 hours, our algorithm becomes very slow if n becomes large. denomcong

3.3.9 The denominator and the congruences

For the following we assume that (3.89) is correct. We discuss the denominator of the Eisenstein class in this special case. In [Talk-Lille] this is discussed in a more abstract way, so here we treat basically the simplest example of 4.3 in [Talk-Lille]. Remember that in this section $\tilde{\mathcal{M}}=\tilde{\mathcal{M}}^\flat_n$, or $\tilde{\mathcal{M}}=\tilde{\mathcal{M}}_n$ for some even positive integer n.

The fundamental exact sequence provides the short exact sequence fuex

$$0\to H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})\to H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})\xrightarrow{r} H^1_{\mathrm{int}}(\partial(\Gamma\backslash\mathbb{H}),\tilde{\mathcal{M}})\to 0 \tag{3.101}$$

It is clear that the restriction map r is surjective because it is surjective if we localise at primes. We have $H^1_{\mathrm{int}}(\partial(\Gamma\backslash\mathbb{H}),\tilde{\mathcal{M}})=\mathbb{Z}e_n$ and $T_2(e_n)=(2^{n+1}+1)e_n$. We get a saturated decomposition into Hecke modules

$$H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})\oplus\mathbb{Z}\tilde{e}_n\subset H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}) \tag{3.102}$$

where $T_2\tilde{e}_n=(2^{n+1}+1)\tilde{e}_n$ and $r(\tilde{e}_n)=\Delta(n)e_n$ and

$$H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})/(H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})\oplus\mathbb{Z}\tilde{e}_n)=\mathbb{Z}/\Delta(n)\mathbb{Z}. \tag{3.103}$$

If $e_n^\dagger \in H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})$ maps to e_n then we can write

$$e_n^\dagger = r(\frac{y'+\tilde{e}_n}{\Delta(n)}) \tag{3.104}$$

and the element $y' \in H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})$ is unique up to an element in $\Delta(n)H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})$.

Hence

Theorem 3.3.1. *The Hecke module $H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})\otimes\mathbb{Z}/\Delta(n)\mathbb{Z}$ contains a cyclic submodule $\mathbb{Z}/\Delta(n)\mathbb{Z}(-1-n)$ on which for all primes p the Hecke operator T_p acts by the eigenvalue $p^{n+1}+1 \mod \Delta(n)$*

Proof. The submodule is simply the cyclic submodule generated by y'. □

We discuss some consequences of this theorem. We anticipate some results from the following chapter and from chapter 5. These results can be formulated in terms of the concepts and the language we used up to here, but the proofs require tools from analysis.

We mentioned already the theorem that the cohomology $H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes\mathbb{Q})$ is semi-simple as module for the Hecke-algebra(Thm.3.1.1). This theorem implies that we can find a finite normal field extension $F/\mathbb{Q}$ such that we have an isotypical decomposition (see3.19) $\boxed{\text{decoFint}}$

$$H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})\otimes F = \bigoplus_{\pi_f} H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes F)(\pi_f). \tag{3.105}$$

Here π_f runs over a finite set of homomorphisms $\pi_f:\mathcal{H}\to\mathcal{O}_F$. We also have the action of the complex conjugation on the cohomology (See sect. (2.84)). The complex conjugation commutes with the action of the Hecke algebra and under this action each eigenspace decomposes into a $+$ and a $-$eigenspace. In the following chapter 4 we will prove the famous multiplicity one theorem which says that the spaces $H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes F)(\pi_f)_\pm$ are one dimensional. Let us denote the set of $\pi_f:\mathcal{H}\to\mathcal{O}_F$ which occur with positive multiplicity (then 2) in the above decomposition by $\mathrm{Coh}_!(n)$.

We know that

$$H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})\otimes F = H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})\otimes F\oplus Fe_n$$

where $T_pe_n=(p^{n+1}+1)e_n$. Let $\pi_f^{\mathrm{Eis}}:\mathcal{H}\to\mathbb{Z}$ be the homomorphism $\pi_f^{\mathrm{Eis}}: T_p\to p^{n+1}+1$, then $\mathrm{Coh}(n)=\mathrm{Coh}_!(n)\cup\{\pi_f^{\mathrm{Eis}}\}$.

We make a list $\{\pi_{1,f},\dots,\pi_{r,f}\}$ of the elements in $\mathrm{Coh}_!(n)$. . This decomposition induces a Jordan-Hölder filtration on the integral cohomology $\boxed{\text{JH}}$

$$(0)\subset\mathcal{JH}^{(1)}H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{\mathcal{O}_F})\subset\mathcal{JH}^{(2)}H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{\mathcal{O}_F})\subset\cdots\subset\mathcal{JH}^{(r)}H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{\mathcal{O}_F}) \tag{3.106}$$

Here the first step $\mathcal{JH}^{(1)}H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{\mathcal{O}_F})=H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes|\mathcal{O}_F)(\pi_{1,f})$, where the subquotients a locally free $\mathcal{O}_F$ modules of rank 2 and after tensoring with F they become isomorphic to the corresponding $\pi_{j,f}$ eigenspace.

We choose a prime ℓ which divides $\Delta(n)$, let $\ell^{\delta_\ell(n)}||\Delta(n)$. Let $\mathfrak{l}$ be a prime in $\mathcal{O}_F$ which lies above ℓ. If e_ℓ is the ramification index then we have

$$\{0\} \subset \mathcal{O}_F/\mathfrak{l}^{e_\ell\delta_\ell(n)}(-1-n) \subset H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\mathcal{O}_F}) \otimes \mathcal{O}_F/\mathfrak{l}^{e_\ell\delta_\ell(n)} \tag{3.107}$$

The above Jordan-Hölder filtration induces a Jordan-Hölder filtration on the cohomology mod $\mathfrak{l}^{e_\ell\delta_\ell(n)}$ we have JHmod

$$\{0\} \subset \mathcal{JH}^{(1)}H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\mathcal{O}_F}) \otimes \mathcal{O}_F/\mathfrak{l}^{e_p\delta_\ell(n)} \subset \mathcal{JH}^{(2)}\dots \tag{3.108}$$

where again the successive subquotients $\overline{\mathcal{JH}^{(\nu)}}H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\mathcal{O}_F})$ are free $\mathcal{O}_F/\mathfrak{l}^{e_p\delta_\ell(n)}$ modules of rank 2.

cong1

Theorem 3.3.2. *We can find $\pi_{f,1}, \pi_{f,2}\dots, \pi_{f,r}$ and numbers $f_1 > 0, f_2 > 0, \dots, f_r > 0$ in the above filtration such that $\sum f_i = e_\ell\delta_\ell(n)$ and we have the congruence*

$$\pi_{f,i}(T_p) \equiv p^{n+1} + 1 \mod \mathfrak{l}^{f_i} \tag{3.109}$$

for all primes p.

Proof. We look at the map from our cyclic submodule into the top Jordan-Hölder quotient

$$\mathcal{O}_F/\mathfrak{l}^{e_\ell\delta_\ell(n)}(-1-n) \to \overline{\mathcal{JH}^{(r)}}H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\mathcal{O}_F}) \tag{3.110}$$

This map has a kernel $\mathfrak{l}^{g_r}$ and the image in the Jordan-Hölder quotient is the cyclic sub module $\mathfrak{l}^{e_\ell\delta_\ell(n)-g_r} = \mathfrak{l}^{f_r}$. The Hecke operator T_p acts on the Jordan-Hölder quotient quotient by multiplication by $\pi_{f,r}(T_p)$ and on the cyclic submodule by multiplication by $p^{n+1}+1$. Hence we get $\pi_{f,r}(T_p) \equiv p^{n+1}+1 \mod \mathfrak{l}^{f_r}$ Now we get an embedding $\mathfrak{l}^{g_r} \hookrightarrow \mathcal{JH}^{(r-1)}H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\mathcal{O}_F}) \otimes \mathcal{O}_F/\mathfrak{l}^{e_\ell\delta_\ell(n)}$ and we apply the same reasoning to this embedding. This process stops if the embedded cyclic sub module becomes trivial. This proves the claim. □

But now we have to be aware that the Jordan-Hölder filtration does not split, if we define

$$H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\mathcal{O}_F})(\pi_f) = H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\mathcal{O}_F}) \cap H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes F)(\pi_f)$$

then we get a saturated decomposition (decomposition up to isogeny) satdeco

$$H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\mathcal{O}_F}) \supset \bigoplus_{\pi_f \in \mathrm{Coh}_!(n)} H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\mathcal{O}_F})(\pi_f) \tag{3.111}$$

Here we encounter another interesting problem:

What can we say about the structure of the quotient if we divide the left hand side by the right hand side.

We can formulate some more or less plausible assertions which we can verify experimentally, but which are very difficult to prove. We definitely have to use methods which go far beyond the very elementary tools we used so far. For instance we verified experimentally (3.89) for a certain range of values of n but our proof in Chapter 5 requires some analysis.

in the following we choose a prime p, the role of the two primes p and ℓ will be exchanged. In a first step we consider the cohomology $\mod p$ we are mainly interested in the ordinary part. We start from the exact sequence of Γ modules

$$0 \to \mathcal{M}_n \xrightarrow{\times p} \mathcal{M}_n \to \mathcal{M}_n \otimes \mathbb{F}_p \to 0. \tag{3.112}$$

Here we want to assume that $p > 3$ then we get the resulting exact sequence of sheaves and hence a long exact sequence of cohomology groups

$$\begin{gathered} 0 \to (\mathcal{M}_n^\Gamma)_{\text{ ord}} \xrightarrow{\times p} (\mathcal{M}_n^\Gamma)_{\text{ ord}} \to (\mathcal{M}_n \otimes \mathbb{F}_p)_{\text{ ord}} \to \\ \to H^1_{\text{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n) \xrightarrow{\times p} H^1_{\text{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n) \to H^1_{\text{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{F}_p) \to 0 \end{gathered} \tag{3.113}$$

and we can break this sequence into pieces

$$0 \to (\mathcal{M}_n^\Gamma)_{\text{ ord}} \xrightarrow{\times p} (\mathcal{M}_n^\Gamma)_{\text{ ord}} \to (\mathcal{M}_n \otimes \mathbb{F}_p)^\Gamma_{\text{ord}} \to H^1_{\text{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n)[p] \to 0 \tag{3.114}$$

and

$$0 \to H^1_{\text{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n)[p] \to H^1_{\text{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n) \xrightarrow{\times p} H^1_{\text{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n) \to H^1_{\text{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{F}_p) \to 0 \tag{3.115}$$

where of course $\dots[p]$ means kernel of the multiplication by p and the far most 0 on the right is the vanishing of H^2.

We analyse these two sequences and get

$\boxed{\text{ordtorfree}}$

Theorem 3.3.3. *The cohomology $H^1_{ord}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n)$ is p- torsion free unless we have $n > 0$ and $n \equiv 0 \mod p(p-1)$. The cohomology groups $H^1_{c,\ ord}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n)$ are always torsion free and $H^2_{c,\ ord}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n) = 0$*

Proof. We consider the polynomial ring in two variables $\mathbb{F}_p[X, Y]$. On this ring we have the action of $\mathrm{Sl}_2(\mathbb{Z})$. It is an old theorem of L.E. Dickson that the ring of invariants is generated by the two polynomials

$$f_1 = X^pY - XY^p \text{ and } f_2 = \frac{X^{p^2-1} - Y^{p^2-1}}{X^{p-1} - Y^{p-1}} = X^{(p-1)p} + X^{(p-1)(p-1)}Y^{p-1} + \dots \tag{3.116}$$

Now every element in $(\mathcal{M}_n \otimes \mathbb{F}_p)^\Gamma_{\text{ord}}$ is a sum of monomials $f_1^a f_2^b$ where $a(p+1) + bp(p-1) = n$. We see that

$$u_{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}} = u_\alpha : \mathcal{M}_n^{(\alpha)} \to \mathcal{M}_n$$

multiplies f_1 with a multiple of p and hence we see that all the monomials with $a > 0$ are multiplied by a multiple of p. This means that $(\mathcal{M}_n \otimes \mathbb{F}_p)^\Gamma_{\text{ord}} \neq 0$

if and only if $n = bp(p-1)$. If $n = 0$ the map $\mathcal{M}_n^\Gamma = \mathbb{Z}_p \to (\mathcal{M}_n \otimes \mathbb{F}_p)^\Gamma$ is surjective if $n > 0$ we have $\mathcal{M}_n^\Gamma = 0$ and hence the theorem.

For the assertions concerning the compactly supported cohomology we have to recall that $H_c^2(\Gamma\backslash\mathbb{H}, \mathcal{M}_n) = (\mathcal{M}_n)_\Gamma = \mathcal{M}_n/I_\Gamma\mathcal{M}_n$ [book vol I, section 2 and 4.8.5]. We check easily that $X^n, Y^n \in I_\Gamma\mathcal{M}_n$ and the assertion is clear. $\square$

We now briefly discuss some interesting questions concerning the cohomology groups $H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n\otimes\mathbb{F}_p)$. We assume that we are not in the exceptional case that $n \equiv 0 \mod p(p-1)$ hence we know that

$$H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{F}_p) = H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n) \otimes \mathbb{F}_p \tag{3.117}$$

We can find a finite extension $\mathbb{F}_{p^r}/\mathbb{F}_p$ such that $\boxed{\text{decomodp}}$

$$H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{F}_{p^r}) = \bigoplus_{\bar\pi_f} H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{F}_{p^r})\{\bar\pi_f\} \tag{3.118}$$

where $\bar\pi_f : \mathcal{H} \to \mathbb{F}_{p^r}$ is a homomorphism and

$$H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{F}_{p^r})\{\bar\pi_f\} = \{x \in H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{F}_{p^r}) \mid (T_\ell - \bar\pi_f(T_\ell))^N x = 0\} \tag{3.119}$$

is a generalised eigenspace. Such an generalised eigenspace has a *socle*, this is the space

$$H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{F}_{p^r})(\pi_f) = \{x \in H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{F}_{p^r}) \mid (T_\ell - \bar\pi_f(T_\ell))x = 0\} \tag{3.120}$$

(Note the difference between ( ) and { }.) We know that the inner cohomology $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Q})$ is a semi simple Hecke module, but we can not expect anymore that the inner cohomology $H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{F}_{p^r})$ is semi-simple. For the restriction oft the decomposition (3.118) to the inner cohomology semi simplicity means that the generalised eigenspaces are always equal to their socle.

We choose a prime $\mathfrak{p} \subset \mathcal{O}_F$ above p and assume $\mathcal{O}_F/\mathfrak{p} = \mathbb{F}_{p^r}$. Let $F_\mathfrak{p}$ be the completion of F at $\mathfrak{p}$, let $\mathcal{O}_\mathfrak{p}$ its ring of integers. We consider the reduction maps $\boxed{\text{redukdiag}}$

$$\begin{array}{ccc} H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathcal{O}_\mathfrak{p}) & \to H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{F}_{p^r}) \\ \downarrow & \downarrow \\ H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathcal{O}_\mathfrak{p}) & \to H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{F}_{p^r}) \end{array} \tag{3.121}$$

under this map an eigenspace maps into the socle,, i.e.

$$r_{\pi_f} : H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathcal{O}_\mathfrak{p})(\pi_f) \to H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{F}_{p^r})(\bar\pi_f). \tag{3.122}$$

The image $r_{\pi_f}(H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathcal{O}_\mathfrak{p})(\pi_f))$ is a $\mathbb{F}_{p^r}$ vector space of dimension 2.

We are interested in the fibres of the surjective map

$$R_\mathfrak{p} : \mathrm{Coh}_!(n) \to \mathrm{Cohmodp}_!(n)\ ;\ \pi_f \mapsto \bar\pi_f$$

For a subset $\Sigma \subset \{R_{\mathfrak{p}}^{-1}\bar{\pi}_f\}$ we define

$$H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathcal{O}_{\mathfrak{p}})\{\Sigma\} = (\bigoplus_{\pi_f\in\Sigma} H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n\otimes F)(\pi_f))\cap H^1_{!\,ord}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n\otimes\mathcal{O}_{\mathfrak{p}}). \tag{3.123}$$

Given Σ we put $\Sigma' = R_{\mathfrak{p}}^{-1}(\bar{\pi}_f)\setminus\Sigma$ and we say that Σ (or Σ') is *closed* if we get a direct sum decomposition

$$H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n\otimes\mathcal{O}_{\mathfrak{p}}) = H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n\otimes\mathcal{O}_{\mathfrak{p}})\{\Sigma\}\oplus H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n\otimes\mathcal{O}_{\mathfrak{p}})\{\Sigma'\} \tag{3.124}$$

and consequently we say that the fibre $R_{\mathfrak{p}}^{-1}(\bar{\pi}_f)$ (or simply $\bar{\pi}_f$) is *connected* if $\emptyset$ and the fibre itself are the only closed subsets. If π_f and $\pi'_f \in R_p^{-1}(\bar{\pi}_f)$ then we say that they are *inner congruent* . We have an easy proposition

Proposition 3.3.4. *If $R_{\mathfrak{p}}^{-1}(\bar{\pi}_f)$ is not connected, then the dimension of the socle $H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n\otimes\mathbb{F}_{p^r})^{(1)}\{\bar{\pi}_f\}$ over $\mathbb{F}_{p^r}$ is ≥ 4*

Proof. This is rather clear. If we have a non trivial direct sum decomposition as above then we also get a direct sum decomposition

$$H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n)\otimes\mathbb{F}_{p^r} = H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n)\otimes\mathbb{F}_{p^r}\{\Sigma\}\oplus H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n)\otimes\mathbb{F}_{p^r}\{\Sigma'\} \tag{3.125}$$

Now any $\pi_f\in\Sigma$ (resp.$\pi'_f\in\Sigma'$) provides provides a two dimensional $\mathbb{F}_{p^r}$ vector space $r_{\pi_f}(H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n\otimes\mathcal{O}_F)(\pi_f))$ (resp $r_{\pi'_f}(H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n\otimes\mathcal{O}_F)(\pi'_f))$. These two vector spaces lie in the socle and in two different summands. □

We say that $\bar{\pi}_f$ occurs *with weak multiplicity one* if the dimension of the socle $\dim H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n\otimes\mathbb{F}_{p^r})(\bar{\pi}_f) = 2$, this socle is the direct sum of the $\pm$ eigenspaces under the complex conjugation. Our above theorem implies that then the fibre of $\bar{\pi}_f$ must be connected.

The (plain) multiplicity of $\bar{\pi}_f$ is just the number of elements in the fibre $R_{\mathfrak{p}}^{-1}(\bar{\pi}_f)$ this is the number $m(\bar{\pi}_f) = \frac{1}{2}\dim H^(_!\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes\mathbb{F}_2)\{\bar{\pi}_f\}$. One might expect that a $\bar{\pi}_f$ occurs with weak multiplicity one happens "very frequently". But it seems to be very difficult to say something substantial in this direction. We will come back to this issue.(See discussion of the Wieferich dilemma).

We are especially interested in the case of the *Eisenstein homomorphism* $\bar{\pi}_f^{\mathrm{Eis}} : T_\ell \to \ell^{n+1}+1 \mod p$. It certainly occurs in the cohomology $\mod p$ and it follows from theorems 3.3.1 and 5.1.2 that $\bar{\pi}_f^{\mathrm{Eis}}$ occurs in $\mathrm{Cohnodp}_!(n)$ if $p|\zeta(-1-n)$.

Here it is very tempting to ask whether or not $\bar{\pi}_f^{\mathrm{Eis}}$ always occurs with weak multiplicity one in the inner cohomology.

This question can be checked experimentally. For any prime ℓ we look at the operator T_ℓ on $H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n\otimes\mathbb{F}_p)$. We compute the characteristic polynomial

$$P_\ell(X) = \det(X\mathrm{Id} - ((\ell^{n+1}+1)\mathrm{Id} - T_\ell) \mid H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n)\otimes\mathbb{F}_p) = Q_\ell(X)^2 \tag{3.126}$$

The characteristic polynomial is a square because the $\pm$ eigenspaces are isomorphic as Hecke modules. Our computer program with Gangl gives us an explicit expression for $Q_2(X)$ for a large number of pairs (n,p). We are interested in pairs (n,p) with $p \mid \zeta(-1-n)$. Then we find

$$Q_2(X) = a_1(n,p)X + a_2(n,p)X^2 \dots \tag{3.127}$$

Then we found $a_1(n,p) \not\equiv 0 \mod p$ for $n \leq 200$. This means that in these cases $\bar{\pi}_f^{\mathrm{Eis}}$ occurs with multiplicity one. This is in no way surprising, we expect that $p|\zeta(-1-n)$ and $p|a_1(n,p)$ will be a very rare event.

Tobias Berger drew my attention to the paper [3] where the authors consider the same problem in a slightly different context. They show that $p|a_1(n,p)$ happens only once for $p < 10^5$ and this is the case $p = 547, n = 484$.

Assume we have such a pair (p,n) and we know that in addition that $p \not| a_2(n,p)$. Then the Hecke operator $T_\ell^{\mathrm{Eis}} = T_\ell - (\ell^{n+1}+1)\mathrm{Id}$ acts nilpotently on the 4 dimensional space $H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n) \otimes \mathbb{F}_p)\{\bar{\pi}_f^{\mathrm{Eis}}\}$ and we can ask the next question:

Is T_ℓ^{Eis} the zero operator? (For all choices of ℓ). Then this means that $\bar{\pi}_f^{\mathrm{Eis}}$ has weak multiplicity 2.

If we restrict T_ℓ^{Eis} to one of the two-dimensional $\pm$ subspaces then T_ℓ^{Eis} is a nilpotent endomorphism, i.e. a nilpotent (2,2) matrix with entries in $\mathbb{F}_p$. If I am not mistaken then there are exactly p^2 such matrices and the zero matrix is just one of them. So one might argue that the probability for $T_\ell^{\mathrm{Eis}} = 0$ is roughly $\frac{1}{p^2}$. So with a high probability the answer to the above question is NO. Since the probability that $p|a_1(n,p)$ is also very small it may be safe to conjecture that $\bar{\pi}_f^{\mathrm{Eis}}$ always occurs weakly with multiplicity one.

Of course I checked the case $(484, 547)$ for the operator T_2 and indeed the answer was NO!

We briefly return to theorem 3.3.2. Assume we have a pair (n,p) with $p^\delta|\zeta(-1-n)$ and in addition $p|a_1(n,p)$, we also assume $p \not| a_2(n,p)$, Then we expect that $\bar{\pi}_f^{\mathrm{Eis}}$ occurs weakly with multiplicity one. Let $F_{\mathfrak{p}}/\mathbb{Q}_p$ a smallest extension such that we get a decomposition

$$H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes F_{\mathfrak{p}}) = H^1_1(\mathbb{H}, \tilde{\mathcal{M}}_n \otimes F_{\mathfrak{p}})(\pi_{f,1}) \oplus H^1_!(\mathbb{H}, \tilde{\mathcal{M}}_n \otimes F_{\mathfrak{p}})(\pi_{f,2}).$$

This extension is either trivial or a ramified quadratic extension of $\mathbb{Q}_p$. The probability that we are in the first case is again very low, so let us assume that $F_{\mathfrak{p}} = \mathbb{Q}_p[\sqrt{p}]$ We have the inclusion $j : \mathbb{Z}/p^\delta(-n-1) \hookrightarrow H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_p/p^\delta)$, The filtration has 2-steps

$$\{0\} \subset H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_p/p^\delta)(\pi_{f,1}) \subset H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_p/p^\delta) \tag{3.128}$$

and let f_1 the smallest integer such that $p^{f_1}\mathbb{Z}/p^\delta(-n-1) \hookrightarrow H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_p)(\pi_{f,1})$. Then our previous argument yields the congruence $\pi_{f,1}(T_\ell) \equiv \ell^{n+1}+1 \mod p^{\delta-f_1}$. Then the inclusion j yields the cyclic submodule $\mathbb{Z}/p^{f_1}$ in the quotient. This yields the congruence $\pi_{f,2}(T_\ell) \equiv \ell^{n+1}+1 \mod p^{f_1}$ Now we invoke our assumption that $\bar{\pi}_f$ occurs weakly with multiplicity one and this implies that we can find a Hecke operator $T^{\rm Eis}_{\ell_1}$ which maps the above cyclic module $\mathbb{Z}/p^{f1}$ into $H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n\otimes\mathbb{Z}_p/p^\delta)(\pi_{f,1})$ and hence to the cyclic submodule $p^{f_1}Z/p^\delta\subset\mathbb{Z}/p^\delta)$ Hence we can conclude that $2f_1\le\delta$.

Hence we see we see that under the above assumptions we have a congruence

$$\pi_{f,1}(T_\ell)\equiv \ell^{n+1}+1 \mod \mathfrak{p}^\delta$$

Of course the same applies to $\pi_{f,2}$.

With a little bit of luck we can check the assumptions using the explicit computation of T_2.

It is not very difficult to produce examples of $\bar\pi_f\in \mathrm{Coh}_!(n)$ where we have $\dim H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n\otimes\mathbb{F}_{p^r})\{\bar\pi_f\}>2$. If we take $n=22$, then we know that $H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{12}\otimes\mathbb{Q})$ is of dimension four, then the $\pm$ eigenspaces are of dimension 2. We can compute T_2 and find for the smallest field that decomposes the cohomology (see 3.105) $F=\mathbb{Q}(\sqrt{144169})$, this was of course known to Hecke. In this case $\mathrm{Coh}_!(22)$ consists of 2 elements, which are conjugate under the Galois group $\mathrm{Gal}(F/\mathbb{Q})$. If π_f is one of the elements in $\mathrm{Coh}_!(22)$ then our program yields $\pi_f(T_2)=-12(-45+\sqrt{144169})$ (of course this value can be looked up in any table for modular forms) hence we see that for any prime $p>3$ the image of ring $\mathbb{Z}_{(p)}[T_2]=\mathcal{O}_F\otimes\mathbb{Z}_{(p)}.$ This implies that we do not have inner congruences except for $p=144169$.

But now our program with Gangl provides an explicit matrix for $T_2 \mod 144169$ and this matrix has only one eigenvalue $\mod 144169$.

This matrix for T_2 is not a diagonal matrix, and this implies that the fibre is connected and hence π_f occurs weakly with multiplicity one.

Essentially the same happens if we look at the six values $n=22,26,28,30,32,36$ for which the degree of the splitting field F is 2. It will happen that the discriminant is not a prime, so we will have inner congruences modulo several primes.

3.3.10 L-values, weak multiplicity one and connectedness

in the previous section we investigated questions regarding the structure of the integral cohomology as module for the Hecke algebra. We discussed the denominator of the Eisenstein classes and our experimental data suggested an answer in terms of special values of L-functions. (See 3.89 and also (5.1.2).)

We briefly mention some results about the questions concerning inner congruences which we contemplated in the previous section. To state these results we have to anticipate the notion of $L-$ functions attached to a $\pi_f\in\mathrm{Coh}_!(n)$ and we anticipate the theorems on special values. We still localise at a prime p and we only look at the ordinary part. We apply the results from section 5.1.2 .

Assume we have two elements $\pi_f,\pi'_f\in R_{\mathfrak{p}}^{-1}(\bar\pi_f)$ We say that these two elements are *linked* if they lie in the same connected component. I refer to Theorem 5.1.1

Theorem 3.3.4. *The Hecke modules $\pi_f,\pi'_f\in R_{\mathfrak{p}}^{-1}(\bar\pi_f)$ are linked if and only* *if*

$$\frac{Z(\nu)}{\Omega(\epsilon(\nu)\times\pi_f)}\Lambda^{\mathrm{coh}}(\pi,n+1-\nu)\equiv\frac{Z(\nu)}{\Omega(\epsilon(\nu)\times\pi'_f)}\Lambda^{\mathrm{coh}}(\pi',n+1-\nu)\mod\mathfrak{p} \tag{3.129}$$

for all $\nu=0,1,\ldots,n$ *and where* $Z(\nu)=1$ *for* $\nu\neq 0,n$ *and* $Z(0)=Z(n)=$ *numerator*$(\zeta(-1-n))$ *and* $\epsilon(\nu)=\pm$ *depending on the parity of* ν.

(The factor $Z(\nu)$ is needed to make the expressions integers) This is a slightly strengthened version of a theorem of Vatsal ([88]). We do not prove it in this book, it is not to difficult to prove using the results in ...

We have a second theorem which is due to Hida. We pick a $\pi_f\in R_{\mathfrak{p}}^{-1}(\bar\pi_f)$ and we say that π_f is *isolated* if $\{\pi_f\}$ is open. Then Hida's theorem says ([52])

Theorem 3.3.5. *The Hecke module* $\pi_f\in R_{\mathfrak{p}}^{-1}(\bar\pi_f)$ *is isolated if and only if*

$$\frac{\Lambda^{\mathrm{coh}}(\pi_f,\mathrm{Sym}^2,1)}{\Omega(+\otimes\pi_f)\Omega(-\otimes\pi_f)}\notin\mathfrak{p} \tag{3.130}$$

3.3.11 p-adic interpolation

$\boxed{\text{p-adic-zeta}}$

The p-adic ζ-function

Let p be an irregular prime, i.e. $p\mid\zeta(-1-n_0)$, here we assume $0<n_0<p-1$. We consider $\zeta(-1-n)=\zeta(-1-n_0-\alpha(p-1))$ as function in the variable $\alpha\in\mathbb{N}$ and we want to find values $n=-1-n_0-\alpha(p-1)$ such that $\zeta(-1-n)$ is divisible by higher powers of p. We know that there exist a p-adic $\zeta-$ function (See [58],[54],[90]) and this function has an expansion $\boxed{\text{p-appr}}$

$$\zeta(-1-n)=\zeta(-1-n_0-\alpha(p-1))\equiv\zeta(-1-n_0)+a(n_0,1)\alpha p+a(n_0,2)\alpha^2p^2\ldots \tag{3.131}$$

where the coefficients $a(n_0,\nu)\in\mathbb{Z}_p$ they are only defined $\mod p$. If now $p\not|a(n_0,1)$ then we can apply Newton‘s method and we find a converging sequence $\alpha_1,\alpha_2,\ldots$ such that

$$\alpha_\nu\equiv\alpha_{\nu+1}\mod p^\nu \text{ and } \zeta(-1-n_0-\alpha_\nu(p-1))\equiv 0\mod p^{\nu+1} \tag{3.132}$$

The sequence converges to a zero α_∞ of the p-adic $\zeta-$ function.

It is not always possible to raise the power of p which divides $\zeta(--1-n_0-\alpha(p-1))$. If for instance $p^2\not|\zeta(-1-n_0)$ and in addition $p|a(n_0,1)$ then we never find a higher power of p dividing some $\zeta(-1-n_0-\alpha(p-1))$. Again a naive probabilistic argument suggests that is an extremely rare event that this happens, but the argument also suggests that such a prime exists (See section on the Wieferich Dilemma).

Again we write $n = n_0 + (p-1)\alpha$ where we assume $0 < n_0 < p-1$ and we want to study how the cohomology $H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{Z}_p)$ varies with n. We know for instance that the denominator of the Eisenstein class may become larger. We have seen already that

$$H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n) \otimes \mathbb{Z}/p^r\mathbb{Z} \xrightarrow{\sim} H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{Z}/p^r) \tag{3.133}$$

Now we have the following theorem which is due to Hida

$\boxed{\text{interpol}}$

Theorem 3.3.6. *If $n = n_0 + (p-1)\alpha, n' = n_0 + (p-1)\alpha'$ and $\alpha \equiv \alpha' \mod p^{r-1}$, (i.e. $n \equiv n' \mod (p-1)p^{r-1}$) then we have a canonical Hecke invariant isomorphism*

$$\Phi(n,n')_r : H^1_{ord}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}/p^r) \xrightarrow{\sim} H^1_{ord}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n'} \otimes \mathbb{Z}/p^r). \tag{3.134}$$

This system of isomorphisms is consistent with change of the parameter α, α' and r.

Proof. See paper on interpolation. □

We find a finite extension $F_{\mathfrak{p}}/\mathbb{Q}_p$ such that we have a decomposition into eigenspaces

$$H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes F) = \bigoplus_{\pi_f} H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes F)(\pi_f) \oplus Fe_n \tag{3.135}$$

where the first summation on the right hand side goes over those $\pi_f \in \mathrm{Coh}_!(n)$ for which $\pi_f(T_p)$ is a unit in $\mathcal{O}_{\mathfrak{p}}$, the ring of integers in F. Let us denote this set by $\mathrm{Coh}^{(n)}_{!,\mathrm{ord}}$. Then the full summation goes over the set $\mathrm{Coh}^{(n)}_{\mathrm{ord}} = \mathrm{Coh}^{(n)}_{!,\mathrm{ord}} \cup \{\pi_f^{\mathrm{Eis}}\}$. Intersecting this decomposition with $H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathcal{O}_{\mathfrak{p}})$ gives us a submodule of finite index

$$H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathcal{O}_{\mathfrak{p}}) \supset \bigoplus_{\pi_f} H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathcal{O}_{\mathfrak{p}})(\pi_f) \oplus \mathcal{O}_{\mathfrak{p}}e_n \tag{3.136}$$

and this also gives us a Jordan-Hölder filtration as in (3.106).

For $\bar{\pi}_f \in \mathrm{Cohmodp}_!(n)$ we define

$$H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathcal{O}_{\mathfrak{p}})\{\bar{\pi}_f\} := H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathcal{O}_{\mathfrak{p}})\{R_{\mathfrak{p}}^{-1}\{\bar{\pi}_f)\} \tag{3.137}$$

and then we get a direct sum decomposition

$$H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathcal{O}_{\mathfrak{p}}) = \bigoplus_{\bar{\pi}_f} H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathcal{O}_{\mathfrak{p}})\{\bar{\pi}_f\} \tag{3.138}$$

and for any $\bar{\pi}_f$ we get a decomposition up to isogeny

$\boxed{\text{decoEis}}$

$$H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathcal{O}_{\mathfrak{p}})\{\bar{\pi}_f\} \supset \bigoplus_{\pi_f \in R_{\mathfrak{p}}^{-1}(\bar{\pi}_f)} H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathcal{O}_{\mathfrak{p}})(\pi_f) \tag{3.139}$$

We are mainly interested in the Hecke module $H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathcal{O}_{\mathfrak{p}})\{\bar\pi_f^{\mathrm{Eis}}\}$ how this varies if α varies. Our theorem above implies that the Hecke module $H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathcal{O}_{\mathfrak{p}}/\mathfrak{p})$ does not depend on α.

Let us now assume that $\bar\pi_f^{\mathrm{Eis}}$ has multiplicity one. Then we know that the decomposition (3.139) applied to $\bar\pi_f^{\mathrm{Eis}}$ has only one summand $\pi_{!,f}^{\mathrm{Eis}}$, We still assume that $n=n_0+(p-1)\alpha$,

$\boxed{\text{Vand}}$

Theorem 3.3.7. *If $p^{\delta_\ell(n)}\mid\zeta(-1-n)$ we have the congruence*

$$\pi_f(T_\ell)\equiv\ell^{n+1}+1\mod p^{\delta_\ell(n)}\ \forall\ \text{primes } \ell$$

Finally we get $\pi_f(T_\ell)\in\mathbb{Z}_p$ for all primes ℓ and hence we may take $\mathcal{O}_{\mathfrak{p}}=\mathbb{Z}_p$. We can find a basis f_0,f_1,f_2 of $H^1{}_{ord}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n)\{\bar\pi_f^{\mathrm{Eis}}\}$ where

a) f_1,f_2 form a basis of $H^1_{!,\,ord}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n)$ and f_0 maps to a generator of $H^1(\partial(\Gamma\mathbb{H}),\tilde{\mathcal{M}}_n\otimes\mathbb{Z}_p)$

b)The complex conjugation c acts by $c(f_i)=(-1)^{i+1}f_i$

and finally
c) the matrix $T_\ell{}^{ord}$ with respect to this basis satisfies

$$T_\ell{}^{ord}\equiv\begin{pmatrix}\ell^{n+1}+1 & 0 & t^{(\ell)}\\ 0 & \ell^{n+1}+1 & 0\\ 0 & 0 & \ell^{n+1}+1\end{pmatrix}\mod p^{\delta_\ell(n)}$$

Proof. Clear □

Now we assume that $p\not|a(1,n_0)$ and we choose a sequence $\alpha_0=0,\alpha_1,\dots,\alpha_\nu$ as in (3.132) then we get Hecke-module maps

$$H^1_{!,\mathrm{ord}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{n_{\nu+1}}\otimes\mathbb{Z}_p/p^{\nu+1})\to H^1_{!,\mathrm{ord}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{n_{\nu+1}}\otimes\mathbb{Z}_p/p^{\nu})\xrightarrow{\sim}H^1_{!,\mathrm{ord}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{n_\nu}\otimes\mathbb{Z}_p/p^{\nu})\tag{3.140}$$

The sequence n_ν converges to an p-adic integer n_∞, we can form the projective limit and define

$$H^1{}_{\mathrm{ord}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{n_\infty})=\varprojlim H^1{}_{\mathrm{ord}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{n_\nu}\otimes\mathbb{Z}/p^\nu\mathbb{Z})\tag{3.141}$$

Under our assumptions $H^1{}_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{n_\infty})\{\bar\pi_f^{\mathrm{Eis}}\}$ is a free $\mathbb{Z}_p$-module of rank 3. The Hecke operators $T_\ell{}^{\mathrm{ord}}$ acts on $H^1{}_{\mathrm{ord}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{n_\nu}\otimes\mathbb{Z}/p^\nu\mathbb{Z})$ by a matrix of the shape as in theorem 3.3.7, and the eigenvalues on the diagonal are

$$\ell^{n_\nu+1}+1=\ell^{n_0+(p-1)\alpha_\nu}+1\mod p^\nu$$

For $\ell\neq p$ we write

$$\ell^{p-1}=1+p\delta(\ell),\delta_p(\ell)\in\mathbb{N}$$

and then

$$\ell^{n_0+(p-1)\alpha_\nu}=\ell^{n_0}(1+\delta_p(\ell)p)^{\alpha_\nu}=l^{n_0}(1+\alpha_\nu p\delta_p(\ell)+\binom{\alpha}{2}\alpha_\nu^2p^2\delta_p(\ell)^2\dots$$

We see that we can define $\ell^{n_0+\alpha(p-1)}$ for any $\alpha \in \mathbb{Z}_p$ and then clearly $\lim_{\nu\to\infty} \ell^{n_\nu} = \ell^{n_\infty}$. Hence we see that $T_\ell{}^{\mathrm{ord}}$ acts on $H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_\infty})$ by the matrix

$$T_\ell{}^{\mathrm{ord}} \equiv \begin{pmatrix} \ell^{n_\infty+1}+1 & 0 & t^{(\ell)} \\ 0 & \ell^{n_\infty+1}+1 & 0 \\ 0 & 0 & \ell^{n_\infty+1}+1 \end{pmatrix}$$

where we have $t^{(\ell)} \neq 0$. (This follows from earlier arguments)

If we drop the assumption that $\bar{\pi}_f^{\mathrm{Eis}}$ has multiplicity one then the situation becomes definitely much more complicated. We believe that this a rare event, we have seen that for $p < 10^5$ this happens only once. But on the other hand our naive probabilistic argument suggests that it should happen again. But then the same probabilistic argument suggests that it never happens that $m(\bar{\pi}_f^{\mathrm{Eis}}) > 2$.

Therefore we make the assumption that $m(\bar{\pi}_f^{\mathrm{Eis}}) = 2$. We look at that the decomposition (3.139) applied to $\bar{\pi}_f = \bar{\pi}_f^{\mathrm{Eis}}$. We consider the characteristic polynomial

$$\det(T_\ell^{\mathrm{Eis}} - \mathrm{Id}X | H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}_p)\{\bar{\pi}_f^{\mathrm{Eis}})\} = (b_0(n,p) + b_1(n,p)X + X^2)^2. \tag{3.142}$$

Our assumption $m(\bar{\pi}_f^{\mathrm{Eis}}) = 2$ implies that $b_1(n,p), b_0(n,p) \equiv 0 \mod p$. Let us write

$$P^{\mathrm{Eis}}(T_\ell, n, , X) := b_0(n,p) + b_1(n,p)X + X^2 \tag{3.143}$$

If we now find an ℓ such that $p^2 \not| \, b_0(n,p)$, then it is clear that we need a quadratic extension $F_\mathfrak{p} = \mathbb{Q}_p[\sqrt{\epsilon p}]$ (ϵ a unit in $\mathbb{Z}_p^\times$) if we want a decomposition into eigen spaces (3.135). If $\mathcal{O}_\mathfrak{p}$ is the ring of integers in $F_\mathfrak{p}$ then we get a decomposition up to isogeny into two conjugate eigenspaces

$$H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathcal{O}_\mathfrak{p})/\{\bar{\pi}_f^{\mathrm{Eis}}\} \supset H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathcal{O}_\mathfrak{p})(\pi_f) \oplus H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathcal{O}_\mathfrak{p})({}^\sigma\pi_f) \tag{3.144}$$

where σ is the non trivial element in the Galois group of $F_\mathfrak{p}/\mathbb{Q}_p$.

Under our assumptions the quotient of the left hand side by the right hand side is isomorphic to $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathcal{O}_\mathfrak{p}/\mathfrak{p})(\bar{\pi}_f^{\mathrm{Eis}})$. For all ℓ' we have the congruence $\pi_f(T_{\ell'}) \equiv {}^\sigma\pi_f(T_{\ell'}) \mod \mathfrak{p}$. We recall that $n = n_0 + \alpha(p-1)$. It is clear that the condition $p^2 \not| b_0(n,p)$ only depends on $\alpha \mod p$. Therefore we should expect that for a fixed α the "probability" that $p^2 | \, b_2(n,p)$ is $1/p$. but there may be a value of α for which this divisibility holds.

We return to our sequence $\alpha_0, \alpha_1, \dots$ see((3.132)). We assume that $p^2 \not| \, b_0(n_0 + \alpha_1(p-1), p)$. Then it is easy to see that for $\nu \geq 1$ we have the decomposition up to isogeny (3.144)

$$H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_\nu} \otimes \mathcal{O}_\mathfrak{p})\{\bar{\pi}_f^{\mathrm{Eis}}\} \supset H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_\nu} \otimes \mathcal{O}_\mathfrak{p})(\pi_f^{(\nu)}) \oplus H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_\nu} \otimes \mathcal{O}_\mathfrak{p})({}^\sigma\pi_f^{(\nu)}) \tag{3.145}$$

We have the inclusion $\mathbb{Z}_p/(p^{\nu+1})(-n_\nu - 1) \hookrightarrow H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_\nu} \otimes \mathbb{Z}/(p^{\nu+1}))$ and tensoring by $\mathcal{O}_\mathfrak{p}/\mathfrak{p}^{2(n_\nu+1)}$ yields the inclusion

$$j_{\mathfrak{p},\nu} : \mathcal{O}_\mathfrak{p}/\mathfrak{p}^{2(\nu+1)}(-n_\nu - 1) \hookrightarrow H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_\nu} \otimes \mathcal{O}_\mathfrak{p}/(\mathfrak{p}^{2(\nu+1)}))$$

We have $H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n\otimes\mathcal{O}_\mathfrak{p}/(\mathfrak{p}^{(2(\nu+1)})))(\pi_f)\subset H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n\otimes\mathcal{O}_\mathfrak{p}/(\mathfrak{p}^{(2(\nu+1)}))$ this is the first step in the Jordan-Hölder filtration and the quotient by this first step is isomorphic to $H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n\otimes\mathcal{O}_\mathfrak{p}/(\mathfrak{p}^{(2(\nu+1)})))({}^\sigma\pi_f)$. We repeat the argument in the proof of Theorem 3.3.2 and we conclude that we have congruences

$$\pi_f^{(\nu)}(T_\ell)\equiv\ell^{n_\nu+1}+1\mod\mathfrak{p}^{(\nu+1)}\;;{}^\sigma\,\pi_f^{(\nu)}(T_\ell)\equiv\ell^{n_\nu+1}+1\mod\mathfrak{p}^{(\nu+1)}\tag{3.146}$$

Again we can pass to the limit $H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{n_\infty})\{\bar{\pi}_f^{\mathrm{Eis}}\}$, this limit has a three step Jordan-Hölder filtration, the top quotient is $\mathbb{Z}_p(-n_\infty-1)\otimes\mathcal{O}_\mathfrak{p}$ -this is the cohomology of the boundary. For the eigenvalues on the middle step and the bottom we also get the same limit . Hence -if we choose our basis as before, i.e. we take the action of the complex conjugation into account, then we get for the matrix of the Hecke operator-restricted to the Eisenstein part-

$$T_\ell^\infty=\begin{pmatrix}\ell^{n_\infty+1}+1&0&t^{(\ell)}&0&s^{(\ell)}\\0&\ell^{n_\infty+1}+1&0&u^{(\ell)}&0\\0&0&\ell^{n_\infty+1}+1&0&v^{(\ell)}\\0&0&0&\ell^{n_\infty+1}+1&0\\0&0&0&0&\ell^{n_\infty+1}+1\end{pmatrix}\tag{3.147}$$

where the non zero entries are units for a suitable choice of ℓ.

In the special case (484,547) the number $\alpha_1=100$ we have $\zeta(-485-100*546)=\zeta(-55085)\equiv 0\mod 547^2$. Earlier we checked $547^2\not| \;b_0(484,547)$ for the Hecke operator T_2. But how can we ever check $547^2\not|\; b_0(55084,547)$? The matrices become too big.

Our chances that $547^2\mid b_0(55084,547)$ **are 1/547.**

But there is a way out. I think it is possible to prove that we have an expansion

$$b_0(484+\alpha 546,547)\equiv b_0(484,547)+\alpha 547 b_0'(484,457)\mod 547^2.\tag{3.148}$$

(This should follow from the general results which we announced in [41] and which we still hope to prove in a paper with J. Mahnkopf).

Using our program with Gangl for T_2 and with the help of A.Weisse we computed $T_2\mod p^2$ for the cases $\alpha=0,1,2$,the program still works in reasonable time in these cases. We found (of course everything $\mod 547^2$)

$$\begin{aligned}b_0(484)&=547\times 10\\b_0(484+546,547)&=547\times 174=547(10+164),\\b_0(484+2\times 546,547)&=547\times 338=547(10+2\times 164)\end{aligned}\tag{3.149}$$

Hence we expect $b_0'(484,547)=164\mod 547$ and assuming the above linearity we get $b_0(55084,547)=547(10+100\times 164)=547\times 16410$.

To my great surprise $16410=2\times 3\times 5\times 547$!!!

Perhaps I was just stupid and a closer look shows that there is an obvious reason that this must be so

Hence we have to compute $P^{\text{Eis}}(T_2, 484 + a * 546, X) \mod 547^3$ for some small values of a. We computed the zeroes

$$\lambda(T_2, a) = \alpha * 457 \pm \sqrt{\beta * 547} \mod 547^3$$

of the quadratic factor $P^{\text{Eis}}(T_2, 484 + a * 546, X)$ for $a = 0, 1, 2$. We found

$$\begin{aligned}\lambda(T_2, 0) &= 268381 * 547 \pm \sqrt{537 * 547} \mod 547^3\\ \lambda(T_2, 1) &= 189064 * 547 \pm \sqrt{251993 * 547} \mod 547^3\\ \lambda(T_2, 2) &= 13475 * 547 \pm \sqrt{169232 * 547} \mod 547^3\end{aligned} \tag{3.150}$$

we see that these roots lie in a ramified quadratic extension of $Z/(547^3)$. From the roots we get the coefficients of $P^{\text{Eis}}(T_2, 484 + a * 546, X)$

$$\begin{aligned}b_0(484, 547) &= (268381 * 547)^2 - 537 * 547 = 37406595 \mod 547^3\\ b_0(484 + 546, 547) &= (189064 * 547)^2 - 251993 * 547 = 135636855 \mod 547^3\\ b_0(484 + 2 * 546, 547) &= (13475 * 547)^2 - 169232 * 547 = 91742840 \mod 547^3\\ b_1(484, 547) &= 2 * 268381 * 547 \mod 547^3\\ b_1(484 + 546, 547) &= 2 * 189064 * 547 \mod 547^3\end{aligned} \tag{3.151}$$

We now hope that in the still to be written paper with J. Mahnkopf we will show that we have an expansion

$$b_0(484 + \alpha 546, 547) = b_0(484, 547) + b_0'(484)\alpha 547 + b_0''(484)\alpha^2 547^2 \mod 547^3 \tag{3.152}$$

where $b_0'(484), b_0''(484)$ are numbers $\mod 547$, which we can compute from the three values above. We easily get

$$b_0'(484) = 164\ ;\ b_0''(484) = 24 \tag{3.153}$$

We do the same for $b_1(484 + \alpha * 546) = b_1(484) + 543 * 547 \mod 547^2$. Now the discriminant is Discriminant $\Delta(\alpha, 547) = -b_0(484 + \alpha 546, 547) + b_1(484 + \alpha * 546, 457)^2$ and if we believe in the interpolation formula we get

$$\Delta(100, 547) = 286 * 547^2 \mod 547^3 \tag{3.154}$$

Hence we see that

$$\lambda(T_2, 100) = 547(238096 \pm \frac{1}{2}\sqrt{286}) \mod 547^3 \tag{3.155}$$

Now we check easily that 286 is not a square $\mod 547$ and hence we see the roots now lie in the unramified quadratic extension $\mathbb{Z}/(547^3)[\sqrt{2}]$. Now we put again $\mathcal{O}_{\mathfrak{p}} = \mathbb{Z}_{547}[\sqrt{2}]$ let us put $p = 547$ then $\mathfrak{p} = (p)$ We consider the sequence $\alpha_0, \alpha_1, \ldots, \alpha_\nu$ (see 3.132). As before we get for any $\nu \geq 1$ (a decomposition up to isogeny (3.139)

$$\begin{gathered} H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_1} \otimes \mathcal{O}_{\mathfrak{p}})\{\bar{\pi}_f^{\mathrm{Eis}}\} \supset \\ H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_1} \otimes \mathcal{O}_{\mathfrak{p}})(\pi^{(\nu)})_f) \oplus H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_1} \otimes \mathcal{O}_{\mathfrak{p}}))({}^\sigma\pi_f^{(\nu)}). \end{gathered} \tag{3.156}$$

We argue as before, our embedding $\mathbb{Z}/p^{\nu+1}(-1-n_\nu) \hookrightarrow H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_1} \otimes \mathbb{Z}/(p^{\nu+1}))$ provides the Galois invariant embedding $\mathcal{O}_{\mathfrak{p}}/(p^{\nu+1})(-1-n_\nu) \hookrightarrow H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_\nu} \otimes \mathcal{O}_{\mathfrak{p}}/(p^{\nu+1})$. There is a largest number $f_\nu \leq \nu$ such that the submodule $p^{f_\nu}\mathcal{O}_{\mathfrak{p}}/p^{\nu+1}(-1-n_\nu)$ embeds into the submodule $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_1} \otimes \mathcal{O}_{\mathfrak{p}}/(p^{\nu+1}))(\pi_f^{(\nu)})$. But then we get an injection

$$\begin{gathered} \mathcal{O}_{\mathfrak{p}}/(p^{\nu+1})(-1-n_\nu)/p^\mu\mathcal{O}_{\mathfrak{p}}/(p^{f_\nu})(-1-n_\nu) \hookrightarrow \\ H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_\nu} \otimes \mathcal{O}_{\mathfrak{p}}/(p^{\nu+1})/H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_1} \otimes \mathcal{O}_{\mathfrak{p}})(\pi_f^{(\nu)}) \otimes \mathcal{O}_{\mathfrak{p}}/(p^{\nu+1}) \end{gathered} \tag{3.157}$$

The module in the bottom line is isomorphic to $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_1} \otimes \mathcal{O}_{\mathfrak{p}})({}^\sigma\pi_f^{(\nu)}) \otimes \mathcal{O}_{\mathfrak{p}}/(p^\nu)$ and hence we get congruences

$$\pi_f^{(\nu)}(T_\ell) \equiv \ell^{n_\nu+1} + 1 \mod p^{f_\nu} \;,\; {}^\sigma\pi_f^{(\nu)}(T_\ell) \equiv \ell^{n_\nu+1} + 1 \mod p^{\nu+1-f_\nu} \tag{3.158}$$

Since these congruences are invariant under the action of the Galois group we get congruences

$$\pi_f^{(\nu)}(T_\ell) \equiv \ell^{n_\nu+1} + 1 \mod p^{[\frac{\nu+2}{2}]} \;,\; {}^\sigma\pi_f^{(\nu)}(T_\ell) \equiv \ell^{n_\nu+1} + 1 \mod p^{[\frac{\nu+2}{2}]} \tag{3.159}$$

where $[x]$ denotes the usual Gauss bracket.

The Galois group

Viel ausführlicher It is a fundamental fact that we have an action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the modules $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_p/p^\delta), H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathcal{O}_{\mathfrak{p}}), H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_\infty})\{\bar{\pi}_f^{\mathrm{Eis}}\}$ this action commutes with the action of the Hecke algebra. Hence we get interesting representations of the Galois group, these representations have been studied by many people. (See for instance [65], [74],[86] or [46]).

We will be explain this matter in a very cursory manner in section 5.1.5.

3.3.12 The Wieferich dilemma

In 1909 the student Arthur Wieferich proved the following

If for a prime $p > 2$ the equation $x^p + y^p + z^p = 0$ has a solution in integers with $xyz \not\equiv 0 \mod p$, then $2^{p-1} - 1 \equiv 0 \mod p^2$.

Of course we now that $2^{p-1}-1 \equiv 0 \mod p$ but checking a few of small primes suggests that the residue class $w(p) = \frac{2^{p-1}-1}{p} \mod p$ can just be any number mod p. Hence we expect that it is a rare event that this residue class is zero, the "probability" is $\frac{1}{p}$. Later these primes ere called Wieferich primes. At the present moment it seems that there are only two Wieferich primes $< 6.7 \times 10^5$.,

But poor Arthur could not show that he had proved the first case of Fermat for infinitely many prime exponents p because he could not show that there are infinitely many non Wieferich primes (and we still do not know it now):

This kind of phenomenon was not new and well known at the time when Wieferich proved his theorem (Simply Google: The first case of Fermat's Theorem) but Wieferich's case is a striking example because it is so easy to state. That is the reason why we propose to call it the Wieferich dilemma.

In this book we encounter the Wieferich dilemma at several occasions. If $p|\zeta(-1-n)$ we raised the question whether $\bar{\pi}_f^{\mathrm{Eis}}$ occurs with multiplicity one. We saw that this is exactly the case if $a_1(n,p) \not\equiv 0 \mod p$. Again we may argue that the probability that $p|a_1(n,p)$ is $\frac{1}{p}$ so we expect this to be a rare event that $\bar{\pi}_f^{\mathrm{Eis}}$ occurs with higher multiplicity. Actually we learned from [3] that there is only one exception for $p < 10^5$. On the other hand we believe that $\sum_{p\ irregular} \frac{1}{p}$ is divergent so there is a certain chance that some larger such a prime exists.

Assume that there is a prime p for which $\bar{\pi}_f^{\mathrm{Eis}}$ occurs with higher multiplicity. Then we may ask whether it occurs weakly with multiplicity one and again a probability argument shows that the probability that this is not so is $\frac{1}{p^2}$. This now suggests that it may always occurs weakly with multiplicity one.

Here I want to make a metamathematical statement.

It is very well conceivable that $\bar{\pi}_f^{\mathrm{Eis}}$ always occurs with weak multiplicity one, but we will never find a proof. But it is simply true because the probability that $\bar{\pi}_f^{\mathrm{Eis}}$ occurs with higher weak multiplicity is so small. It is simply "true" without a proof.

In this chapter 3 we discuss some questions concerning the structure of the cohomology of arithmetic groups as module under the Hecke algebra. We we execute computations and experiments to support and suggest certain hypotheses. But we only considered a very special example.

But there is much wider range where can ask questions and make hypotheses.

We drop our assumption that we are in the totally unramified situation, this means that we can replace $\Gamma_0 = \mathrm{Sl}_2(\mathbb{Z})$ by a (normal) congruence subgroup $\Gamma \subset \Gamma_0$. We choose a free $\mathbb{Z}-$ module of finite rank $\mathcal{V}$ with an action of Γ_0/Γ, i.e. we have a representation

$$\rho_{\mathcal{V}} : \Gamma_0/\Gamma \to \mathrm{Gl}(\mathcal{V})$$

we assume that the matrix $-\mathrm{Id}$ acts by a scalar $\rho_{\mathcal{V}}(-\mathrm{Id}) = \pm\mathrm{Id}$. The Γ_0-modules $\mathcal{M}_n \otimes \mathcal{V}$ provide sheaves $\widetilde{\mathcal{M}_n \otimes \mathcal{V}}$, here we assume that $\rho_{\mathcal{V}}(-\mathrm{Id}) \equiv n \mod 2$.Again we study the cohomology groups and especially we can study the fundamental exact sequence

$$\to H^1_c(\Gamma_0\backslash\mathbb{H}, \widetilde{\mathcal{M}_n \otimes \mathcal{V}}) \to H^1(\Gamma_0\backslash\mathbb{H}, \widetilde{\mathcal{M}_n \otimes \mathcal{V}}) \xrightarrow{r} H^1(\partial(\Gamma_0\backslash\mathbb{H}), \widetilde{\mathcal{M}_n \otimes \mathcal{V}}) \tag{3.160}$$

On these cohomology groups we have the action of Hecke operators $T(\alpha, u_\alpha)$. Here we have to be a little carful. Our group Γ contains a full congruence group $\Gamma(N)$. Then we take the elements $\alpha \in \mathrm{Gl}_2(\mathbb{Z}_{(N)})$ and our $u_\alpha = u_\alpha^{\mathcal{M}} \otimes u_\alpha^{\mathcal{V}}$ (See section 3.1).

We may for instance choose a positive integer N and we consider the congruence subgroup $\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}_2(\mathbb{Z}) | c \equiv 0 \mod N\}$. Let $\Gamma_1(N) \subset \Gamma_0(N)$ be the subgroup where $a \equiv 1 \mod N$ then $\Gamma_0(N)/\Gamma_1(N) = (\mathbb{Z}/N\mathbb{Z})^\times$. We choose a character $\chi : \Gamma_0(N)/\Gamma_1(N) \to \mathbb{Z}[\zeta_N]^\times$ and consider the induced representation

$$\mathcal{V}_\chi = \mathrm{Ind}_{\Gamma_0(N)}^{\Gamma_0} \chi = \{f : \Gamma_0 \to \mathbb{Z}[\zeta_N] \mid f(\gamma x) = \chi(\gamma) f(x); \gamma \in \Gamma_0(N), x \in \Gamma_0\}.$$

It an interesting task to extend the results and the experimental computations from the previous chapters to these Hecke modules. It should not be too difficult to generalise the results in section 3.3.1 for the Hecke modules $H^1(\partial(\Gamma_0\backslash\mathbb{H}), \widetilde{\mathcal{M}_n \otimes \mathcal{V}_\chi})$. Then we can formulate the denominator question again.

In this case the denominator should be related to the L values $L(\chi, -1-n)$, It is certainly interesting to collect some data, which might allow us to formulate more precise hypotheses. For this purpose one has to extend the algorithm for T_2 to this new situation.

At this point we ignore (or forget) that analytic methods (Eisenstein cohomology) also provide some tools to understand the denominator. (see 5.1.2)

I think it is even more interesting to investigate the multiplicity questions. Now our cohomology groups are $\mathbb{Z}[\zeta_N]$ modules, let us assume that we have a prime ideal $\mathfrak{p} | L(\chi, -1-n)$. We assume $\mathfrak{p} \not| N$ and let $\mathcal{O}_\mathfrak{p} = \mathbb{Z}[\zeta_N]_\mathfrak{p}$ be the completion at $\mathfrak{p}$. Then we can should be able to define the direct summand (see3.119)

$$H^1(\Gamma_0\backslash\mathbb{H}, \mathcal{M}_n \widetilde{\otimes \mathcal{V}} \otimes \mathcal{O}_\mathfrak{p})\{\pi_{f,\chi}^{\mathrm{Eis}}\} \subset H^1_{\mathrm{ord}}(\Gamma_0\backslash\mathbb{H}, \mathcal{M}_n \widetilde{\otimes \mathcal{V}} \otimes \mathcal{O}_\mathfrak{p}) \tag{3.161}$$

and we want this direct summand as Hecke module. Earlier we have done some experimental computation in the unramified case ($N = 1$) and probabilistic arguments let us make some conjectures. But now the we have many more cases (vary the character χ and the primes involved are much smaller so we have some chances to falsify the analogous conjectures once we have some ramification.

I think that here is a wide field for interesting experiments.

Chapter 4

Representation Theory, Eichler-Shimura Isomorphism

$\boxed{\text{HC}}$

4.1 Harish-Chandra modules with cohomology

In Chapter 6 we will give a general discussion of the tools from representation theory and analysis which help us to understand the cohomology of arithmetic groups. Especially in Chapter 6 section 6.1.5 we will recall the results of Vogan-Zuckerman on the cohomology of Harish-Chandra modules.

Here we specialise these results to the specific cases $G = \mathrm{Gl}_2(\mathbb{R})$ (case A)) and $G = \mathrm{Gl}_2(\mathbb{C})$ (case B)). For the general definition of Harish-Chandra modules and for the definition of $(\mathfrak{g}, K_\infty)$ cohomology we refer to (6.1.2)

$\boxed{\text{Mlambda}}$

4.1.1 The finite rank highest weight modules

We consider the case A), in this case our group $G/\mathbb{R}$ is the base extension of the reductive group scheme $\mathcal{G} = \mathrm{Gl}_2/\operatorname{Spec}(\mathbb{Z})$. In principle this a pretentious language. At this point it simply means that for for any commutative ring R with identity we can speak of $\mathcal{G}(R)$-the group of Rvalued points- , and that $\mathcal{G}(R)$ depends functorially on R.

(Sometimes in the following we will replace Spec($\mathbb{Z}$) by $\mathbb{Z}$.) Then $\mathcal{G}^{(1)}/\mathbb{Z}$ is the kernel of the determinant map $\det : \mathcal{G}/\mathbb{Z} \to \mathbb{G}_m/\mathbb{Z}$. We have the standard maximal torus $\mathcal{T}/\mathbb{Z}$ and choose the Borel subgroup $\mathcal{B}/\mathbb{Z} \supset \mathcal{T}/\mathbb{Z}$ to be the group of upper triangular matrices. Let $X^*(\mathcal{T}) = X^*(T \times \mathbb{C})$ be the character module This character module is $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ where

$$e_i : \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mapsto t_i \tag{4.1}$$

Any character can be written as $\lambda = n\gamma + d\det$ where $\gamma = \frac{e_1 - e_2}{2} (\notin X^*(\mathcal{T})\ !), \det = e_1 + e_2$ and where $n \in \mathbb{Z}, d \in \frac{1}{2}\mathbb{Z}$ and $n \equiv 2d \mod 2$. We say that λ is /it dominant iif $n \geq 0$.

To any such character $\lambda = n\gamma + d\det$ we want to attach a highest weight module $\mathcal{M}_\lambda$. We consider the $\mathbb{Z}-$ module of polynomials

$$\mathcal{M}_n = \{P(X,Y) \mid P(X,Y) = \sum_{\nu=0}^{n} a_v X^\nu Y^{n-\nu}, a_\nu \in \mathbb{Z}\}.$$

To a polynomial $P \in \mathcal{M}_n$ we attach the regular function (see 1.1.1)

$$f_P(\begin{pmatrix} x & y \\ u & v \end{pmatrix}) = P(u,v) \det(\begin{pmatrix} x & y \\ u & v \end{pmatrix})^{-\frac{n}{2}+d}, \tag{4.2}$$

then

$$f_P(\begin{pmatrix} t_1 & w \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} x & y \\ u & v \end{pmatrix}) = t_2^n (t_1 t_2)^{-\frac{n}{2}+d} f_P(\begin{pmatrix} x & y \\ u & v \end{pmatrix}) = \lambda^-(\begin{pmatrix} t_1 & w \\ 0 & t_2 \end{pmatrix}) f_P(\begin{pmatrix} x & y \\ u & v \end{pmatrix}) \tag{4.3}$$

where $\lambda^- = -n\gamma + d\det$. On this module of regular functions the group scheme $\mathcal{G}/\mathbb{Z}$ acts by right translations:

$$\rho_\lambda(\begin{pmatrix} a & b \\ c & d \end{pmatrix})(f_P)(\begin{pmatrix} x & y \\ u & v \end{pmatrix})) = f_P(\begin{pmatrix} x & y \\ u & v \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \tag{4.4}$$

This is now the highest weight module $\mathcal{M}_\lambda$ for the group scheme $\mathcal{G}/\mathbb{Z}$. The highest weight vector is f_{X^n}, clearly we have

$$\rho_\lambda(\begin{pmatrix} t_1 & w \\ 0 & t_2 \end{pmatrix})(f_{X^n}) = \lambda(\begin{pmatrix} t_1 & w \\ 0 & t_2 \end{pmatrix}) f_{X^n} = \lambda(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} f_{X^n}$$

In the following we change the notation, instead of f_P we will simply write P.

Comment: When we say that $\mathcal{M}_\lambda$ is a module for the group scheme $\mathcal{G}/\mathbb{Z}$ we mean nothing more than that for any commutative ring R with identity we have an action of $\mathcal{G}(R)$ on $\mathcal{M}_n \otimes R$, which is given by (4.2) and depends functorially on R. We can "evaluate" at $R = \mathbb{Z}$ and get the $\Gamma = \mathrm{Gl}_2(\mathbb{Z})$ module $\mathcal{M}_{\lambda,\mathbb{Z}}$. (Actually we should not so much distinguish between the $\mathrm{Gl}_2(\mathbb{Z})$ module $\mathcal{M}_{\lambda,\mathbb{Z}}$ and $\mathcal{M}_\lambda$) Of course we have have seen these $\mathrm{Gl}_2(\mathbb{Z})$ modules before, they are of course equal to the modules $\mathcal{M}_n[d - \frac{n}{2}]$ in section 1.2.2.

Remark: There is a slightly more sophisticated interpretation of this module. We can form the flag manifold $\mathcal{B}\backslash\mathcal{G} = \mathbb{P}^1/\mathbb{Z}$ and the character λ yields a line bundle $\mathcal{L}_{\lambda^-}$. The group scheme $\mathcal{G}$ is acting on the pair $(\mathcal{B}\backslash\mathcal{G}, \mathcal{L}_{\lambda^-})$ and hence on $H^0(\mathcal{B}\backslash\mathcal{G}, \mathcal{L}_{\lambda^-})$ which is tautologically equal to $\mathcal{M}_\lambda$ (Borel-Weil theorem).

We can do essentially the same in the case B). In this case we start from an imaginary quadratic extension $F/\mathbb{Q}$ and let $\mathcal{O} = \mathcal{O}_F \subset F$ its ring of integers. We form the group scheme $\mathcal{G}/\mathbb{Z} = R_{\mathcal{O}/\mathbb{Z}}(\mathrm{Gl}_2/\mathcal{O})$. Again $\mathcal{G}^{(1)}/\mathbb{Z}$ will be the kernel of $\det : \mathcal{G}/\mathbb{Z} \to \mathcal{Z}/\mathbb{Z} = R_{\mathcal{O}/\mathbb{Z}}(\mathbb{G}_m)$. Then $\mathcal{G}(\mathcal{O}) = \mathrm{Gl}_2(\mathcal{O}\otimes\mathcal{O}) \subset \mathrm{Gl}_2(\mathcal{O}) \times \mathrm{Gl}_2(\mathcal{O})$. The base change of the maximal torus $T/\mathbb{Q} \subset \mathcal{G} \times_{\mathbb{Z}} \mathbb{Q}$ is the product $T_1 \times T_2/F$ where the two factors are the standard maximal tori in the two factors Gl_2/F.

We get for the character module $\boxed{\text{CHMsplit}}$

$$X^*(T \times F) = X^*(T_1) \oplus X^*(T_2) = \{n_1\gamma_1 + d_1 \det\} \oplus \{n_2\gamma_2 + d_2\bar{\det}\} \tag{4.5}$$

where we have to observe the parity conditions $n_1 \equiv 2d_1 \mod 2, n_2 \equiv 2d_2 \mod 2$.

Then the same procedure as in case A) provides a free $\mathcal{O}$- module $\mathcal{M}_\lambda$ with an action of $\mathcal{G}(\mathbb{Z})$ on it. To get this module and to see this action we embed the group $\mathcal{G}(\mathbb{Z}) = \mathrm{Gl}_2(\mathcal{O})$ into $\mathrm{Gl}_2(\mathcal{O}) \times \mathrm{Gl}_2(\mathcal{O})$ by the map $g \mapsto (g, \bar{g})$ where $\bar{g}$ is of course the conjugate. If now our $\lambda = n_1\gamma_1 + d_1 \det_1 + n_2\gamma_2 + d_2\det_2 = \lambda_1 + \lambda_2$ then we have our two $\mathrm{Gl}_2(\mathcal{O})$ modules $\mathcal{M}_{\lambda_1,\mathcal{O}}, \mathcal{M}_{\lambda_2,\mathcal{O}}$ and this provides the $\mathrm{Gl}_2(\mathcal{O}) \times \mathrm{Gl}_2(\mathcal{O})$- module $\mathcal{M}_{\lambda_1,\mathcal{O}} \otimes \mathcal{M}_{\lambda_2,\mathcal{O}}$, our $\mathcal{M}_{\lambda,\mathcal{O}}$ is is now simply the restriction of this tensor product module to $\mathcal{G}(\mathbb{Z})$. Sometimes we will also write our character as the sum of the semi simple component and the central component, i.e.

$$\lambda = \lambda^{(1)} + \delta = (n_1\gamma_1 + n_2\gamma_2) + (d_1\det_1 + d_2\det_2) \tag{4.6}$$

The relevant term is the semi simple component, the central component is not important at all, it only serves to fulfill the parity condition. If we restrict the representation $\mathcal{M}_\lambda$ to $\mathcal{G}^{(1)}/\mathbb{Z}$ then the dependence on d disappears. In other words representations with the same semi simple highest weight component only differ by a twist, the role played by δ is marginal.

At this point we notice that the module $\mathcal{M}_{\lambda,\mathcal{O}}$ is only a module over $\mathcal{O}$. We may also say that $\mathcal{M}_{\lambda,\mathcal{O}} \otimes F$ is an absolutely irreducible highest weight module for the group $\mathcal{G} \otimes_{\mathcal{O}} F = \mathrm{Gl}_2 \times \mathrm{Gl}_2/F$, this representation "is defined " over F. But in the special case that $\lambda_1 = \lambda_2$ we have an action of the Galois group $\mathrm{Gal}(F/\mathbb{Q})$: If c is the non trivial element in this Galois group then

$$c((\sum a_\nu X^\nu Y^{n-\nu}) \otimes (\sum_\mu b_\mu \bar{X}^\mu \bar{Y}^{n-\mu}) = (\sum_\mu c(b_\mu) X^\mu Y^{n-\mu})(\sum c(a_\nu)\bar{X}^\nu \bar{Y}^{n-\nu})$$

and for $g \in \mathcal{G}(\mathcal{O}), m \in \mathcal{M}_\lambda$ we have

$$c(g)c(m) = c(gm)$$

and therefore it is clear that the $\mathbb{Z}$ module $(\mathcal{M}_\lambda)^{(c)}$ is a module for $\mathcal{G}/\mathbb{Z}$.

We return to $\mathrm{Gl}_2/\mathbb{Z}$. Given $\lambda = \lambda^{(1)} + \delta$ we define the dual character as $\lambda^\vee = \lambda^{(1)} - \delta$. For our finite dimensional modules we have

$$\mathcal{M}_\lambda^\vee \otimes \mathbb{Q} \xrightarrow{\sim} \mathcal{M}_{\lambda^\vee} \otimes \mathbb{Q} \tag{4.7}$$

If we consider the modules over the integers the above relation is not true. We define the submodule $\boxed{\text{duallambda}}$

$$\mathcal{M}_n^\flat = \{P(X,Y) \mid P(X,Y) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu X^\nu Y^{n-\nu}, a_\nu \in \mathbb{Z}\}. \tag{4.8}$$

This is a submodule of $\mathcal{M}_n$ and the quotient $\mathcal{M}_n/\mathcal{M}_n^\flat$ is finite. It is also clear that this submodule is invariant under $\mathrm{Sl}_2/\mathbb{Z}$. We introduce some notation

$$e_\nu := X^\nu Y^{n-\nu} \text{ and } e_\nu^\flat := \binom{n}{\nu} X^{n-\nu} Y^\nu, \tag{4.9}$$

then the e_ν(resp. $e_\nu^\flat$) for a basis of $\mathcal{M}_n$(resp. $\mathcal{M}_n^\flat$).

An easy calculation shows that the pairing $\boxed{\text{pairMn}}$

$$< \, , \, >_{\mathcal{M}}: (e_\nu, e_\mu^\flat) \mapsto \delta_{\nu,\mu} \tag{4.10}$$

is non degenerate over $\mathbb{Z}$ and invariant under $\mathrm{Sl}_2/\mathbb{Z}$. We can also define the twisted (?!?!) actions of $\mathcal{G}/\mathbb{Z}$. Of course we can define the twisted modules $\mathcal{M}_\lambda^\vee$ and then we get a $\mathcal{G}/\mathbb{Z}$ invariant non degenerate pairing over $\mathbb{Z}$:

$$< \, , \, >_{\mathcal{M}}: \mathcal{M}_{\lambda^\vee}^\flat \times \mathcal{M}_\lambda \to \mathbb{Z}$$

In other words

$$(\mathcal{M}_\lambda)^\vee = \mathcal{M}_{\lambda^\vee}^\flat$$

We always consider $\mathcal{M}_\lambda^\flat$ as the above submodule of $\mathcal{M}_\lambda$.

$\boxed{\text{prinseries}}$

4.1.2 The principal series representations

We consider the two real algebraic groups $G = \mathrm{Gl}_2/\mathbb{R}($ case A)) and $G = R_{\mathbb{C}/\mathbb{R}}(\mathrm{Gl}_2/\mathbb{C})$ (case B). Let $T/\mathbb{R}$,(resp. $B/\mathbb{R}$) be the standard diagonal torus (resp. Borel subgroup of upper triangular matrices). Let us put $C/\mathbb{R} = \mathbb{G}_m$ (resp. $R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$). We have the determinant $\det : G/\mathbb{R} \to C/\mathbb{R}$ and moreover $C/\mathbb{R} = center(G/\mathbb{R})$. If we restrict the determinant to the center then this becomes the map $z \mapsto z^2$. The kernel of the determinant is denoted by $G^{(1)}/\mathbb{R}$, of course $G^{(1)} = \mathrm{Sl}_2$, resp. $R_{\mathbb{C}/\mathbb{R}}(\mathrm{Sl}_2/\mathbb{C})$. Let us denote by $\mathfrak{g}, \mathfrak{g}^{(1)}, \mathfrak{t}, \mathfrak{b}, \mathfrak{z}$ the corresponding Lie-algebras.

The Cartan decompositions

In both cases we fix a maximal compact compact subgroup $K_\infty \subset G^{(1)}(\mathbb{R})$:

$$K_\infty = e(\phi) = \{\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} |\phi \in \mathbb{R}\} \text{ and } K_\infty = \{\begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} |\alpha\bar\alpha + \beta\bar\beta = 1\} \tag{4.11}$$

We define extensions $\tilde K_\infty = Z(\mathbb{R})^{(0)} K_\infty$, here of course $Z(\mathbb{R})^{(0)}$ is the connected component of the identity. In both cases the group K_∞ is the group of fixed points under the Cartan involution Θ_0 which is given by

$$\Theta_0 : g \mapsto^t g^{-1} \text{ resp. } g \mapsto^t \bar g^{-1} \text{ i.e. } \Theta_0(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} \bar d & -\bar c \\ -\bar b & \bar a \end{pmatrix}. \tag{4.12}$$

This involution induces an involution on $\mathfrak{g}^{(1)}$ we can extend it to an involution acting on $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}^{(1)}$, we let it act trivially on $\mathfrak{z}$. Then the fixed point Lie algebra $\tilde{\mathfrak{k}} = \mathfrak{z} \oplus \mathfrak{k} \subset \mathfrak{z} \oplus \mathfrak{g}^{(1)}$ is the Lie-algebra of $\tilde{K}_\infty$.

Here are some arithmetic considerations, they may not be so relevant, but further down we make some choices of a basis in some of these algebras, and these choices can be justified by these arithmetic considerations.

We can write our group scheme $G/\mathbb{R}$ as a base extension of a group scheme $\mathcal{G}/\mathbb{Z}$, i.e. $G/\mathbb{R} = \mathcal{G} \times_{\mathbb{Z}} \mathbb{R}$. For this we simply take $\mathcal{G}/\mathbb{Z} = \mathrm{Gl}_2/\mathbb{Z}$ in case A). In case B) we take $\mathcal{G}/\mathbb{Z} = R_{\mathbb{Z}[\mathbf{i}]/\mathbb{Z}}(\mathrm{Gl}_2/\mathbb{Z}[\mathbf{i}])$. In the case A) this gives a reductive group scheme over $\mathbb{Z}$, in case B) it is only a flat group scheme, but the base extension $\mathcal{G} \times_{\mathbb{Z}} \mathbb{Z}[1/2]$ is reductive. (This group scheme over $\mathbb{Z}$ is not semi-simple since $\mathbb{Z}[\mathbf{i}]$ is ramified at the prime 2.)

Now it is clear that Θ_0 is actually an automorphism of $\mathcal{G}/\mathbb{Z}$ and then it follows that the scheme of fixed points is again a group scheme $\mathcal{K}/\mathbb{Z}$. If we define $R = \mathbb{Z}[1/2]$ then $\mathcal{K} \times_{\mathbb{Z}} R$ is actually eductive. (If we replace $\mathbb{Z}[\mathbf{i}]$ by the ring of integers of another imaginary quadratic extension, we have to modify R accordingly.)

Consequently we see that the all the above Lie-algebras are defined over R, hence they actually are free R modules, we denote them by $\mathfrak{g}_R$ and so on.

The Cartan Θ_0 involution induces an involution on the Lie algebras $\mathfrak{g}_R, \mathfrak{g}_R^{(1)}$, the module decomposes into a + and a − eigenspace $\boxed{\text{CaDec}}$

$$\mathfrak{g}_R = \tilde{\mathfrak{k}}_R \oplus \mathfrak{p}_R \text{ and } \mathfrak{g}_R^{(1)} = \mathfrak{k}_R \oplus \mathfrak{p}_R, \tag{4.13}$$

The + eigenspaces $\tilde{\mathfrak{k}}_R, \mathfrak{k}_R$ are the Lie-algebras of $\tilde{\mathcal{K}}, \mathcal{K}$, both summands in the decompositions are $\tilde{\mathcal{K}}$-modules.

The Lie-algebra $\mathfrak{b}_R$ is not stable under Θ_0, it is clear that the intersection

$$\mathfrak{b}_R \cap \Theta_0(\mathfrak{b}_R) = \mathfrak{t}_R,$$

where $\mathfrak{t}_R$ is the Lie-algebra of the standard maximal torus $\mathcal{T}/R \subset \mathcal{G}/R$. This torus is a product (up to isogeny) $\mathcal{T}/R = \mathcal{Z} \cdot \mathcal{T}^{(1)}/R$.

In case A) the torus $\mathcal{T}^{(1)}/R \xrightarrow{\sim} \mathbb{G}_m/R$ and the Cartan involution Θ_0 acts by $t \mapsto t^{-1}$. Therefore it acts by -1 on $\mathfrak{t}_R^{(1)}$. We write

$$\mathfrak{t}_R^{(1)} = R \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = RH \tag{4.14}$$

the generator H is unique up to an element in $R^\times$, i.e. up to a sign and a power of 2.

In case B) the torus $\mathcal{T}^{(1)}/R$ is (up to isogeny) a product $\mathcal{T}_s^{(1)} \cdot \mathcal{T}_c^{(1)}/R$ the Cartan involution Θ_0 acts by $t \to t^{-1}$ on the split component $\mathcal{T}_s^{(1)}$ and by the identity on $\mathcal{T}_c^{(1)}$. The Lie-algebra decomposes accordingly into two summands of rank one:

$$\mathfrak{t}_R^{(1)} = R \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus R \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = RH \oplus RH_i.$$

In both cases the group scheme $\mathcal{K}$ acts on $\mathfrak{p}_R$ by the adjoint action, we can describe this action explicitly.

In case A) the group scheme $\mathcal{K}$ is the following group of matrices

$$\mathcal{K} = \{\alpha = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} | a^2 + b^2 = 1\}$$

this is a torus over R which splits over $R[i]$. We have

$$\mathfrak{p}_R = RH \oplus R \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = RH \oplus RV$$

and $\mathrm{Ad}(\alpha)(H) = (a^2 - b^2)H - 2abV,\ \mathrm{Ad}(\alpha)(V) = 2abH + (a^2 - b^2)V$. Since the torus splits over $\mathbb{Z}[\mathbf{i}]$ we can decompose $\mathfrak{p} \otimes R[i]$ into weight spaces, we introduce the basis elements

$$P_+ := H - V \otimes i,\ P_- := H + V \otimes i \in \mathfrak{p} \otimes R[i]$$

then $\boxed{\text{Ppm}}$

$$\mathrm{Ad}(\alpha)P_+ = (a + bi)^2 P_+,\ \mathrm{Ad}(\alpha)P_- = (a - bi)^2 P_- \tag{4.15}$$

Hence we get - in case A) -the decomposition

$$\mathfrak{g}_R^{(1)} = \mathfrak{k}_R \oplus \mathfrak{p}_R = R \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus RP_+ \oplus RP_- = RY \oplus RP_+ \oplus RP_- \tag{4.16}$$

where the generators are unique up to an element in $R[i]^\times$.

In case B) the group scheme $\mathcal{K}/R$ is semi simple, it contains $\mathcal{T}_c^{(1)}/R$ as maximal torus. The two $\mathcal{K}/R$ modules $\mathfrak{k}_R$ and $\mathfrak{p}_R$ are highest weight modules of rank 3, since 2 is invertible in R they are even isomorphic. Again we can decompose them into rank one weight spaces and give almost canonical generators for these weight spaces. $\boxed{\text{basisfkfp}}$ The Lie algebra

$$\mathfrak{k}_R = RH_i \oplus R \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus R \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = RH_i \oplus RY \oplus RF_i. \tag{4.17}$$

We introduce the elements $P_{c+} = Y - F_i \otimes i,\ P_{c-} = Y + F_i \otimes i$ and then

$$\mathfrak{k}_R \otimes R[i] = R[i]H_i \oplus R[i]P_{c+} \oplus R[i]P_{c-}. \tag{4.18}$$

This is the decomposition into weight spaces under the action of $\mathcal{T}_c^{(1)}/R$, the element $\alpha = \begin{pmatrix} x & 0 \\ 0 & \bar{x} \end{pmatrix}$ acts via the adjoint action

$$\mathrm{Ad}(\alpha)P_{c+} = x^2 P_{c+}\ ,\ \mathrm{Ad}(\alpha)H_i = H_i\ ,\ \mathrm{Ad}(\alpha)P_{c-} = x^{-2}P_{c-}.$$

Essentially the same can be done for $\mathfrak{p}_R \otimes R[i]$. We define

$$P_{\mathfrak{p},+} = V - \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \otimes i,\ P_{\mathfrak{p},-} = V + \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \otimes i$$

then we get the weight decomposition $\boxed{\text{basisfp}}$

$$\mathfrak{p}_R \otimes R[i] = R[i]P_{\mathfrak{p},+} \oplus R[i]H \oplus R[i]P_{\mathfrak{p},-} \tag{4.19}$$

Rational characters vs continuous characters

Our aim is to to construct certain irreducible (differentiable) representations of $G(\mathbb{R})$ together with their "algebraic skeleton" the associated *Harish-Chandra-modules.*

For any torus $T/\mathbb{R}$ we consider the group of (continuous) characters $\mathrm{Hom}(T(\mathbb{R}),\mathbb{C}^\times)$, we write this group multiplicatively, i.e. $\chi_1\cdot\chi_2(x)=\chi_1(x)\chi_2(x)$. We also have defined the group of (rational) characters $X^*(T\times_\mathbb{R}\mathbb{C},\mathbb{G}_m)$ (See Chap. 1, 1.5), and we have the evaluation map

$$X^*(T\times_\mathbb{R}\mathbb{C},\mathbb{G}_m)\xrightarrow{ev}\mathrm{Hom}(T(\mathbb{R}),\mathbb{C}^\times);\ ev:\gamma\mapsto\gamma_\mathbb{R}=\{x\mapsto\gamma(x)\}\tag{4.20}$$

Since we wrote the group of (rational) characters additively we have

$$(\gamma_1+\gamma_2)_\mathbb{R}=\gamma_{1\mathbb{R}}\cdot\gamma_{2\mathbb{R}}.$$

We also introduce the character $|\gamma|:=\{x\mapsto|\gamma_R(x)|_\mathbb{C}\}$ where of course $|a|_\mathbb{C}=a\bar{a}$.

We can also introduces the characters $\gamma\otimes\mathbb{C}$ we simply put $\gamma\otimes\mathbb{C}(x)=|\gamma|^z$.

4.1.3 The induced representations

We start from a continuous homomorphism (a character) $\chi:T(\mathbb{R})\to\mathbb{C}^\times$, of course this can also be seen as a character $\chi:B(\mathbb{R})\to\mathbb{C}^\times$. This allows us to define the induced module

$$I_B^G\chi:=\{f:G(\mathbb{R})\to\mathbb{C}\mid f\in\mathcal{C}_\infty(G(\mathbb{R})),f(bg)=\chi(b)f(g),\ \forall\, b\in B(\mathbb{R}),g\in G(\mathbb{R})\}\tag{4.21}$$

where we require that f should be $\mathcal{C}_\infty$. Then this space of functions is a $G(\mathbb{R})$ - module, the group $G(\mathbb{R})$ acts by right translations: For $f\in I_B^G\chi, g\in G(\mathbb{R})$ we put

$$R_g(f)(x)=f(xg)$$

If modify our character χ by a character $\delta\circ\det$ where $\delta:Z(\mathbb{R})\to\mathbb{C}^\times$ then the central character gets multiplied by δ^2.

We know that $G(\mathbb{R})=B(\mathbb{R})\cdot\tilde{K}_\infty$. This implies that a function $f\in I_B^G\chi$ is determined by its restriction to K_∞. In other words we have an identification of vector spaces $\boxed{\text{Iwasawa}}$

$$I_B^G\chi=\{f:\tilde{K}_\infty\to\mathbb{C}\mid f(t_ck)=\chi(t_c)f(k),t_c\in\tilde{K}_\infty\cap B(\mathbb{R}),k\in\tilde{K}_\infty\}.\tag{4.22}$$

The center acts by the central character ω_χ, the restriction of χ to $Z(\mathbb{R})$.

We put $T_c=B(\mathbb{R})\cap\tilde{K}_\infty$ and define χ_c to be the restriction of χ to T_c. Then the module on the right in the above equation can be written as $I_{T_c}^{\tilde{K}_\infty}\chi_c$. By its very definition $I_{T_c}^{\tilde{K}_\infty}\chi_c$ is only a K_∞ module. Inside $I_{T_c}^{\tilde{K}_\infty}\chi_c$ we have the submodule of vectors of finite type

$${}^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c:=\{f\in I_{T_c}^{\tilde{K}_\infty}\chi_c\mid\ \text{the translates } R_k(f) \text{ lie in a finite dimensional subspace}\}\tag{4.23}$$

Here it suffices to consider only the translates $R_k(f)$ for $k \in K_\infty$ because $Z(\mathbb{R})^{(0)}$ acts by the character ω_χ. The famous Peter-Weyl theorem tells us that all irreducible representations (satisfying some continuity condition) are finite dimensional and occur with finite multiplicity in $I_{T_c}^{\tilde{K}_\infty}\chi_c$ and therefore we get

$$^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c = \bigoplus_{\vartheta\in\hat{K}_\infty} V_\vartheta^{m(\vartheta)} = \bigoplus_{\vartheta\in\hat{K}_\infty} {}^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c[\vartheta] \tag{4.24}$$

where $\hat{K}_\infty$ is the set of isomorphism classes of irreducible representations of K_∞, where V_ϑ is an irreducible module of type ϑ and where $m(\vartheta)$ is the multiplicity of ϑ in $^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c$. Of course $^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c$ is a submodule $I_B^G\chi$, but this submodule is not invariant invariant under the operation of $G(\mathbb{R})$, in other words if $0 \neq f \in {}^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c$ and $g \in G(\mathbb{R})$ a sufficiently general element then $R_g(f) \notin {}^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c$.

We can differentiate the action of $G(\mathbb{R})$ on $I_B^G\chi$. We have the well known exponential map $\exp : \mathfrak{g} = \mathrm{Lie}(G/\mathbb{R}) \to G(\mathbb{R})$ and for $f \in I_B^G, X \in \mathfrak{g}$ we define

$$Xf(g) = \lim_{t\to 0}\frac{f(g\exp(tX)) - f(g)}{t} \tag{4.25}$$

and it is well known and also easy to see, that this gives an action of the Lie-algebra on I_B^G, we have $X_1(X_2f) - X_2(X_1f) = [X_1, X_2]f$. The Lie-algebra is a K_∞ module under the adjoint action and is obvious that for $f \in {}^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c[\vartheta]$ the element Xf lies in $\bigoplus_{\vartheta\in\hat{K}_\infty} {}^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c[\vartheta']$ where ϑ' runs over the finitely many isomorphism types occurring in $V_\vartheta \otimes \mathfrak{g}$. Hence

Proposition 4.1.1. *The submodule* $^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c \subset I_B^G\chi$ *is invariant under the action of* $\mathfrak{g}$.

The submodule $^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c$ together with this action of $\mathfrak{g}$ will now be denoted by $\mathfrak{I}_B^G\chi$. We should think of this module as the algebraic skeleton of $I_B^G\chi$.

Such a module will be called a $(\mathfrak{g}, K_\infty)$ - module or a Harish-Chandra module this means that we have an action of the Lie-algebra $\mathfrak{g}$, an action of K_∞ and these two actions satisfy some obvious compatibility conditions.

We also observe that $^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c$ is also invariant under right translation R_z for $z \in Z(\mathbb{R})$. Hence we can extend the action of K_∞ to the larger group $\tilde{K}_\infty = K_\infty \cdot Z(\mathbb{R})$. Then $\mathfrak{I}_B^G\chi$ becomes a $(\mathfrak{g}, \tilde{K}_\infty)$ module. Finally observe that in the case A) the element $\boxed{\text{complexcon}}$

$$\mathbf{c} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \notin \tilde{K}_\infty, \tag{4.26}$$

clearly $R_\mathbf{c}$ induces an involution on $\mathfrak{I}_B^G$. We could also say that we can enlarge K_∞(resp. $\tilde{K}_\infty$) to subgroups $K_\infty^*(resp.\tilde{K}_\infty^*)$ which contain $\mathbf{c}$ and contain K_∞ resp. $\tilde{K}_\infty$ as subgroups of index two. Then $\mathfrak{I}_B^G\chi$ also becomes a $(\mathfrak{g}, \tilde{K}_\infty^*)$ module.

These $(\mathfrak{g}, \tilde{K}_\infty)$ modules $\mathfrak{I}_B^G\chi$ are called the *principal series modules.* We have the following important

Theorem 4.1.1. *For any irreducible Harish-Chandra module*$(\mathfrak{g},\tilde{K}_\infty)$ *we can find a* χ *such that we have an embedding of* $(\mathfrak{g},\tilde{K}_\infty)$*-modules*

$$i : \mathcal{V} \hookrightarrow \mathfrak{I}_B^G\chi$$

This is actually a special case of a much more general theorem and applies *mutatis mutandis* to all reductive groups over $\mathbb{R}$. In the following we will see, that in our special cases we only have a very short list of submodules of the $\mathfrak{I}_B^G\chi$ and hence we get a complete list of irreducible Harish-Chandra modules.

We denote the restriction of χ to the central torus $Z = \{\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}\}$ by ω_χ. Then $Z(\mathbb{R})$ acts on $\mathfrak{I}_B^G\chi$ by the *central character* character ω_χ, i.e. $R_z(f) = \omega_\chi(z)f$. Once we fix the central character, then there is no difference between $(\mathfrak{g},\tilde{K}_\infty)$ and $(\mathfrak{g},K_\infty)$ modules. Hence we always assume that ω_χ is fixed.

Unitary induction

In the general theory of representations of real ((-or p-adic groups-) people work with the so called *unitary induction.* We introduce the special character

$$|\rho|_\mathbb{R} : \begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix} \to |\frac{t_1}{t_2}|^{\frac{1}{2}}, \tag{4.27}$$

here the absolute value $|t|$ is the usual absolute value if we are in case A) and $|z| = z\bar{z}$ for $z \in \mathbb{C}$, i.e if we are in case B).

Now we define the unitarily induced module (see4.27).

$$\text{Indunit}_B^G\chi := I_B^G\chi \cdot |\rho|_\mathbb{R} \tag{4.28}$$

This concept of induction is of course equivalent to the previous one, it has certain advantages, some statements have a more elegant formulation. But in this book we are also interested in the "arithmetic properties" of our modules and the naive concept of inductions has its own virtues.

The decomposition into K_∞-types

$\boxed{\text{Kutypes}}$

We look briefly at the K_∞-module ${}^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c$. In case A) the group

$$K_\infty = \mathrm{SO}(2) = \{\begin{pmatrix} \cos(\varphi) & \sin(\phi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix} = e(\varphi)\} \tag{4.29}$$

and $T_c = K_\infty^T = T(\mathbb{R}) \cap K_\infty$ is cyclic of order two with generator $e(\pi)$. Then χ_c is given by an integer $\mod 2$, i.e. $\chi_c(e(\varphi)) = (-1)^m$. For any $n \equiv m \mod 2$ we define $\psi_n \in \mathfrak{I}_B^G\chi$ by

$$\psi_n(e(\phi))) = e^{in\phi} \tag{4.30}$$

and then $\boxed{\text{decoKuA}}$

$$\mathfrak{I}_B^G\chi = \bigoplus_{k\equiv m \mod 2} \mathbb{C}\psi_k \tag{4.31}$$

In the case B) the maximal compact subgroup is

$$U(2) \subset G(\mathbb{R}) = R_{\mathbb{C}/\mathbb{R}}(\mathrm{Gl}_2/\mathbb{C})(\mathbb{R}) \subset \mathrm{Gl}_2(\mathbb{C}) \times \mathbb{G}_2(\mathbb{C})$$

this is the group of real points of the reductive group $U(2)/\mathbb{R}$. The intersection

$$T_c = K_\infty^T = T(\mathbb{R}) \cap K_\infty = \{\begin{pmatrix} e^{2\pi i\varphi_1} & 0 \\ 0 & e^{2\pi i\varphi_2} \end{pmatrix} = e(\underline{\phi})\}.$$

The base change $U(2) \times \mathbb{C} = \mathrm{Gl}_2/\mathbb{C}$ and $T_c \times \mathbb{C}$ becomes the standard maximal compact torus. The irreducible finite dimensional $U(2)$-modules are labelled by dominant highest weights $\lambda_c = n\gamma_c + d\det \in X^*(T_c \times \mathbb{C})$ (See section (4.1.1), here again $n \geq 0, n \in \mathbb{Z}, n \equiv 2d \mod 2$ and $\gamma_c(e(\underline{\phi})) = e^{i(\phi_1 - \phi_2)/2}$.)

We denote these modules by $\mathcal{M}_{\lambda_c}$ after base change to $\mathbb{C}$ they become the modules $\mathcal{M}_{\lambda,\mathbb{C}}$.

As a subgroup of $G(\mathbb{R}) \subset \mathrm{Gl}_2(\mathbb{C}) \times \mathbb{G}_2(\mathbb{C})$ our torus is

$$T_c = \{\begin{pmatrix} e^{2\pi i\varphi_1} & 0 \\ 0 & e^{2\pi i\varphi_2} \end{pmatrix} \times \begin{pmatrix} e^{-2\pi i\varphi_1} & 0 \\ 0 & e^{-2\pi i\varphi_2} \end{pmatrix}\} \xrightarrow{\sim} \{\begin{pmatrix} e^{2\pi i\varphi_1} & 0 \\ 0 & e^{2\pi i\varphi_2} \end{pmatrix}\} \tag{4.32}$$

and the restriction of χ to T_c is of the form

$$\chi_c(e(\underline{\phi})) = e^{ia\phi_1 + ib\phi_2} = e^{\frac{a-b}{2}(\phi_1-\phi_2)}e^{\frac{a+b}{2}(\phi_1+\phi_2)}. \tag{4.33}$$

and this character is $(a-b)\gamma_c + \frac{a+b}{2}\det$. Then we know

decoKuB

$$^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c = \mathfrak{I}_B^G\chi = \bigoplus_{\mu_c = k\gamma_c + \frac{a+b}{2}\det; k \equiv (a-b) \mod 2; k \geq |a-b|} \mathcal{M}_{\mu_c} \tag{4.34}$$

IndInt

4.1.4 Intertwining operators

Let $N(T)$ the normalizer of $T/\mathbb{R}$, the quotient $W = N(T)/T$ is a finite group scheme. The in our case the group $W(\mathbb{R})$ is cyclic of order 2 and generated by

$$w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

In case A) we have $W(\mathbb{R}) = W(\mathbb{C})$, in case B) we have

$$G \times_{\mathbb{R}} \mathbb{C} = (\mathrm{Gl}_2 \times \mathrm{Gl}_2)/\mathbb{C}\ ;\ T \times_{\mathbb{R}} \mathbb{C} = T_1 \times T_2\ ; \text{ and } W(\mathbb{C}) = \mathbb{Z}/2 \times \mathbb{Z}/2,$$

where the two factors are generated by $s_1 = (w_0, 1), s_2 = (1, w_0)$. The group $W(\mathbb{R})$- the group of real points of the Weyl group- is the cyclic group of order two generated by (w_0, w_0). We call this element also w_0. The group $W(\mathbb{R})$ acts on $T(\mathbb{R})$ by conjugation and hence it also acts on the group $\mathrm{Hom}(T(\mathbb{R}), \mathbb{C}^\times)$ of characters, we denote this action by $\chi \mapsto \chi^w$. We write this group of characters multiplicatively and we define the twisted action

$$w \cdot \chi = (\chi|\rho|)^w |\rho_{\mathbb{R}}|^{-1} \tag{4.35}$$

We recall some well known facts

i) We have a non degenerate $(\mathfrak{g}, K_\infty)$ invariant pairing

$$\mathfrak{I}_B^G\chi \times \mathfrak{I}_B^G\chi^{w_0}|\rho|_{\mathbb{R}}^2 \to \mathbb{C}\omega_\chi^2 \text{ given by } (f_1, f_2) \mapsto \int_{K_\infty} f_1(k) f_2(k) dk \tag{4.36}$$

We define the dual $\mathfrak{I}_B^{G,\vee}\chi$ of a Harish-Chandra as a submodule of $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{I}_B^G\chi, \mathbb{C})$, it consists of those linear maps which vanish on almost all K_∞ types. It is clear that this is again a $(\mathfrak{g}, K_\infty)$-module. The above assertion can be reformulated

ii) We have an isomorphism of $(\mathfrak{g}, K_\infty)$ modules

$$\mathfrak{I}_B^G\chi(\omega_\chi \circ \det)^{-1} \to \mathfrak{I}_B^{G,\vee}\chi^{w_0}|\rho|_{\mathbb{R}}^2 \tag{4.37}$$

The group $T(\mathbb{R}) = T_c \times (\mathbb{R}_{>0}^\times)^2$ and hence we can write any character χ in the form $\boxed{\text{char}}$

$$\chi(t) = \chi_c(t)|t_1|^{z_1}|t_2|^{z_2} = |\frac{t_1}{t_2}|^{\frac{z_1-z_2}{2}}|t_1t_2|^{\frac{z_1+z_2}{2}} \tag{4.38}$$

where $z_1, z_2 \in \mathbb{C}$. We put $z = z_1 - z_2$ and $\zeta = z_1 + z_2$. The relevant variable is z.

For $f \in \mathfrak{I}_B^G\chi, ;\ g \in G(\mathbb{R})$ we consider the integral

$$T_\infty^{\mathrm{loc}}(f)(g) = \int_{U(\mathbb{R})} f(w_0ug)du. \tag{4.39}$$

It is well known and easy to check that these integrals converge absolutely and locally uniformly for $\Re(z) >> 0$ and provide an intertwining operator

$$T_\infty^{\mathrm{loc}}(\chi^{w_0}, z) : \mathfrak{I}_B^G\chi^{w_0}|\rho|_{\mathbb{R}}^z \to \mathfrak{I}_B^G\chi|\rho|_{\mathbb{R}}^2|\rho|_{\mathbb{R}}^{-z}. \tag{4.40}$$

It is also not hard to see that they extend to meromorphic functions in the entire $\mathbb{C}^2$. To see this we recall the decomposition into K_∞ types

$$\mathfrak{I}_B^G\lambda_{\mathbb{R}}^{w_0}|\rho|_{\mathbb{R}}^z = \bigoplus_{\vartheta \in \hat{K}_\infty} {}^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c[\vartheta] = \bigoplus_{\vartheta \in \hat{K}_\infty} \mathfrak{I}_B^G\lambda_{\mathbb{R}}^{w_0}|\rho|_{\mathbb{R}}^z[\vartheta]$$

and our intertwining operator is a direct sum of linear maps between finite dimensional vector spaces

$$c(\lambda_{\mathbb{R}}^{w_0}, z, \vartheta) : \mathfrak{I}_B^G\chi^{w_0}|\rho|_{\mathbb{R}}^z[\vartheta] \to \mathfrak{I}_B^G\chi|\rho|_{\mathbb{R}}^2|\rho|_{\mathbb{R}}^{-z}[\vartheta]$$

The finite dimensional vector spaces do not depend on z and the $c(\lambda_{\mathbb{R}}^{w_0}|\rho|_{\mathbb{R}}^z, \vartheta)$ can be expressed in terms of values of the $\Gamma-$ function. Especially they are meromorphic functions in the variable z (See sl2neu.pdf,). For any $z_0 \in \mathbb{C}$ where we have a pole we can find an integer $m \geq 0$ such that

$$(z - z_0)^m T_\infty^{\mathrm{loc}}(\chi^{w_0}, z)|_{z=z_0} : \mathfrak{I}_B^G\chi^{w_0} \to \mathfrak{I}_B^G\chi|\rho|_{\mathbb{R}}^2$$

is a non zero intertwining operator and this is now our regularized operator $T_\infty^{\mathrm{loc,reg}}(\chi^{w_0})$.

iii) The regularized values define non zero intertwining operators

$$T_\infty^{\mathrm{loc},reg}(\chi^{w_0}, z) : \mathfrak{I}_B^G\chi \to \mathfrak{I}_B^G\chi^{w_0}|\rho|_{\mathbb{R}}^2 \tag{4.41}$$

These operators span the one dimensional space of intertwining operators

$$\mathrm{Hom}_{(\mathfrak{g},K_\infty)}(\mathfrak{I}_B^G\chi, \mathfrak{I}_B^G w_0\cdot\chi).$$

Of course we can translate this into the language of unitary induction, then the intertwining operators map

$$T^{\mathrm{\ loc},un}(\chi) : \mathrm{Indunit}_B^G\chi \to \mathrm{Indunit}_B^G\chi^{w_0}. \tag{4.42}$$

This is definitely a more elegant formulation.

Finally we discuss the question which of these representations are unitary. This means that we have to find a pairing

$$\psi : \mathfrak{I}_B^G\chi \times \mathfrak{I}_B^G\chi \to \mathbb{C} \tag{4.43}$$

which satisfies

a) it is linear in the first and conjugate linear in the second variable

b) It is positive definite, i.e. $\psi(f,f) > 0\,\forall f \in \mathfrak{I}_B^G\chi$

c) It is invariant under the action of K_∞ and Lie-algebra invariant under the action of $\mathfrak{g}$, i.e. we have

For $f_1, f_2 \in \mathfrak{I}_B^G\chi$ and $X \in \mathfrak{g}$ we have $\psi(Xf_1, f_2) + \psi(f_1, Xf_2) = 0.$

We are also interested in quasi-unitatry modules. This is notion is perhaps best explained if and instead of c) we require

d) There exists a continuous homomorphism (a character) $\eta : G(\mathbb{R}) \to \mathbb{R}^\times$ such that for $X \in \mathfrak{g}$ $\psi(Xf_1, gf_2) + \psi(f_1, Xf_2) = d\eta(Xg)\psi(f_1, f_2),\ \forall g \in G(\mathbb{R}), f_1 f_2 \in \mathfrak{I}_B^G\chi$.

It is clear that a non zero pairing ψ which satisfies a) and c) is the same thing as a non zero $(\mathfrak{g}, K_\infty)$-module linear map

$$i_\psi : \mathfrak{I}_B^G\chi \to \overline{(\mathfrak{I}_B^G\chi)^\vee} \tag{4.44}$$

this means that i_ψ is a conjugate linear map from $\mathfrak{I}_B^G\chi$ to $(\mathfrak{I}_B^G\chi)^\vee$. The map i_ψ and the pairing ψ are related by the formula $\psi(v_1, v_2) = i_\psi(v_2)(v_1)$.

Of course we know that (See (4.37))

$$\overline{(\mathfrak{I}_B^G\chi)^\vee} \xrightarrow{\sim} \mathfrak{I}_B^G\overline{\chi^{w_0}|\rho|_{\mathbb{R}}^2\delta_\chi^{-1}} \tag{4.45}$$

and we find such an i_ψ if

$$\chi = \overline{\chi^{w_0}|\rho|_{\mathbb{R}}^2\delta_\chi^{-1}} \text{ or } \chi^{w_0}|\rho|_{\mathbb{R}}^2 = \overline{\chi^{w_0}|\rho|_{\mathbb{R}}^2\delta_\chi^{-1}} \tag{4.46}$$

We write our χ in the form (4.38). A necessary condition for the existence of a hermitian form is of course that all $|\omega_\chi(x)| = 1$ for $x \in Z(\mathbb{R})$ and this means that $\Re(z_1 + z_2) = 0$, hence we write

$$z_1 = \sigma + i\tau_1, z_2 = -\sigma + i\tau_2 \tag{4.47}$$

Then the two conditions in (4.46) simply say

$$(\mathrm{un}_1) : \sigma = \frac{1}{2} \text{ or } (\mathrm{un}_2) : \tau_1 = \tau_2 \text{ and } \chi_c = \chi_c^{w_0} \tag{4.48}$$

In both cases we can write down a pairing which satisfies a) and c). We still have to check b). In the first case, i.e. $\sigma = \frac{1}{2}$ we can take the map $i_\psi = \mathrm{Id}$ and then we get for $f_1, f_2 \in \mathfrak{I}_B^G\chi$ the formula

$$\psi(f_1, f_2) = \int_{K_\infty} f_1(k)\overline{f_2(k)}dk \tag{4.49}$$

and this is clearly positive definite. These are the representation of the *unitary principal series.*

In the second case we have to use the intertwining operator in (4.41) and write

$$\psi(f_1, f_2) = T_\infty^{\mathrm{loc,reg}}(f_2)(f_1) \tag{4.50}$$

Now it is not clear whether this pairing satisfies b). This will depend on the parameter σ. We can twist by a character $\eta : Z(\mathbb{R}) \to \mathbb{C}^\times$ and achieve that $\chi_c = 1, \tau_1 = \tau_2 = 0$. We know that for $\sigma = \frac{1}{2}$ the intertwining operator T_∞^{loc} is regular at χ and since in addition under these conditions $\mathfrak{I}_B^G\chi$ is irreducible we see that

$$T_\infty^{\mathrm{loc}}(\chi) = \alpha \text{ Id with } \alpha \in \mathbb{R}_{>0}^\times \tag{4.51}$$

Since we now are in case A) and B) at the same time we see that the two pairings defined by the rule in case (un_1) and (un_2) differ by a positive real number hence the pairing defined in (4.50) is positive definite if $\sigma = \frac{1}{2}$.

But now we can vary σ. It is well known that $\mathfrak{I}_B^G\chi$ stays irreducible as long as $0 < \sigma < 1$ (See next section) and since $T_\infty^{\mathrm{loc}}(\chi)(f)(f)$ varies continuously we see that (4.50) defines a positive definite hermitian product on $\mathfrak{I}_B^G\chi$ as long as $0 < \sigma < 1$. This is the supplementary series. What happens if we leave this interval will be discussed in the next section.

$\boxed{\text{nontriv}}$

4.1.5 Reducibility and representations with non trivial cohomology

As usual we denote by $\rho \in X^*(T) \otimes \mathbb{Q}$ the half sum of positive roots we have $\rho = \gamma$(resp. $\rho = \gamma_1 + \gamma_2 \in X^*(T) \otimes \mathbb{Q}$) in case A) (resp. B)).

For any character $\lambda \in X^*(T \times \mathbb{C})$ the character $\lambda_\mathbb{R}$ provides a homomorphism $B(\mathbb{R}) \to T(\mathbb{R})$ and hence we get the Harish-Chandra modules $\mathfrak{I}_B^G\lambda_\mathbb{R}$, which are of special interest for us because these are the only ones with non trivial cohomology. We just mention the fact that $\mathfrak{I}_B^G\chi$ is always irreducible unless $\chi = \lambda_{\mathbb{R}}$ for some λ. (See sl2neu.pdf, Condition (red)).

We return to the situation discussed in section (4.1.1), especially we reintroduce the field $F/\mathbb{Q}$. Then we have $X^*(T\times F) = X^*(T\times\mathbb{C})$ and hence $\lambda\in X^*(T\times F)$. We assume that λ is dominant, i.e. $n\geq 0$ in case A) or $n_1, n_2 \geq 0$ in case B). Now we realise our modules $\mathcal{M}_\lambda$ as submodules in the algebra of regular functions on $\mathcal{G}/\mathbb{Z}$: If we look at the definition (See (4.3)) we see immediately that $\mathcal{M}_{\lambda,\mathbb{C}} \subset \mathfrak{I}_B^G\lambda_{\mathbb{R}}^{w_0}$ and hence we get an exact sequence of $(\mathfrak{g}, K_\infty)$ modules $\boxed{\text{seq}}$

$$0 \to \mathcal{M}_{\lambda,\mathbb{C}} \to \mathfrak{I}_B^G\lambda_{\mathbb{R}}^{w_0} \xrightarrow{r} \mathcal{D}_\lambda \to 0 \tag{4.52}$$

Hence we see that $\mathfrak{I}_B^G\lambda_{\mathbb{R}}^{w_0}$ is not irreducible. We can also look at the dual sequence. Here we recall that we wrote $\lambda = n\gamma + d\det$. We consider the dual sequence. Clearly $\mathcal{M}_{\lambda,\mathbb{C}}^\vee = \mathcal{M}_{\lambda-2d\det,\mathbb{C}}$, if we twist the dual sequence by $\det^{2d}$ then dual sequence becomes

$$0 \to \mathcal{D}_\lambda^\vee\otimes\det_{\mathbb{R}}^{2d} \to (\mathfrak{I}_B^G\lambda_{\mathbb{R}}^{w_0})^\vee\otimes\det_{\mathbb{R}}^{2d} \to \mathcal{M}_{\lambda,\mathbb{C}} \to 0 \tag{4.53}$$

Equation (4.37) yields $(\mathfrak{I}_B^G\lambda_{\mathbb{R}}^{w_0})^\vee\otimes\det_{\mathbb{R}}^{2d} \xrightarrow{\sim} \mathfrak{I}_B^G\chi|\rho|_{\mathbb{R}}^2$ and our second sequence becomes

$$0 \to \mathcal{D}_\lambda^\vee\otimes\det_{\mathbb{R}}^{2d} \to \mathfrak{I}_B^G\lambda_{\mathbb{R}}|\rho|_{\mathbb{R}}^2 \to \mathcal{M}_{\lambda,\mathbb{C}} \to 0, \tag{4.54}$$

we put $\mathcal{D}_{\lambda^\vee} := \mathcal{D}_\lambda^\vee\otimes\det_{\mathbb{R}}^{2d}$.

Now we consider the two middle terms in the two exact sequences (4.52,4.54) above. The equation (4.41) claims that we have two non zero *regularized* intertwining operators

$$T_\infty^{\text{loc,reg}}(\lambda_{\mathbb{R}}^{w_0}) : \mathfrak{I}_B^G\lambda_{\mathbb{R}}^{w_0} \to \mathfrak{I}_B^G\lambda_{\mathbb{R}}|\rho|_{\mathbb{R}}^2 \;\; ; T_\infty^{\text{loc,reg}}(\lambda_{\mathbb{R}}|\rho|_{\mathbb{R}}^2) : \mathfrak{I}_B^G\lambda_{\mathbb{R}}|\rho|_{\mathbb{R}}^2 \to \mathfrak{I}_B^G\lambda_{\mathbb{R}}^{w_0} \tag{4.55}$$

If we now look more carefully at our two regularized intertwining operators above then a simple computation yields (see sl2neu.pdf)

Proposition 4.1.2. *The kernel of* $T_\infty^{loc,\text{reg}}(\lambda_{\mathbb{R}}^{w_0})$ *is* $\mathcal{M}_{\lambda,\mathbb{C}}$ *and this operator induces an isomorphism*

$$\bar{T}(\lambda_R) : \mathcal{D}_\lambda \xrightarrow{\sim} \mathcal{D}_\lambda^\vee\otimes\det_{\mathbb{R}}^{2d}$$

Remember λ *is dominant.*

The kernel of $T_\infty^{loc,\text{reg}}(\lambda_{\mathbb{R}}|\rho|_{\mathbb{R}}^2)$ *is* $\mathcal{D}_\lambda^\vee\otimes\det_{\mathbb{R}}^{2d}$ *and it induces an isomorphism of* $\mathcal{M}_{\lambda,\mathbb{C}}$.

The module $\mathfrak{I}_B^G\chi$ *is reducible if* $T_\infty^{loc,\text{reg}}(\chi)$ *not an isomorphism and this happens if an only if* $\chi=\lambda_{\mathbb{R}}$ *or* $\lambda_{\mathbb{R}}^{w_0}|\rho|_{\mathbb{R}}^2$ *and* λ *dominant. (There is one exception to the converse of the above assertion, namely in the case A) and* $\sigma=\frac{1}{2}$ *and* $\chi_c^{w_0}\neq\chi_c$.*) bf Etwas genauery*

Unitarity

For us it is of relevance to know whether we have a positive definite hermitian form on the $(\mathfrak{g}, K_\infty)$-modules $\mathcal{D}_\lambda$. To discuss this question we treat the cases A) and B) separately.

We look at the decomposition into K_∞-types. (See (4.31)) In case A) (See (4.31)) it is clear that $\mathcal{M}_{\lambda,\mathbb{C}}$ is the direct sum of the K_∞ types $\mathbb{C}\psi_l$ with $|l| \leq n$. Hence $\boxed{\text{KTA}}$

$$\mathcal{D}_\lambda = \bigoplus_{k\leq -n-2, k\equiv d(2)} \mathbb{C}\psi_k \oplus \bigoplus_{k\geq n+2, k\equiv d(2)} \mathbb{C}\psi_k = \mathcal{D}_\lambda^- \oplus \mathcal{D}_\lambda^+ \tag{4.56}$$

Proposition 4.1.3. *The representations $\mathcal{D}_\lambda^-, \mathcal{D}_\lambda^+$ are irreducible, these are the discrete series representations. The element* $\mathbf{c}$ *interchanges $\mathcal{D}_\lambda^-, \mathcal{D}_\lambda^+$, hence $\mathcal{D}_\lambda$ is an irreducible $(\mathfrak{g}, \tilde{K}_\infty^*)$ module.*

The operator $\bar{T}(\lambda_R)$ induces a quasi-unitary structure on the $(\mathfrak{g}, \tilde{K}_\infty)$-module $\mathcal{D}_\lambda$. The two sets : K_∞ types occurring in $\mathcal{M}_{\lambda,\mathbb{C}}$ and K_∞ types occuring in $\mathcal{D}_\lambda$ (resp.) are disjoint.

Proof. Remember that as a vector space $\mathcal{D}_\lambda^\vee \otimes \det_\mathbb{R}^{2d} = \mathcal{D}_\lambda^\vee$, only the way how $\tilde{K}_\infty$ acts is twisted by $\det_\mathbb{R}^{2d}$ . .Then the form

$$h_\psi(f_1, f_2) = T_\infty^{\text{loc,reg}}(\lambda_\mathbb{R}^{w_0})(f_2)(f_1) \tag{4.57}$$

defines a quasi invariant hermitian form. It is positive definite (for more details see sl2neu.pdf). □

A similar argument works in case B).We restrict the $\mathrm{Gl}_2(\mathbb{C})\times\mathrm{Gl}_2(\mathbb{C})$ module $\mathcal{M}_{\lambda,\mathbb{C}}$ to $U(2)\times U(2)$ then it becomes the highest weight module $\mathcal{M}_{\lambda_c} = \mathcal{M}_{\lambda_{1,c}} \otimes \mathcal{M}_{\lambda_{2,c}}$. (See4.1.1) Under the action of $U(2) \subset U(2) \times U(2)$ it decomposes into $U(2)$ types according to the Clebsch-Gordan formula $\boxed{\text{CG}}$

$$\mathcal{M}_{\lambda_c}|_{U(2)} = \bigoplus_{\mu_c = k\gamma_c + \frac{d_1+d_2}{2}\det;\ k\equiv(n_1-n_2) \mod 2;\ n_1+n_2\geq k\geq|n_1-n_2|} \mathcal{M}_{\mu_c} \tag{4.58}$$

Hence we get $\boxed{\text{KTB}}$

$$\mathcal{D}_{\lambda_c}|_{U(2)} = \bigoplus_{\mu_c = k\gamma_c + \frac{d_1+d_2}{2}\det;\ k\equiv(n_1-n_2) \mod 2;\ k\geq n_1+n_2+2} \mathcal{M}_{\mu_c} \tag{4.59}$$

Again we have $\boxed{\text{unit}}$

Proposition 4.1.4. *The operator $T_\infty^{loc,\text{reg}}(\lambda_\mathbb{R}^{w_0})$ induces an isomorphism*

$$\bar{T}(\lambda_R) : \mathcal{D}_\lambda \xrightarrow{\sim} \mathcal{D}_\lambda^\vee \otimes \det_\mathbb{R}^{2d}$$

The $(\mathfrak{g}, K_\infty)$ modules are irreducible.

The operator $T_\infty^{loc,\text{reg}}(\lambda_\mathbb{R}^{w_0})$ induces the structure of a quasi-unitary module on $\mathcal{D}_\lambda$ if and only if $n_1 = n_2$. This is the only case when we have a quasi-unitary structure on $\mathcal{D}_\lambda$. The two sets of K_∞ types occurring in $\mathcal{M}_{\lambda,\mathbb{C}}$ and in $\mathcal{D}_\lambda$ (resp.) are disjoint.

The Weyl W group acts on T by conjugation, hence on $X^*(T\times\mathbb{C})$ and we define the *twisted action* by

$$s\cdot\lambda = s(\lambda+\rho)-\rho \tag{4.60}$$

Given a dominant λ we may consider the four characters $w\cdot\lambda, w\in W(\mathbb{C}) = W$ and the resulting induced modules $\mathfrak{I}_B^G w\cdot\lambda_{\mathbb{R}}$. We observe (notation from (4.1.1))

$$\begin{aligned} s_1\cdot(n_1\gamma+d_1\det+n_2\bar\gamma+d_2\overline{\det}) &= (-n_1-2)\gamma+d_1\det+n_2\bar\gamma+d_2\overline{\det})\\ s_2\cdot(n_1\gamma+d_1\det+n_2\bar\gamma+d_2\overline{\det}) &= n_1\gamma+d_1\det+(-n_2-2)\bar\gamma+d_2\overline{\det}) \end{aligned} \tag{4.61}$$

Looking closely we see that the K_∞ types occurring in $\mathfrak{I}_B^G s_1\cdot\lambda$ or $\mathfrak{I}_B^G s_2\cdot\lambda$ are exactly those which occur in $\mathcal{D}_\lambda$. This has a simple explanation, we have $\boxed{\text{exiso}}$

Proposition 4.1.5. *For a dominant character λ we have isomorphisms between the $(\mathfrak{g},K_\infty)$ modules*

$$\mathcal{D}_\lambda\xrightarrow{\sim}\mathfrak{I}_B^G s_1\cdot\lambda,\ \mathcal{D}_\lambda\xrightarrow{\sim}\mathfrak{I}_B^G s_2\cdot\lambda. \tag{4.62}$$

The resulting isomorphism $\mathfrak{I}_B^G s_1\cdot\lambda_{\mathbb{R}}\xrightarrow{\sim}\mathfrak{I}_B^G s_2\cdot\lambda_{\mathbb{R}}$ is of course given by $T_\infty^{\mathrm{loc}}(s_1\cdot\lambda)$.

Interlude: Here we see a fundamental difference between the two cases A) and B). In the second case the infinite dimensional subquotients of the induced representations are again induced representations. In the case A) this is not so, the representations $\mathcal{D}_\lambda^\pm$ are not isomorphic to representations induced from the Borel subgroup.

These representation $\mathcal{D}_\lambda^\pm$ are called *discrete series* representations and we want to explain briefly why. Let G be the group of real points of a reductive group over $\mathbb{R}$ for example our $G=G(\mathbb{R})$, here we allow both cases. Let Z be the center of G, it can be written as $Z_0(\mathbb{R})\cdot Z_c$ where Z_c is maximal compact and $Z_0=(\mathbb{R}^\times_{>0})^t$. Let $\omega^{(0)}:Z_0\to\mathbb{R}^\times_{>0}$ be a character. Then we define the space

$$\mathcal{C}_\infty(G,\omega_R) := \{f\in\mathcal{C}(G)\mid f(zg)=\omega^{(0)}(z)f(g)\ ;\forall z\in Z_0, g\in G\} \tag{4.63}$$

and we define the subspace

$$L^2_\infty(G,\omega_R) := \{f\in\mathcal{C}_\infty(G,\omega_R)\mid \int_G f(g)\overline{f(g)}(\omega^{(0)}(g))^{-2}dg<\infty\} \tag{4.64}$$

where of course dg is a Haar measure. As usual $L^2(G,\omega_R)$ will be the Hilbert space obtained by completion. This Hilbert space only depends in a very mild way on the choice of $\omega^{(0)}$ we can find a character $\delta: G\to\mathbb{R}^\times_{>0}$ such that $\omega^{(0)}\delta|_{Z_0}=1$. Then $f\mapsto f\delta$ provides an isomorphism $L^2(G,\omega^{(0)})\xrightarrow{\sim}L^2(G/Z_0)$.

We have an action of $G\times G$ on $L^2(G,\omega^{(0)})$ by left and right translations. Then Harish-Chandra has investigated the question how this "decomposes" into irreducible submodules. Let $\hat{G}_{\omega^{(0)}}$ be the set of isomorphism classes of irreducible unitary representations of G.

Harish-Chandra shows that there exist a positive measure μ on $\hat{G}_{\omega^{(0)}}$ and a measurable family H_ξ of irreducible unitary representations of G such that

$$L^2(G,\omega_{\mathbb{R}}) = \int_{\hat{G}_{\omega_{\mathbb{R}}}} H_\xi \otimes \overline{H_\xi}\ \mu(d\xi) \tag{4.65}$$

(If instead of a semi simple Lie group we take a finite group G then this is the fundamental theorem of Frobenius that the group ring $\mathbb{C}[G] = \oplus_\theta V_\theta \otimes V_\theta^\vee$ where V_θ are the irreducible representations.)

If we are in the case A), the sets consisting of just one point $\{\mathcal{D}_\lambda^\pm\}$ have strictly positive measure, i.e. $\mu(\{\mathcal{D}_\lambda^\pm\}) > 0$. This means that the irreducible unitary $G \times G$ modules $\mathcal{D}_\lambda^\pm \otimes \mathcal{D}_{\lambda^\vee}^\pm$ occur as direct summand (i.e. discretely in $L^2(G)$.).

Such irreducible direct summands do not exist in the case B), in this case for any $\xi \in \hat{G}$ we have $\mu(\{\xi\}) = 0$.

End Interlude

We return to the sequences (4.52),(4.54). We claim that both sequences do do not split as sequences of $(\mathfrak{g}, K_\infty)$-modules. Of course it follows from the above proposition that these sequences split canonically as sequence of K_∞ modules. But one sees easily that complementary summand is not invariant under the action of $\mathfrak{g}$. This means that the sequence provides a non trivial classes in $\mathrm{Ext}^1_{(\mathfrak{g},K_\infty)}(\mathcal{D}_\lambda, \mathcal{M}_{\lambda,\mathbb{C}})$.

The general principles of homological algebra teach us that we can understand these extension groups in terms of relative Lie-algebra cohomology. Let $\mathfrak{k}$ resp. $\tilde{\mathfrak{k}}$ be the Lie-algebras of K_∞ resp. $\tilde{K}_\infty$ the group $\tilde{K}_\infty$ acts on $\mathfrak{g}, \tilde{\mathfrak{k}}$ via the adjoint action (see 1.1.4)

We start from a $(\mathfrak{g}, \tilde{K}_\infty)$ module $\mathfrak{I}_B^G\chi$ and a module $\mathcal{M}_{\lambda,\mathbb{C}}$. Our first goal is to compute the cohomology $H^\bullet(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}})$ which is defined as the cohomology of the complex (See 6.1.2, (6.3))

$$H^\bullet(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}}) :=\ \mathrm{Hom}_{\tilde{K}_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathfrak{I}_B^G\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}}). \tag{4.66}$$

Here we only assume that $\chi : T(\mathbb{R}) \to \mathbb{C}^\times$ is any character, we will see that there is only one χ for which we have non trivial cohomology.

There is an obvious condition for the complex to be non zero. The group $Z(\mathbb{R}) \subset \tilde{K}_\infty$ acts trivially on $\mathfrak{g}/\mathfrak{k}$ and hence we see that the complex is trivial unless we have

$$\omega_\chi^{-1} = \lambda_{\mathbb{R}}|_{Z(\mathbb{R})^{(0)}} \tag{4.67}$$

we assume that this relation holds.

We will derive a formula for these cohomology modules. This formula is a special case of a formula of Delorme. which will be discussed in greater generality in Chapter 9.

An element $\omega \in\ \mathrm{Hom}_{\tilde{K}_\infty}(\Lambda^n(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathfrak{I}_B^G\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}})$ attaches to any n tuple $v_1, \ldots, v_n$ of elements in $\mathfrak{g}/\tilde{\mathfrak{k}}$ an element

$$\omega(v_1, \ldots, v_n) \in \mathfrak{I}_B^G\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}} \tag{4.68}$$

such that $\omega(\mathrm{Ad}(k)v_1,\dots,\mathrm{Ad}(k)v_n) = k\omega(v_1,\dots,v_n)$ for all $k \in \tilde{K}_\infty$.

By construction

$$\omega(v_1,\dots,v_n) = \sum f_\nu \otimes m_\nu \text{ where } f_\nu \in \mathfrak{I}_B^G\chi, m_\nu \in \mathcal{M}_{\lambda,\mathbb{C}}$$

and f_ν is a function in $\mathcal{C}_\infty$ which is determined by its restriction to $\tilde{K}_\infty$ (and this restriction is $\tilde{K}_\infty$ finite). We can evaluate this function at the identity $e_G \in G(\mathbb{R})$ and then

$$\omega(v_1,\dots,v_n)(e_G) = \sum f_\nu(e) \otimes m_\nu \in \mathbb{C}\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}}.$$

The $\tilde{K}_\infty$ invariance (4.68) implies that ω is determined by this evaluation at e_G. Let $\tilde{K}_\infty^T = T(\mathbb{R}) \cap \tilde{K}_\infty = Z(\mathbb{R}) \cdot T_c$. Then it is clear that

$$\omega^* : \{v_1,\dots,v_n\} \mapsto \omega(v_1,\dots,v_n)(e_G) \tag{4.69}$$

is an element in

$$\omega^* \in \mathrm{Hom}_{\tilde{K}_\infty^T}(\Lambda^n(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathbb{C}\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}}) \tag{4.70}$$

and we have: The map $\omega \mapsto \omega^*$ is an isomorphism of complexes $\boxed{\text{iso1}}$. **Besser machen**

$$\mathrm{Hom}_{\tilde{K}_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathfrak{I}_B^G\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}}) \xrightarrow{\sim} \mathrm{Hom}_{\tilde{K}_\infty^T}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathbb{C}\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}}) \tag{4.71}$$

The Lie algebra $\mathfrak{g}$ can be written as a sum of $\mathfrak{c}$ invariant submodules

$$\mathfrak{g} = \mathfrak{b} + \tilde{\mathfrak{k}} = \mathfrak{t} + \mathfrak{u} + \tilde{\mathfrak{k}} \tag{4.72}$$

in case B) this sum is not direct, we have $\mathfrak{b} \cap \tilde{\mathfrak{k}} = \mathfrak{t} \cap \tilde{\mathfrak{k}} = \mathfrak{c}$ and hence we get the direct sum decomposition into $\tilde{K}_\infty^T$-invariant subspaces

$$\mathfrak{g}/\tilde{\mathfrak{k}} = \mathfrak{t}/\mathfrak{c} \oplus \mathfrak{u}. \tag{4.73}$$

We get an isomorphism of complexes $\boxed{\text{isodel}}$

$$\mathrm{Hom}_{\tilde{K}_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathfrak{I}_B^G\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}}) \xrightarrow{\sim} \mathrm{Hom}_{\tilde{K}_\infty^T}(\Lambda^\bullet(\mathfrak{t}/\tilde{\mathfrak{k}}), \mathbb{C}\chi \otimes \mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}), \mathcal{M}_{\lambda,\mathbb{C}})) \tag{4.74}$$

the complex on the left is isomorphic to the total complex of the double complex on the right. The next step is the computation of the cohomology of the complex $\mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}), \mathcal{M}_{\lambda,\mathbb{C}})$.

Case A). We have $\mathfrak{u} = \mathbb{Q}E_+$ where $E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and our module $\mathcal{M}_{\lambda,\mathbb{Q}}$ has a decomposition into weight spaces

$$\mathcal{M}_{\lambda,\mathbb{Q}} = \bigoplus_{\nu=0}^{\nu} \mathbb{Q}X^{n-\nu}Y^\nu = \bigoplus_{\mu=-n,\mu\equiv n(2)}^{\mu=n} \mathbb{Q}e_\mu. \tag{4.75}$$

The torus $T^{(1)} = \{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\}$ acts on $e_\mu = X^{n-\nu}Y^\nu$ by

$$\rho_\lambda(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix})e_\mu = t^\mu e_\mu \tag{4.76}$$

We also have the action of the Lie algebra on $\mathcal{M}_{\lambda,\mathbb{Q}}$ and by definition we get

$$d(\rho_\lambda)(E_+)e_\mu = E_+e_\mu = \frac{n-\mu}{2}e_{\mu+2} \tag{4.77}$$

Now we can write down our complex $\mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}),\mathcal{M}_{\lambda,\mathbb{C}})$ very explicitly. Let $E_+^\vee \in \mathrm{Hom}(\mathfrak{u},\mathbb{Q})$ be the element $E_+^\vee(E_+)=1$ then the complex becomes

$$0 \to \bigoplus_{\mu=-n,\mu\equiv n(2)}^{\mu=n} \mathbb{Q}e_\mu \xrightarrow{d} \bigoplus_{\mu=-n,\mu\equiv n(2)}^{\mu=n} \mathbb{Q}E_+^\vee \otimes e_\mu \to 0 \tag{4.78}$$

where $d(e_\mu) = \frac{n-\mu}{2}E_+^\vee \otimes e_{\mu+2}$. This gives us a decomposition of our complex into two sub complexes

$$\mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}),\mathcal{M}_{\lambda,\mathbb{C}}) = \mathbb{H}^\bullet(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}}) \oplus AC^\bullet \tag{4.79}$$

where $AC^\bullet$ is acyclic (it has no cohomology) and

$$\mathbb{H}^\bullet(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}}) = \{0 \to \mathbb{Q}\, e_n \xrightarrow{d} \mathbb{Q}\, E_+^\vee \otimes e_{-n} \to 0\}, \tag{4.80}$$

where the differential $d=0$. Hence we get

$$H^\bullet(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}}) = H^\bullet(\,\mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}),\mathcal{M}_{\lambda,\mathbb{Q}})) = \mathbb{H}^\bullet(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}}). \tag{4.81}$$

We notice that the torus T acts on $H^\bullet(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}})$ (The Borel subgroup B acts on the complex $\mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}),\mathcal{M}_{\lambda,\mathbb{Q}})$ but since the Lie algebra cohomology is the derived functor of taking invariants under U (elements annihilated by $\mathfrak{u}$) it follows that this action is trivial on U). Now it is clear that (4.74) yields

$$H^\bullet(\mathfrak{g},K_\infty,\mathfrak{I}_B^G\chi\otimes\mathcal{M}_{\lambda,\mathbb{C}}) \xrightarrow{\sim} H^\bullet(\mathfrak{t},K_\infty^T,\mathbb{C}\chi\otimes\mathbb{H}^\bullet(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}})) \tag{4.82}$$

Hence we see that T acts by the character λ on $\mathbb{Q}\ e_n = H^0(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}})$ and by the character $\lambda^- - \alpha = w_0\cdot\lambda = \lambda^{w_0} - 2\rho$ on $\mathbb{Q}\ E_+^\vee \otimes e_{-n} = H^1(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}})$. Here we see the simplest example of the famous theorem of Kostant which will be discussed in Chap. 8 section **??**

Then our cohomology groups $H^\bullet(\mathfrak{t},K_\infty^T,\mathbb{C}\chi\otimes\mathbb{H}^\bullet(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}}))$ are given as the cohomology groups of the double complex with entries $\mathrm{Hom}_{K_\infty^T}(\Lambda^p(\mathfrak{t}/\mathfrak{k})\mathbb{C}\chi\otimes \mathbb{H}^q(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}})$ where $p=0,1, q=0,1$ and where the differentials in direction q are zero. We have to compute the cohomology of the complexes

$$0 \to \mathrm{Hom}_{K_\infty^T}(\Lambda^0(\mathfrak{t}/\mathfrak{k}),\mathbb{C}\chi\otimes\mathbb{H}^q(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}})) \xrightarrow{d} \mathrm{Hom}_{K_\infty^T}(\Lambda^1(\mathfrak{t}/\mathfrak{k}),\mathbb{C}\chi\otimes\mathbb{H}^q(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}})) \to 0 \tag{4.83}$$

In this complex we drop the subscript K_∞^T then both spaces in the complex are one dimensional and the differential is up to a non zero factor multiplication by $d\chi(H) + d(w\cdot\lambda)(H)$ and hence we have zero cohomology unless we have $d\chi(H)+d(w\cdot\lambda)(H)=0$. Hence we see (observe that $q=l(w)$)

$$H^\bullet(\mathfrak{t},K_\infty^T,\mathbb{C}\chi\otimes\mathbb{H}^q(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}})) \neq 0 \implies \chi|T(\mathbb{R})^{(0)} = (w\cdot\lambda)_\mathbb{R}^{-1}|T(\mathbb{R})^{(0)}.$$

We now reintroduce the subscript K_∞^T. Since clearly $K_\infty^T\cdot T(\mathbb{R})^{(0)} = T(\mathbb{R})$ we see that we have non trivial cohomology if and only if $\chi = (w\cdot\lambda)_\mathbb{R}^{-1}$. Putting everything together we see

$$H^{\bullet+l(w)}(\mathfrak{g},K_\infty,\mathfrak{I}_B^G\chi\otimes\mathcal{M}_{\lambda,\mathbb{Q}})=\begin{cases}\mathbb{H}^{l(w)}(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}}))\wedge\Lambda^\bullet(\mathfrak{t}/\mathfrak{k})^\vee & \text{if } \chi=(w\cdot\lambda)_{\mathbb{R}}^{-1}\\ 0 & \text{else}\end{cases}\tag{4.84}$$

Now we tensorize the sequence (4.52) with the dual $\mathcal{M}_{\lambda^\vee}$ we get an exact sequence of $(\mathfrak{g},K_\infty)$ modules and we look at the resulting long exact sequence in cohomology. We know that $H^1(\mathfrak{g},K_\infty,\mathcal{M}_\lambda\otimes M_{\lambda^\vee})=0$ and then we look at the piece

$$0\to H^1(\mathfrak{g},K_\infty,\mathfrak{I}_B^G\lambda^{w_0}\otimes\mathcal{M}_{\lambda^\vee})\to H^1(\mathfrak{g},K_\infty,\mathcal{D}_\lambda\otimes\mathcal{M}_{\lambda^\vee})\to H^2(\mathfrak{g},K_\infty,\mathcal{M}_\lambda\otimes\mathcal{M}_{\lambda^\vee})\to 0\tag{4.85}$$

We have seen and we know that the two extreme terms are equal to $\mathbb{C}$ and then we get easily

$$H^1(\mathfrak{g},K_\infty,\mathcal{D}_\lambda\otimes\mathcal{M}_{\lambda^\vee})=\mathbb{C}\oplus\mathbb{C}\tag{4.86}$$

and vanishes in all other degrees.

Of course we can get this last result easily if we look at the complex $\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),\mathcal{D}_\lambda\otimes\mathcal{M}_{\lambda^\vee})$ which in this situation collapses to

$$0\to\ \mathrm{Hom}_{K_\infty}(\Lambda^1(\mathfrak{g}//\mathfrak{k},\mathcal{D}_\lambda\otimes\mathcal{M}_{\lambda^\vee})\to 0\to ..,\tag{4.87}$$

in section 4.1.11 we give explicit elements $\omega_\pm^\dagger\in\ \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),\mathcal{D}_\lambda\otimes\mathcal{M}_{\lambda^\vee})$ which form a basis for this space.

We discuss briefly the case B). Again we want that our group $G/\mathbb{R}=R_{\mathbb{C}/\mathbb{R}}(\mathrm{Gl}_2/\mathbb{C})$ is a base change from a group $G/\mathbb{Q}$ denoted by the same letter. We need an imaginary quadratic extension $F/\mathbb{Q}$ and put $G/\mathbb{Q}=R_{F/\mathbb{Q}}(\mathrm{Gl}_2/F)$. We choose a dominant weight $\lambda=\lambda_1+\lambda_2=n_1\gamma_1+d_1\det_1+n_2\gamma_2+d_2\det_2$ and then $\mathcal{M}_{\lambda,F}=\mathcal{M}_{\lambda_1,F}\otimes\mathcal{M}_{\lambda_2,F}$ is an irreducible representation of $G\times_{\mathbb{Q}}F=\mathrm{Gl}_2\times\mathrm{Gl}_2/F$. Now we have $\mathfrak{u}\otimes F=FE_+^1\oplus FE_+^2$. Then basically the same computation yields:

The cohomology $H^\bullet(\mathfrak{u},\mathcal{M}_{\lambda,F})$ is equal the complex

$$\begin{aligned}\mathbb{H}^\bullet(\mathfrak{u},\mathcal{M}_{\lambda,F})=\{0\to Fe_{n_1}^{(1)}\otimes Fe_{n_2}^{(2)}\xrightarrow{d}FE_+^{1,\vee}\otimes e_{-n_1}^{(1)}\otimes e_{n_2}^{(2)}\oplus FE_+^{1,\vee}\otimes e_{n_1}^{(1)}\otimes E_+^{2,\vee}\otimes e_{-n_2}^{(2)}\\ \xrightarrow{d}FE_+^{1,\vee}\otimes e_{-n_1}^{(1)}\otimes E_+^{2,\vee}\otimes e_{-n_2}^{(2)}\to 0\}\end{aligned}\tag{4.88}$$

where all the differentials are zero. The torus T acts by the weights

$$\lambda\text{ in degree }0,\ s_1\cdot\lambda,s_2\cdot\lambda\text{ in degree }1,\ w_0\cdot\lambda\text{ in degree }2\tag{4.89}$$

and we have a decomposition into one dimensional weight spaces

$$H^\bullet(\mathfrak{u},\mathcal{M}_{\lambda,F})=\bigoplus_{w\in W(\mathbb{C})}\mathbb{H}^\bullet(\mathfrak{u},\mathcal{M}_{\lambda,F})(w\cdot\lambda)$$

We go back to (4.74) and get a homomorphism of complexes

$$\mathrm{Hom}_{\mathfrak{c}}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathbb{C}\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}}) \to \mathrm{Hom}_{\tilde{K}_\infty \otimes T}(\Lambda^\bullet(\mathfrak{t}/\tilde{\mathfrak{k}}), \mathbb{C}\chi \otimes \mathbb{H}^\bullet(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{C}})) \tag{4.90}$$

which induces an isomorphism in cohomology so that finally

$$H^\bullet(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G \chi \otimes \mathcal{M}_{\lambda,\mathbb{C}}) \xrightarrow{\sim} H^\bullet(\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{t}/\tilde{\mathfrak{k}}), \mathbb{C}\chi \otimes H^\bullet(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{C}})) \tag{4.91}$$

and combining this with the results above we get $\boxed{\text{cohlam}}$

Theorem 4.1.2. *If we can find an element $w \in W(\mathbb{C})$ such that $\chi^{-1} = w \cdot \lambda_{\mathbb{R}}$ then*

$$H^{l(w)}(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{C}})(w \cdot \lambda) \otimes \Lambda^\bullet(\mathfrak{t}/\tilde{\mathfrak{k}})^\vee \xrightarrow{\sim} H^\bullet(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G \chi \otimes \mathcal{M}_{\lambda,\mathbb{C}})$$

If there is no such w then the cohomology is zero.

Proof. Our torus $T(\mathbb{R}) = \mathfrak{c} \times \{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} ;\ t \in \mathbb{R}_{>0}^\times\} = \mathfrak{c} \times A$. Hence we see that $\dim \mathfrak{t}/\tilde{\mathfrak{k}} = 1$, and the element $H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Of course we must have that $\chi^{-1} \cdot \lambda_{\mathbb{R}}|_{\mathfrak{c}}$ is the trivial character. The second factor A does acts on $\mathbb{C}\chi$ by the character $\chi(t) = t^z$ and on $H^{l(w)}(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{C}})(w \cdot \lambda)$ by $t \mapsto t^{m(w)}$. Differentiating we get for the complex

$$0 \to H^{l(w)}(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{C}})(w \cdot \lambda) \to \mathbb{C} \otimes H_0^\vee \otimes H^{l(w)}(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{C}})(w \cdot \lambda) \to 0 \tag{4.92}$$

where the differential is multiplication by $m(w) + z$. Hence we see that the cohomology is trivial unless $m(w) + z = 0$, but this means $\chi^{-1} = w \cdot \lambda_{\mathbb{R}}$. $\square$

4.1.6 The cohomology of the modules $\mathcal{M}_{\lambda,\mathbb{C}}$, $\mathcal{D}_\lambda$ and the cohomology of unitary modules

Let $\mathrm{Irr}(G, K_\infty)$ be the set of isomorphism classes of irreducible $(\mathfrak{g}, K_\infty)$-Harish-Chandra-modules, we are a little bit pedantic, if $\mathcal{V}$ is such an irreducible module, then its isomorphism class is $[\mathcal{V}]$. For any dominant λ we define the sets

$$\mathrm{Coh}(\lambda) = \{[\mathcal{V}] \in \mathrm{Irr}(G, K_\infty) \mid H^\bullet(\mathfrak{g}, K_\infty, \mathcal{V} \otimes \mathcal{M}_{\lambda,\mathbb{C}}) \neq 0\} \tag{4.93}$$

We also define $\mathrm{Coh}_2(\lambda)$, this are those $[\mathcal{V}]$ which in addition are unitary. This definition makes sense in greater generality (see 6.25). In our special case there these sets are very small. Remember that we have a fixed central character ω.

At first we determine the finite dimensional elements in $\mathrm{Coh}(\lambda)$. Of course $\mathcal{M}_{\lambda,\mathbb{C}}$ itself is a Harish-Chandra module and it follows from Wigner's lemma that $H^\bullet(\mathfrak{g}, K_\infty, \mathcal{M}_{\lambda,\mathbb{C}})) = 0$ unless $\lambda^{(1)} = 0$, i.e. $\mathcal{M}_{\lambda,\mathbb{C}}$ is one dimensional. Then it follows from Clebsch- Gordan that

Proposition 4.1.6. *In case A)*

$$\begin{gathered} H^0(\mathfrak{g}, K_\infty, \mathcal{M}_{\lambda^\vee,\mathbb{C}} \otimes \mathcal{M}_{\lambda,\mathbb{C}}) = H^2(\mathfrak{g}, K_\infty, \mathcal{M}_{\lambda^\vee,\mathbb{C}} \otimes \mathcal{M}_{\lambda,\mathbb{C}}) = \mathbb{C}, \\ H^1(\mathfrak{g}, K_\infty, \mathcal{M}_{\lambda^\vee,\mathbb{C}} \otimes \mathcal{M}_{\lambda,\mathbb{C}}) = 0 \end{gathered} \tag{4.94}$$

In case B)

$$\begin{aligned} H^0(\mathfrak{g}, K_\infty, \mathcal{M}_{\lambda^\vee,\mathbb{C}} \otimes \mathcal{M}_{\lambda,\mathbb{C}}) = H^3(\mathfrak{g}, K_\infty, \mathcal{M}_{\lambda^\vee,\mathbb{C}} \otimes \mathcal{M}_{\lambda,\mathbb{C}}) = \mathbb{C}, \\ H^1(\mathfrak{g}, K_\infty, \mathcal{M}_{\lambda^\vee,\mathbb{C}} \otimes \mathcal{M}_{\lambda,\mathbb{C}}) = H^2(\mathfrak{g}, K_\infty, \mathcal{M}_{\lambda^\vee,\mathbb{C}} \otimes \mathcal{M}_{\lambda,\mathbb{C}}) = 0 \end{aligned} \tag{4.95}$$

Here we take notice of a point, which plays a role if it comes to questions concerning orientability. In case A) we can twist the $G(\mathbb{R})$ module $\mathcal{M}_{\lambda^\vee,\mathbb{C}}$ by the sign character $\eta : g \mapsto \operatorname{sgn}(\det(g))$, it has the same central character. Obviously the twisted module $\mathcal{M}_{\lambda^\vee,\mathbb{C}} \otimes \eta$ provides the same $(\mathfrak{g}, K_\infty)$-module. But this depends on the choice of K_∞, if we replace K_∞ by the larger group K_∞^* (see section 4.1.3) then the $(\mathfrak{g}, K_\infty^*)$ modules $\mathcal{M}_{\lambda^\vee,\mathbb{C}}$ and $\mathcal{M}_{\lambda^\vee,\mathbb{C}} \otimes \eta$ are not isomorphic. If we replace in the above proposition K_∞ by K_∞^* and $\mathcal{M}_{\lambda^\vee,\mathbb{C}}$ by $\mathcal{M}_{\lambda^\vee,\mathbb{C}} \otimes \eta$, then the cohomology vanishes in all degrees.

Small remark: In general it is sapient to work with a connected K_∞ or $\tilde{K}_\infty$ and then keep track of the action of K_∞^* on $H^\bullet(\mathfrak{g}, K_\infty, \mathcal{V} \otimes \mathcal{M}_{\lambda,\mathbb{C}})$.

Again we start from a dominant character λ. Then our considerations yield that in case A)

$$\operatorname{Coh}(\lambda^\vee) = \{\mathcal{M}_{\lambda,\mathbb{C}}, \mathcal{D}_\lambda^+, \mathcal{D}_\lambda^-\} \tag{4.96}$$

we even have $\mathcal{D}_\lambda^+, \mathcal{D}_\lambda^- \in \operatorname{Coh}_2(\lambda^\vee)$ and $\mathcal{M}_{\lambda,\mathbb{C}} \in \operatorname{Coh}_2(\lambda^\vee)$ if and only if $\lambda^{(1)} = 0$.

For some reason we call $\{\mathcal{D}_\lambda^+, \mathcal{D}_\lambda^-\} = \operatorname{Coh}_{\text{cusp}}(\lambda^\vee)$ and $\{\mathcal{M}_{\lambda,\mathbb{C}}\} = \operatorname{Coh}_{\text{Eis}}(\lambda^\vee)$

in case B) we take the tensor product of the exact sequence (4.52) by $\mathcal{M}_{\lambda^\vee,\mathbb{C}}$ and we get a long exact sequence of $(\mathfrak{g}, K_\infty)$ cohomology modules (we insert the values for $H^\bullet(\mathfrak{g}, K_\infty, \mathcal{M}_{\lambda,\mathbb{C}} \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}})$)

$$\begin{gathered} 0 \to \mathbb{C} \to H^0(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G \lambda_\mathbb{R}^{w_0} \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \xrightarrow{r^0} H^0(\mathfrak{g}, K_\infty, \mathcal{D}_\lambda \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}})(= 0) \\ \to 0 \to H^1(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G \lambda_\mathbb{R}^{w_0} \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \xrightarrow{r^1} H^1(\mathfrak{g}, K_\infty, \mathcal{D}_\lambda \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \to \\ 0 \to H^2(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G \lambda_\mathbb{R}^{w_0} \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \xrightarrow{r^2} H^2(\mathfrak{g}, K_\infty, \mathcal{D}_\lambda \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \\ \to \mathbb{C} \to H^3(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G \lambda_\mathbb{R}^{w_0} \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \xrightarrow{r^3} 0 \end{gathered} \tag{4.97}$$

The homomorphisms r^1, r^2 are isomorphisms and all the $H^1, H^2 = \mathbb{C}$. Hence we see that in this case

$$\operatorname{Coh}(\lambda^\vee) = \{\mathcal{M}_{\lambda,\mathbb{C}}, \mathcal{D}_\lambda\} \tag{4.98}$$

and

$$\operatorname{Coh}_2(\lambda^\vee) = \begin{cases} \{\mathcal{M}_{\lambda,\mathbb{C}}, \mathcal{D}_\lambda\} & \text{if } \lambda^{(1)} = 0 \\ \{\mathcal{D}_\lambda\} & \text{if } n_1 = n_2 > 0 \end{cases} \tag{4.99}$$

$\boxed{\text{EiShiso}}$

4.1.7 The Eichler-Shimura Isomorphism

We want to apply these facts about representation theory to the study of cohomology groups $H^\bullet(\Gamma\backslash X, \mathcal{M}_{\lambda,\mathbb{C}})$ where now Γ is a congruence subgroup of $\mathrm{Gl}_2(\mathbb{Z})$ or $\mathrm{Gl}_2(\mathcal{O})$.

We start again from a dominant weight $\lambda = n\gamma + d\det \in X^*(T \times \mathbb{C})$. Every $(\mathfrak{g}, K_\infty)$ invariant homomorphism $\Psi_\lambda : \mathfrak{I}_B^G w \cdot \lambda_{\mathbb{R}} \to \mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R}))$ induces a homomorphism

$$\Psi_\Lambda : \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathfrak{I}_B^G w \cdot \lambda_{\mathbb{R}} \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \to \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R})) \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \tag{4.100}$$

We will show in section 6.1.3 Proposition 6.1.1 that the complex on the right is isomorphic to the de-Rham complex:

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R}) \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \xrightarrow{\sim} \Omega^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_{\lambda^\vee,\mathbb{C}}) \tag{4.101}$$

This de-Rham complex computes the cohomology and hence we get an homomorphism $\boxed{\text{gkdeR}}$

$$\Psi_\lambda^\bullet : H^\bullet(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G w \cdot \lambda_{\mathbb{R}} \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \to H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_{\lambda^\vee,\mathbb{C}}) \tag{4.102}$$

We denote by $\omega^{(0)}$ the restriction of the central character of $\mathfrak{I}_B^G w \cdot \lambda_{\mathbb{R}}$ to the subgroup Z_0. and we introduce the spaces

$$\begin{array}{rl} \mathcal{E}^{(\infty)}(\lambda, w, \Gamma) = & \mathrm{Hom}_{(\mathfrak{g},K_\infty)}(\mathfrak{I}_B^G w \cdot \lambda_{\mathbb{R}},\ \mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R}), \omega^{(0)}) \\ \cup & \\ \mathcal{E}^{(2)}(\lambda, w, \Gamma) = & \mathrm{Hom}_{(\mathfrak{g},K_\infty)}(\mathfrak{I}_B^G w \cdot \lambda_{\mathbb{R}},\ \mathcal{C}^{(2)}_\infty(\Gamma\backslash G(\mathbb{R}), \omega^{(0)}) \end{array} \tag{4.103}$$

where the superscript $^{(2)}$ means square integrable.(See 6.14). It is clear from the results in Chapter 6 that the spaces $\mathcal{E}^{(?)}$ are finite dimensional.

We still have another subspace of $\mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R}))$ namely the space of *cusp forms*

$$\mathcal{C}_\infty^{(\mathrm{cusp})}(\Gamma\backslash G(\mathbb{R})) = \{f \ \in \ \mathcal{C}_\infty^{(2)}(\Gamma\backslash G(\mathbb{R})|;| \int_{\Gamma\cap U(\mathbb{R})} f(ug) = 0\} \tag{4.104}$$

here U runs over the set of unipotent radicals of Borel subgroups over $\mathbb{Q}$ and f should satisfy some mild growth condition (see **??**). It is well known that cusp forms are rapidly decreasing and hence we have $\mathcal{C}_\infty^{(\mathrm{cusp})}(\Gamma\backslash G(\mathbb{R})) \subset \mathcal{C}_\infty^{(2)}(\Gamma\backslash G(\mathbb{R}))$

For $? = \infty, (2), (\mathrm{cusp})$ we e get maps in cohomology

$$\Phi^? : \mathcal{E}^?(\lambda, w, \Gamma) \otimes H^\bullet(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G w \cdot \lambda_{\mathbb{R}} \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \to H^\bullet(\Gamma\backslash X, \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \tag{4.105}$$

Of course the module $\mathcal{E}^{(2)}(\lambda, w, \lambda) = 0$ unless $\mathfrak{I}_B^G w \cdot \lambda_{\mathbb{R}}$ has a non trivial quotient module which admits a positive definite quasi unitary $(\mathfrak{g}, K_\infty)$ invariant metric. This means that $\mathcal{E}^{(2)}(\lambda, w\cdot\lambda) \neq 0$ implies that in case B) the coefficients satisfy $\boxed{\text{ul}}$

$$n_1 = n_2,\ \text{i.e.}\lambda = n(\gamma_1 + \gamma_2) + d_1 \det + d_2 \det, \tag{4.106}$$

we will say that λ is unitary if this condition is fulfilled. Then the results in section (4.1.5) yield that these irreducible quasi unitary quotient modules are $\mathcal{D}_\lambda^\pm$ in case A) and $\mathcal{D}_\lambda$ in case B) .

Hence it is clear that a $\Psi_\lambda \in \mathcal{E}^{(2)}(\lambda, w\cdot\lambda)$ must vanish on the finite dimensional submodule $\mathcal{M}_\lambda$ if $n>0$ and hence under this condition we have

$$\mathcal{E}^{(2)}(\lambda, w\cdot\lambda) = \mathrm{Hom}_{(\mathfrak{g},K_\infty)}(\mathcal{D}_\lambda, \mathcal{C}_\infty^{(2)}(\Gamma\backslash G(\mathbb{R}), \omega^{(0)})$$

We have the fundamental
$\boxed{\text{ESI}}$

Theorem 4.1.3. *(Eichler-Shimura Isomorphism) For $\lambda = n\gamma$ unitary the map*

$$\Phi_\lambda^{(2)} : \mathcal{E}^{(2)}(\lambda, w, \Gamma)\otimes H^q(\mathfrak{g}, K_\infty, \mathcal{D}_\lambda\otimes\mathcal{M}_{\lambda^\vee,\mathbb{C}}) \to H_!^q(\Gamma\backslash X, \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \tag{4.107}$$

is an isomorphism for $q=1$ in case A) and for $q=1,2$ in case B).

This is well known and proofs can be found everywhere in the literature. it is a special case of theorem 6.1.1. It says a little bit more, it says that the image of $\Phi_\lambda^{(2)}$ lies in the inner cohomology and not only in $H^\bullet_{(2)}$. But this is very easy to see, if we apply our considerations from section **??** to this special case.

We still have the special case *llambda*$^{(1)}=0$ in this case $\mathcal{M}_\lambda$ is one dimensional and isomorphic to the one dimensional subspace $\mathbb{C}[\lambda]\subset\mathcal{C}_\infty^{(2)}(\Gamma\backslash G(\mathbb{R})$.. Then and the map

$$H^\bullet(\mathfrak{g}, K_\infty, \mathbb{C}[\tilde\lambda]\otimes\mathbb{C}[\tilde\lambda^\vee]) \to H^\bullet(\Gamma\backslash X, \mathcal{M}_{\lambda^\vee,\mathbb{C}}\otimes\mathbb{C}) \tag{4.108}$$

is an isomorphism in degree zero and zero in all other degrees.

For the case A).we want to relate this to the classical formulation. The group $\mathrm{Sl}_2(\mathbb{R})$ acts transitively on the upper half plane $\mathbb{H} = \mathrm{Sl}_2(\mathbb{R})/\mathrm{SO}(2)$. For $g = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$ and $z\in\mathbb{H}$ we put $j(g,z) = cz+d$. To any

$$\Phi \in \mathrm{Hom}_{(\mathfrak{g},K_\infty)}(\mathcal{D}_\lambda^+,\ \mathcal{C}_\infty^{(2)}(\Gamma\backslash G(\mathbb{R}), \omega^{(0)}))$$

we attach a function $f_{n+2}^\Phi : \mathbb{H}\to\mathbb{C}$: We write $z = gi$ with $g\in\mathrm{Sl}_2(\mathbb{R})$ and put
$\boxed{\text{holWh}}$

$$f_{n+2}^\Phi(z) = \Phi(\psi_{n+2})(g) j(g,i)^{n+2} \tag{4.109}$$

An easy calculation shows that f_{n+2}^Φ is well defined and holomorphic (slzweineu.pdf)p.25-26) and for $\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{Sl}_2(\mathbb{Z})$ it satisfies

$$f_{n+2}^\Phi(\gamma z) = (cz+d)^{n+2} f_{n+2}^\Phi(z) \tag{4.110}$$

The condition that $\Phi(\psi_{n+2})(g)$ is square integrable implies that f_{n+2} is a holomorphic cusp form of weight $n+2=k$. It is a special case of the theorem of Gelfand-Graev that this provides an isomorphism $\boxed{\text{GelfGraev}}$

$$\mathrm{Hom}_{(\mathfrak{g},K_\infty)}(\mathcal{D}_\lambda^+,\ \mathcal{C}_\infty^{(2)}(\Gamma\backslash G(\mathbb{R})) \xrightarrow{\sim} S_k(\Gamma) \tag{4.111}$$

where of course $S_k(\Gamma)$ is the space of holomorphic cusp forms for Γ.

We can do the same thing with $\mathcal{D}_\lambda^-$ then we land in the spaces of anti holomorphic cusp forms, these two spaces are isomorphic under conjugation. Combining this with our results above gives the classical formulation of the Eichler-Shimura theorem:

We have a canonical isomorphism

$$S_k(\Gamma) \oplus \overline{S_k(\Gamma)} \xrightarrow{\sim} H^1_! \Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\lambda^\vee,\mathbb{C}}) \tag{4.112}$$

Of course the involution $\mathbf{c}^{(1)}$ interchanges the two summands. It is also clear that $\mathbf{c}^{(1)}$ is the extension of the involution

There is an analogous formulation in case where we have to work with Bianchi modular forms.

4.1.8 Petersson scalar product and semi simplicity

Earlier in chapter 3 we stated a general theorem 3.1.1 which in this case says that $H^1_! \Gamma\backslash\mathbb{H}, \mathcal{M}_{\lambda^\vee,\mathbb{C}})$ is a semi-simple module for the Hecke algebra, we gave an outline of the proof. In this case the hermitian scalar product is obtained from the Petersson scalar product on $S_k(\Gamma)$. For two cusp forms $f, g \in S_k(\Gamma)$ this scalar product is given by

$$< f, g >= \int_{\Gamma\backslash\mathbb{H}} f(z)\overline{g(z)} y^{n+2} i \frac{dz \wedge d\bar{z}}{y^2}$$

For this metric the Hecke operators are self adjoint, and from this it follows that $S_k(\Gamma)$ is semi simple as Hecke module.

We can decompose into eigenspaces

$$H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\lambda^\vee,F}) = \bigoplus_{\pi_f} H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\lambda^\vee,F})(\pi_f) \tag{4.113}$$

where $\pi_f : \mathcal{H} \to F$ is a homomorphism. In this case we know that each π_f which occurs actually occurs with multiplicity 2 (it occurs with multiplicity one in $S_k(\Gamma)$ and $\overline{S_k(\Gamma)}$)

For any embedding $\iota : F \hookrightarrow \mathbb{C}$ we know the Ramanujan-Petersson conjecture, which says

$$\text{For all primes } p \text{ we have } |\iota(\pi_f(T_p))| \leq 2\, p^{\frac{n+1}{2}} \tag{4.114}$$

and again we can conclude that we get a canonical splitting of Hecke-modules

$$H^1 \Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\lambda^\vee,F}) = H^1_! \Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\lambda^\vee,F}) \oplus F \text{ Eis}_n \tag{4.115}$$

where $T_p(\text{ Eis}_n) = (p^{n+1}+1) \text{ Eis}_n$. (The eigenvalue of T_p on Eis_n is different from the eigenvalues of T_p on $H^1_! \Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\lambda^\vee,F})$ (Manin-Drinfeld principle) and then a standard linear argument gives us the splitting.) Of course we could also say that the Hecke-module $H^1_! \Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\lambda^\vee,F})$ is complete in $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\lambda^\vee,F})$.

$\boxed{\text{Whittloc}}$

4.1.9 Local Whittaker models

How do we get such Ψ_λ? In our special situation we get them from Fourier-expansions of Whittaker functions and this will be explained next. We recall some fundamental results from representation theory of groups $\mathrm{Gl}_2(\mathbb{Q}_p)$. Let $F/\mathbb{Q}$ be a finite extension $\mathbb{Q}$. An admissible representation of $\mathrm{Gl}_2(\mathbb{Q}_p)$ is an action of $\mathrm{Gl}_2(\mathbb{Q}_p)$ on a F-vector space V which fulfills the following two additional requirements

a) For any open subgroup $K_p \subset \mathrm{Gl}_2(\mathbb{Z}_p)$ the space of fixed vectors V^{K_p} is finite dimensional.

b) For any $v \in V$ we find an open subgroup $K_p \subset \mathrm{Gl}_2(\mathbb{Z}_p)$ such that $v \in V^{K_p}$.

We say that V is a $\mathrm{Gl}_2(\mathbb{Q}_p)$ module, we denote the action of $\mathrm{Gl}_2(\mathbb{Q}_p)$ on V by $(g, v) \mapsto gv$. In addition we want to assume that our module has a central character, this means that the center $Z(\mathbb{Q}_p) = \mathbb{Q}_p^\times$ acts by a character $\omega_V : Z(\mathbb{Q}_p) \to F^\times$. Such a module is called irreducible if it does not contain a non trivial invariant submodule.

Again we dispose of a Hecke algebra, given K_p we consider the space of functions

$$\mathcal{H}_{K_p} = \{f : \mathrm{Gl}_2(\mathbb{Q}_p) \to F \mid f(zg) = \omega_V^{-1}(z) f(g)\ ;\ f \text{ has compact support} \mod Z(\mathbb{Q}_p)\} \tag{4.116}$$

this gives as an algebra by convolution and this algebra acts on V^{K_p} by

$$f * v = \int_{\mathrm{Gl}_2(\mathbb{Q}_p)/Z(\mathbb{Q}_p)} f(x) x v dx.$$

We normalize the measure dx such that it gives volume one to K_p.

We recall - and explain the meaning of - the fundamental fact that each isomorphism class of admissible irreducible modules has a unique Whittaker model. We assume that $F \subset \mathbb{C}$, then we define the (additive) character $\boxed{\text{PSI}}$

$$\psi_p : \mathbb{Q}_p \to \mathbb{C}^\times;\ \psi_p\ :\ a/p^m \mapsto e^{\frac{2\pi i a}{p^m}} \tag{4.117}$$

it is clear that the kernel of ψ_p is $\mathbb{Z}_p$. Since we have $U(\mathbb{Q}_p) = \mathbb{Q}_p$ we can view ψ_p as a character $\psi_p : U(\mathbb{Q}_p) \to \mathbb{C}^\times$. We introduce the space

$$\mathcal{C}_{\psi_p}(\mathrm{Gl}_2(\mathbb{Q}_p)) = \{f : \mathrm{Gl}_2(\mathbb{Q}_p) \to \mathbb{C} | f(ug) = \psi_p(u) f(g)\}$$

where in addition we require that our f is invariant under a suitable open subgroup $K_f \subset \mathrm{Gl}_2(\mathbb{Z}_p)$. The group $\mathrm{Gl}_2(\mathbb{Q}_p)$ acts on this space by right translation the action is not admissible but satisfies the above condition b) .

Now we can state the theorem about existence and uniqueness of the Whittaker model

$\boxed{\text{Whittp}}$

Theorem 4.1.4. *For any infinite dimensional, absolutely irreducible admissible* $\mathrm{Gl}_2(\mathbb{Q}_p)$ *-module* V *we find a non trivial (of course invariant under* $\mathrm{Gl}_2(\mathbb{Q}_p)$*) homomorphism*

$$\Psi : V \to \mathcal{C}_{\psi_p}(\mathrm{Gl}_2(\mathbb{Q}_p)), \tag{4.118}$$

it is unique up to multiplication by a non zero scalar.

Proof. We refer to the literature, [55], [32] $\square$

Spherical representations, their Whittaker model and the Euler factor

An absolutely irreducible $\mathrm{Gl}_2(\mathbb{Q}_p)$ module is called spherical or unramified if for $K_p = \mathrm{Gl}_2(\mathbb{Z}_p)$ we have $V^{K_p} \neq \{0\}$. In this case it is known that (*Reference*)

$$\dim_F(V^{\mathrm{Gl}_2(\mathbb{Z}_p)}) = 1; V^{\mathrm{Gl}_2(\mathbb{Z}_p)} = Fh_0. \tag{4.119}$$

The Hecke algebra $\mathcal{H}_{K_p}$ is commutative and generated by the two double cosets

$$T_p = \mathrm{Gl}_2(\mathbb{Z}_p)\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\mathrm{Gl}_2(\mathbb{Z}_p) \text{ and } C_p = \mathrm{Gl}_2(\mathbb{Z}_p)\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}. \tag{4.120}$$

The space $V^{\mathrm{Gl}_2(\mathbb{Z}_p)}$ is an absolutely irreducible module for $\mathcal{H}_{K_p}$ hence it is of rank one, let ψ_0 be a generator. Our two operators act by scalars on V^{K_p}, we write

$$T_p(h_0) = \pi_V(T_p)h_0 \text{ and } C_p(h_0) = \pi_V(C_p)h_0 \tag{4.121}$$

The module V is completely determined by these two eigenvalues, of course $\pi_V(C_p) = \omega_V(C_p)$.

We can formulate this a little bit differently. Let π_p an isomorphism type of our $\mathrm{Gl}_2(\mathbb{Q}_p)$ module V. Then our theorem above asserts that there is a unique $\mathrm{Gl}_2(\mathbb{Q}_p)$ -module

$$\mathcal{W}(\pi_p) \subset \mathcal{C}_{\psi_p}(\mathrm{Gl}_2(\mathbb{Q}_p)) \tag{4.122}$$

with isomorphism-type equal to $\pi_p \times_F \mathbb{C}$. We call this module the Whittaker realization of π_p. If our isomorphism type is unramified then the resulting homomorphism of $\mathcal{H}_p$ to F is also denoted by π_p.

We have the spherical vector $h^{(0)}_{\pi_p} \in \mathcal{W}(\pi_p)^{\mathrm{Gl}_2(\mathbb{Z}_p)}$ which is unique up to a scalar. Since $\mathrm{Gl}_2(\mathbb{Q}_p) = U(\mathbb{Q}_p)T(\mathbb{Q}_p)\mathrm{Gl}_2(\mathbb{Z}_p)$ this spherical vector is determined by its restriction to $T(\mathbb{Q}_p)$. We have a formula for this restriction. First of all we observe that

$$h^{(0)}_{\pi_p}(\begin{pmatrix} p^n & 0 \\ 0 & p^m \end{pmatrix}) = \pi_p(C_p^m)h^{(0)}_{\pi_p}(\begin{pmatrix} p^{n-m} & 0 \\ 0 & 1 \end{pmatrix}). \tag{4.123}$$

We claim that $h^{(0)}_{\pi_p}(\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}) = 0$ if $n < 0$. To see this we look at the equalities

$$h^{(0)}_{\pi_p}(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}) = \psi_p(u)h^{(0)}_{\pi_p}(\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}) = h^{(0)}_{\pi_p}(\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & p^{-n}u \\ 0 & 1 \end{pmatrix})$$

and we can find an element $u \in \mathbb{Q}_p$ such that $p^{-n}u \in \mathbb{Z}_p$ and $\psi_p(u) \neq 1$, this implies the claim. We exploit the eigenvalue equation $T_p(h^{(0)}_{\pi_p}) = \pi_p(T_p)h^{(0)}_{\pi_p}$, we write the double coset $K_p\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}K_p$ as union of right K_p cosets

$$K_p\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}K_p = \bigcup_{x \in \mathbb{Z}/p\mathbb{Z}}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}K_p \bigcup \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}K_p.$$

Clearly

$$h^{(0)}_{\pi_p}(\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) = h^{(0)}_{\pi_p}(\begin{pmatrix} p^{n+1} & 0 \\ 0 & 1) \end{pmatrix})$$

$$h^{(0)}_{\pi_p}(\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) = \pi_p(C_p)h^{(0)}_{\pi_p}(\begin{pmatrix} p^{n-1} & 0 \\ 0 & 1 \end{pmatrix})$$

and this implies the recursion formula $\boxed{\text{recurs}}$

$$\pi_p(T_p)h^{(0)}_{\pi_p}(\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}) = \pi_p(C_p)h^{(0)}_{\pi_p}(\begin{pmatrix} p^{n-1} & 0 \\ 0 & 1 \end{pmatrix}) + \begin{cases} ph^{(0)}_{\pi_p}(\begin{pmatrix} p^{n+1} & 0 \\ 0 & 1) \end{pmatrix}) & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases} \tag{4.124}$$

We can normalize $h^{(0)}_{\pi_p}(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = 1$, then the values for $n > 0$ follow from the recursion.

There is a more elegant way writing this recursion. For our unramified π_p we define the local Euler factor $\boxed{\text{Euler}}$

$$L(\pi_p, s) = \frac{1}{1 - \pi_p(T_p)p^{-s} + p\pi_p(C_p)p^{-2s}} \tag{4.125}$$

We expand this into a power series in p^{-s} and an elementary calculation shows that $\boxed{\text{Mellin}}$

$$L(\pi_p, s) = \sum_{n=0}^{\infty} h^{(0)}_{\pi_p}(\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix})p^n p^{-ns} \tag{4.126}$$

Whittaker models for Harish-Chandra modules

We also have a theory of Whittacker models for the irreducible Harish-Chandra modules studied in section 4.1. The unipotent radical $U(\mathbb{R}) = \mathbb{R}$ resp. $U(\mathbb{R}) = \mathbb{C}$. Again we fix characters $\psi_\infty : U(\mathbb{R}) \to \mathbb{C}^\times$ we put

$$\psi_\infty(x) = \begin{cases} e^{-2\pi i x} & \text{in case A)} \\ e^{-2\pi i(x+\bar{x})} & \text{in case B)} \end{cases} \tag{4.127}$$

and as in the p-adic case we define

$$\mathcal{C}_{\psi_\infty}(G(\mathbb{R})) = \{f : G(\mathbb{R}) \to \mathbb{C} \mid f(ug) = \psi_\infty(u)f(g), f \text{ is } \mathcal{C}_\infty\} \tag{4.128}$$

Then we have again $\boxed{\text{Whittinf}}$

Theorem 4.1.5. *For any infinite dimensional, absolutely irreducible admissible* $\mathrm{Gl}_2(\mathbb{R})$ *-module* V *we find a non trivial (of course invariant under* $\mathrm{Gl}_2(\mathbb{R})$*) homomorphism*

$$\Psi : V \to \mathcal{C}_{\psi_\infty}(G(\mathbb{R})), \tag{4.129}$$

This homomorphism is unique up to a scalar. The image of V *under the homomorphism* Ψ *will be denoted by* $\tilde{V}$.

Proof. Again we refer to the literature. [32]. □

Hence we can say that for any isomorphism class π_∞ of irreducible infinite dimensional Harish-Chandra modules we have a unique Whittaker model $\mathcal{W}(\pi_\infty) \subset \mathcal{C}_{\psi_\infty}(G(\mathbb{R}))$. In the book of Godement we find explicit formulae for these Whittaker functions.

Actually it is easy to write down such maps $\tilde{\Psi}_\pm$ resp. $\tilde{\Psi}$ explicitly for our induced modules, we start from a dominant weight $\lambda = n\gamma + \delta$ (resp. $n_1\gamma_1 + n_2\gamma_2 + \delta$ where $n \geq 0, n_1, n_2 \geq 0$. We define

$$\mathcal{F} : \mathfrak{I}_B^G \lambda_{\mathbb{R}} |\rho|^2_{\mathbb{R}} \to \mathcal{C}_{\psi_\infty}(G(\mathbb{R}))$$

by the integral

$$\mathcal{F}(f)(g) = \int_{U(\mathbb{R})} f(wug)\psi_\infty(-u)du,$$

there is no problem with convergence as long $n > 0, n_1, n_2 > 0$. If one of these numbers is zero then there is a tiny difficulty to overcome, we ignore it. In any case we get an isomorphism

$$\mathcal{F} : \mathfrak{I}_B^G \lambda_{\mathbb{R}} |\rho|^2_{\mathbb{R}} \xrightarrow{\sim} \mathfrak{I}_B^{G,\dagger} \lambda_{\mathbb{R}} |\rho|^2_{\mathbb{R}} \tag{4.130}$$

i.e. we will denote elements or spaces which lie in a Whittaker model by $?^\dagger$.

We consider the case A). Let n be even. We consider induced module $\mathfrak{I}_B^G \lambda_{\mathbb{R}} \rho^2_{\mathbb{R}} = \bigoplus_{\nu \equiv 0(2)} \mathbb{C}\phi_{\lambda,\nu}$, (See 4.31we have the exact sequence (See seqd

$$0 \to \mathcal{D}^+_{\lambda^\vee} \oplus \mathcal{D}^-_{\lambda^\vee} \to \mathfrak{I}_B^G \lambda_{\mathbb{R}} \rho^2_{\mathbb{R}} \to \mathcal{M}_\lambda \to 0$$

We have the Whittaker map

$$\mathcal{F} : \mathfrak{I}_B^G \lambda_{\mathbb{R}} \rho^2_{\mathbb{R}} \to \mathcal{C}_\psi(G(\mathbb{R}))$$

which is defined by

$$\mathcal{F}(\phi_{\lambda,\nu})(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) := \int_{-\infty}^{\infty} \phi_{\lambda,\nu}(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) e^{-2\pi i x} dx$$

We change variables $x \to -x$ then the Iwaswa decomposition gives

$$w \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ t & x \end{pmatrix} = \begin{pmatrix} \frac{t}{\sqrt{t^2+x^2}} & * \\ 0 & \sqrt{t^2+x^2} \end{pmatrix} \begin{pmatrix} \frac{x}{\sqrt{t^2+x^2}} & \frac{t}{\sqrt{t^2+x^2}} \\ \frac{-t}{\sqrt{t^2+x^2}} & \frac{x}{\sqrt{t^2+x^2}} \end{pmatrix}$$

and a straightforward computation gives us that we have to evaluate

$$\mathcal{F}(\phi_{\lambda,\nu})(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = t^{\frac{n}{2}+1} \int_{-\infty}^{\infty} \frac{e^{2\pi i x}}{(-x+ti)^{n/2+\nu/2+1}(-x-ti)^{n/2-\nu/2+1}} dx$$

We apply the Residue theorem. We consider the case $t > 0$. Let $R >> 0$ we consider the path from $-R$ to R on the real line followed by the half arc C_R in the upper half plane from from R back to $_R$. The integral along this path is the sum of the residues in the interior of this closed path. On the other hand it it is easy to see that for $R \to \infty$ the integral along C_R goes to zero, basically because $|e^|2\pi iz|$ becomes very small if $\mathrm{Im}(z)$ becomes large. Our function has only one pole in the upper half plane, namely for $x = ti$ and therefore

$$\int_{-\infty}^{\infty} \phi_{\lambda,\nu}(w \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) e^{2\pi ix} dx = t^{\frac{n}{2}+1} \operatorname{Res}_{x=ti} \frac{e^{2\pi ix}}{(-x+ti)^{n/2+\nu/2+1}(-x-ti)^{n/2-\nu/2+1}}$$

If we put $z := x - ti$ then our integral becomes

$$(2i)^{-n/2-\nu/2-1} t^{-\nu/2} e^{-2\pi t} \operatorname{Res}_{z=0} \frac{e^{2\pi iz}}{(1+\frac{z}{2ti})^{n/2+\nu/2+1} z^{n/2+\nu/2+1}} = P_{\lambda,\nu}(t) e^{-2\pi t},$$

where $P_{\lambda,\nu}(t)$ is a Laurent polynomial in $\mathbb{C}[t, t^{-1}]$.

If $t > 0$ then there is no pole for$\nu \leq -n-2$ and this implies that $\mathcal{F}$ maps $\mathcal{D}^-_{\lambda^\vee}$ to zero. If we restrict to $t > 0$ our map $\mathcal{F}$ induces an injection

$$\mathfrak{I}_B^G \lambda_{\mathbb{R}} \rho^2_{\mathbb{R}} / \mathcal{D}^-_{\lambda^\vee} \hookrightarrow \mathcal{C}_\psi(G(\mathbb{R}))$$

this is of course an intertwining operator. The module $\mathcal{D}^+_{\lambda^\vee} \subset \mathfrak{I}_B^G \lambda_{\mathbb{R}} \rho^2_{\mathbb{R}} / \mathcal{D}^+_{\lambda^\vee}$ it has $\phi_{\lambda,n+2}$ as a lowest weight vector. We compute $\mathcal{F}(\phi_{\lambda,n+2})$, then the nasty factor $(1+\frac{z}{2ti})^{n/2+\nu/2+1}$ is equal to one in this case and hence we have up to a non zero constant

$$\mathcal{F}(\phi_{\lambda,n+2})(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = c_\lambda t^{\frac{n}{2}+1} e^{-2\pi t}.$$

If we now restrict to $t < 0$ then bsidcally the same computation shows that $\mathcal{F}$ sends $|Dsm$ to zero-

In the case A) the we the two discrete irreducible series representations $\mathcal{D}^+_{\lambda^\vee}, \mathcal{D}^-_{\lambda^\vee}$ attached to a dominant weight λ. We have their Whittaker model

$$\mathcal{F}_\pm : \mathcal{D}^\pm_{\lambda^\vee} \hookrightarrow \mathcal{C}_{\psi_\infty}(\mathrm{Gl}_2(\mathbb{R})). \tag{4.131}$$

The group $(\mathrm{Gl}_2(\mathbb{R})$ has the two connected components $\mathrm{Gl}_2(\mathbb{R})^+, \mathrm{Gl}_2(\mathbb{R})^-$,($\det > 0, \det < 0$) and we have

$$\mathcal{F}_+(\mathcal{D}^+_{\lambda^\vee}) = \mathcal{D}^{+,\dagger}_{\lambda^\vee} \text{ is supported on } \mathrm{Gl}_2(\mathbb{R})^+, \mathcal{D}^{-,\dagger}_{\lambda^\vee} \text{ is supported on } \mathrm{Gl}_2(\mathbb{R})^- \tag{4.132}$$

Under the isomorphism $\tilde{\Psi}_\pm$ the elements $\psi_{\pm(n+2)}$ (See (4.30)) are mapped to functions $\psi^\dagger_{\pm(n+2)}$. We can normalize $\tilde{\Psi}_\pm$ such that $\boxed{\text{tpsin}}$

$$\psi^\dagger{}_{n+2}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = \begin{cases} t^{\frac{n}{2}+1} e^{-2\pi t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases} \tag{4.133}$$

and $\psi^\dagger{}_{-n-2}$ is given by the corresponding formula.

We discuss the same issue for the group $\mathrm{Gl}_2(\mathbb{C})$ later in section 4.1.11 $\boxed{\text{Whitt}}$

Global Whittaker models, Fourier expansions and multiplicity one

We also have global Whittaker models. To define them we recall some results from Tate's thesis ([87]). We introduce the ring of adeles $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$, we write it as a product $\mathbb{A} = \mathbb{Q}_\infty \times \mathbb{A}_f = \mathbb{R} \times \mathbb{A}_f$. The ring of finite adeles contains the compact subring $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ of integral adeles.

We define a global character $\psi : U(\mathbb{A})/U(\mathbb{Q}) = \mathbb{A}/\mathbb{Q} \to \mathbb{C}^\times$ as the product $\boxed{\text{psiq}}$

$$\psi(x_\infty, \dots, x_p, \dots) = \psi_\infty(x_\infty) \prod_p \psi_p(x_p) \tag{4.134}$$

where the local components ψ_v are as above, we have to check that ψ is trivial on $U(\mathbb{Q})$. (See [87], "note the minus sign") For any $a \in \mathbb{Q}$ we define $\psi^{[a]}(x) = \psi(ax)$, so $\psi = \psi^{[1]}$. In ([87]) it is shown that the map

$$\mathbb{Q} \to \operatorname{Hom}(\mathbb{A}/\mathbb{Q}, \mathbb{C}^\times);\ a \mapsto \psi^{[a]} \tag{4.135}$$

is an isomorphism between $\mathbb{Q}$ and the character group of $\mathbb{A}/\mathbb{Q}$. Hence we know that for any reasonable function $h : \mathbb{A}/\mathbb{Q} \to \mathbb{C}$ we have a Fourier expansion $\boxed{\text{Fouex}}$

$$h(\underline{u}) = \sum_{a \in \mathbb{Q}} \hat{h}(a)\psi(a\underline{u}) \tag{4.136}$$

where $\hat{h}(a) = \int_{\mathbb{A}/\mathbb{Q}} h(\underline{u})\psi(-a\underline{u})d\underline{u}$, and where $\operatorname{vol}_{d\underline{u}}(\mathbb{A}/\mathbb{Q}) = 1$. Then we put

$$\mathcal{C}_\psi(\mathrm{Gl}_2(\mathbb{R}) \times \mathrm{Gl}_2(\mathbb{A}_f)/K_f)) = \{f : \mathrm{Gl}_2(\mathbb{R}) \times \mathrm{Gl}_2(\mathbb{A}_f)/K_f \to \mathbb{C} | f(\underline{u}\underline{g}) = \psi(\underline{u})f(\underline{g})\}$$

this is a module for $\mathrm{Gl}_2(\mathbb{R}) \times \bigotimes' \mathcal{H}_p$

Let us start from the Harish-Chandra module $\pi_\infty = \mathcal{D}_\lambda^+$ and a homomorphism $\pi_f = \otimes' \pi_p : \otimes' \mathcal{H}_p \to F$ from the unramified Hecke algebra to F. Here $F/\mathbb{Q}$ is a finite extension of $\mathbb{Q}$. We assume it comes with an embedding $\iota : F \hookrightarrow \mathbb{C}$, i.e. we also may it consider as a subfield of $\mathbb{C}$.

We still assume for simplicity that $K_f = \mathrm{Gl}_2(\hat{\mathbb{Z}})$. The results on Whittaker-models imply that we have a unique Whittaker-model

$$\mathcal{W}(\pi) = \mathcal{W}(\pi_\infty) \otimes \mathbb{C}h_{\pi_f}^{(0)} \subset \mathcal{C}_\psi(\mathrm{Gl}_2(\mathbb{R}) \times \mathrm{Gl}_2(\mathbb{A}_f)/K_f) \tag{4.137}$$

for our isomorphism class $\pi = \pi_\infty \times \pi_f$. Here of course $h_{\pi_f}^{(0)} = \otimes h_{\pi_p}^{(0)}$.

We return to Theorem 4.1.3. On the space $\mathcal{C}_\infty^{(2)}(\Gamma\backslash G(\mathbb{R}), \omega^{(0)}))$ we have the action of the unramified Hecke algebra. To see this action we start from the observation that the map $\mathrm{Gl}_2(\mathbb{Q}) \to \mathrm{Gl}_2(\mathbb{A}_f)/K_f$ (Chap. III , 1.5) is surjective and hence

$$\mathrm{Gl}_2(\mathbb{Z})\backslash\mathrm{Gl}_2(\mathbb{R}) \xrightarrow{\sim} \mathrm{Gl}_2(\mathbb{Q})\backslash\mathrm{Gl}_2(\mathbb{R}) \times \mathrm{Gl}_2(\mathbb{A}_f)/K_f \tag{4.138}$$

and hence

$$\mathcal{C}_\infty^{(2)}(\mathrm{Gl}_2(\mathbb{Z})\backslash\mathrm{Gl}_2(\mathbb{R})) = \mathcal{C}_\infty^{(2)}(\mathrm{Gl}_2(\mathbb{Q})\backslash\mathrm{Gl}_2(\mathbb{R}) \times \mathrm{Gl}_2(\mathbb{A}_f)/K_f) \tag{4.139}$$

and the space on the right is a $\mathrm{Gl}_2(\mathbb{R}) \times \bigotimes' \mathcal{H}_p$ module. Now we consider the $\pi = \pi_\infty \times \pi_f$ isotypical submodule $\mathcal{C}_\infty^{(2)}(\mathrm{Gl}_2(\mathbb{Q})\backslash \mathrm{Gl}_2(\mathbb{R}) \times \mathrm{Gl}_2(\mathbb{A}_f)/K_f)(\pi) \subset \mathcal{C}_\infty^{(2)}(\mathrm{Gl}_2(\mathbb{Q})\backslash \mathrm{Gl}_2(\mathbb{R}) \times \mathrm{Gl}_2(\mathbb{A}_f)/K_f)$.

We have the famous Theorem which in the case $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$ is due to Hecke

$\boxed{\text{multone}}$

Theorem 4.1.6. *If $\mathcal{C}_\infty^{(2)}(\mathrm{Gl}_2(\mathbb{Q})\backslash \mathrm{Gl}_2(\mathbb{R}) \times \mathrm{Gl}_2(\mathbb{A}_f)/K_f)(\pi) \neq 0$ then have a canonical isomorphism*

$$\mathcal{F} : \mathcal{W}(\pi) \xrightarrow{\sim} \mathcal{C}_\infty^{(2)}(\mathrm{Gl}_2(\mathbb{Q})\backslash \mathrm{Gl}_2(\mathbb{R}) \times \mathrm{Gl}_2(\mathbb{A}_f)/K_f)(\pi) \tag{4.140}$$

especially we know that π occurs with multiplicity one.

Proof. We give the inverse of $\mathcal{F}$. Given a function

$$h \in \mathcal{C}_\infty^{(2)}(\mathrm{Gl}_2(\mathbb{Q})\backslash \mathrm{Gl}_2(\mathbb{R}) \times \mathrm{Gl}_2(\mathbb{A}_f)/K_f)(\pi)$$

we define

$$h^\dagger((g_\infty, \underline{g}_f)) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} h(\underline{u}\underline{g})\overline{\psi(\underline{u})}d\underline{u} \tag{4.141}$$

it is clear that $h^\dagger(g_\infty, \underline{g}_f) \in \mathcal{W}(\pi)$. It follows from the theory of automorphic forms that h is actually in the space of cusp forms, this means that the constant Fourier coefficient $\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} h(\underline{u}\underline{g})d\underline{u} = 0$ and hence our Fourier expansion yields ((4.136), evaluated at $u = 0$)

$$h(\underline{g}) = \sum_{a \in \mathbb{Q}^\times} \int_{U(\mathbb{A})/U(\mathbb{Q})} h(\underline{u}\underline{g})\psi^{[a]}(\underline{u})d\underline{u} \tag{4.142}$$

The measure $d\underline{u}$ is invariant under multiplication by $a \in \mathbb{Q}^\times$ and hence a individual term in the summation is

$$\int_{U(\mathbb{A})/U(\mathbb{Q})} h(\begin{pmatrix} 1 & \underline{u} \\ 0 & 1 \end{pmatrix} \underline{g})\psi(\begin{pmatrix} 1 & a\underline{u} \\ 0 & 1 \end{pmatrix})d\underline{u} = \int_{U(\mathbb{A})/U(\mathbb{Q})} h(\begin{pmatrix} 1 & a^{-1}\underline{u} \\ 0 & 1 \end{pmatrix} \underline{g})\psi(\begin{pmatrix} 1 & \underline{u} \\ 0 & 1 \end{pmatrix})d\underline{u} \tag{4.143}$$

Now

$$\begin{pmatrix} 1 & a^{-1}\underline{u} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \underline{u} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

Since h is invariant under the action of $G(\mathbb{Q})$ from the left we find

$$\int_{U(\mathbb{A})/U(\mathbb{Q})} h(\underline{u}\underline{g})\psi^{[a]}(\underline{u})d\underline{u} = h^\dagger(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})g_\infty)h_f^\dagger(\underline{a}_f)\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}(g_\infty, \underline{g}_f)) \tag{4.144}$$

We evaluate at $\underline{g} = (g_\infty, e)$ then

$$h^\dagger(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}(g_\infty, e)) = h^\dagger(\begin{pmatrix} a_\infty & 0 \\ 0 & 1 \end{pmatrix}g_\infty, \begin{pmatrix} \underline{a}_f & 0 \\ 0 & 1 \end{pmatrix}) \tag{4.145}$$

For a fixed g_∞ the function $\underline{g}_f \mapsto h^\dagger(g_\infty, \underline{g}_f)$ is up to a factor equal to $h^{(0)}_{\pi_f} = \bigotimes_p' h^{(0)}_{\pi_p}$ and hence we find

$$h^\dagger(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}(g_\infty, e)) = h^\dagger(\begin{pmatrix} a_\infty & 0 \\ 0 & 1 \end{pmatrix} g_\infty, e) h^{(0)}_{\pi_f}(\begin{pmatrix} \underline{a}_f & 0 \\ 0 & 1 \end{pmatrix}) \tag{4.146}$$

The recursion formulae (4.124),(4.126) imply that $h^{(0)}_{\pi_f}(\begin{pmatrix} \underline{a}_f & 0 \\ 0 & 1 \end{pmatrix}) = 0$ unless $a \in \mathbb{Z}$.

We restrict our functions to $\mathrm{Gl}_2^+(\mathbb{R})$, i.e. we take $g_\infty \in \mathrm{Gl}_2(\mathbb{R})^+$ and we remember that our representationπ_∞ is $\mathcal{D}^+_{\lambda^\vee}$. Then we know that for $h_\infty \in \mathcal{D}^+_{\lambda^\vee}$ the value $h^\dagger(\begin{pmatrix} a_\infty & 0 \\ 0 & 1 \end{pmatrix} g_\infty, e) = 0$ if $a_\infty < 0$ and hence

$$h^\dagger(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}(g_\infty, e)) = h^\dagger(\begin{pmatrix} a_\infty & 0 \\ 0 & 1 \end{pmatrix} g_\infty, \begin{pmatrix} \underline{a}_f & 0 \\ 0 & 1 \end{pmatrix}) = 0 \text{ unless } a > 0, a \in \mathbb{Z},$$

and our Fourier expansion (4.136) becomes $\boxed{\text{Fexpl}}$

$$h(\underline{g}) = \sum_{a=1}^{\infty} h^\dagger(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}(g_\infty, e)) h^{(0)}_{\pi_f}(\begin{pmatrix} \underline{a}_f & 0 \\ 0 & 1 \end{pmatrix}) \tag{4.147}$$

□

We notice that there is never any problem with convergence. The Whittaker functions $h^\dagger_\infty$ always decay very rapidly at infinity. We write $g_\infty = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} k$ with $k \in K_\infty$, then it is easy to see

$$|h^\dagger_\infty(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} g_\infty| < P(t)e^{-2\pi t}$$

where $P(t)$ is a polynomial in t. This implies that the series is really very rapidly converging (See remark below).

Now we choose for the component at infinity the function $h^\dagger_\infty = \tilde{\psi}_{n+2}$ and we compute the corresponding holomorphic cusp form h^Φ under the Eichler-Shimura isomorphism. We have the formula (4.109)

$$h^\Phi(z) = h^\Phi(x+iy) = h(\begin{pmatrix} y^{\frac12} & \frac{x}{y^{\frac12}} \\ 0 & y^{-\frac12} \end{pmatrix}) j(\begin{pmatrix} y^{\frac12} & \frac{x}{y^{\frac12}} \\ 0 & y^{-\frac12} \end{pmatrix}, i)^{n+2} = h(\begin{pmatrix} y^{\frac12} & \frac{x}{y^{\frac12}} \\ 0 & y^{-\frac12} \end{pmatrix}) y^{-\frac{n}{2}-1}$$

and hence our Fourier expansion (4.147) becomes$\boxed{\text{FouH}}$

$$h^\Phi(z) = y^{-\frac{n}{2}-1} \sum_{a=1}^{\infty} \tilde{\phi}_{n+2}(\begin{pmatrix} ay & ax \\ 0 & 1 \end{pmatrix}) h^{(0)}_{\pi_f}(\begin{pmatrix} \underline{a} & 0 \\ 0 & 1 \end{pmatrix}) \tag{4.148}$$

We have the formula (4.133) for $\tilde{\phi}_{n+2}$ and then this becomes

$$h^\Phi(z) = \sum_{a=1}^{\infty} a^{\frac{n}{2}+1} h^{(0)}_{\pi_f}(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}) e^{2\pi i z a} \tag{4.149}$$

This is now the classical Fourier expansion of a holomorphic cusp eigenform of weight $k = n+2$, ([50]). The numbers $c(\pi_f, a) = a^{\frac{n}{2}+1} h^{(0)}_{\pi_f}(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})$ are the

Fourier coefficients and they also the eigenvalues of the operator T_a -defined in by Hecke in [50]- on h^Φ. If we apply the Eichler-Shimura isomorphism and interpret h^Φ as a cohomology class then it is an eigenclass in $H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n\otimes\mathbb{C})$ and for any prime p the number $c(\pi_f,p)$ is the eigenvalue of the operator T_p defined in 3.9.

We briefly come back to the question of convergence. Hecke proves in [50] that $\boxed{\text{Estone}}$

$$|c(\pi_f,a)|\le Ca^{n+1+\epsilon} \tag{4.150}$$

and with this estimate the convergence becomes obvious.

Actually there is a much better estimate, which will be discussed in the "probably removed" section. $\boxed{\text{Lfu}}$

4.1.10 The L-functions

We still assume that $K_f=\mathrm{Gl}_2(\hat{\mathbb{Z}})$ or what amounts to the same that $\Gamma=\mathrm{Sl}_2(\mathbb{Z})$. We start from an eigenspace $H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_\lambda\otimes F)(\pi_f)$, now π_f is simply a homomorphism $\pi_f:\mathcal{H}_{K_f}\to\mathcal{O}_F$. To this homomorphism we attach the *cohomological L-function*

$$L^{\mathrm{coh}}(\pi_f,s)=\prod_p\frac{1}{1-\pi_p(T_p)p^{-s}+p^{1+n-2s}} \tag{4.151}$$

here T_p is the Hecke operator defined in 3.9, it differs from the Hecke operator defined by convolution by a factor $p^{\frac{n}{2}}$ in front. If we expand this product over all primes we get

$$L^{\mathrm{coh}}(\pi_f,s)=\sum_{a=1}^{\infty}\frac{c(\pi_f,a)}{a^s} \tag{4.152}$$

and this is exactly the L-function Hecke attaches to the cusp form provided by π_f. But we want to stress that this cohomological L-function is defined in purely combinatorial terms (See section 3.2.1, and Chapter 7).

At this moment this L function is a formal expression, it is a formal Dirichlet series with coefficients in our field F, which is simply a finite extension of $\mathbb{Q}$. If we assume that $F\subset\mathbb{C}$. then we may interpret s as a complex variable and the above estimate of the size of the coefficients implies that this series converges absolutely and locally uniformly for $\Re(s)>n+2$ and hence gives a holomorphic function in this halfspace. But something much better is true. We define the completed L function

$$\Lambda^{\mathrm{coh}}(\pi_f,s)=\frac{\Gamma(s)}{(2\pi)^s}L^{\mathrm{coh}}(\pi_f,s), \tag{4.153}$$

for this completed L-function Hecke proved $\boxed{\text{HFu}}$

Theorem 4.1.7. *The function $\Lambda^{\mathrm{coh}}(\pi_f,s)$ has holomorphic continuation into the entire complex plane and satisfies the functional equation*

$$\Lambda^{\mathrm{coh}}(\pi_f,s)=(-1)^{\frac{n}{2}+1}\Lambda^{\mathrm{coh}}(\pi_f,n+2-s)$$

Proof. We could refer to Hecke, but for some reason we give an outline of the argument. We have the integral representation (Mellin-transform)

$$\Lambda^{\mathrm{coh}}(\pi_f,s)=\int_0^\infty\sum_{a=1}^\infty c(\pi_f,a)e^{-2\pi ay}y^s\frac{dy}{y}=\int_0^\infty h^\Phi(iy)y^s\frac{dy}{y}$$

of course here we have to be courageous (or stupid) enough to exchange integration and summation. But since $e^{-2\pi ay}$ goes rapidly to zero if $y\to\infty$ there is no problem with the upper integration limit ∞. If $\Re(s)>>0$ the y^s also tends to zero fast enough, so that we do not have a problem with the lower integration limit. But now we can split the integration into two parts

$$\int_0^\infty\sum_a^\infty c(\pi_f,a)e^{-2\pi ay}y^s\tfrac{dy}{y}=$$

$$\int_0^1\sum_a^\infty c(\pi_f,a)e^{-2\pi ay}y^s\tfrac{dy}{y}+\int_1^\infty\sum_a^\infty c(\pi_f,a)e^{-2\pi ay}y^s\tfrac{dy}{y}$$

the second integration is converging for all values of s. To handle the first integral we observe that $h^\Phi(-\frac{1}{z})=z^{n+2}h^\Phi(z)$, Hence we can substitute $y\to\frac{1}{y}$ in the first integral and get

$$\Lambda^{\mathrm{coh}}(\pi_f,s)=$$

$$\sum_a^\infty\Big(\frac{1}{(2\pi)^s}\frac{c(\pi_f,a)}{a^s}\Gamma(s,2\pi a)+\frac{(-1)^{\frac{n}{2}+1}}{(2\pi)^{n+2-s}}\frac{c(\pi_f,a)}{a^{n+2-s}}\Gamma(n+2-s,2\pi a)\Big). \tag{4.154}$$

Here $\Gamma(,)$ is the incomplete Γ function, which defined by $\Gamma(s,A)=\int_A^\infty e^{-y}y^s\frac{dy}{y}$, it has the virtue that for any given value of s it decays rapidly if A goes to infinity.

Therefore we see that $\Lambda^{\mathrm{coh}}(\pi_f,s)$ can be written as a sum of two infinite series which are very rapidly converging, hence it follows that $\Lambda^{\mathrm{coh}}(\pi_f,s)$ is holomorphic in the entire s plane and the functional equation also becomes obvious. □

We included the proof of the above theorem, because the above formula also gives us a very effective procedure to compute the numerical value of $\Lambda^{\mathrm{coh}}(\pi_f,s_0)$ with high accuracy. We will come back to this issue in section 5.6.

$\boxed{\text{periods}}$

4.1.11 The Periods

Together with the map $\mathcal{F}$ comes the map

$$\tilde{\mathcal{F}}=\mathrm{Id}\otimes\mathcal{F}\otimes\mathrm{Id}:\ \mathrm{Hom}_{\tilde{K}_\infty}(\Lambda(\mathfrak{g}/\tilde{\mathfrak{k}}),\mathcal{W}(\pi)\otimes\tilde{\mathcal{M}}_\lambda)\to$$
$$\mathrm{Hom}_{\tilde{K}_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}),\mathcal{C}_\infty(\mathrm{Gl}_2(\mathbb{Q})\backslash(\mathrm{Gl}_2(\mathbb{R})\times\mathrm{Gl}_2(\mathbb{A}_f)/K_f)\otimes\tilde{\mathcal{M}}_\lambda)$$

The purpose of the following computations is to fix a specific choice of basis elements $\omega_\pm^\dagger\in\ \mathrm{Hom}_{\tilde{K}_\infty}(\Lambda^1(\mathfrak{g}/\mathfrak{k}),\mathcal{D}_{\lambda^\vee}{}^\dagger\otimes\tilde{\mathcal{M}}_\lambda)$ (in case A) $\omega_{1,2}^\dagger\in\ \mathrm{Hom}_{\tilde{K}_\infty}(\Lambda^{1,2}(\mathfrak{g}/\mathfrak{k}),\mathcal{D}_{\lambda^\vee}^\dagger\otimes\tilde{\mathcal{M}}_\lambda)$ (in case B)) These "canonical" generators will serve us to define the periods.

In case A) we have

$$\mathfrak{g}/\tilde{\mathfrak{k}} \xrightarrow{\sim} \mathbb{Q}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \oplus \mathbb{Q}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \mathbb{Q}H \oplus \mathbb{Q}V = \mathfrak{p} \tag{4.155}$$

If we put $P = H + V \otimes i, \bar{P} = H - V \otimes i \in \mathfrak{g}/\tilde{\mathfrak{k}} \otimes \mathbb{Q}(i)$ then

$$\mathfrak{g}/\tilde{\mathfrak{k}} \otimes \mathbb{Q}(i) = \mathbb{Q}(i)P \oplus \mathbb{Q}(i)\bar{P} \text{ and } e(\phi)Pe(-\phi) = e^{22\pi i\varphi}P; e(\phi)\bar{P}e(-\phi) = e^{-22\pi i\varphi}\bar{P} \tag{4.156}$$

Let $P^\vee, \bar{P}^\vee \in \operatorname{Hom}(\mathfrak{g}/\tilde{\mathfrak{k}}, \mathbb{Q}(i))$ be the dual basis. Then we check easily that $\boxed{\text{Pvee}}$

$$P^\vee(H) = \bar{P}^\vee(H) = \frac{1}{2} \text{ and } P^\vee(V) = -\frac{i}{2}, \bar{P}^\vee(V) = \frac{i}{2} \tag{4.157}$$

The module $\tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q}(i)$ decomposes under the action of $\tilde{K}_\infty$ into eigenspaces under $\tilde{K}_\infty$

$$\mathcal{M}_{\lambda^\vee} \otimes \mathbb{Q}(i) = \bigoplus_{\nu}^{n} \mathbb{Q}(i)(X + Y \otimes i)^{n-\nu}(X - Y \otimes i)^\nu \tag{4.158}$$

where

$$e(\phi)((X + Y \otimes i)^{n-\nu}(X - Y \otimes i)^\nu) = e^{\pi i(n-2\nu)\phi} \cdot (X + Y \otimes i)^{n-\nu}(X - Y \otimes i)^\nu.$$

Then we define the basis elements

$$\omega^\dagger = P^\vee \otimes \tilde{\psi}_{n+2} \otimes (X - Y \otimes i)^n \; ; \; \bar{\omega}^\dagger = \bar{P}^\vee \otimes \tilde{\psi}_{-n-2} \otimes (X + Y \otimes i)^n \tag{4.159}$$

We still have our involution $\mathbf{c} \in \tilde{K}^*_\infty$(See (4.26)) and clearly we have $\mathbf{c}\omega^\dagger = i^n\bar{\omega}^\dagger$ (Remember $n \equiv 0 \mod 2$.)

Now we put $\boxed{\text{OPM}}$

$$\omega^\dagger_+ = \frac{1}{2}(\omega^\dagger + i^n\bar{\omega}^\dagger) \; ; \; \omega^\dagger_- = \frac{1}{2}(\omega^\dagger - i^n\bar{\omega}^\dagger) \tag{4.160}$$

then these elements

$$\omega^\dagger_\pm = \frac{1}{2}(\omega^\dagger \pm i^n\bar{\omega}^\dagger) \; \in \; \operatorname{Hom}_{K_\infty}(\Lambda^1(\mathfrak{g}/\mathfrak{k}), \tilde{\mathcal{D}}_\lambda \otimes \mathcal{M}_\lambda)_\pm$$

and they are generators of these one dimensional spaces. The choice of these generators seems to be somewhat arbitrary, in [**?**] we give some motivation for this choice.

There is an alternative way to select $\omega^\dagger_\pm$. If we evaluate $\omega^\dagger_\pm$ on the element $H \in \mathfrak{g}/\mathfrak{k} = \mathfrak{p}$ then

$$\omega^\dagger_\pm(H) = \frac{1}{4}(\psi^\dagger_{n+2} \otimes (X - Y \otimes i)^n \pm i^n(\psi^\dagger_{-n-2} \otimes (X + Y \otimes i)^n) \in \mathcal{D}^\dagger_\lambda \otimes \mathcal{M}_\lambda$$

These are functions on $\mathrm{Gl}_2(\mathbb{R})$ with values in $\mathcal{M}_\lambda$. We pair these functions with an $\mathcal{M}_\lambda \otimes \mathbb{C}$ valued function, more precisely we consider the function $g \mapsto <\omega^\dagger_\pm(\mathrm{Ad}(g)H)(g), \rho_\lambda(g)X^\nu Y^\nu>$.

We restrict these scalar valued functions to the real points of the split torus

$$< \omega^\dagger_\pm(H)(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}), \rho_\lambda(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix})X^\nu Y^{n-\nu} >=$$

$$< \tfrac{1}{4}(\psi^\dagger_{n+2}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) \otimes (X - Y\otimes i)^n \pm i^n \psi^\dagger_{-n-2}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) \otimes (X + Y \otimes i)^n), X^\nu Y^{n-\nu} > t^{-\frac{n}{2}+\nu}$$

Now let ϵ be a variable which can take the values $+,-$, then $\epsilon = +1, -1$. Our formula (4.10) gives us $< (X - \epsilon Y \otimes i)^n, X^\nu Y^{n-\nu} >= (-\epsilon i)^{n-\nu}$ and combining this with the explicit formula (4.133) for the values of $\psi^\dagger_{\epsilon(n+2)}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix})$ we get

$$< \omega^\dagger_\epsilon(H)(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}), \rho_\lambda(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix})X^\nu Y^{n-\nu} >= \begin{cases} (-i)^{n-\nu} t^{\frac{n}{2}+1} e^{-2\pi t} t^{-\frac{n}{2}+\nu} & \text{for } t > 0 \\ \epsilon i^{n-\nu}(-t)^{\frac{n}{2}+1} e^{2\pi t}(-t)^{-\frac{n}{2}+\nu} & \text{for } t < 0 \end{cases}.$$

(Here we use that n is even, but with suitable minor modifications we can also treat the case n odd.) Then a straight forward computation yields $\boxed{\text{Mellinone}}$

$$\int_{T^{\mathrm{ad}}(\mathbb{R})} < \omega^\dagger_\epsilon(H)(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}), \rho_\lambda(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix})X^\nu Y^{n-\nu} > \frac{dt}{t} = \frac{1}{2}\begin{cases} \frac{\Gamma(n+1-\nu)}{(2\pi)^{n+1-\nu}} & \text{if } (-1)^{\frac{n}{2}-\nu} = \mathrm{sg}(\epsilon) \\ 0 & \text{else} \end{cases} \tag{4.161}$$

For each choice of the sign $\epsilon = \pm 1$ one of these equation determines the generator $\Omega^\dagger_\pm$. This formula will be of importance when we discuss the special values of L-functions.

In case B) we do basically the same, in some sense it is even simpler because K_∞ is maximal compact in this case, i.e. $K_\infty = K^*_\infty$. But on the other hand we need some very explicit information about the theory of irreducible representations of K_∞ and also about the decomposition of tensor products of these representations. We will also use some explicit formulas for Bessel functions.

A small arithmetic consideration

The quotient $\mathfrak{g}/\tilde{\mathfrak{k}}$ is a three-dimensional vector space over $\mathbb{Q}$ the group K_∞ acts by the adjoint representation and this gives us the standard three dimensional representation of $K_\infty = U(2)$, which in addition is trivial on the center. (See 4.1.2). This module is given by the highest weight $2\gamma_c$. We must have $\lambda = n(\gamma + \bar\gamma) + ..$, if we want $\mathcal{E}^{(2)}(\lambda, w, \Gamma) \neq 0$, and then the formulae 4.1.6 and 4.59 imply that for $\bullet = 1, 2$

$$\dim_{\mathbb{C}} \ \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathcal{D}^\dagger_{\lambda^\vee} \otimes \mathcal{M}_{\lambda^\vee}) = 1 \tag{4.162}$$

Now we recall that we have defined a structure of a $R = \mathbb{Z}[\frac{1}{2}]$ module on all the modules on the stage, hence we see that

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathcal{D}_{\lambda^\vee} \otimes \mathcal{M}_{\lambda^\vee}) = \ \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}})_R, \mathcal{D}_{\lambda^\vee R} \otimes \mathcal{M}_{\lambda^\vee R}) \otimes \mathbb{C}, \tag{4.163}$$

here we are a little bit sloppy: The first subscript K_∞ is the compact group and the second subscript is a smooth groups scheme over R. For both choices of $\bullet$

the second term in the above equation is a free R module of rank 1. We choose generators

$$\omega_{\lambda}^{\dagger,\bullet} \in \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}})_R, \mathcal{D}_{\lambda^\vee R} \otimes \mathcal{M}_{\lambda^\vee R}).$$

These generators $\omega^{\dagger,1}, \omega^{\dagger,2}$ are well defined up to an element in $R^\times$.

End of the small consideration

The quotient $\mathfrak{g}/\tilde{\mathfrak{k}}$ is a three-dimensional vector space over $\mathbb{Q}$ the group K_∞ acts by the adjoint representation and this gives us the standard three dimensional representation of $K_\infty = U(2)$, which in addition is trivial on the center. (See 4.1.2). This module is given by the highest weight $2\gamma_c$. We must have $\lambda = n(\gamma+\bar{\gamma}) + ..$, if we want $\mathcal{E}^{(2)}(\lambda, w, \Gamma) \neq 0$, and then the formulae 4.1.6 and 4.59 imply that for $\bullet = 1, 2$

$$\dim_{\mathbb{C}}\ \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathcal{D}^\dagger_{\lambda^\vee} \otimes \mathcal{M}_{\lambda^\vee}) = 1 \tag{4.164}$$

We fix these generators by prescribing values of certain Mellin transforms. To do this we need a little bit of representation theory. Of course we may replace K_∞ by SU(2) because the action of the center on the different modules cancels out. The modules $\mathfrak{g}/\mathfrak{k} \otimes \mathbb{C}$, $w\mathcal{D}^\dagger_{\lambda^\vee}$ and $\mathcal{M}_{\lambda^\vee} \otimes \mathbb{C}$ extend naturally to $\mathrm{Sl}_2(\mathbb{C})$ modules and hence we have to find an explicit generator in

$$\mathrm{Hom}_{\mathrm{Sl}_2(\mathbb{C})}(\mathfrak{g}/\mathfrak{k} \otimes \mathbb{C}, \mathcal{D}^\dagger_{\lambda^\vee} \otimes \mathcal{M}_{n\gamma} \otimes \mathcal{M}_{n\bar{\gamma}}).$$

We have an explicit basis for $\mathfrak{g}/\mathfrak{k} \otimes \mathbb{C}$ (See (4.19), our module $\mathcal{M}_{\lambda^\vee} = \mathcal{M}^\flat_{n\gamma} \otimes \mathcal{M}^\flat_{n\gamma} \otimes_{\mathcal{O}} \mathbb{C}$ is given explicitly to us.

Our module $\mathcal{D}^\dagger_{\lambda^\vee} \subset \mathfrak{I}_B^G \lambda_{\mathbb{R}} \rho^2_{\mathbb{R}}$, and this last module decomposes into SU(2) -types (See(4.34). These SU(2) modules canonically extend to $\mathrm{Sl}_2(\mathbb{C})$-modules, we have $\boxed{\text{decU}}$

$$\mathfrak{I}_B^G \lambda_{\mathbb{R}} \rho^2_{\mathbb{R}} = \bigoplus_{\nu=0}^{\infty} \mathcal{M}_{2\nu\gamma} = \bigoplus_{\nu=0}^{\infty} \mathfrak{I}_B^G \lambda_{\mathbb{R}} \rho^2_{\mathbb{R}}(2\nu) \tag{4.165}$$

and

$$(\mathfrak{I}_B^G \lambda_{\mathbb{R}} \rho^2_{\mathbb{R}}(2(n+1)))^\dagger = \mathcal{D}^\dagger_{\lambda^\vee}(2(n+1))$$

Now it is clear that we have the problem to select a specific generator in

$$\mathrm{Hom}_{\mathrm{Sl}_2(\mathbb{C})}(\mathfrak{g}/\mathfrak{k} \otimes \mathbb{C}, \mathcal{D}^\dagger_{\lambda^\vee}(2(n+1)) \otimes \mathcal{M}^\flat_{n\gamma} \otimes \mathcal{M}^\flat_{n\bar{\gamma}} \otimes \mathbb{C}).$$

The modules $\mathfrak{g}/\mathfrak{k} \otimes \mathbb{C}, \mathcal{M}^\flat_{n\gamma}, \mathcal{M}^\flat_{n\bar{\gamma}}$ come with an explicit basis (See 4.19), if we want to write down a specific generator $\omega^{\dagger,\bullet}$ we have to write down a basis of $\mathcal{D}^\dagger_{\lambda^\vee}(2(n+1))$.

Again we start from our exact sequence

$$0 \to \mathcal{D}_{\lambda^\vee} \to \mathfrak{I}_B^G \to \mathcal{M}_\lambda \to 0 \tag{4.166}$$

we apply the map $\mathcal{F}$ to it and get an exact sequence of Whittaker modules

$$0 \to \mathcal{D}^\dagger_{\lambda^\vee} \to \mathfrak{I}_B^{G,\dagger} \to \mathcal{M}_\lambda \to 0 \tag{4.167}$$

We recall the definition of $\mathfrak{I}_B^G\lambda_{\mathbb{R}}\rho_{\mathbb{R}}^2$ as an induced representation, the space of K_∞ invariant vectors is spanned by the spherical function

$$\psi_{\lambda,0}(bk) = \psi_{\lambda,0}(\begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}) = \lambda_{\mathbb{R}}\rho_{\mathbb{R}}^2(b).$$

We map the induced representation to its Whittaker model by

$$\mathcal{F} : \psi \mapsto \{g \mapsto \int \psi(w \begin{pmatrix} 1 & x+iy \\ 0 & 1 \end{pmatrix} g)e^{2\pi i x} dxdy\} \tag{4.168}$$

our basis element will be $\phi^\dagger_{\lambda,0} = \mathcal{F}(\psi_{\lambda,0})$. A straightforward computation yields

$$\phi^\dagger_{\lambda,0}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = \mathcal{F}(\psi_{\lambda,0})(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = \int_\infty^\infty \frac{t^{n+2}}{(t^2+x^2+y^2)^{n+2}} e^{2\pi i x} dxdy$$

The educated reader knows that this function in the variable t is well known, we have

$$\phi^\dagger_{\lambda,0}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = \frac{2\pi^{n+2}}{\Gamma(n+2)} tK_{n+1}(2\pi t)$$

where $K_n(2\pi t)$ is the modified Bessel function. Of course $phi^\dagger_{\lambda,0}$ is a function on $G(\mathbb{R}) = \mathrm{Gl}_2(\mathbb{C})$, it is right invariant under K_∞ and of course

$$\phi^\dagger_{\lambda,0}(\begin{pmatrix} 1 & x+iy \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = e^{2\pi i x}\phi^\dagger_{\lambda,0}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix})$$

hence it is defined by its restriction to $T^{\mathrm{ad}}(\mathbb{R})_{>0}$.

Starting from this function we construct the desired basis of $\mathcal{D}^\dagger_{\lambda^\vee}(2(n+1))$. The Lie-algebra $\mathfrak{g}$ acts on $\mathfrak{I}_B^G\lambda_{\mathbb{R}}\rho_{\mathbb{R}}^2$, we restrict this action to $\mathfrak{p}$ and it is clear that under this action

$$\mathfrak{p} \otimes \mathfrak{I}_B^G\lambda_{\mathbb{R}}\rho_{\mathbb{R}}^2(2\nu) \to \mathfrak{I}_B^G\lambda_{\mathbb{R}}\rho_{\mathbb{R}}^2(2\nu+2) \oplus \mathfrak{I}_B^G\lambda_{\mathbb{R}}\rho_{\mathbb{R}}^2(2\nu) \oplus \mathfrak{I}_B^G\lambda_{\mathbb{R}}\rho_{\mathbb{R}}^2(2\nu-2)$$

and if we extend this action to the tensor algebra we get a map

$$\mathfrak{U}_{n+1} : \mathfrak{p}^{\otimes(n+1)} \otimes \mathfrak{I}_B^G\lambda_{\mathbb{R}}\rho_{\mathbb{R}}^2(0) \to \bigoplus_{\nu=0}^{n+1} \mathfrak{I}_B^G\lambda_{\mathbb{R}}\rho_{\mathbb{R}}^2(2\nu). \tag{4.169}$$

here we may replace $n+1$ by any positive integer k.

The group K_∞ acts on $\mathfrak{p}^{\otimes(n+1)}$ by the adjoint action and the above map is of course a K_∞ homomorphism. On the right hand side we can project to the highest K_∞ type $\mathfrak{I}_B^G\lambda_{\mathbb{R}}\rho_{\mathbb{R}}^2(2n+2) = \mathcal{D}^\dagger_{\lambda^\vee}(2(n+1))$, i.e. we get a surjective homomorphism

$$\Pi_{n+1} : \mathfrak{p}^{\otimes(n+1)} \otimes \mathfrak{I}_B^G\lambda_{\mathbb{R}}\rho_{\mathbb{R}}^2(0) \to \mathcal{D}^\dagger_{\lambda^\vee}(2(n+1)), \tag{4.170}$$

again we may replace $n+1$ by any positive integer k.

We have the standard surjective homomorphism $\mathfrak{p}^{\otimes(n+1)} \to \mathrm{Sym}^{n+1}(\mathfrak{p})$, let us denote its kernel by I_{n+1}. For any $f \in \mathfrak{I}_B^G\lambda_{\mathbb{R}}\rho_{\mathbb{R}}^2$ and $X', X'' \in \mathfrak{p}$ we have

$$(X'X'' - X''X')f = [X', X'']f.$$

Since the Lie bracket $[X_1, X_2] \in \mathfrak{k}$ it follows easily that Π_{n+1} vanishes on the kernel I_{n+1}. Hence our homomorphism Π_{n+1} factors over the quotient, i.e.

$$\Pi_{n+1} : \mathrm{Sym}^{n+1}(\mathfrak{p}) \to \mathcal{D}^{\dagger}_{\lambda^\vee}(2(n+1)).$$

We change our notation for the basis of $\mathfrak{p} \otimes \mathbb{C}$ (see 4.19) and put

$$\begin{gathered} X_1 = \tfrac{1}{2}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - i \otimes \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}); X_0 = \tfrac{1}{\sqrt{2}}(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) \\ X_{-1} = \tfrac{1}{2}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i \otimes \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}) \end{gathered} \tag{4.171}$$

We have the following proposition

Proposition 4.1.7. *The $2n+3$ elements*

$$\{X_1^{n+1}, X_0X_1^n, \dots, X_0^{n+1}, X_0^nX_{-1}, \dots, X_{-1}^{n+1}\}$$

form a basis of a K_∞ invariant subspace of Sym$^{n+1}(\mathfrak{p}) \otimes \mathbb{C}$. This subspace is irreducible, it is isomorphic to $\mathcal{M}_{2n+2}$. These basis elements are weight eigenvectors for the action of T_c.

Proof. The representation of the algebraic group K_∞ on $\mathfrak{p}$ extends to a representation of the algebraic group $\mathrm{Sl}_2/\mathbb{C}$ on $\mathfrak{p} \otimes \mathbb{C}$. As such it is isomorphic to the symmetric square $\mathrm{Sym}^2(\mathbb{C}^2)$ of the tautological representation, i.e. to the module $\mathcal{M}_2$ of polynomials $aU^2 + bUV + cV^2$. We get an isomorphism $\mathcal{M}_2 \xrightarrow{\sim} \mathfrak{p} \otimes \mathbb{C}$ by sending $U^2 \mapsto X_1, UV \mapsto X_0, V^2 \mapsto X_1$.. Now $\mathrm{Sym}^{2n+2}(\mathcal{M}_2) \subset \mathrm{Sym}^{n+1}(\mathrm{Sym}^2(\mathbb{C}^2)) = \mathrm{Sym}^{n+1}(\mathfrak{p} \otimes \mathbb{C})$ is an invariant submodule. It has the basis $U^{2n+2-\nu}V^\nu$ and clearly

$$U^{2n+2-\nu}V^\nu = X_1^{n+1-\nu}X_0^\nu \text{ if } \nu \leq n+1 \text{ and } X_0^{n+1\nu}X_1$$

and this implies the assertion. ∎

This implies that the elements

$$\begin{gathered} \{\Pi_{n+1}(X_1^{n+1}\phi^\dagger_{\lambda,0}), \Pi_{n+1}(X_0X_1^n\phi^\dagger_{\lambda,0}), \dots, \Pi_{n+1}(X_0^nX_1\phi^\dagger_{\lambda,0}), \\ \Pi_{n+1}(X_0^{n+1}, \phi^\dagger_{\lambda,0}), \Pi_{n+1}(X_0^nX_{-1}\phi^\dagger_{\lambda,0}), \dots, \Pi_{n+1}(X_{-1}^{n+1}\phi^\dagger_{\lambda,0})\} \end{gathered} \tag{4.172}$$

form a basis of $\mathcal{D}^\dagger_{\lambda^\vee}(2(n+1))$.

We change our notation slightly. For $m < 0$ we put $X_1^m := X_{-1}^{-m}$ and for $0 \leq \nu \leq 2n+2$ we put $[\nu] = \nu$ if $\nu \leq n+1$ and $[\nu] = 2n+2-\nu$ if $\nu \geq n+1$. Then our above basis can be written as

$$\{\dots, \Pi_{n+1}(X_0^{[\nu]}X_1^{n+1-\nu}\phi^\dagger_{\lambda,0}), \dots\}_{\nu=0,\dots,\nu=2n+2}, \tag{4.173}$$

these are the weight vectors of weight $2(n+1-\nu)\gamma$. We introduce the notation

$$\phi^\dagger_{\lambda,n+1-\nu} := \Pi_{n+1}(X_0^{[\nu]}X_1^{n+1-\nu}\phi^\dagger_{\lambda,0})$$

These functions $\phi^\dagger_{\lambda,\mu}$ are Whittaker functions they satisfy

$$\phi^\dagger_{\lambda,\mu}(\begin{pmatrix} 1 & x+iy \\ 0 & 1 \end{pmatrix} g) = e^{2\pi i x}\phi^\dagger_{\lambda,\mu}(g).$$

They are not K_∞ invariant, but they are weight vectors for the torus, we have

$$\phi^\dagger_{\lambda,\mu}(g\begin{pmatrix} e^{2\pi i\varphi} & 0 \\ 0 & 1\end{pmatrix}) = e^{4\mu\pi i\varphi}\phi^\dagger(g) \tag{4.174}$$

and more generally $\phi^\dagger_{\lambda,\nu}(gk) = \sum_\mu a_{\nu,\mu}(k)\phi^\dagger_{\lambda,\mu}(g)$ where the $a_{\nu,\mu}(k)$ are the matrix coefficients of $\mathcal{M}_{2n+2}$. (above proposition).

We consider the restriction of the functions $\phi^\dagger_{\lambda,\nu}$ to the maximal torus $T(\mathbb{R})$. Since $\mathfrak{I}_B^G\lambda_\mathbb{R}\rho^2_\mathbb{R}(2\nu)$ has a central character, it suffices to consider the restriction

$$\phi^\dagger_{\lambda,\nu} \to \{z \mapsto \phi^\dagger_{\lambda,\nu}(\begin{pmatrix} z & 0 \\ 0 & 1\end{pmatrix}),\}$$

we write $z = te^{2\pi i\varphi}$. This means that we map the module $\mathfrak{I}_B^G\lambda_\mathbb{R}\rho^2_\mathbb{R}$ to its Kirillow realisation $\mathfrak{I}_B^{G,\kappa}\lambda_\mathbb{R}\rho^2_\mathbb{R} \subset \mathcal{C}_\infty(\mathbb{C}^\times)$. (See [32] , §2 5.), especially this map is injective.

We express the restriction of these functions $\phi^\dagger_{\lambda,\nu}$ to the torus $T^{\mathrm{ad}}(\mathbb{R})_{>0}$ in terms of Bessel functions. We introduce the notation

$$\mathfrak{I}_B^G[2k] := \bigoplus_{\nu=0}^{k} \mathfrak{I}_B^G\lambda_\mathbb{R}\rho^2_\mathbb{R}(2\nu) \tag{4.175}$$

For any Whittaker function $\phi^\dagger \in \mathfrak{I}_B^G[2k]^\dagger$ we have

$$\Pi_{k+1}(X_1\phi^\dagger)(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}) = \frac{\phi^\dagger(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}\exp(\epsilon X_1)) - \phi^\dagger(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix})}{\epsilon}$$

We write $X_1 = \frac{1}{2}(\begin{pmatrix} 0 & 2 \\ 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & 2i \\ 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix})$ the last two matrices are in $\mathfrak{k}$ so they preserve the K_∞ type and

$$\phi^\dagger(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}\exp(\epsilon(\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & i \\ 0 & 0\end{pmatrix}))) =$$

$$\phi^\dagger(\exp(\epsilon(\begin{pmatrix} 0 & t \\ 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & it \\ 0 & 0\end{pmatrix})\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}) = \phi^\dagger(((\begin{pmatrix} 1 & \epsilon(t+it) \\ 0 & 1\end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}) =$$
$$e^{2\pi i\epsilon t}\phi^\dagger(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}) = \phi^\dagger(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix})(1+it\epsilon))$$

and hence

$$\Pi_{k+1}(X_1\phi^\dagger)(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}) = 2it\phi^\dagger(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix})$$

If $\phi^\dagger$ is a weight vector, i.e. $\phi^\dagger(\begin{pmatrix} te^{2\pi i\varphi} & 0 \\ 0 & 1\end{pmatrix}) = e^{2\pi i\mu\varphi}\phi^\dagger(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix})$ then $X_1\phi^\dagger$ is also a weight vector with weight $e^{2\pi i(\mu+2)\varphi}$.

This gives us

$$\phi^\dagger_{\lambda,n+1}(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}) = (X_1^{n+1}\phi^\dagger_{\lambda,0})(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}) = \frac{2^{2n+3}\pi^{2n+3}}{\Gamma(n+2)}t^{n+2}K_{n+1}(2\pi t) \tag{4.176}$$

Since this function is of weight $2n+2$ we can forget the projection Π_{n+1}.

We have recursion formulas for the Bessel functions

$$\begin{aligned}\tfrac{d}{dt}K_n(t) &= -\tfrac{1}{2}(K_{n-1}(t)+K_{n+1}(t))\\ K_{n+1}(t) &= K_{n-1}(t)+\tfrac{2n}{t}K_n(t)\end{aligned} \tag{4.177}$$

A straightforward calculation yields

$$t\frac{d}{dt}t^{\mu}K_{\nu}(2\pi t) = (\mu-\nu)t^{\mu}K_{\nu}(2\pi t) - 2\pi t^{\mu+1}K_{\nu-1}(2\pi t) \tag{4.178}$$

Then $\phi^{\dagger}_{\lambda,n+1-\nu}(\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}) = \Pi_{n+1}(X_0^{[\nu]}X_1^{n+1-\nu}\phi^{\dagger}_{\lambda,n+1}(\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}))$. We get

$$X_1^{n+1-\nu}\phi^{\dagger}_{\lambda,n+1}(\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix})) = \frac{(2\pi)^{n+2+|n+1-\nu|}}{\Gamma(n+2)}t^{1+|n+1-\nu|}K_{n+1}(2\pi t).$$

To this we apply $X_0^{[\nu]}$. The operator X_0 is $t\frac{d}{dt}$, then the above formula gives

$$\Pi_{n+1}(X_0^{[\nu]}X_1^{n+1-\nu}\phi^{\dagger}_{\lambda}(\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix})) = \Pi_{n+1}(\cdots + \frac{2^{2n+3}\pi^{2n+3}}{\Gamma(n+2)}t^{n+2}K_{n+1-\nu}(2\pi t)) \tag{4.179}$$

where the dots $\cdots$ are a sum of those terms which are in the image of $\mathfrak{I}_B^{G,\kappa}\lambda_{\mathbb{R}}\rho_{\mathbb{R}}^2[2n]$ hence they vanish under $\Pi_{n+1.}$ and consequently

$$\phi^{\dagger}_{\lambda,\mu}(\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}) = \frac{2^{2n+3}\pi^{2n+3}}{\Gamma(n+2)}t^{n+2}K_{\mu}(\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}) \tag{4.180}$$

where μ runs from $n+1$ to $-n-1$ and of course $K_{\mu} = K_{-\mu}$.

Decompositions of tensor products

If $\lambda_1 = n_1\gamma, \lambda_2 = n_2\gamma$ are two highest weights and if we consider the highest weight modules $\mathcal{M}_{\lambda_1,\mathbb{Q}}, \mathcal{M}_{\lambda_2,\mathbb{Q}}$ then it is a classical theorem that

$$\mathcal{M}_{\lambda_1,\mathbb{Q}}\otimes\mathcal{M}_{\lambda_2,\mathbb{Q}} = \mathcal{M}_{(n_1+n_2)\gamma,\mathbb{Q}}\oplus\mathcal{M}_{(n_1+n_2-2)\gamma,\mathbb{Q}}\oplus\cdots\oplus\mathcal{M}_{(n_1-n_2)\gamma,\mathbb{Q}}\cdots$$

where we assume $n_1\geq n_2$, we put $n = n_1+n_2$. Our next aim is to give an explicit homomorphism

$$j_{n_1,n_2} : \mathcal{M}^{\flat}_{(n_1+n_2)\gamma}\hookrightarrow\mathcal{M}^{\flat}_{n_1\gamma}\otimes\mathcal{M}^{\flat}_{n_2\gamma} \tag{4.181}$$

in other words we want to write explicit tensors for the images of $e^{\flat}_{\mu}, \mu = n_1+n_2, n_1+n_2-2,\dots,-n_1-n_2$. Of course we send the highest weight vector $e^{\flat}_{n_1+n_2}\mapsto {}'e^{\flat}_{n_1}\otimes{}''e^{\flat}_{n_2}$, this vector is the highest weight vector in the direct summand $\mathcal{M}^{\flat}_{(n_1+n_2)\gamma,\mathbb{Q}}\subset\mathcal{M}_{(n_1+n_2)\gamma,\mathbb{Q}}\oplus\cdots\oplus\mathcal{M}_{(n_1-n_2)\gamma,\mathbb{Q}}$. In terms of the explicit realisation of these modules we can say

$$X^{n_1+n_2}\mapsto {}'X^{n_1}\otimes{}''X^{n_2} \tag{4.182}$$

Now we apply the matrix $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ to it, here we may think of t as an in determinant. Then we see

$$(X+tY)^{n_1+n_2} \mapsto ({}'X+t\,{}'Y)^{n_1} \otimes ({}''X+t\,{}''Y)^{n_2} \tag{4.183}$$

We expand on both sides and find

$$\textstyle\sum_{\mu=0}^{n_1+n_2} \binom{n_1+n_2}{\mu} t^\mu X^{n_1+n_2-\mu} Y^\mu \mapsto$$

$$\textstyle\sum_{\mu=0}^{n_1+n_2} t^\mu (\sum_{\mu_1,\mu_2:\mu_1+\mu_2=\mu} \binom{n_1}{\mu_1} {}'X^{n_1-\mu_1}{}'Y^{\mu_1} \otimes \binom{n_2}{\mu_2} {}''X^{n_2-\mu_2} \otimes {}''X^{n_2-\mu_2}{}''Y_{\mu_2}) \tag{4.184}$$

We remember the definition of the basis elements $e^\flat_\mu$, the formula above gives us

$$j_{n_1,n_2} : e^\flat_\mu \mapsto \sum_{\mu_1+\mu_2=\mu} {}'e^\flat_{\mu_1} \otimes {}''e^\flat_{\mu_2} \tag{4.185}$$

We apply this to the SU(2) -module

$$(\mathfrak{g}/\mathfrak{k})^\vee_{\mathbb{F}} \otimes \mathcal{M}_{n\gamma} \otimes_F \mathcal{M}_{n\bar\gamma},$$

this module contains a unique copy of $\mathcal{M}^\flat_{2n+2}$. We write

$$\mathfrak{g}/\mathfrak{k}^\vee_{\mathbb{F}} = F\,{}^0e^\flat_2 \oplus F\,{}^0e^\flat_0 \oplus F\,{}^0e^\flat_{-2},\ \mathcal{M}_{n_1\gamma,F} = \bigoplus_{\mu_1} F e^\flat_{\mu_1},\ \mathcal{M}_{n_2\gamma,F} = \bigoplus_{\mu_2} F \bar e^\flat_{\mu_2} \tag{4.186}$$

where of course μ_i run from n_i to $-n_i$ and $\mu_i \equiv n_i \mod 2$. Then our copy of $\mathcal{M}^\flat_{2n+2}$ comes with the basis

$$\tilde e^\flat_\mu = \sum_{\mu_0+\mu_1+\mu_2=\mu} {}^0e^\flat_{\mu_0} \otimes e^\flat_{\mu_1} \otimes \bar e^\flat_{\mu_2}$$

We have the invariant pairing (4.10) and this tells us that we can choose as our generator $\boxed{\text{cangen}}$

$$\omega^{\dagger,\bullet}_\lambda = \sum_{\mu=0}^{n_1+n_2+2} \phi^\dagger_{\lambda,\mu} \otimes (\sum_{\mu_0+\mu_1+\mu_2=n+1-\mu} {}^0e^\flat_{\mu_0} \otimes e^\flat_{\mu_1} \otimes \bar e^\flat_{\mu_2}) \tag{4.187}$$

This generator is only determined up to a scalar.

The "canonical" choice of the generator

Again we can fix the generator by requiring that certain Mellin transforms have a prescribed value at certain prescribed arguments.

We do essentially the same as in the case A). We can interpret $\omega^{\dagger,1}$ as a differential 1- form on $G(\mathbb{R})$ with values in $\mathcal{M}^\flat_\lambda \otimes \mathbb{C}$. We can restrict this 1-form to the torus $T^{\mathrm{ad}}(\mathbb{R})_{>0} = \{\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} |t>0\}$. We have the "cycles" $e_{\mu_1} \otimes e_{\mu_2} \in \mathcal{M}_\lambda^\vee$. We evaluate $\omega^{\dagger,1}(X_0)$ on these "cycles" and get

$$< \omega_\lambda^{\dagger,\bullet}(X_0), e_{\mu_1} \otimes e_{\mu_2} > (\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = \phi^\dagger_{\lambda,n-\mu_1-\mu_2}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) t^{\mu_1+\mu_2} = c_n' t^{n+2+\mu_1+\mu_2} K_{n-\mu_1-\mu_2}(2\pi t) \tag{4.188}$$

Later -when we study the special values of L-functions- we need to know the value

$$\int_0^\infty < \omega^{\dagger,\bullet}(X_0), e_{\mu_1} \otimes e_{\mu_2} > (\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) \frac{dt}{t} = c_n' \int_0^\infty t^{n+2+\mu_1+\mu_2} K_{n-\mu_1-\mu_2}(2\pi t) \frac{dt}{t} \tag{4.189}$$

We also will need formulas for the Mellin transforms of these Bessel functions. Here we quote [1] .p.331,334 and recall two of them (the second one for later use)

$$\int_0^\infty K_\nu(2\pi t) t^s \tfrac{dt}{t} = 2^{s-2}(2\pi)^{-s}\Gamma(\tfrac{s-\nu}{2})\Gamma(\tfrac{s+\nu}{2})$$
$$\int_0^\infty K_\mu(2\pi t) K_\nu(2\pi t) t^s \tfrac{dt}{t} = 2^{s-3}(2\pi)^{-s}\Gamma(\tfrac{s-\mu-\nu}{2})\Gamma(\tfrac{s-\mu+\nu}{2})\Gamma(\tfrac{s+\mu-\nu}{2})\Gamma(\tfrac{s+\nu+\mu}{2}) \tag{4.190}$$

the first one gives us

$$\int_0^\infty t^{n+2+\mu_1+\mu_2} K_{n-\mu_1-\mu_2}(2\pi t) \frac{dt}{t} = \frac{\Gamma(n+1)}{4\pi} \frac{\Gamma(\mu+1)}{\pi^{\mu+1}} \tag{4.191}$$

We observe that the first factor in front does not depend on μ_1, μ_2. So we renormalise our generator and for $\mu = -n-1, -n, \dots, n+1$ we now put $\boxed{\text{Phi}}$

$$\phi^\dagger_{\lambda,\mu}(t) = \frac{4\pi}{\Gamma(n+1)} t^{n+2} K_{n+1-\mu}(t) \tag{4.192}$$

and with this choice of $\phi^\dagger_{\lambda,\nu}$ the $\omega^{\dagger,1}$ (4.187) is now our canonical generator. Now our formula (4.188) becomes

$$< \omega^{\dagger,\bullet}(X_0), e_{\mu_1} \otimes e_{\mu_2} > (\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = \frac{\Gamma(\mu+1)}{\pi^{\mu+1}} \tag{4.193}$$

Hence we may just choose $\mu_1 = \mu_2 = 0$ to nail down $\omega^{\dagger,\bullet}$, it is not clear to me whether or not it is a "miracle" that the above relation holds for all values of μ_1, μ_2.

The definition of the periods

The inner cohomology with rational coefficients is a semi-simple module under the action of the Hecke algebra (See Theorem 3.1.1). We find a finite Galois-extension $F/\mathbb{Q}$ such that we get a decomposition into absolutely irreducible modules

$$H^\bullet_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_\lambda \otimes F) = \bigoplus_{\pi_f} H^\bullet_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_\lambda \otimes F)(\pi_f) \tag{4.194}$$

Since we assume that $\Gamma = \mathrm{Gl}_2(\mathbb{Z})$, hence the π_f are homomorphisms $\pi_f : \mathcal{H} \to \mathcal{O}_F$. (See see 3.20) In the case A) such an isotypical piece is a direct sum

$$H^\bullet_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_\lambda \otimes F)(\pi_f) = H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_\lambda \otimes F)(\pi_f)_+ \oplus H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_\lambda \otimes F)(\pi_f)_- \tag{4.195}$$

where both summands are of dimension one over F.

In case B) we get

$$H^\bullet_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_\lambda \otimes F)(\pi_f) = H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_\lambda \otimes F)(\pi_f) \oplus H^2_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_\lambda \otimes F)(\pi_f) \tag{4.196}$$

and again the summands are one dimensional.

We have defined the module of integral classes $H^1_{!,\,\mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_\lambda \otimes \mathcal{O}_F) \subset H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_\lambda \otimes F)$ (See 2.72) and we consider the intersection

$$H^\bullet_{!,\,\mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_\lambda \otimes \mathcal{O}_F)(\pi_f)_\epsilon = H^\bullet_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_\lambda \otimes F)(\pi_f)_\epsilon \cap H^1_{!,\,\mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_\lambda \otimes \mathcal{O}_F)$$

is a locally free $\mathcal{O}_F$ -module of rank 1, here $\epsilon = \pm, \bullet = 1$(resp. $\epsilon = 1, \bullet \in \{1, 2\}$). We assume for simplicity that it is actually free, otherwise the formulation of the following becomes slightly more complicated. (See below). On the set of π_f which occur in this decomposition we have an action of the Galois group (See (3.21)) and the Galois action yields canonical isomorphisms

$$\Phi_{\sigma,\tau} : H^\bullet_{!,\,\mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_\lambda \otimes \mathcal{O}_F)({}^\sigma\pi_f)_\epsilon \xrightarrow{\sim} H^\bullet_{!,\,\mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_\lambda \otimes \mathcal{O}_F)({}^\tau\pi_f)_\epsilon \tag{4.197}$$

We choose generators ${}^\sigma e^\bullet_\epsilon(\pi_f)$ and a simple argument using Hilbert theorem 90 shows that we can assume the consistency condition $\boxed{\text{H90}}$

$$\Phi_{\sigma,\tau}(e^\bullet_\epsilon({}^\sigma\pi_f)) = e^\bullet_\epsilon({}^\tau\pi_f) \tag{4.198}$$

We get isomorphisms

$$\mathcal{F}^\bullet(\omega^\dagger_\epsilon) : \mathcal{W}({}^\sigma\pi_f) \otimes_F \mathbb{C} \xrightarrow{\sim} H^\bullet_\epsilon(\Gamma\backslash\mathbb{H}, \mathcal{M}_{\lambda^\vee})({}^\sigma\pi_f) \otimes_F \mathbb{C} \tag{4.199}$$

which are defined by $\boxed{\text{Armand1}}$

$$\mathcal{F}^\bullet(\omega^\dagger_\epsilon) : h_{{}^\sigma\pi_f} \mapsto [\mathcal{F}(\omega^\dagger_\epsilon \times h_{{}^\sigma\pi_f})], \tag{4.200}$$

here $\mathcal{F}(\omega^\dagger_\epsilon \times h_{{}^\sigma\pi_f})$ is viewed as a closed $\mathcal{M}_\lambda \otimes \mathbb{C}$ valued differential via the identification 4.101, and $[\dots]$ is its class in cohomology.

Since we assume that π_f is unramified everywhere $\mathcal{W}(\pi_f)$ we have the canonical basis element $h^{(0)}_f = \prod_p h^{(0)}_{{}^\sigma\pi_p}$ where $h^{(0)}_{\sigma\pi_p}$ is defined by the equality 4.126. Then we have obviously $\sigma(h^{(0)}_{\pi_p}) = h^{(0)}_{{}^\sigma\pi_p}$.

Then we define the *periods* by the relation

$$\mathcal{F}(\omega^\dagger_\epsilon)(h^{(\dagger,0)}_{{}^\sigma\pi_f}) = \Omega^\bullet(\epsilon \times \pi_f,)e^\bullet(\epsilon \times \pi_f) \tag{4.201}$$

These periods depend of course on our choice of the "canonical" generator $\omega_\epsilon^\dagger$. We see that the numbers $\Omega^\bullet({}^\sigma\pi_f,\epsilon)$ are well defined up to an element in $\mathcal{O}_F^\times$.

If $H^\bullet_{!,\,\mathrm{int}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}^\flat_\lambda\otimes\mathcal{O}_F)(\pi_f)_\epsilon$ is not a free $\mathcal{O}_F$ module, then we can find a covering by two open subsets U_1,U_2 of $\mathrm{Spec}(\mathcal{O}_F)$ such that $H^\bullet_{!,\,\mathrm{int}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}^\flat_\lambda\otimes$ $\otimes\mathcal{O}_F(U_i))(\epsilon\times\pi_f)$ is free. We can apply the above procedure and we get periods $\Omega_1(\epsilon\times\pi_f,),\Omega_2(\epsilon\times\pi_f)$, they are well defined up to an element in $\mathcal{O}_F(U_1)^\times,\mathcal{O}_F(U_2)^\times$ respectively. The ratio of these periods is an element in $\mathcal{O}_F(U_1\cap U_2)^\times$.

Perhaps at this point we should introduce the sheaf $\mathcal{P}$ of periods over F. For any open subset $U\subset\ \mathrm{Spec}(\mathcal{O}_F)$ we put $\mathcal{P}_F^*(U):=\mathbb{C}^\times/\mathcal{O}_F(U)^\times$, this is a Zariski preasheaf on $\mathrm{Spec}(\mathcal{O}_F)$, the associated sheaf is our sheaf of periods $\mathcal{P}_F$.

Now we can interpret the generators $e^\bullet(\epsilon\times\pi_f)$ as (the unique) section in the sheaf of generators modulo $\mathcal{O}_F^\times$ and then the equation (4.201) makes sense without the assumption on the class number.

These considerations will play a role in the following chapter.

Some little subtleties

We should notice that these periods are defined with respect to the "small" sheaves $\tilde{\mathcal{M}}^\flat_\lambda$. We have $\tilde{\mathcal{M}}^\flat_\lambda\subset\tilde{\mathcal{M}}_\lambda$ and therefore the map

$$H^\bullet_{!,\,\mathrm{int}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}^\flat_\lambda\otimes\mathcal{O}_F)(\pi_f)_\epsilon\to H^\bullet_{!,\,\mathrm{int}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_\lambda\otimes\mathcal{O}_F)(\pi_f)_\epsilon \tag{4.202}$$

may not be surjective. (The reader should not be puzzled by the fact that $\tilde{\mathcal{M}}^\flat_\lambda\otimes F=\tilde{\mathcal{M}}_\lambda\otimes F$.) Therefore, if we would work with $\tilde{\mathcal{M}}_\lambda$ instead and define the periods $\Omega^{\bullet,\#}({}^\sigma\pi_f,\epsilon)$ by the same procedure. Then we will get a relation

$$\Omega^{\bullet,\#}({}^\sigma\pi_f,\epsilon)=d(\pi_f,\epsilon)\Omega^\bullet({}^\sigma\pi_f,\epsilon)$$

where $d(\pi_,,\epsilon)$ is a non zero factor in $\mathcal{O}_F$. The primes in these factors are the divisors of the binomial coefficients.

But we could also with the module $H^\bullet(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}^\flat_\lambda\otimes\mathcal{O}_F)_{\mathrm{int},!}(\pi_f)_\epsilon$ and define the periods with respect to this module. Again these periods will integral multiples of the periods $\Omega^\bullet(\pi_f,\epsilon)$.

In the following Chapter 5 we will discuss the rationality results (Manin and Shimura) which relate these periods to special values of the $L-$ function (see section 5.6). But we also want to discuss this method not only for cuspidal classes but also for the Eisenstein cohomology classes, therefore we close this Chapter with a brief account of these Eisenstein classes.

4.1.12 The Eisenstein cohomology class

In section 3.3.6 we claimed the existence of the specific cohomology class $\mathrm{Eis}_n\in H^1(\Gamma\backslash\mathbb{H},\mathcal{M}_n)$. In this section we give s construction of this class on transcendental level, i.e. we construct a cohomology class $\mathrm{Eis}(\omega_n)\in H^1(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathbb{C})$ whose restriction to the boundary $H^1(\partial(\Gamma\backslash\mathbb{H}),\mathcal{M}_n\otimes\mathbb{C})$ is a given class ω_n. For the general theory of Eisenstein cohomology we refer to Chapter 9.

We start from our highest weight module $\mathcal{M}_\lambda$ and we observe that by definition we have an inclusion

$$i_0 : \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} \hookrightarrow \mathcal{C}_\infty(\Gamma_\infty^+ \backslash G^+(\mathbb{R}))$$

where

$$\Gamma_\infty^+ = \{\begin{pmatrix} t_1 & m \\ 0 & t_1 \end{pmatrix} | m \in \mathbb{Z} \, ; t_1 = \pm 1\}.$$

Therefore we get an isomorphism

$$H^1(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} \otimes \mathcal{M}_\lambda \otimes \mathbb{C}) \xrightarrow{\sim} H^1(\Gamma_\infty^+ \backslash \mathbb{H}, \mathcal{M}_\lambda \otimes \mathbb{C}) = H^1(\partial(\Gamma \backslash \mathbb{H}), \mathcal{M}_\lambda \otimes \mathbb{C})$$

The inclusion i_0 sends the module $\mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0}$ into a space of functions which are Γ_∞^+ invariant under left translations. Therefore we get a homomorphism

$$\mathrm{Eis} : \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} \to \mathcal{C}_\infty(\Gamma \backslash \mathrm{Sl}_2(\mathbb{R}))$$

if we make it invariant by summation, i.e. for $f \in \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0}$ we define $\boxed{\text{ESeries}}$

$$\mathrm{Eis}(f)(x) = \sum_{\Gamma_\infty^+ \backslash \mathrm{Sl}_2(\mathbb{Z})} f(\gamma x) \tag{4.203}$$

Of course we have to discuss the convergence of this infinite series. We could quote H. Jacquet: "Let us speak about convergence later", but here is a short interlude discussing this issue.

Interlude: Here is the point: We twist our module, for any complex number $z \in \mathbb{C}$ we consider the induced module

$$\mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} |\rho|^z \subset \mathcal{C}_\infty(\Gamma_\infty^+ \backslash \mathrm{Sl}_2(\mathbb{R}))$$

and again we write down the Eisenstein series. Now it an elementary exercise to show that the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (c, d)$$

provides a bijection

$$\Gamma_\infty^+ \backslash \mathrm{Sl}_2(\mathbb{Z}) \xrightarrow{\sim} \{(c, d) \in \mathbb{Z} \times \mathbb{Z} \mid (c, d) \text{ coprime } \}/\{\pm 1\} = \mathbb{P}^1(\mathbb{Q}).$$

An element $x \in \mathrm{Sl}_2(\mathbb{R})$ can be written as $x = \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} k$ with $k \in K_\infty$. Then for $f \in \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} |\rho|^z$

$$f(\gamma x, z) =$$

$$f(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} k, z) =$$

$$f(\begin{pmatrix} (c^2 t^2 + (cv + dt^{-1})^2)^{-1/2} & * \\ 0 & (c^2 t^2 + (cv + dt^{-1})^2)^{1/2} \end{pmatrix}) f(k(\gamma g) k, z) =$$

$$(c^2 t^2 + (cv + dt^{-1})^2)^{-n-2-z} f(k(\gamma g) k).$$

Since $|f(k(\gamma)k)|$ is bounded the series

$$\mathrm{Eis}(f,z)(x)=\sum_{\Gamma_\infty^+\backslash \mathrm{Sl}_2(\mathbb{Z})} f(\gamma x,z)$$

is converging if $\Re(z) >> 0$ and then it is also holomorphic in z. Selberg and others showed that it can be extended to a meromorphic function in the entire complex plane, it is now a special case of a theorem of Langlands [61]. If now the function $x \mapsto \mathrm{Eis}(f,z)(x)$ is holomorphic at $z=0$ then we do not care about convergence and we simply define

$$\mathrm{Eis}(f)(x)=\sum_{\Gamma_\infty^+\backslash \mathrm{Sl}_2(\mathbb{Z})} f(\gamma x) = \mathrm{Eis}(f,0)(x).$$

In our special case it is easy to see that the series is convergent at $z=0$ provided we have $n>0$ and this is the only case where we will apply this construction. *End interlude*

This provides a homomorphism

$$\mathrm{Eis}^\bullet : H^1(\mathfrak{g},K_\infty,\mathfrak{I}_B^G\lambda_{\mathbb{R}}^{w_0}\otimes\mathcal{M}_\lambda\otimes\mathbb{C}))\to H^1(\Gamma\backslash\mathbb{H},\mathcal{M}_\lambda\otimes\mathbb{C}) \tag{4.204}$$

In 4.80 we wrote down a distinguished generator $\omega_n = E_+^\vee \otimes e_{-n} \in H^1(\mathfrak{g},K_\infty,\mathfrak{I}_B^G\lambda_{\mathbb{R}}^{w_0}\otimes \mathcal{M}_\lambda\otimes\mathbb{C})$ and we define

$$\mathrm{Eis}_n = \mathrm{Eis}(\omega_n)$$

Proposition 4.1.8. *The restriction of Eis_n to $H^1(\partial(\Gamma\backslash\mathbb{H}),\mathcal{M}_\lambda\otimes\mathbb{C})$ is the class $[Y^n]$*

We have a brief look at the Eisenstein cohomology in case B). We refer to the Final remark at the end of chapter 2. For our imaginary field we take again $F=\mathbb{Q}[\mathbf{i}],\Gamma=\mathrm{Sl}_2(\mathbb{Z}[\mathbf{i}])$ and $\mathcal{M}_\lambda=\mathcal{M}_{n_1}\otimes\mathcal{M}_{n_2}$. We assume the parity condition $n_1\equiv n_2+2\equiv 0 \mod 4$. In chapter 4 we get from a rather elementary computation

$$H^1(\partial(\Gamma\backslash\mathbb{H}_3),\tilde{\mathcal{M}}_\lambda)_{\text{ int}}=\mathbb{Z}[\mathbf{i}]e_{01}\oplus\mathbb{Z}[\mathbf{i}]e_{10}. \tag{4.205}$$

If we extend the scalars to $\mathbb{C}$ we can represent these classes by differential forms. To do so we apply the two reflections s_1,s_2 to our highest weight $\lambda=n_1\gamma_1+n_2\gamma_2$ and get the two characters

$$s_1\cdot\lambda=(-n_1-2)\gamma_1+n_2\gamma_2;\ s_2\cdot\lambda=n_1\gamma_1+(-n_2-2)\gamma_2$$

These two characters yield characters $s_i\cdot\gamma : B(\mathbb{R})\to\mathbb{C}^\times$ and the we have seen that the two classes $E_+^{1,\vee}\otimes e^{(1)}_{-n_1-2}\otimes e^{(2)}_{n_2},\ e^{(1)}_{n_1}\otimes E_+^{2,\vee}\otimes e^{(2)}_{-n-2}$ provide differential forms $e_{s_i\cdot\lambda}$ and cohomology classes in $\mathbb{H}^1(\mathfrak{g},K_\infty,\mathfrak{I}_B^G s_i\cdot\lambda\otimes\mathcal{M}_\lambda)$ (see Thm.4.1.2 Since we have $\mathfrak{I}_B^G s_i\cdot\lambda\subset\mathcal{C}_\infty(B(\mathbb{Z})\backslash G(\mathbb{R}))$ we get the two classes $e_{s_i\cdot\lambda}\in H^1(\partial(\Gamma\backslash\mathbb{H}_3),\mathcal{M}_\lambda\otimes\mathbb{C})$. Hence we know

$$H^1(\partial(\Gamma\backslash\mathbb{H}_3),\tilde{\mathcal{M}}_\lambda\otimes\mathbb{C})=\mathbb{C}e_{s_1\cdot\lambda}\oplus\mathbb{C}e_{s_2\cdot\lambda} \tag{4.206}$$

but a close inspection shows that the $e_{s_1\cdot\lambda}$ are equal to the $e[i,j]$.

Again we invoke the theory of Eisenstein series, we assume $n_1 > n_2$ and define the Eisenstein intertwining operator

$$\mathrm{Eis}: \mathfrak{I}_B^G s_1 \cdot \lambda \to (\mathfrak{g}, K_\infty)_\infty(B(\mathbb{Z})\backslash G(\mathbb{R}))) \,; \{g \mapsto f(g\} \mapsto \{g \mapsto \sum_{\alpha \in B(\mathbb{Z})\backslash G(\mathbb{Z})} f(ag)\}. \tag{4.207}$$

It is not difficult to check that this infinite series is convergent (locally uniformly on compact sets),but we can also define the Eisenstein operator if $n_2 > n_1$ by analytic continuation.

Hence we get the Eisenstein cohomology class

$$\mathrm{Eis}(e_{s_1 \cdot \lambda}) \in H^1(\Gamma\backslash\mathbb{H}_3, \mathcal{M}_\lambda \otimes \mathbb{C}). \tag{4.208}$$

and if we restrict again to the boundary we get by a standard computation

$$r((\mathrm{Eis}(e_{s_1 \cdot \lambda})) = e_{s_1 \cdot \lambda} + c(\lambda) e_{s_2 \cdot \lambda} \tag{4.209}$$

where a priori $c(\lambda) \in \mathbb{C}$. Hence we conclude that $\dim r(H^1(\Gamma\backslash\mathbb{H}_3, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C}) \geq 1$ but then it follows from proposition 2.1.2 that this dimension is equal to one.Therefore

$$r(H^1(\Gamma\backslash\mathbb{H}_3, \mathcal{M}_\lambda \otimes \mathbb{C})) = \mathbb{C} \cdot (e_{s_1 \cdot \lambda}) + c(\lambda) e_{s_2 \cdot \lambda}) = r(H^1(\Gamma\backslash\mathbb{H}_3, \mathcal{M}_\lambda) \otimes_{\mathbb{Q}(\mathbf{i})} \mathbb{C} \tag{4.210}$$

and then we can conclude that $c(\lambda) \in \mathbb{Q}[\mathbf{i})$ actually it lies in $\mathbb{Q}$.

The computation of the constant term of the Eisenstein series yields an explicit and very simple formula for $c(\lambda)$, which we explain next. (see for instance [**?**]:

Let $\mathcal{I}_{\mathbb{Q}(\mathbf{i})}$ be the group of fractional ideals of $\mathbb{Q}(\mathbf{i})$, we can view our characters $s_i \cdot \lambda \in X^*(T \times \mathbb{Q}[\mathbf{i})$ also as Hecke characters $s_i \cdot \lambda : \mathcal{I}_{\mathbb{Q}(\mathbf{i})} \to \mathbb{Q}(\mathbf{i})^\times$, where we exploit the fact that every fractional ideal is principal and define

$$s_1 \cdot \lambda(\mathfrak{a}) = s_i \cdot \lambda((\alpha)) = \alpha^{-n_1 - n_2} \bar{\alpha}^{n_2}; \; s_2 \cdot \lambda(\mathfrak{a}) = s_i \cdot \lambda((\alpha)) = \alpha^{n_1} \bar{\alpha}^{-n_2 - 2} \tag{4.211}$$

Here we need the above parity condition.

To these Hecke characters we attach (completed) Hecke L functions

$$\Lambda(s_i \cdot \lambda, z) := \frac{\Gamma(z)}{(2\pi)^z} \prod_{\mathfrak{p}} \frac{1}{1 - \frac{s_i \cdot \lambda(\mathfrak{p})}{N\mathfrak{p}^z}} = \frac{\Gamma(z)}{(2\pi)^z} \sum_{\mathfrak{a}} \frac{s_i \cdot \lambda(\mathfrak{a})}{N\mathfrak{a}^z}, \tag{4.212}$$

here the $\mathfrak{p}$ run over all prime ideals and $\mathfrak{a} = (a + b\mathbf{i})$ runs over all integral ideals in $\mathbb{Z}[\mathbf{i}]$ and of course $N\mathfrak{a} = (a^2 + b^2)$. It is well known that these Hecke L functions are meromorphic in the entire complex z-plane with possible first order pole at $z = 0, -1, \dots$

Then our simple formula for the restriction of the Eisenstein class to the boundary cohomology is

$$r(H^1(\Gamma\backslash\mathbb{H}_3, \mathcal{M}_\lambda \otimes \mathbb{C})) = \mathbb{C} \cdot (e_{s_1 \cdot \lambda} + \frac{\Lambda(s_1 \cdot \lambda, -1)}{\Lambda(s_1 \cdot \lambda, 0)} e_{s_2 \cdot \lambda}) \tag{4.213}$$

here we have to observe that the two poles at $z=-1$ and $z=0$ cancel out.

We comment a little bit on this result. We put $4k = n_1+n_2+2$. and consider the Hecke character $\phi_k : \mathfrak{a} = (a+b\mathbf{i}) \to (a-b\mathbf{i})^{4k}$ and the completed Hecke $L-$ function

$$\Lambda(\phi_k, z) := \frac{\Gamma(z)}{(2\pi)^z} \prod_{\mathfrak{p}} \frac{1}{1-\frac{\phi_k(\mathfrak{p})}{N\mathfrak{p}^z}} = \frac{\Gamma(z)}{(2\pi)^z} \sum_{\mathfrak{a}} \frac{\phi_k(\mathfrak{a})}{N\mathfrak{a}^z} = \frac{\Gamma(z)}{(2\pi)^z} L(\phi_k, z). \tag{4.214}$$

It is well known that the series is converging for $\Re(z) >> 0$ and can be analytically continued into the entire z-plane. It satisfies the functional equation

$$\Lambda(\phi_k, z) = 2^{4k+1-z}\Lambda(\phi_k, 4k+1-z) \tag{4.215}$$

In his paper [53] Hurwitz considered the period integral

$$\Omega = \int_0^1 \frac{1}{\sqrt{1-x^4}} dx = 2.62206.....$$

and proved that for all values of $k>0$ the number

$$E_{4k} = \frac{L(\phi_k, 4k)}{\Omega^{4k}} \in \mathbb{Q} \tag{4.216}$$

Hurwitz proved many more things about these numbers, his main concern was an analogue of the von Staudt-Clausen theorem.

We now observe that for for $s_1 \cdot \lambda = (-n_1-2)\gamma_1 + n_2\gamma_2$ we have the equality

$$L(\phi_k, 4k-n_2) = L(s_1 \cdot \lambda, 0) \tag{4.217}$$

and hence we get

$$\frac{\Lambda(s_1 \cdot \lambda, -1)}{\Lambda(s_1 \cdot \lambda, 0)} = 2\pi \frac{L(s_1 \cdot \lambda, -1)}{L(s_1 \cdot \lambda, 0)} = 2\pi \frac{L(\phi_k, 4k-1-n_2)}{L(\phi_k, 4k-n_2)} \in \mathbb{Q} \tag{4.218}$$

and this implies that

$$\pi^\nu \frac{L(\phi_k, 4k-\nu)}{\Omega^{4k}} \in \mathbb{Q} \text{ for } \nu = 0, 1, \ldots 4k-2 \tag{4.219}$$

This has been proved by Damerell [24] and is one of the first instances of Deligne's conjecture [26].

Chapter 5

Application to Number Theory

5.1 Modular symbols, L-values and denominators of Eisenstein classes.

In this chapter we want to restrict to the case $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$ or $\Gamma = \mathrm{Sl}_2(\mathcal{O})$ where $\mathcal{O}$ is the ring of integers of an imaginary quadratic extension. We refer to section 4.1.1 then this means that $\Gamma = \mathcal{G}(\mathbb{Z})$. Our coefficient systems will be obtained from the modules $\mathcal{M}_\lambda$. We assume that we have $d = 0$ and hence $n \equiv 0 \mod 2$ in case A), and $d_1 = d_2 = 0,\ n_1 = n_1$ in case B). This has the effect that $\lambda^\vee = \lambda$.

We want to study the pairing

$$H^1_c(\Gamma\backslash X, \tilde{\mathcal{M}}^\flat_\lambda) \times H_1(\Gamma\backslash X, \partial(\Gamma\backslash X), \underline{\mathcal{M}}_\lambda) \to \mathbb{Z}, \tag{5.1}$$

5.1.1 Modular symbols attached to a torus in Gl_2.

In a first step we construct (relative) cycles in $C_1(\Gamma\backslash X, \underline{\mathcal{M}}_\lambda), C_1(\Gamma\backslash X, \partial(\Gamma\backslash X), \underline{\mathcal{M}}_\lambda)$. Our starting point is a maximal torus $T/\mathbb{Q} \subset G/\mathbb{Q}$ and we assume that it is split over a real quadratic extension $F/\mathbb{Q}$. Then the group of real points

$$T(\mathbb{R}) = \mathbb{R}^\times \times \mathbb{R}^\times$$

act on $\mathbb{H}$ and $\bar{\mathbb{H}}$ and it has two fixed points $r, s \in \mathbb{P}^1(F)$. There is a unique geodesic (half) circle $\bar{C}_{r,s} \subset \bar{\mathbb{H}}$ joining these two points. Then $T(\mathbb{R})$ acts transitively on $C_{r,s} = \bar{C}_{r,s} \setminus \{r, s\}$. We have two cases:

a) The torus $T/\mathbb{Q}$ is split. Then the two points $r, s \in \mathbb{P}^1(\mathbb{Q})$. Here for instance we can take $r = 0, s = \infty$, then the geodesic circle is the line $\{iy, y > 0\}$ and the torus is the standard diagonal split torus.

b) Here $\{r, s\} \in \mathbb{P}^1(F) \setminus \mathbb{P}^1(\mathbb{Q})$, then r, s are Galois-conjugates of each other. Our torus $T/\mathbb{Q}$ is given by a suitable embedding

$$j : R_{F/\mathbb{Q}}(\mathbb{G}_m/F) = T \hookrightarrow \mathrm{Gl}_2/\mathbb{Q}.$$

In case a) we can choose any reasonable homeomorphism $[0,1] \xrightarrow{\sim} [0,\infty]$ - for instance $x \mapsto x/(1-x)$− and then we get a one chain

$$\sigma : [0,1] \xrightarrow{\sim} \bar{C}_{r,s} = \mathbb{R}_{>0} \cup \{0\} \cup \{\infty\}, \sigma(0) = r, \sigma(1) = s \in \partial(\bar{\mathbb{H}}),$$

and for any $m \in \mathcal{M}$ we can consider the image of $\sigma \otimes m \in C_1(\bar{\mathbb{H}}) \otimes \mathcal{M}$ in $C_1(\Gamma\backslash\bar{\mathbb{H}}, \partial(\Gamma\backslash\bar{\mathbb{H}}), \underline{\mathcal{M}})$. By definition this is a cycle and hence we get a (relative) homology class

$$[\bar{C}_{r,s} \otimes m] \in H_1(\Gamma\backslash\bar{\mathbb{H}}, \partial(\Gamma\backslash\bar{\mathbb{H}}), \underline{\mathcal{M}}_\lambda), \tag{5.2}$$

it is easy to see that it does not depend on the choice of σ.

In case b) we have $T(\mathbb{Q}) \xrightarrow{\sim} F^\times$. Then the group $T(\mathbb{Q}) \cap \Gamma$ is a subgroup of finite index in the group of units $\mathcal{O}_F^\times = \{\epsilon_0\} \times \{\pm 1\}$, where ϵ_0 is a fundamental unit. Hence

$$\Gamma_T = T(\mathbb{Q}) \cap \Gamma = \{\epsilon_T\} \times \mu_T \tag{5.3}$$

where ϵ_T is an element of infinite order and μ_T is trivial or $\{\pm 1\}$. This element ϵ_T induces a translation on $C_{r,s}$. The quotient $C_{r,s}/\Gamma_T$ is a circle. If we pick any point $x \in C_{r,s}$ then $[x, \epsilon_T x] \subset C_{r,s}$ is an interval and as above we can find a $\sigma : [0,1] \xrightarrow{\sim} [x, \epsilon_T x], \sigma(0) = x, \sigma(1) = \epsilon_T x$, As before we can consider the 1-chain $\sigma \otimes m \in C_1(\mathbb{H}) \otimes \mathcal{M}$. Its boundary boundary is the zero chain $\{x\} \otimes m - \{\epsilon_T x\} \otimes m$. If we look at the images in $C_\bullet(\Gamma\backslash\mathbb{H}, \underline{\mathcal{M}}_\lambda)$ then

$$\partial_1(\sigma \otimes m) = \sigma(0) \otimes (m - \epsilon_T m) = r \otimes (m - \epsilon_T m) \tag{5.4}$$

Hence we see that $\sigma \otimes m$ is a 1 -cycle if and only if $m = \epsilon_T m$ and hence $m \in \mathcal{M}^T$. We have constructed homology classes

$$[C_{r,s} \otimes m] \in H_1(\Gamma\backslash\mathbb{H}, \underline{\mathcal{M}}_\lambda) \text{ for all } m \in \mathcal{M}_\lambda^{<\epsilon_T>} = \mathcal{M}_\lambda^T \tag{5.5}$$

PDualsec

5.1.2 Evaluation of cuspidal classes on modular symbols

The following issue will also be discussed in greater generality and more systematically in part II-

We start from a highest weight $\lambda = n\gamma$ for simplicity we assume n to be even and $d = 0$. Then $\lambda = \lambda^\vee$, we consider the two modules $\mathcal{M}_\lambda$ and $\mathcal{M}^\flat_{\underline{\lambda}}$. Then we have the pairings

$$\begin{gathered} H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}}) \times H_1(\Gamma\backslash\mathbb{H}, \underline{\mathcal{M}}_\lambda) \to \mathbb{Z} \\ H^1_c(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}}) \times H_1(\Gamma\backslash\mathbb{H}, \partial(\Gamma\backslash\mathbb{H}), \underline{\mathcal{M}}_\lambda) \to \mathbb{Z} \end{gathered} \tag{5.6}$$

These two pairings are non degenerate if we invert 6 and divide by the torsion on both sides. (See [book]).

We have the surjective homomorphism $H^1_c(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}^\flat_{\underline{\lambda}}) \to H^1_!((\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}^\flat_{\underline{\lambda}})$ and over a suitably large finite extension $F/\mathbb{Q}$ we have the isotypical decomposition

$$H^1_!((\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}^\flat_{\underline{\lambda}}\otimes F) = \bigoplus_{\pi_f} H^1_!((\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}^\flat_{\underline{\lambda}}\otimes F)(\pi_f) \tag{5.7}$$

where the π_f are absolutely irreducible. (See Theorem 5.7, of course here it does not matter whether we work with $\mathcal{M}_\lambda$ or $\mathcal{M}^\flat_{\underline{\lambda}}$) . We choose an embedding $\iota : F \hookrightarrow \mathbb{C}$, in section 4.1.11 we constructed the isomorphism

$$\mathcal{F}^1_1(\omega^\dagger_\epsilon) : \mathcal{W}(\pi_f)\otimes_{F,\iota}\mathbb{C} \to H^1_{\epsilon,!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}^\flat_{\underline{\lambda}}\otimes F)({}^\iota\pi_f) \tag{5.8}$$

The space $\mathcal{W}(\pi_f)$ is a very explicit space. Since we want to stick to the case $K_f = K_f^{(0)}$ it is of dimension one and is generated by the element

$$h^{\dagger,0}_{\pi_f} = \prod_p h^{\dagger,0}_p \in \prod_p \mathcal{W}(\pi_p) \text{ where } h^{\dagger,0}_p(e) = 1 \tag{5.9}$$

Now we want to compute the value

$$< \mathcal{F}^1_1(\omega^\dagger_\epsilon)(h^{\dagger,0}_{\pi_f})), \bar{C}_{r,s}\otimes m) > . \tag{5.10}$$

here we assume that the torus is split, i.e. $r,s\in\mathbb{P}^1(\mathbb{Q})$. Then this expression is problematic. The argument $C_{r,s}$ on the left lives in the relative homology group, hence the argument on the right should be in $H^1_c(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathbb{C})$. Of course we can lift the class $\mathcal{F}^1_1(\omega^\dagger_\epsilon)(h^{\dagger,0}_{\pi_f})$ to a class

$$\mathcal{F}^1_1\widetilde{(\omega^\dagger_\epsilon)(}h^{\dagger,0}_{\pi_f}) \in H^1_c(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathbb{C}).$$

Then

$$< \mathcal{F}^1_1(\widetilde{\omega^\dagger_\epsilon\times h}^{\dagger,0}_{\pi_f}), C_{r,s}\otimes m >$$

makes sense, but the result may depend on the lift. We have $\boxed{\text{paircusp}}$

Proposition 5.1.1. *If $\partial(C_{r,s}\otimes m)$ gives the trivial class in $H_0(\partial(\Gamma\backslash\bar{\mathbb{H}}),\tilde{\mathcal{M}}_\lambda\otimes\mathbb{C})$ then $< \mathcal{F}^1_1\widetilde{(\omega^\dagger_\epsilon)(}h^{\dagger,0}_{\pi_f}), C_{r,s}\otimes m >$ does not depend on the lift, i.e. the value $< \mathcal{F}^1_1(\omega^\dagger_\epsilon)(h^{\dagger,0}_{\pi_f}), C_{r,s}\otimes m >$ is well defined.*

Proof. This is rather clear, we refer to the systematic dscussion in 2.1.10. □

Now we compute the value of the pairing. We realised the relative homology class by a $\mathcal{M}_\lambda$ valued 1-chain $\sigma\otimes m$. The cohomology class $\mathcal{F}^1_1(\omega^\dagger_\epsilon)(h^{\dagger,0}_{\pi_f})$ is represented by $\mathcal{F}^1(\widetilde{\omega^\dagger_\epsilon\times h}^{\dagger,0}_{\pi_f})$. (See 4.101 ,6.4). We consider the pullback $\sigma^*(\mathcal{F}^1(\widetilde{\omega^\dagger_\epsilon\times h}^{\dagger,0}_{\pi_f}))$, since $\mathcal{F}^1(\omega^\dagger_\epsilon\times h^{\dagger,0}_{\pi_f})$ is rapidly decaying if $x\to 0$ or $x\to 1$ this gives us a 1-form with values in $\mathcal{M}_\lambda\otimes\mathbb{C}$ on the closed interval $[0,1]$.

We claim - under the assumption $[\partial(C_{r,s}\otimes m)]=0$-that

$$< \mathcal{F}^1_1(\omega^\dagger_\epsilon)(h^{\dagger,0}_{\pi_f}), C_{r,s}\otimes m > = \int_0^1 < \sigma^*(\mathcal{F}^1(\widetilde{\omega^\dagger_\epsilon\times h}^{\dagger,0}_{\pi_f}), m > . \tag{5.11}$$

We have to be a little bit careful at this point. Of course our assumption implies that the integral class $[\partial(C_{r,s}\otimes m)] \in H_0(\partial(\Gamma\backslash\bar{\mathbb{H}}),\tilde{\mathcal{M}}_\lambda)$ is a torsion class, Let $\delta_{r,s}(m)$ be the order of this torsion class, hence we can write
$\boxed{\text{defdel}}$

$$\delta_{r,s}(m)\partial C_{r,s}\otimes m = \partial c_{r,s} \text{ with } c_{r,s}\in C_1(\partial(\Gamma\backslash\bar{\mathbb{H}},\underline{\mathcal{M}_\lambda}). \tag{5.12}$$

This 1-chain lies in the boundary of the Borel-Serre compactification (see section 1.2.7). We consider the special case that T is the standard split diagonal torus, this means that $\{r,s\}=\{0,\infty\}$. We can pull the cycle $\delta_{r,s}(m)C_{r,s}\otimes m-c_{r,s}$ into the interior $\Gamma\backslash\mathbb{H}$ by a simple homotopy, this means we replace it by $\delta_{r,s}(m)[iy_0^{-1},iy_0]\otimes m-\partial\delta_{r,s}(m)(y_0)$ where $y_0>>1$ and $\delta_{r,s}(m)(y_0)$ is the 1-chain $c_{r,s}$ on the level y_0. Then

$$\delta_{r,s}(m) < \mathcal{F}^1_1(\omega^\dagger_\epsilon)(h^{\dagger,0}_{\pi_f}),C_{r,s}\otimes m >=< \mathcal{F}^1_1(\omega^\dagger_\epsilon)(h^{\dagger,0}_{\pi_f}),\delta_{r,s}(m)[iy_0^{-1},iy_0]\otimes m-c_{r,s}(y_0)> . \tag{5.13}$$

where now the value on the right hand side is an integral over the truncated cycle. Since the differential form $\mathcal{F}^1_1(\omega^\dagger_\epsilon)(h^{\dagger,0}_{\pi_f})$ is rapidly decreasing if $y_0\to\infty$ we get $\delta_{r,s}(m) < \mathcal{F}^1_1(\omega^\dagger_\epsilon)(h^{\dagger,0}_{\pi_f}),C_{r,s}\otimes m >= \lim\limits_{y_0\to\infty} < \mathcal{F}^1_1(\omega^\dagger_\epsilon)(h^{\dagger,0}_{\pi_f}),\delta_{r,s}(m)[iy_0^{-1},iy_0]\otimes m > .$

We use the above identification $[0,1]=[0,\infty]$ and our 1- chain is given by the map

$$\sigma:[0,\infty]\to\bar{\mathbb{H}}: t\mapsto \begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix} i = ti\in\bar{\mathbb{H}}, \tag{5.14}$$

especially $\sigma(0)=0$ and $\sigma(\infty)=i\infty$. The group $T(\mathbb{R})$ acts transitively on the open part $C_{0,i\infty}$. This action can be used to trivialize the tangent bundle. The tangent space at $i\in\mathbb{H}$ is identified to the subspace $\mathfrak{p}\subset\mathfrak{g}$ (see 4.1.11) and $\frac{H}{2}$ is a generator of the tangent space of $C_{0,i\infty}$ at one. Using the translations by $T(\mathbb{R})$ we get an invariant vector field on $C_{0,i\infty}$. If we identify $C_{0,i\infty}=\mathbb{R}_{>0}$, an easy calculation shows that this vector field is $t\frac{d}{dt}=D^*$.

Now an easy calculation (See 6.4) shows that (here e_f is the identity element in $G(\mathbb{A}_f)$)

$$\mathcal{F}^1(\widetilde{\omega^\dagger_\epsilon\times h^{\dagger,0}_{\pi_f}})(D^*)((\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix} e_f)=\rho_\lambda(\begin{pmatrix} t^{-1} & 0\\ 0 & 1\end{pmatrix})\mathcal{F}^1(\omega^\dagger_\epsilon\times h^{\dagger,0}_{\pi_f})(\frac{H}{2})((\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix},e_f))$$

and our integral in the formula above becomes

$$\int_0^\infty < \rho_\lambda(\begin{pmatrix} t^{-1} & 0\\ 0 & 1\end{pmatrix})\mathcal{F}^1(\omega^\dagger_\epsilon(\frac{H}{2})\times h^{\dagger,0}_{\pi_f})(\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix},e_f),m > \frac{dt}{t}. \tag{5.15}$$

Our formulas in 4.1.11 give

$$\omega^\dagger_\pm(\frac{H}{2})=\frac{1}{8}(\tilde\psi_{n+2}\otimes(X-Y\otimes i)^n\pm\tilde\psi_{-n-2}\otimes(X+Y\otimes i)^n \tag{5.16}$$

this is an element in $\tilde{\mathcal{D}}^\pm_\lambda\otimes\mathcal{M}_\lambda$. We apply $\mathcal{F}^1$ to $\omega^\dagger_\pm(\frac{H}{2})\times h^{\dagger,0}_{\pi_f})$ and evaluate at $(\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix},e_f)$. Applying $\mathcal{F}^1$ means that we have to sum over $a\in\mathbb{Q}^\times$ but since

$h^{\dagger,0}_{\pi_f}$) is the Whittaker function attached to the unramified spherical function only the terms with $a \in \mathbb{Z}$ can be non zero. Hence get

$$\mathcal{F}^1(\omega^{\dagger}_{\pm}(\tfrac{H}{2}) \times h^{\dagger,0}_{\pi_f})(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}, e_f) =$$

$$\frac{1}{8} \sum_{a\in\mathbb{Z};a\neq 0} (\tilde{\psi}_{n+2}(\begin{pmatrix} at & 0 \\ 0 & 1 \end{pmatrix}) \otimes (X - Y\otimes i)^n \pm \tilde{\psi}_{-n-2}(\begin{pmatrix} at & 0 \\ 0 & 1 \end{pmatrix}) \otimes (X + Y\otimes i)^n h^{\dagger,0}_{\pi_f}(a) \tag{5.17}$$

We have seen that $\tilde{\psi}_{n+2}(\begin{pmatrix} at & 0 \\ 0 & 1 \end{pmatrix}) = 0$ if $at < 0$ and $\tilde{\psi}_{n+2}(\begin{pmatrix} -at & 0 \\ 0 & 1 \end{pmatrix}) = \tilde{\psi}_{-n-2}(\begin{pmatrix} at & 0 \\ 0 & 1 \end{pmatrix})$ and therefore our Fourier expansion becomes

$$\frac{1}{8}\sum_{a=1}^{\infty} \tilde{\psi}_{n+2}(\begin{pmatrix} at & 0 \\ 0 & 1 \end{pmatrix}) \otimes ((X - Y\otimes i)^n \pm i^n (X + Y\otimes i)^n) h^{\dagger,0}_{\pi_f}(a) \tag{5.18}$$

We have

$$\begin{gathered}\rho_\lambda(\begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix})((X - Y\otimes i)^n \pm i^n (X + Y\otimes i)^n) = \\ \sum_{\nu=0}^{n} \binom{n}{\nu} t^{\frac{n}{2}-\nu} X^\nu Y^{n-\nu} (i^{n+\nu} \pm i^{-\nu}),\end{gathered} \tag{5.19}$$

we remember that n is even, then the last factor is equal to $i^{-\nu}((-1)^{\frac{n}{2}+\nu} \pm 1)$. and this is $i^{-\nu}$ times 2 or 0 or -2, depending on the choices of signs and the parity of $\frac{n}{2}$ and ν. The elements $e_\nu = X^\nu Y^{n-\nu}$ form the dual basis to the basis $\binom{n}{n-\nu} X^{n-\nu} Y^\nu$ of $\mathcal{M}^\flat_\lambda$, this implies: If we choose $m = e_{n-\nu}$ in our expression above then $\boxed{\text{pairinf}}$

$$< \rho_\lambda(\begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix})((X - Y\otimes i)^n \pm i^n (X + Y\otimes i)^n), m >= t^{\frac{n}{2}-\nu}(i^{n-\nu} \pm i^{-\nu}) \tag{5.20}$$

and hence we have to compute

$$\frac{i^{n+\nu} \pm i^{-\nu}}{8} \int_0^\infty \sum_{a=1}^{\infty} \tilde{\psi}_{n+2}(\begin{pmatrix} at & 0 \\ 0 & 1 \end{pmatrix}) t^{\frac{n}{2}-\nu} h^{\dagger,0}_{\pi_f}(a) \frac{dt}{t}. \tag{5.21}$$

We remember $\tilde{\psi}_{n+2}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = t^{\frac{n}{2}+1} e^{-2\pi t}$, we exchange summation and integration and after some innocent substitutions we get

$$\frac{i^{n+\nu} \pm i^{-\nu}}{8} \int_0^\infty \frac{t^{n-\nu+1}}{(2\pi)^{n-\nu+1}} e^{-t} \frac{dt}{t} \sum_{a=1}^{\infty} \frac{h^{\dagger}_{\pi_f}(a) a^{\frac{n}{2}}}{a^\nu} \tag{5.22}$$

We refer to the discussion of the L -function attached to π_f and get

$$\int_0^\infty \frac{t^{n-\nu+1}}{(2\pi)^{n-\nu+1}} e^{-t}\frac{dt}{t}\sum_{a=1}^\infty \frac{h^\dagger_{\pi_f}(a)a^{\frac{n}{2}}}{a^\nu} = \Lambda^{\mathrm{coh}}(\pi, n+1-\nu) \tag{5.23}$$

Of course some question concerning convergence have to be discussed, for this we refer to the proof of Theorem 4.1.7.

In the case that $\nu \neq 0, n$ we know that $\partial(C_{0,\infty}\otimes X^{n-\nu}Y^\nu)$ is a torsion element in $H^0(\partial(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$ and therefore the value of the integral is also the evaluation of the cohomology class $\mathcal{F}_1^1(\omega_\epsilon^\dagger)(h^{\dagger,0}_{\pi_f})$ on a integral homology class. We get

$$< \mathcal{F}_1^1(\omega_\epsilon^\dagger)(h^{\dagger,0}_{\pi_f}, C_{0,\infty}\otimes X^\nu Y^{n-\nu}) >= \frac{i^{n+\nu}\pm i^{-\nu}}{8}\Lambda^{\mathrm{coh}}(\pi, n+1-\nu) \tag{5.24}$$

In the factor in front on the right side we have $\epsilon = \pm 1$, this factor is zero unless we have $\epsilon = (-1)^{\frac{n}{2}-\nu}$ (see 4.161) and then it is simply $\pm\frac{1}{4}$.

If the class number of $\mathcal{O}_F$ is one we defined the periods $\Omega(\epsilon\times\pi_f)$, (see 4.1.11) we then know that

$$\frac{1}{\Omega(\epsilon\times\pi_f)}\mathcal{F}_1^1(\omega_\epsilon^\dagger)(h^{\dagger,0}_{\pi_f}) \in H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathcal{O}_F) \tag{5.25}$$

and hence we can conclude for $\nu\neq 0, n$ $\boxed{\text{ratint}}$

$$\frac{\delta_{0,\infty}(e_\nu)}{\Omega(\epsilon\times\pi_f)}\Lambda^{\mathrm{coh}}(\pi, n+1-\nu) \in \mathcal{O}_F \tag{5.26}$$

If the class number is not one we have to interpret $\Omega(\epsilon\times\pi_f)$ as section in the sheaf of periods and $\mathcal{O}_F$ has to be replaced by the monoid of integral ideals in $\mathcal{O}_F$. Notice that the term $\delta_{0,\infty}(e_\nu)$ has only prime factors $< n$. We will improve this term after the following discussion of the cases $\nu = 0, \nu = n$.

This argument fails for $\nu = 0, n$ because $\partial(C_{0,\infty}\otimes X^n) = \infty\otimes(X^n - Y^n)$ is not a torsion class in $H_0(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_\lambda)$ (See section 3.2.1). We apply the Manin-Drinfeld principle to show that the rationality statement also holds for $\nu = 0, n$ but we will get a denominator.

We pick a prime p then we know that the class $[\partial(C_{0,\infty}\otimes X^n)]$ is an eigenclass modulo torsion for T_p, i.e.

$$T_p([\partial(C_{0,\infty}\otimes X^n]) = (p^{n+1}+1)[\partial(C_{0,\infty}\otimes X^n)] \tag{5.27}$$

This implies that $\partial(T_p([\mathbb{C}_{0,\infty}\otimes X^n]) - (p^{n+1}+1)[(C_{0,\infty}\otimes X^n]))$ is a torsion class, hence we can apply proposition 5.1.1 and get that the value of the pairing is equal to the integral against the modular symbol. If we exploit the adjointness formula for the Hecke operator then we get

$$< T_p([\mathbb{C}_{0,\infty}\otimes X^n]) - (p^{n+1}+1)[(C_{0,\infty}\otimes X^n]), \mathcal{F}_1^1(\omega_\epsilon^\dagger\otimes h^{\dagger,0}_{\pi_f}) >$$

$$= \int_0^\infty (< C_{0,\infty}\otimes X^n, \mathcal{F}_1^1(\omega_\epsilon^\dagger\otimes T_p(h_{\pi_f})^{\dagger,0}) > \tag{5.28}$$

$$-(p^{n+1}+1) < C_{0,\infty}\otimes X^n, \mathcal{F}_1^1(\omega_\epsilon^\dagger)\otimes((h^{\dagger,0}_{\pi_f}) >))\tfrac{dt}{t}$$

We have $T_p(h^{\dagger,0}_{\pi_f}) = a_p h^{\dagger,0}_{\pi_f}$ where $a_p \in \mathcal{O}_F$ and hence we get

$$\begin{aligned} &< T_p([\mathbb{C}_{0,\infty} \otimes X^n]) - (p^{n+1}+1)[(C_{0,\infty} \otimes X^n]), \mathcal{F}_1^1(\omega_\epsilon^\dagger \otimes h^{\dagger,0}_{\pi_f}) > \\ &= (a_p - (p^{n+1}+1))\Lambda^{\mathrm{coh}}(\pi_f, n+1) \end{aligned} \tag{5.29}$$

It is again the Manin-Drinfeld principle that tells us that for almost all primes p the number $a_p - (p^{n+1}+1) \neq 0$. Let $(Z(n))$ be the ideal in $\mathcal{O}_F$ generated by these numbers. of these numbers. We will see (Theorem 5.1.2) that

$$(\mathrm{numerator}(\zeta(-1-n))) \subset (Z(n)) \tag{5.30}$$

Ribet gives an argument in [74] that yields even equality.

Now we can conclude: *For $\nu = 0, n+1$* $\boxed{ratintE}$

$$\frac{Z(n)}{\Omega(\epsilon \times \pi_f)} \Lambda^{\mathrm{coh}}(\pi, n+1-\nu) \in \mathcal{O}_F \tag{5.31}$$

We want to have an estimate of the denominator ideal of

$$\frac{\Lambda^{\mathrm{coh}}(\pi, n+1-\nu)}{\Omega(\epsilon \times \pi_f)}$$

for all values of ν. For $\nu = 0, \nu = n$ we have the estimate $Z(n)$. For the other values of ν we have the $\delta_{0,\infty}(e_\nu)$, but we can do much better. Notice that this denominator ideal is an ideal in $\mathcal{O}_F$. We pick a prime $p < n$ which then may divide $\delta_{0,\infty}(e_\nu)$. We work locally at p and replace $\mathbb{Z}$ by $\mathbb{Z}_{(p)}$, the local ring at p. It follows from proposition 3.3.1 that for $0 < \nu < n$ the torsion element $[\partial(C_{0,\infty} \otimes e_\nu^\vee))]$ is annihilated by a sufficiently high power of the Hecke operator T_p^m. Hence we see that $T_p^m(c)$ can be lifted to an element $\widetilde{T_p^m(c)} \in H_1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}^\flat_{\underline{\lambda}} \otimes \mathbb{Z}_{(p)})$. Hence we can lift $T_p^m(C_{0,\infty} \otimes e_\nu^\vee))$ to an element $T_p^m(\widetilde{C_{0,\infty}} \otimes e_\nu^\vee)) \in H_1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}} \otimes \mathbb{Z}_{(p)})$. We know that

$$< \mathcal{F}_1^1(\omega_\epsilon^\dagger)(h^{\dagger,0}_{\pi_f}, T_p^m(\widetilde{C_{0,\infty} \otimes} e_\nu^\vee)) \in \mathcal{O}_F \otimes \mathbb{Z}_{(p)}. \tag{5.32}$$

Again we can use the adjointness property of T_p and we get

$$\pi_f(T_p)^m < F_1^1(\omega_\epsilon^\dagger)(h^{\dagger,0}_{\pi_f}, (C_{0,\infty} \otimes e_\nu^\vee)) >= \frac{\pi_f(T_p)^m}{\Omega(\epsilon \times \pi_f)} \Lambda^{\mathrm{coh}}(\pi, n+1-\nu) \in \mathcal{O}_F \otimes \mathbb{Z}_{(p)} \tag{5.33}$$

We consider the ideal $\mathfrak{n}(p,\nu,\pi_f) = (\delta_{0,\infty}(e_\nu), \pi_f(T_p)^m) \subset \mathcal{O}_F \otimes \mathbb{Z}_{(p)}$. This ideal may be much larger than $(\delta_{0,\infty}(e_\nu)$. We put $\mathfrak{n}(\nu,\pi_f) = \prod_p \mathfrak{n}(p,\nu,\pi_f)$ for $\nu \neq 0, n$ and for convenience $\mathfrak{n}(n) = \mathfrak{n}(0) = Z(n)$

Then we get the final result: **Noch mal genauer diskutieren und vorher sagen was $\delta_{0,\infty}(e_\nu)$ ist**

Theorem 5.1.1. *For any π_f which occurs in (5.7) and any $\nu = 0 \ldots n$ the ideal*

$$\frac{\mathfrak{n}(\nu,\pi_f)}{\Omega(\epsilon \times \pi_f)} \Lambda^{\mathrm{coh}}(\pi, n+1-\nu)) \tag{5.34}$$

is an integral ideal in $\mathcal{O}_F$. The primes $\mathfrak{p}$ dividing $\mathfrak{n}(\nu,\pi_f)$ lie over primes $p < n$. Furthermore these primes are not ordinary for π_f, i.e if $\mathfrak{p}$ divides $\mathfrak{n}(\nu,\pi_f)$ then $\pi_f(T_p) \equiv 0 \mod \mathfrak{p}$.

These rationality results go back to Manin and Shimura, In principle we may say that also the integrality assertion goes back to these authors, but here we have to take into account the fine tuning of the periods. (Deligne conjecture? Later if we speak about motives)

It is clear that this compatible with the action of the Galois group $\mathrm{Gal}(F/\mathbb{Q})$, for $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ we have

$$\sigma(\frac{1}{\Omega(\epsilon\times\pi_f)}\Lambda^{\mathrm{coh}}(\pi,n+1-\nu)) = \frac{1}{\Omega(\epsilon\times\pi_f)}\Lambda^{\mathrm{coh}}({}^{\sigma}\pi,n+1-\nu)) \tag{5.35}$$

There is still a slightly different way to look at the theorem above. For each choice of $\epsilon=\pm$ we can look at the array of numbers

$$\{\dot{\Lambda}^{\mathrm{coh}}(\pi,n+1-\nu)),\dots\}_{\nu=0,\dots n;(-1)^{\frac{n}{2}-\nu}=\epsilon} \tag{5.36}$$

Since we may assume that $n\geq 10$ it is easy to see that not all of the entries entries can be zero, hence we can project the arrays to a point $\mathbf{\Lambda}(\epsilon,\pi_f)$ in the projective space $\mathbb{P}^{d(\epsilon,n)}(\mathbb{C})$. Then a slightly weakened form of our results asserts

$$\mathbf{\Lambda}(\epsilon,\pi_f)\in\mathbb{P}^{d(\epsilon,n)}(F)=\mathbb{P}^{d(\epsilon,n}(\mathcal{O}_F) \text{ and } \sigma(\mathbf{\Lambda}(\epsilon,\pi_f))=\mathbf{\Lambda}(\sigma(\epsilon,\pi_f))) \tag{5.37}$$

In this formulation we do not see the period. But now we can fix the period as a section in the period sheaf: We require that the arrays of ideals

$$\{\dots,\frac{\mathfrak{n}(\nu,\pi_f)}{\Omega(\epsilon\times\pi_f)}\Lambda^{\mathrm{coh}}(\pi,n+1-\nu),\dots\}_{\nu=0,\dots n;(-1)^{\frac{n}{2}-\nu}=\epsilon} \tag{5.38}$$

is an ideal of integral and coprime ideals. This period is not necessarily equal to our period we defined earlier, but they may only differ at primes $\mathfrak{p}$ dividing $\mathfrak{n}(\nu,\pi_f)$.

We pay so much attention to the careful choice of the periods because we conjecture that the factorisation of the numbers $\frac{\mathfrak{n}(\nu,\pi_f)}{\Omega(\epsilon\times\pi_f)}\Lambda^{\mathrm{coh}}(\pi,n+1-\nu))$ has influence on the structure of the integral cohomology of some other groups. We expect that prime ideals $\mathfrak{p}\subset\mathcal{O}_F$ which divide an ideal $\frac{\mathfrak{n}(\nu,\pi_f)}{\Omega(\epsilon\times\pi_f)}\Lambda^{\mathrm{coh}}(\pi,n+1-\nu))$ will also divide the denominator of an Eisenstein class on the symplectic group. A prototype of such an assertion has been discussed in [42]. We will resume this discussion in part 2.

In the following section we discuss another (simpler) example, where we see the relationship between divisibility of certain L-values and denominators of Eisenstein classes.

$\boxed{\text{EvalEis}}$

5.1.3 Evaluation of Eisenstein classes on capped modular symbols

In the following we consider cohomology with coefficients in $\mathcal{M}_n$. We have seen that $\boxed{\text{MDEis}}$

$$H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}^{\flat}_{\underline{\lambda}}\otimes\mathbb{Q})=H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}^{\flat}_{\underline{\lambda}}\otimes\mathbb{Q})\oplus\mathbb{Q}\mathrm{Eis}_{\mathrm{n}} \tag{5.39}$$

where $\mathrm{Eis_n}$ is defined by the two conditions

$$r(\mathrm{Eis_n}) = [Y^n] \text{ and } T_p(\mathrm{Eis_n}) = (p^{n+1}+1)\mathrm{Eis_n}, \tag{5.40}$$

for all Hecke operators T_p, in our special situation it suffices to check the second condition for $p = 2$. Earlier we raised the question to determine the denominator of the class $\mathrm{Eis_n}$, i.e. we want to determine the smallest integer $\Delta(n) > 0$ such that $\Delta(n)\mathrm{Eis_n}$ becomes an integral class.

To achieve this goal we compute the evaluation of $\mathrm{Eis_n}$ on the first homology group, i.e we compute the value $< c, \mathrm{Eis_n} >$ for $c \in H_1(\Gamma\backslash\mathbb{H}, \underline{\mathcal{M}}_\lambda)$. We have the exact sequence

$$H_1(\partial(\Gamma\backslash\mathbb{H}), \underline{\mathcal{M}}_\lambda) \xrightarrow{j} H_1(\Gamma\backslash\mathbb{H}, \underline{\mathcal{M}}_\lambda) \to H_1(\Gamma\backslash\mathbb{H}, \partial(\Gamma\backslash\mathbb{H}), \underline{\mathcal{M}}_\lambda) \xrightarrow{\delta} H_0(\partial(\Gamma\backslash\mathbb{H}), \underline{\mathcal{M}}_\lambda) \tag{5.41}$$

It follows from the construction of $\mathrm{Eis_n}$ that $< c, \mathrm{Eis_n} >\in \mathbb{Z}$ for all the elements the image of j. Therefore we only have to compute the values $< \tilde{c}_\nu, \mathrm{Eis_n} >$, where $\tilde{c}_\mu$ are lifts of a system of generators $\{c_\mu\}$ of $\ker(\delta)$.

In our special case the elements $C_{0,\infty} \otimes e_\nu$, where $\nu = 0, 1 \ldots, n$ form a set of generators of $H_1(\Gamma\backslash\mathbb{H}, \partial(\Gamma\backslash\mathbb{H}), \underline{\mathcal{M}}_\lambda)$. (Diploma thesis Gebertz). We observe:

The boundary of the element $C_{0,\infty} \otimes e_n^\vee (= \pm C_{0,\infty} \otimes e_0^\vee)$ *is an element of infinite order in* $H^0(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}^\flat_{\underline{\lambda}})$,

The boundary of an elements $C_{0,\infty} \otimes e_\nu^\vee$ with $0 < \nu < n$ are torsion elements in $H^0(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}^\flat_{\underline{\lambda}})$, This implies

Proposition 5.1.2. *The elements* $C_{0,\infty} \otimes m \in H_1(\Gamma\backslash\mathbb{H}, \partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}^\flat_{\underline{\lambda}})$ *with* $\partial(C_{0,\infty} \otimes m) = 0$ *are of the form*

$$c = C_{0,\infty} \otimes (\sum_{\nu=1}^{\nu=n-1} a_\nu e_\nu^\vee); \quad \textit{with } a_\nu \in \mathbb{Z}$$

Now it seems to be tempting to choose for our generators above the $C_{0,\infty} \otimes e_\nu^\vee$, but this is not possible because for $\delta(C_{0,\infty} \otimes e_\nu^\vee)$ is not necessarily zero, it is only a torsion element. So we see that it is not clear how to find a suitable system of generators.

To overcome this difficulty we use the Hecke operators. If we want to determine the denominator $\Delta(n)$ we can localize, i.e. for each prime p we have to determine the highest power $p^{d(n,p)}$ which divides $\Delta(n)$. As usual we write $d(n,p) = \mathrm{ord}_p(\Delta(n))$. We replace the ring $\mathbb{Z}$ by its localization $\mathbb{Z}_{(p)}$ and replace all our cohomology and homology groups by he localized groups. In other words we have to check we have to find a set of generators $\{\ldots, \tilde{c}_\nu \ldots\}_\nu \subset H_1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}^\flat_{\underline{\lambda}} \otimes \mathbb{Z}_{(p)})$ and compute the denominator $< \tilde{c}_\nu, \mathrm{Eis}_n >\in \mathbb{Z}_{(p)}$.

It follows from proposition 3.3.1 that for $0 < \nu < n$ the torsion element $\partial(c) = \partial(C_{0,\infty} \otimes (\sum_{\nu=1}^{\nu=n-1} a_\nu e_\nu^\vee))$ is annihilated by a sufficiently high power of the Hecke operator T_p^m and hence we see that $T_p^m(c)$ can be lifted to an element $\widetilde{T_p^m(c)} \in H_1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}^\flat_{\underline{\lambda}} \otimes \mathbb{Z}_{(p)})$. Now

$$< \widetilde{T_p^m(c)}, \mathrm{Eis}_n >=< c, T_p^m(\mathrm{Eis}_n) >= (p^{n+1}+1)^m < c, \mathrm{Eis}_n > \tag{5.42}$$

and hence $\mathrm{ord}_p(<\widetilde{T_p^m(c)},\ \mathrm{Eis}_n>) = \mathrm{ord}_p(<c,\ \mathrm{Eis}_n>)$. Hence we get

Proposition 5.1.3. *If ν runs from 1 to $n-1$ and if $T_p^m(\widetilde{C_{0,\infty}} \otimes e_\nu^\vee)$ is any lift of $T_p^m(e_\nu^\vee)$ then*

$$d(n,p) = -\min\big(\min_\nu(\ ord_p(<T_p^m(\widetilde{C_{0,\infty}} \otimes e_\nu^\vee),\ Eis_n>)), 0\big)$$

Proof. This is now obvious. ☐

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$\boxed{\text{CMS}}$

5.1.4 The capped modular symbol

Therefore we have to compute $<T_p^m(\widetilde{C_{0,\infty}} \otimes e_\nu),\ \mathrm{Eis}_n>)$. At this point some meditation is in order. Our cohomology class Eis_n is represented by a closed differential form $\mathrm{Eis}(\omega_n)$ (See (???)) and this differential form lives on $\Gamma\backslash\mathbb{H}$ a hence provides a cohomology class in $\Gamma\backslash\mathbb{H}$. But we know that the inclusion provides an isomorphism

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}}) \xrightarrow{\sim} H^1(\Gamma\backslash\bar{\mathbb{H}}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}})$$

and since $T_p^m(\widetilde{C_{0,\infty}} \otimes e_\nu) \in H_1(\Gamma\backslash\bar{\mathbb{H}}, \underline{\mathcal{M}}_\lambda)$ we can evaluate the cohomology class $\mathrm{Eis}(\omega_n)$ on the cycle. But we want get this value $<T_p^m(\widetilde{C_{0,\infty}} \otimes e_\nu),\ \mathrm{Eis}_n>$ by integration of the differential form against the cycle. This is problematic because the cycle has non trivial support in $\partial(\Gamma\backslash\mathbb{H})$, and on this circle at infinity the differential form is not really defined.

There are certainly several ways out of this dilemma. The Borel-Serre boundary is a circle $\Gamma_\infty\backslash\mathbb{R}$ where $\Gamma_\infty = \{\pm\mathrm{Id}\} \times \{\mathcal{T}_\infty^m\}$ and $T_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The cycle is the sum of two 1-chains:

$$T_p^m(\widetilde{C_{0,\infty}} \otimes e_\nu) = C_{0,\infty} \otimes m_\nu + [i\infty, T_\infty i\infty] \otimes P_\nu$$

(recall definition of Borel-Serre construction from earlier chapters) where

$$\partial(C_{0,\infty} \otimes m_\nu) = \infty \otimes (m_\nu - w m_\nu) + \infty \otimes (1 - T_\infty)P_\nu = 0$$

One possibility is to deform the cycle $T_p^m(\widetilde{C_{0,\infty}} \otimes e_\nu)$ and "pull" it into the interior $\Gamma\backslash\mathbb{H}$. Recall that $C_{0,\infty}$ is the continuous extension of $t \mapsto \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} i$ from $\mathbb{R}^\times_{>0}$ to $\mathbb{H}$ to a map from $[0,\infty] \to \bar{\mathbb{H}}$. We choose a sufficiently large $t_0 \in \mathbb{R}^\times_{>0}$ and restrict $C_{0,\infty}$ to $[t_0^{-1}, t_0]$ we get the one chain $C_{0,\infty}(t_0) \otimes m_\nu$. The boundary of this 1-chain is $\partial(C_{0,\infty}(t_0) \otimes m_\nu) = t_0 \otimes (m_\nu - w m_\nu)$. Now we can do at this level the same thing as what we do at infinity we get a 1-cycle

$$\widetilde{C_{0,\infty}(t_0)} \otimes m_\nu = C_{0,\infty}(t_0) \otimes m_\nu + [t_0, T_\infty t_0] \otimes P_\nu$$

This 1-cycle clearly defines the same class as $T_p^m(\widetilde{C_{0,\infty} \otimes e_\nu})$ and since it is a cycle in $C_1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$ we get

$$< T_p^m(\widetilde{C_{0,\infty} \otimes e_\nu}),\ \mathrm{Eis}_n >= \int_{C_{0,\infty}(t_0)\otimes m_\nu + [t_0, Tt_0]\otimes P_\nu} \mathrm{Eis}_n \tag{5.43}$$

The value of this integral does not depend on t_0 and we check easily that for both summands the limit for $t_0 \to \infty$ exists. We find that $\boxed{\text{Nenner1}}$

$$< T_p^m(\widetilde{C_{0,\infty} \otimes e_\nu^\vee}),\ \mathrm{Eis}(\omega_n) >=$$

$$\textstyle\int_0^\infty < T_p^m(C_{0,\infty} \otimes e_\nu^\vee),\ \mathrm{Eis}_n > \frac{dt}{t} + \lim_{t_0\to\infty} \int_0^1 < [it_0, it_0 + x] \otimes P_\nu,\ \mathrm{Eis}_n > dx \tag{5.44}$$

For the first integral we have

$$\int_0^\infty < T_p^m(C_{0,\infty} \otimes e_\nu^\vee),\ \mathrm{Eis}_n > \frac{dt}{t} = (1 + p^{n+1})^m \int_0^\infty < C_{0,\infty} \otimes e_\nu^\vee,\ \mathrm{Eis}_n > \frac{dt}{t}$$

and (handwritten notes page 49)

$$\int_0^\infty < C_{0,\infty} \otimes e_\nu^\vee,\ \mathrm{Eis}_n > \frac{dt}{t} = \frac{\zeta(-\nu)\zeta(\nu - n)}{\zeta(-1-n)} \tag{5.45}$$

remember this holds for $0 < \nu < n$.

For the second term we have to observe that it depends on the choice of P_ν. We can replace P_ν by $P_\nu + V$ where $V^T = V$. (This means of course that $V = aX^n$) Then $[V] \in H^0(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}_\lambda)$ and

$$\lim_{t_0\to\infty} \int_0^1 < [it_0, it_0 + x] \otimes (P_\nu + V),\ \mathrm{Eis}_n > dx = \lim_{t_0\to\infty} \int_0^1 < [it_0, it_0 + x] \otimes P_\nu,\ \mathrm{Eis}_n > dx + < V, \omega_n > .$$

Therefore the second term is only defined up to a number in $\mathbb{Z}_{(p)}$ but this is ok because we are interested in the p-denominator in (5.44).

We have to evaluate the expression $< [it_0, it_0 + x] \otimes (P_\nu + V),\ \mathrm{Eis}_n >$. Using the formula (6.4) we find

$$< [it_0, it_0 + x] \otimes (P_\nu + V),\ \mathrm{Eis}_n >=< \begin{pmatrix} t_0 & x \\ 0 & 1 \end{pmatrix} P_\nu,\ \mathrm{Eis}(\omega_n)(E_+)(\begin{pmatrix} t_0 & x \\ 0 & 1 \end{pmatrix} > \tag{5.46}$$

We know that for $t_0 >> 1$ the Eisenstein series is approximated by its constant term, i.e.

$$\mathrm{Eis}(\omega_n)(E_+)(\begin{pmatrix} t_0 & x \\ 0 & 1 \end{pmatrix}) = t_0^{-n} Y^n + O(e^{-t_0}) \tag{5.47}$$

On the other hand we can write $P_\nu(X, Y) = \sum p_\mu^{(\nu)} X^{n-\mu} Y^\mu$ with $p_\mu^{(\nu)} \in \mathbb{Z}$. Then

$$\begin{pmatrix} t_0 & x \\ 0 & 1 \end{pmatrix} P_\nu = t_0^n p_0^{(\nu)} X^n + \dots \tag{5.48}$$

and

$$< \begin{pmatrix} t_0 & x \\ 0 & 1 \end{pmatrix} P_\nu,\ \mathrm{Eis}(\omega_n)(E_+)(\begin{pmatrix} t_0 & x \\ 0 & 1 \end{pmatrix}) >= p_0^{(\nu)} + O(e^{-t_0}) \tag{5.49}$$

and hence we see that the limit exists and we get

$$\lim_{t_0\to\infty} \int_0^1 < [it_0, it_0 + x] \otimes (P_\nu + V),\ \mathrm{Eis}_n > dx = p_0^{(\nu)} = P_\nu(1,0) \tag{5.50}$$

and hence we have the final formula

$$< T_p^m(\widetilde{C_{0,\infty}} \otimes e_\nu),\ \mathrm{Eis}_n >= \frac{\zeta(-\nu)\zeta(\nu-n)}{\zeta(-1-n)} + P_\nu(1,0) \mod \mathbb{Z}_{(p)}. \tag{5.51}$$

Therefore we have to compute $P_\nu(1,0) \mod \mathbb{Z}_{(p)}$. Recall that for any $\nu, \nu \neq 0, n$ we have to choose a very large $m > 0$ such that the zero chain $T_p^m(e_\nu)$ is homologous to

$$T_p^m(e_\nu) \sim \{\infty\} \otimes L_\nu = \{\infty\} \otimes (1-T)Q_\nu \tag{5.52}$$

with $Q_\nu \in \mathcal{M}_n$. Then we find $P_\nu = Q_\nu \pm Q_{n+1-\nu}$.

Hence we have to compute $T_p^m(e_\nu)$. A straightforward but lengthy computation yields

$$Q_\nu(1,0) \in \begin{cases} \mathbb{Z}_{(p)} & \text{if } (p-1) \not| \, \nu+1 \\ \frac{1}{p^{\frac{\nu+1}{p-1}}} + \mathbb{Z}_{(p)} & \text{else} \end{cases} \tag{5.53}$$

Now we are ready to compute $d(n,p)$, it is the maximum over all ν $\boxed{\text{denomest}}$

$$d(n,p,\nu) = -\operatorname{ord}_p\left(\frac{\zeta(-\nu)\zeta(\nu-n)}{\zeta(-1-n)} + (Q_\nu(1,0) + Q_{n-\nu}(1,0)) \mod \mathbb{Z}_{(p)}\right). \tag{5.54}$$

We analyse this expression. We exploit the old theorems of Kummer and of von Staudt-Clausen. For an odd positive integer m the number $\zeta(-m)$ is a rational number. The theorem of von Staudt-Clausen asserts

$$\begin{cases} \zeta(-m) \in \mathbb{Z}_{(p)} & \text{if } p-1 \not| \, m+1 \\ \zeta(-m) + \frac{1}{p^{\frac{m+1}{p-1}}} \in \mathbb{Z}_{(p)} & \text{if } p-1 | m+1 \end{cases} \tag{5.55}$$

We distinguish cases.

I) We have $(p-1) \not| \, n+2$, then $\operatorname{ord}_p(\zeta(-1-n)) = \operatorname{ord}_p(\text{Numerator}\,\zeta(-1-n))$, and $p-1$ can divide at most one of the two numbers $\nu+1$ or $n+1-\nu$.

Ia) Let us assume it divides neither of them. Then in (5.54)

$$d(n,p,\nu) = -\operatorname{ord}_p((\zeta(-\nu)\zeta(\nu-n)) + \operatorname{ord}_p(\zeta(-1-n)) \tag{5.56}$$

Ib) Alternatively we assume that $p-1|\nu+1$ we write $\nu+1=p^{\alpha-1}\nu_0$, with $p^{\alpha-1}||\nu+1$. Then the p-denominator of $\zeta(-\nu)$ is p^α and $\nu-n\equiv -n-1 \mod (p-1)p^{\alpha-1}$. The Kummer congruences imply

$$\zeta(\nu-n)=\zeta(-n-1)+p^\alpha Z(\nu,n)\ ;\ \text{with } Z(\nu,n)\in\mathbb{Z}_{(p)} \tag{5.57}$$

and then $\mod \mathbb{Z}_{(p)}$

$$\begin{gathered}\tfrac{\zeta(-\nu)\zeta(\nu-n)}{\zeta(-1-n)}+(Q_\nu(1,0)+Q_{n-\nu}(1,0))=\\ \zeta(-\nu)(1+p^\alpha\tfrac{Z(\nu,n)}{\zeta(-1-n)})+Q_\nu(1,0)=\zeta(-\nu)p^\alpha\tfrac{Z(\nu,n)}{\zeta(-1-n)}\end{gathered} \tag{5.58}$$

This implies that

$$d(n,p,\nu)=\ \mathrm{ord}_p(\mathrm{Numerator}(\zeta(-1-n))-\ \mathrm{ord}_p(Z(\nu,n),$$

the factor in front is a unit.

II) We have $p-1|n+2$. Then p does not divide Numerator$(\zeta(-1-n))$ and hence we have to prove $d(n,p,\nu)=0$ for all ν. This is obvious if $p-1$ does not divide $\nu+1$ and hence also does not divide $n+1-\nu$.

Therefore assume $p-1|\nu+1$. We write $\nu+1=(p-1)xp^{a-1}, n+1-\nu=(p-1)yp^{b-1}$ with $a>0, b>0$ and x,y prime to p. We assume $a\le b$ and compute

$$\frac{\zeta(1-(p-1)xp^{a-1})\zeta(1-(p-1)yp^{b-1})}{\zeta(1-(p-1)p^{a-1}(x+yp^{b-a}))}\quad \mod \mathbb{Z}_{(p)} \tag{5.59}$$

For a value $\zeta(1-m)$ with $p-1|m$ we write $m=(p-1)xp^{k-1}$ with $(x,p)=1$. We apply again the von Staudt-Clausen theorem

$$\zeta(1-m)=\zeta(1-(p-1)xp^{k-1}=-\frac{1}{xp^k}+Z(x)\text{ where } Z(x)\in\mathbb{Z}_{(p)}$$

In our case this gives -let us assume $a<b$ - for our expression above

$$\frac{-\frac{1}{(xp^a}+Z(x))(-\frac{1}{(yp^b}+Z(y))}{-\frac{1}{(x+yp^{b-a})p^a}+Z(x+yp^{b-a}))}=-\frac{(x+yp^{b-a})(\frac{1}{x}+p^aZ(x))(\frac{1}{yp^b}+Z(y))}{1+p^a(x+yp^{b-a})Z(x+p^{b-a}y)} \tag{5.60}$$

The denominator is a unit, we need to know it modulo p^b, the numerator is a sum of eight terms we can forget all the terms in $\mathbb{Z}_{(p)}$. Then the above expression simplifies

$$\frac{\frac{1}{yp^b}+\frac{1}{xp^a}+\frac{p^{a-b}xZ(x)}{y}}{1+p^axZ(x+yp^{b-a})} \tag{5.61}$$

We want this to be equal to $\frac{1}{yp^b}+\frac{1}{xp^a}$. Hence we have to verify the equality

$$\frac{1}{yp^b}+\frac{1}{xp^a}+\frac{p^{a-b}xZ(x)}{y}=(\frac{1}{yp^b}+\frac{1}{xp^a})(1+p^axZ(x+yp^{b-a})) \tag{5.62}$$

and this comes down to

$$p^{a-b}\frac{xZ(x)}{y} \equiv p^{a-b}\frac{xZ(x+yp^{b-a})}{y} \mod \mathbb{Z}_{(p)} \tag{5.63}$$

and this means

$$Z(x) \equiv Z(x+yp^{b-a}) \mod p^{b-a}$$

and this congruence is easy to verify.

Basically the same argument works if $a = b$. Then it can happen that $x+y \equiv 0 \mod p$. Then we have to write $x+y = p^c z$. Then (5.60) changes into

$$\frac{(-\frac{1}{xp^a}+Z(x))(-\frac{1}{yp^a}+Z(y))}{-\frac{1}{zp^{a+c}}+Z(z))} = -\frac{zp^c(\frac{1}{x}+p^aZ(x))(\frac{1}{yp^a}+Z(y))}{1+p^{a+c}zZ(z)}. \tag{5.64}$$

We ignore the denominator then the only non integral term is

$$(x+y)\frac{1}{x}\frac{1}{yp^a} = \frac{1}{xp^a}+\frac{1}{yp^a}$$

We see that in case $p-1 \mid n+2$ the prime p does not divide the numerator of $\zeta(-1-n)$ and that the prime p does not divide the denominator $\Delta(n)$.

If $p-1 \nmid n+2$ then p must be an irregular prime. We look at the maximal value of $d(n,p,\nu)$ in (5.54), this means we look for the minimum value of $\operatorname{ord}_p((\zeta(-\nu)\zeta(\nu-n))$ for $\nu = 1, 3, \dots \frac{n}{2}$. We claim that this minimum value is actually equal to zero. Now it is extremely likely that this is true, because simply too many random integers have to be divisible by p. But as always it is not easy to prove.

For our given prime p the index of irregularity of p is the number of even numbers k with $2 \le k \le p-3$ such that $p|\zeta(1-k) = \frac{B_k}{k}$, it is denoted by $i(p)$. Probabilistic considerations suggest that $i(p) = O(\log(p)/\log\log(p))$, but this can not be proved at the present time. (Again a Wieferich dilemma). Therefore it seems to be very plausible that always $i(p) < \frac{n}{4}$. Then not all of the $\frac{n}{4}$ numbers $\zeta(-\nu)\zeta(\nu-n)$ can be divisible by p. The above assertion that $i(p) < \frac{n}{4}$ is certainly true for all primes $p \le 163577833$. (See [20]). In the same paper the authors assert that for the above set of primes the largest the index of irregularity $i(p) \le 7$ and $i(32012327) = 7$.

There is a way out of this dilemma. In his paper [21] L. Carlitz proves a very crude estimate for the index of irregularity. This estimate says that

$$i(p) < \frac{p+3}{4} - \frac{\log(2)}{\log(p)}\frac{p-1}{4} \tag{5.65}$$

and this implies that $i(p) < \frac{p-3}{4} - 2$ provided $p > 100$.

If we now assume assume $n > p$ then we see that not all the $\frac{p-3}{2}$ numbers $\zeta(-\nu)\zeta(\nu-n)$ can be divisible by p and hence we proved $d(n,p) = \operatorname{ord}_p(\zeta(-1-n)$ and hence the theorem below under this assumption.

$\boxed{\text{denomEis}}$

Theorem 5.1.2. *If $\Gamma = Sl_2(\mathbb{Z})$ then the denominator of the Eisenstein class in $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}})$ is the numerator of $\zeta(-1-n)$.*

Proof. We have to remove the assumption $p < n$. We use Hida's method of p-adic interpolation, we refer to the approach in [41]. In section 3.3.12 we explain how the fact $p^\delta || \Delta(n)$ is reflected in the structure of the Hecke-module $H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}} \otimes \mathbb{Z}/p^\delta\mathbb{Z})$. In [41] we prove that we have an isomorphism of Hecke modules

$$H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}} \otimes \mathbb{Z}/p^\delta\mathbb{Z}) \xrightarrow{\sim} H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_{\lambda'} \otimes \mathbb{Z}/p^\delta\mathbb{Z})$$

provided we have $\lambda \equiv \lambda' \mod p^\delta$ i.e. $n \equiv n' \mod p^\delta$. Hence we can replace n by an $n' > p$ and apply the previous argument. □

A slightly weaker version of this theorem has been proved by Haberland in [33]. Somewhat later C. Kaiser proved a more general version in his Diploma thesis and in about the same time the theorem was proved in my class.

The above theorem can be generalised: For instance we may pass to congruence subgroups of $\mathrm{Gl}_2(\mathbb{Z})$, then the special values of the $\zeta-$ function have to be replaced by special values of Dirichlet L functions. Another situation where the above method might lead to some success is provided by Hilbert modular varieties, i.e. $G/\mathbb{Q} = R_{F/\mathbb{Q}}(\mathrm{Gl}_2/F)$ and $F/\mathbb{Q}$ a totally real field.

The denominators of the Eisenstein cohomology classes can be studied for arbitrary semi simple (reductive) groups $G/\mathbb{Q}$. Roughly our general expectation is that there is a connection between the prime factorisation of certain special values of L-functions and denominators of Eisenstein classes.

A first example is discussed in [42], where we consider then cohomology of the group $\mathrm{Sp}_2(\mathbb{Z})$ with coefficients in a very specify coefficient system and we make a conjecture about the denominator of an Eisenstein class. But it is only a conjecture, our method to determine the denominator (-integrating the Eisenstein class against a cycle-) seems to fail. Nevertheless we give some heuristic speculations about mixed Tate motives which support this expectation.

On the other hand we know that the denominators create congruences and for the special case above and some others these congruences have been checked experimentally in v.d. Geer's article in [19]. In many other cases the congruences have been checked experimentally -for some finite number of Hecke operators- [6], [8], but they never check the denominators. Later G. Chenevier and J. Lannes prove the congruences in some cases [22].

If we want to check experimentally the conjectures about the denominators, we have to broaden our program with Gangl, which we wrote for the operator T_2 and the cohomology of $\mathrm{Sl}_2(\mathbb{Z})$. We have to write algorithms for $\mathrm{Sp}_2(\mathbb{Z})$ or some other groups, see also [44]. Several attempts have been made, but this seems to be a very difficult task. Since we believe that know the interesting values of ℓ we hope that some $\mod \ell$ version might work. (See Chap. 3). .

The case B)

We may discuss the denominator issue also in case B), here the situation is slightly different. We go back to the end of chapter 4 . Even if we start from the integral class $e_{s_1\cdot\lambda}$ we can not expect that the the restriction of the the Eisenstein class is integral, simply because the number $c(\lambda)$ (see 4.213) may not be integral. We have to to multiply the Eisenstein class by the denominator $\delta(\lambda)$ of $c(\lambda)$ and then the restriction $r(\delta(\lambda)\mathrm{Eis}(e_{s_1\cdot\lambda})) \in H^1(\partial(\Gamma\backslash\mathbb{H}_3), \tilde{\mathcal{M}}_\lambda)$. This class generates a direct summand in the boundary cohomology and it plays now the role of the class ω_n in case A) I think that only now it is a reasonable to ask

What is the denominator $D(\lambda)$ of the class $\delta(\lambda)\mathrm{Eis}(e_{s_1\cdot\lambda})$? Does this denominator tell us something about the structure of the cohomology as Hecke module?

. We modify the period Ω^{4k} by a rational factor N_k such that the array

$$\{...., \pi^\nu \frac{L(\phi_k, 4k-\nu)}{\Omega^{4k}N_k}, ...\}_{\nu=1,...4k-3}.$$

is an array of coprime integers. To simplify the notation we put

$$L^{\mathrm{ar}}(\phi_k,\nu) := \frac{\pi^\nu L(\phi_k,\nu)}{\Omega^{4k}N_k}$$

and call this the *arithmetic part* of the L value. If now

$$\Delta(\lambda) = gcd(L^{\mathrm{ar}}(\phi_k, 4k-n_2-1), L^{\mathrm{ar}}(\phi_k, 4k-n_2))$$

then

$$\delta(\lambda) = \frac{1}{\Delta(\lambda)} L^{\mathrm{ar}}(\phi_k, 4k-n_2)).$$

This says that the above restriction is equal to

$$r(\delta(\lambda)\mathrm{Eis}(e_{s_1\cdot\lambda})) = \frac{1}{\Delta(\lambda)} L^{\mathrm{ar}}(\phi_k, 4k-n_2))e_{s_1\cdot\lambda} + \frac{1}{\Delta(\lambda)} L^{\mathrm{ar}}(\phi_k, 4k-n_2-1))e_{s_2\cdot\lambda} \tag{5.66}$$

We want to compute (or better estimate) the denominator. We apply the techniques from case A), namely testing $\delta(\lambda)\mathrm{Eis}(e_{s_1\cdot\lambda})$ against certain modular symbols (see section 5.1.4). We have a certain supply of (capped) modular symbols $\mathcal{MS}$, these $\tilde{\mathfrak{z}} \in \mathcal{MS}$ yield homology classes $\tilde{\mathfrak{z}} \in H_1(\Gamma\backslash\mathbb{H}_3, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q})$, which have a "small" denominator $\mathfrak{n}(\tilde{\mathfrak{z}})$, here small means that $\mathfrak{n}(\tilde{\mathfrak{z}})$ is an integer which in its prime factorisation has only primes $p < 4k$ (here it is very desirable to have more precise information). We can evaluate for $\tilde{\mathfrak{z}} \in \mathcal{MS}$ the integral

$$\int_{\tilde{\mathfrak{z}}} \mathrm{Eis}(e_{s_1\cdot\lambda}) = \frac{\mathcal{L}(\tilde{\mathfrak{z}})}{L^{\mathrm{ar}}(\phi_k, 4k-n_2))}, \tag{5.67}$$

, here $\mathcal{L}(\tilde{\mathfrak{z}})$ is a number which is given as expression in terms of special L- values. We formulate an assumption, which can be verified with very high probability by a quick evaluation of L-value

$$\textit{For any (not too small) prime } \ell \textit{ we can find a } \tilde{\mathfrak{z}} \in \mathcal{MS} \textit{ suchthat} \ell \not| \mathcal{L}(\tilde{\mathfrak{z}}) \tag{5.68}$$

class $\mathfrak{n}(\tilde{\mathfrak{z}})\tilde{\mathfrak{z}}$ we get

$$< \delta(\lambda)\mathrm{Eis}(e_{s_1\cdot\lambda}), \mathfrak{n}(\tilde{\mathfrak{z}})\tilde{\mathfrak{z}} > \; = \frac{\mathfrak{n}(\tilde{\mathfrak{z}})}{\Delta(\lambda)} \times \text{ unit in } \mathbb{Z}_{(\ell)} \tag{5.69}$$

and we conclude

$$\ell|\frac{\Delta(\lambda)}{\mathfrak{n}(\tilde{\mathfrak{z}})} \implies \ell|D(\lambda), \tag{5.70}$$

hence we see again that primes which divide L values also may divide denominators of Eisenstein classes.

We call a prime ℓ large (with respect to $4k$) if $\ell > 4k$, otherwise it is small. For large primes we know that they do not divide $\mathfrak{n}(\tilde{\mathfrak{z}})$. A highest weight λ is called unitary if $n_1 = n_2 (= 2k-1)$. If this is the case the functional equation (see 4.215) tells us that $L^{\mathrm{ar}}(\phi_k, 4k-n_2-1) = 2 \times L^{\mathrm{ar}}(\phi_k, 4k-n_2)$ and hence we get

$$\textit{For a unitary } \lambda \textit{ we have } \Delta(\lambda) = L^{\mathrm{ar}}(\phi_k, 4k-n_2)) \tag{5.71}$$

and since the value of the L^{ar} function will very large we see a very large denominator, if k becomes large.

But if λ is not unitary we should not expect "large" denominators. The two integers $L^{\mathrm{ar}}(\phi_k, 4k-n_2-1), L^{\mathrm{ar}}(\phi_k, 4k-n_2)$ will become very large if k grows, but they are not entangled by the functional equation. Hence it should be a rare event if a large prime ℓ divides both of them. I produced a table of values $\Delta(\lambda)$ for $k = 1, \dots, k = 20$ and I did not find any large prime ℓ dividing a $\Delta(\lambda)$ for a non unitary λ. I found some small primes which divide $\Delta(\lambda)$ but then I did not investigate the influence of $\mathfrak{n}(\tilde{\mathfrak{z}})$, hence I do not know whether they divide the denominator.

This leads us to the second half of the question above.

If λ is not unitary we know that the inner cohomology with rational coefficients is trivial, i.e. we have $H^\bullet_!(\Gamma\backslash\mathbb{H}_3, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q}) = 0$. We will see in the next chapter that this phenomenon happens quite frequently (see 6.1.1). More generally we can say that $H^\bullet_!(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q}) = 0$ if λ is not conjugate self dual. N. Bergeron and A. Venkatesh proposed to look at the torsion of the cohomology, and they formulated some conjectures which in a certain sense say that the torsion of the cohomology becomes very large. In our special case we have a theorem by W. Müller and F. Rochon which says

Let $\Gamma \subset Sl_2(\mathbb{Z}[\mathbf{i}])$ a torsion free congruence subgroup, then for any $\epsilon > 0$ we find an $n(\epsilon)$ such that for all $n_1 > n(\epsilon)$ and $\lambda = n_1\gamma_1$ we have

$$(1-\epsilon)e^{\frac{vol(\Gamma\backslash\mathbb{H}_3)}{2\pi}n_1^2} < \#H^2_{\mathrm{tor}}(\Gamma\backslash\mathbb{H}_3), \tilde{\mathcal{M}}_\lambda) < (1+\epsilon)e^{\frac{vol(\Gamma\backslash\mathbb{H}_3)}{\pi}n_1^2} \tag{5.72}$$

This is Thm. 1.11 in [67], the authors claim that they can also prove a similar result for weights λ where also $n_2 \neq 0$. I believe that the assumption of torsion freeness of Γ is not necessary and I expect that $\#H^2(\Gamma\backslash\mathbb{H}_3), \tilde{\mathcal{M}}_\lambda)_{\mathrm{tors}}$ has some kind of exponential growing in the variable λ even for $\Gamma = \mathrm{Sl}_2(\mathbb{Z}[\mathbf{i}])$. But these methods only give some information about the archimedian size of the torsion, we do not get information about the primes dividing $\#H^2\Gamma\backslash\mathbb{H}_3, \tilde{\mathcal{M}}_\lambda)_{\mathrm{tors}}$.

Small primes will occur in $\#H^2(\Gamma\backslash\mathbb{H}_3), \tilde{\mathcal{M}}_\lambda)_{\mathrm{tors}}$, even with high multiplicity. This is very plausible because, we apply the same arguments as in section (3.3.1) and get an analogous statement to proposition (3.3.1).This implies that

$H^1(\partial(\Gamma\backslash\mathbb{H}_3),\tilde{\mathcal{M}}_\lambda)_{\rm tors}$ is a finite abelian group and its order is only divisible by small primes. Then the image of $H^1(\partial(\Gamma\backslash\mathbb{H}_3),\tilde{\mathcal{M}}_\lambda)_{\rm tors}$ under the boundary operator in the fundamental exact sequence provides a notable contribution to $H^2_c(\Gamma\backslash\mathbb{H}_3,\tilde{\mathcal{M}}_\lambda)_{\rm tors}$. But if the ℓ torsion in $H^2_c(\Gamma\backslash\mathbb{H}_3)_{\rm tors}$ is non zero then it is also non zero in $H^2(\Gamma\backslash\mathbb{H}_3,\tilde{\mathcal{M}}_\lambda)_{\rm tors}$.

To get further information we have to analyse the following diagram

$$\begin{array}{ccccc}
 & & H^1(\partial(\Gamma\backslash\mathbb{H}_3),\tilde{\mathcal{M}}_\lambda)_{\text{ tors}} & & \\
 & & \downarrow & & \\
H^1(\Gamma\backslash\mathbb{H}_3,\tilde{\mathcal{M}}_\lambda) & \xrightarrow{r} & H^1(\partial(\Gamma\backslash\mathbb{H}_3),\tilde{\mathcal{M}}_\lambda) & \xrightarrow{\delta_1} & H^2_c(\Gamma\backslash\mathbb{H}_3) \\
\downarrow & & \downarrow \pi_1 & & \\
H^1(\Gamma\backslash\mathbb{H}_3,\tilde{\mathcal{M}}_\lambda)_{\text{ int}} & \xrightarrow{r_{\text{ int}}} & H^1(\partial(\Gamma\backslash\mathbb{H}_3),\tilde{\mathcal{M}}_\lambda)_{\text{ int}} & & \\
 & & \cup & & \\
 & & \mathbb{Z}[\mathbf{i}](r(\delta(\lambda)\mathrm{Eis}(e_{s_1\cdot\lambda})) & & \\
 & & \cup & & \\
 & & \mathbb{Z}[\mathbf{i}](L^{\rm ar}(\phi_k,4k-n_2)\mathrm{Eis}(e_{s_1\cdot\lambda})) & &
\end{array} \tag{5.73}$$

But let us assume we found is a large prime ℓ which divides $\Delta(\lambda)$ for a non unitary λ. We tensorize the above diagram by the local ring $\mathbb{Z}[\mathbf{i}]\otimes\mathbb{Z}_{(\ell)}$. Then π_1 becomes an isomorphism, hence the class $[\delta(\lambda)\mathrm{Eis}(e_{s_1\cdot\lambda})]$ is an element in $H^1(\partial(\Gamma\backslash\mathbb{H}_3),\tilde{\mathcal{M}}_\lambda)\otimes\mathbb{Z}_{(\ell)}$. Then our assumption above implies that

$$\mathbb{Z}_{(\ell)}[\mathbf{i}](r(\delta(\lambda)\mathrm{Eis}(e_{s_1\cdot\lambda}))/\mathbb{Z}_{(\ell)}[\mathbf{i}](L^{\rm ar}(\phi_k,4k-n_2)\mathrm{Eis}(e_{s_1\cdot\lambda}))=\mathbb{Z}_{(\ell)}[\mathbf{i}]/\Delta(\lambda)\mathbb{Z}_{(\ell)}[\mathbf{i}]..$$

Therefore the boundary operator δ_1 yields an injection

$$\mathbb{Z}_{(\ell)}[\mathbf{i}]/\Delta(\lambda)\mathbb{Z}_{(\ell)}[\mathbf{i}]\hookrightarrow H^2_c(\Gamma\backslash\mathbb{H}_3). \tag{5.74}$$

We have constructed a ℓ torsion class, which owes its existence some divisibility properties of special $L-$ values.

I remind the reader that as long as λ is not unitary- for $k\le 20$ I did not find an $\ell|\Delta(\lambda)$ with $\ell>4k$.

The situation changes dramatically if λ is unitary. In this case $\Delta(\lambda)$ is will become very large. For example if we take $k=20$ and λ unitary. then the two primes

$$\ell=27006373,\ell=12621663529147 \tag{5.75}$$

are divisors of $\Delta(\lambda)$. The map $r_{\text{ int}}$ is not necessarily injective anymore, and it can happen, that the image of $r_{\text{ int}}$ is larger than just $\mathbb{Z}[\mathbf{i}](L^{\rm ar}(\phi_k,4k-n_2)\mathrm{Eis}(e_{s_1\cdot\lambda}))$. To be more precise we take such a prime ℓ and and localise our diagram 5.73 at ℓ, then the element $[r(\mathrm{Eis}(\delta(\lambda)e_{s_1\cdot\lambda}))]\in H^1(\partial(\Gamma\backslash\mathbb{H}_3),\tilde{\mathcal{M}}_\lambda)\otimes\mathbb{Z}_{(\ell)}$ has two options

i) *The boundary operator δ_1 maps it to a non zero ℓ -torsion class.*

ii) *The boundary operator sends it to zero and hence this class is in the image of r.*

In case ii) we get $H^1_!(\Gamma\backslash\mathbb{H}_3),\tilde{\mathcal{M}}_\lambda)\neq 0$. Our previous discussion in case A) also yield that we find an eigenclass, which is congruent to the Eisenstein class mod ℓ.

Now I believe that the case i) almost never occurs. At this point we should return remorsefully to our algorithm (2.33) and carry out some experimental computations. Actually H. Sengun provides some data in his paper [76] in section 5.1. , he gives a complete list of large primes which divide $\#H^2(\Gamma\backslash\mathbb{H}_3,\tilde{\mathcal{M}}_\lambda)_{\text{tors}}$ for $k=1,\dots,13$ (His large primes may be smaller than ours). A simple computation using Poincare duality and some simple exact sequences shows that for a large prime

$$\ell|\#H^2(\Gamma\backslash\mathbb{H}_3,\tilde{\mathcal{M}}_\lambda)_{\text{tors}} \iff \ell|\#H^2_c(\Gamma\backslash\mathbb{H}_3,\tilde{\mathcal{M}}_\lambda)_{\text{tors}}.$$

Now we can compare this list with the list of large prime dividing $\Delta(\lambda)$ and we see that these two list do not have any member in common. This shows that for these few values of k we always have option ii). It would be nice if Sengun's list could be extended up to $k=20$ his parameter n and our parameter k are related by $2k=n+1$.

5.1.5 The Deligne-Eichler-Shimura theorem

In this section the material is not presented in a satisfactory form. One reason is that it this point we should start using the language of adeles, but there are also other drawbacks. So in a final version of these notes this section will probably be removed.

Begin of probably removed section

In this section I try to explain very briefly some results which are specific for Gl_2 and a few other low dimensional algebraic groups. These results concern representations of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which can be attached to irreducible constituents Π_f in the cohomology. These results are very deep and reaching a better understanding and more general versions of these results is a fundamental task of the subject treated in these notes. The first cases have been tackled by Eichler and Shimura, then Ihara made some contributions and finally Deligne proved a general result for $\mathrm{Gl}_2/\mathbb{Q}$.

We start from the group $G=\mathrm{Gl}_2/\mathbb{Q}$, this is now only a reductive group and its centre is isomorphic to $\mathbb{G}_m/\mathbb{Q}$. Its group of real points is $\mathrm{Gl}_2(\mathbb{R})$ and the centre $\mathbb{G}_m(\mathbb{R})$ considered as a topological group has two components, the connected component of the identity is $\mathbb{G}_m(\mathbb{R})^{(0)}=\mathbb{R}^\times_{>0}$. Now we enlarge the maximal compact connected subgroup $\mathrm{SO}(2)\subset\mathrm{Gl}_2(\mathbb{R})$ to the group $K_\infty=SO(2)\cdot\mathbb{G}_m(\mathbb{R})^{(0)}$. The resulting symmetric space $X=\mathrm{Gl}_2(\mathbb{R})/K_\infty$ is now a union of a upper and a lower half plane: We write $X=\mathbb{H}_+\cup H_-$.

We choose a positive integer $N>2$ and consider the congruence subgroup $\Gamma(N)\subset\mathrm{Gl}_2(\mathbb{Q}))$. We modify our symmetric space: This modification may look a little bit artificial at this point, it will be justified in the next chapter and is in fact very natural. At this point I want to avoid to use the language of adeles.)

We replace the symmetric space by

$$X=(\mathbb{H}_+\cup\mathbb{H}_-)\times\mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z}).$$

On this space we have an action of $\Gamma = \mathrm{Gl}_2(\mathbb{Z})$, on the second factor it acts via the homomorphism $\mathrm{Gl}_2(\mathbb{Z}) \to \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z})$ by translations from the left. Again we look at the quotient of this space by the action of $\mathrm{Gl}_2(\mathbb{Z})$. This quotient space will have several connected components. The group $\mathrm{Gl}_2(\mathbb{Z})$ contains the group $\mathrm{Sl}_2(\mathbb{Z})$ as a subgroup of index two, because the determinant of an element is ± 1. The element $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ interchanges the upper and the lower half plane and hence we see

$$\mathrm{Gl}_2(\mathbb{Z})\backslash X = \mathrm{Gl}_2(\mathbb{Z})\backslash((\mathbb{H}_+ \cup \mathbb{H}_-) \times \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z})) = \mathrm{Sl}_2(\mathbb{Z})\backslash(\mathbb{H}_+ \times \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z})),$$

the connected components of $(\mathbb{H}_+ \times \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z}))$ are indexed by elements $g \in \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z})$. The stabilizer of such a component is the full congruence subgroup

$$\Gamma(N) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, d \equiv 1 \mod N, b, c \equiv 0 \mod N\}$$

this group is torsion free because we assumed $N > 2$.

The image of the natural homomorphism $\mathrm{Sl}_2(\mathbb{Z}) \to \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z})$ is the subgroup $\mathrm{Sl}_2(\mathbb{Z}/N\mathbb{Z})$ (strong approximation), therefore the quotient is by this subgroup is $(\mathbb{Z}/N\mathbb{Z})^\times$.

We choose as system of representatives for the determinant the matrices $t_a = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, a \in (\mathbb{Z}/N\mathbb{Z})^\times$. The stabiliser of then we get an isomorphism

$$\mathcal{S}_N = \mathrm{Gl}_2(\mathbb{Z})\backslash(\mathbb{H} \times \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z})) \xrightarrow{\sim} (\Gamma(N)\backslash\mathbb{H}) \times (\mathbb{Z}/N\mathbb{Z})^\times.$$

We consider the cohomology groups $H_c^\bullet(\mathcal{S}_N, \tilde{\mathcal{M}}_n), H^\bullet(\mathcal{S}_N, \tilde{\mathcal{M}}_n), H^\bullet(\partial\mathcal{S}_N, \tilde{\mathcal{M}}_n)$, again we have the fundamental long exact sequence and we define $H_!^\bullet(\mathcal{S}_N, \tilde{\mathcal{M}}_n)$ as before.

To any prime p, which does not divide N we can again attach Hecke operators. Again we can attach Hecke operators

$$T_{p^r} = T\left(\begin{pmatrix} p^r & 0 \\ 0 & 1 \end{pmatrix}, u_{\begin{pmatrix} p^r & 0 \\ 0 & 1 \end{pmatrix}}\right)$$

these to the double cosets and using strong approximation we can prove the recursion formulae (for this and the following see the next chapter 6). We define $\mathcal{H}_p := \mathbb{Z}[T_p]$. We also have a Hecke algebra $\mathcal{H}_p$ for the primes $p|N$, but this will not be commutative.anymore. We get an action of a larger Hecke algebra

$$\mathcal{H}_N^{\text{large}} = \bigotimes_p{}' \mathcal{H}_p.$$

We apply 3.1.1 and find a finite normal extension $F/\mathbb{Q}$ such that we get an isotypical decomposition

$$H_!^\bullet(\mathcal{S}_N, \tilde{\mathcal{M}}_n \otimes F) = \bigoplus H_!^\bullet(\mathcal{S}_N, \tilde{\mathcal{M}}_n \otimes F)(\pi_f) \tag{5.76}$$

where $\pi_f = \otimes' \pi_p$ and the π_p are isomorphism types of absolutely irreducible $\mathcal{H}_p$ modules. For $p \not| N$ this $\mathcal{H}_p$-module is a one dimensional F-vector space

$H_{\pi_p} = F$ and π_p is simply a homomorphism $\pi_p : \mathcal{H}_p \to \mathcal{O}_F$. If $p|N$ then the $\mathcal{H}_p$ module is $F^{d(\pi_p)}$ with $d(\pi_p) \geq 1$ and the theory of semi-simple algebras tells us that the map $\mathcal{H}_p \to \operatorname{End}_F(H_{\pi_p})$ is surjective. Hence we know the isomorphism type π_p once we know the two sided ideal $I(\pi_p)$ of this map.

Now we have some input from the theory of automorphic forms

Theorem 5.1.3. *The isomorphism type π_f is determined by its restriction to the central subalgebra $\otimes_{p \not| N}\mathcal{H}_p$. Under the action of the group $\pi_0(\mathrm{Gl}_2(\mathbb{R})) = \{\pm 1\}$ decomposes into two eigenspaces*

$$H^\bullet_!(\mathcal{S}_N, \tilde{\mathcal{M}}_n \otimes F)(\pi_f) = H^\bullet_!(\mathcal{S}_N, \tilde{\mathcal{M}}_n \otimes F)_+(\pi_f) \oplus H^\bullet_!(\mathcal{S}_N, \tilde{\mathcal{M}}_n \otimes F)_-(\pi_f) \tag{5.77}$$

and these two eigenspaces are absolutely irreducible of type π_f. (These assertions are summarised under "strong multiplicity one")

Of course we have the action of the Galoisgroup $\mathrm{Gal}(F/\mathbb{Q})$ on the cohomology groups $H^\bullet_!(\mathcal{S}_N, \tilde{\mathcal{M}}_n \otimes F)$ and it is clear that this induces an action on the isomorphism types π_f. For $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ we have

$$\sigma(H^\bullet_!(\mathcal{S}_N, \tilde{\mathcal{M}}_n \otimes F)(\pi_f)) = H^\bullet_!(\mathcal{S}_N, \tilde{\mathcal{M}}_n \otimes F)(\sigma(\pi_f)). \tag{5.78}$$

I want to discuss some applications.

A) To any isotypical component π_f we can attach an (so called automorphic) L function

$$L(\pi_f, s) = \prod_p L(\pi_p, s)$$

where for $p \not| N$ we define

$$L(\pi_p, s) = \frac{1}{1 - \lambda(\pi_p)p^{-s} + p^{n+1}\omega(\pi_f)(p)p^{-2s}}$$

and for $p|N$ we have

$$L(\pi_p, s) = \begin{cases} \frac{1}{1-p^{n+1}\omega(\Pi_f)(p)p^{-s}} & \text{if } \pi_p \text{ is a Steinberg module} \\ 1 & \text{else} \end{cases}$$

This L-function, which is defined as an infinite product is holomorphic for $\Re(s) >> 0$ it can written as the Mellin transform of a holomorphic cusp form F of weight $n+2$ and this implies that

$$\Lambda(\pi, s) = \frac{\Gamma(s)}{2\pi^s} L(\pi_f, s)$$

has a holomorphic continuation into the entire complex plane and satisfies a funtional equation

$$\Lambda(\pi_f, s) = W(\pi_f)(N(\pi_f))^{s-1-n/2}\Lambda(\pi_f, n+2-s)$$

Here $W(\Pi_f)$ is the so called root number, it can be computed from the π_p where $p|N$, its value is ± 1, the number $N(\pi_f)$ is the conductor of π_f it is a positive integer, whose prime factors are contained in the set of prime divisors of N.

Now we exploit the fact, that the disjoint union of Riemann surfaces $\Gamma(N)\backslash X$. is in fact the space of complex points of the moduli scheme $M_N \to \operatorname{Spec}(\mathbb{Z}[1/N])$. This has been explained at several places in the literature. I refer to the second edition of my book [39] section 5.2.5 where I try to explain that the functor schemes $S \to \operatorname{Spec}(\mathbb{Z}[1/N])$ to elliptic curves over S with N-level structure is representable, provided $N \geq 3$, More precisely we have a smooth quasiprojective scheme $M_N \to \operatorname{Spec}(\mathbb{Z}[1/N])$ with one dimensional fibers and we have the universal elliptic curve with N level structure

$$\begin{array}{ll} \mathcal{E}; & \{e_1, e_2\} \\ \downarrow \pi & \\ M_N & \end{array} \tag{5.79}$$

where $e_i : M_N \to \mathcal{E}$ are sections which yield a pair of generators of the group of N-division points. The group $\mathrm{Gl}_2(\mathbb{Z}/NZ)$ acts on the group of N-division points, this gives an action of $\mathrm{Gl}_2(\mathbb{Z}/NZ)$ on M_N. We can define the moduli stack $M_1 \to \operatorname{Spec}(\mathbb{Z})$ of elliptic curves without level structure. For any $N \geq 3$ we have $M_1 \times \operatorname{Spec}(Z[\frac{1}{N}]) = M_N/\mathrm{Gl}_2(\mathbb{Z}/\mathbb{N}\mathbb{Z})$.

On $\mathcal{E}$ we have the constant ℓ-adic sheaf $\mathbb{Z}_\ell$. For $i = 0, 1, 2$ we can consider the ℓ- adic sheaves $R^i\pi_*(\mathbb{Z}_\ell)$ on M_N. We have the spectral sequence

$$H^p(M_N \times \bar{\mathbb{Q}}, R^q\pi_*(\mathbb{Z}_\ell)) \Rightarrow H^n(\mathcal{E} \times \bar{\mathbb{Q}}, \mathbb{Z}_\ell).$$

We can take the fibered product of the universal elliptic curve

$$\mathcal{E}^{(n)} = \mathcal{E} \times_{M_N} \mathcal{E} \times \cdots \times_{M_N} \mathcal{E} \xrightarrow{\pi_N} M_N$$

where n is the number of factors. This gives us a more general spectral sequence

$$H^p(M_N \times \bar{\mathbb{Q}}, R^q\pi_{N,*}(\mathbb{Z}_\ell)) \Rightarrow H^n(\mathcal{E}^{(n)} \times \bar{\mathbb{Q}}, \mathbb{Z}_\ell).$$

The stalk $R^q\pi_{N,*}(\mathbb{Z}_\ell)_y$) of the sheaf $R^q\pi_{N,*}(\mathbb{Z}_\ell)$ in a geometric point y of M_N is the q-th cohomology $H^q(\mathcal{E}_y^{(n)}, \mathbb{Z}_\ell)$ and this can be computed using the Kuenneth formula

$$H^q(\mathcal{E}_y^{(n)}, \mathbb{Z}_\ell) \xrightarrow{\sim} \bigoplus_{a_1, a_2 \ldots, a_n} H^{a_1}(\mathcal{E}_y, \mathbb{Z}_\ell) \otimes H^{a_2}(\mathcal{E}_y, \mathbb{Z}_\ell) \cdots \otimes H^{a_n}(\mathcal{E}_y, \mathbb{Z}_\ell),$$

where the $a_i = 0, 1, 2$ and sum up to q. We have $H^0(\mathcal{E}_y, \mathbb{Z}_\ell) = \mathbb{Z}_\ell(0), H^2(\mathcal{E}_y, \mathbb{Z}_\ell) = \mathbb{Z}_\ell(-1)$ and the most interesting factor is $H^1(\mathcal{E}_y, \mathbb{Z}_\ell)$ which is a free $\mathbb{Z}_\ell$ module af rank 2.

This tells us that the sheaf decomposes into a direct sum according to the type of Kuenneth summands. We also have an action of the symmetric group S_n which is obtained from the permutations of the factors in $\mathcal{E}^{(n)}$ which also permutes the Kuenneth summands. We are mainly interested in the case $q = n$ and then we have the special summand where $a_1 = a_2 \cdots = a_n = 1$. This summand is invariant under S_n and contains a summand on which S_n acts by the signature character $\sigma : S_n \to \{\pm 1\}$. This defines a unique subsheaf $R^n\pi_{*,n}(\mathbb{Z}_\ell)(\sigma) \subset R^n\pi_{*,n}(\mathbb{Z}_\ell)$ and hence we get an inclusion

$$H^1(M_N \times \bar{\mathbb{Q}}, R^n\pi_{*,n}(\mathbb{Z}_\ell)(\sigma) \hookrightarrow H^{n+1}(\mathcal{E}^{(n)} \times \bar{\mathbb{Q}}, \mathbb{Z}_\ell) \tag{5.80}$$

and we can do the same thing for the cohomology with compact supports.

Now I claim that

A) The restriction of the etale sheaf $R^n\pi_{*,n}(\mathbb{Z}_\ell)(\sigma)$ on $M_N \times \mathbb{C}$ to the topological space $\mathcal{S}_N = M_N(\mathbb{C})$ is isomorphic to $\mathcal{M}_n \otimes \mathbb{Z}_\ell$. Then the comparison theorem gives us

$$H^1(M_N(\mathbb{C}), \mathcal{M}_n \otimes \mathbb{Z}_\ell) \xrightarrow{\sim} H^1(M_N \times \bar{\mathbb{Q}}, R^n\pi_{*,n}(\mathbb{Z}_\ell)(\sigma))$$

B) The Hecke operators T_p for $p \not| N$ are coming from algebraic correspondences $T_p \subset M_N \times M_N$ and induce endomorphisms $T_p : H^1(M_N \otimes \bar{\mathbb{Q}}, R^n\pi_{*,n}(\mathbb{Z}_\ell)(\sigma)) \to H^1(M_N \otimes \bar{\mathbb{Q}}, R^n\pi_{*,n}(\mathbb{Z}_\ell)(\sigma))$ which commute with the action of Gal$(\bar{\mathbb{Q}}/\mathbb{Q})$ on the cohomology.

This gives us the structure of a Gal$(\bar{\mathbb{Q}}/\mathbb{Q}) \times \mathcal{H}_\Gamma$ on $H^1(M_N(\mathbb{C}), \mathcal{M}_n \otimes \mathbb{Z}_\ell)$.

C) The operation of the Galois group on $H^1(M_N(\mathbb{C}), \mathcal{M}_n \otimes \mathbb{Z}_\ell)$ is unramified outside N and ℓ therefore we have the conjugacy class Φ_p^{-1} for all $p \not| N$ as endomorphism of $H^1(M_N(\mathbb{C}), \mathcal{M}_n \otimes \mathbb{Q}_\ell)$.

We choose our normal extension $F/\mathbb{Q}$ and a prime $\mathfrak{l}$ above ℓ. Then an isotypical component $H^1(M_N(\mathbb{C}), \mathcal{M}_n \otimes F_\mathfrak{l})(\pi_f)$ is a Galois module. Let H_{π_f} be a vector space over F which is an irreducible $\mathcal{H}_\Gamma$ module which is of isomorphism type π_f. Then $W(\pi_f) = \mathrm{Hom}_{\mathcal{H}_\gamma}(H_{\pi_f} \otimes E_\mathfrak{l}, H^1_!(M_N(\mathbb{C}), \mathcal{M}_n \otimes F_\mathfrak{l})$ is a Galois module which is unramified outside N and ℓ

We now apply our theorem 2 to the cohomology $H^1_!(M_N(\mathbb{C}), \mathcal{M}_n \otimes Z_\ell)$, as a module under this large Hecke algebra. Then the isotypical summands will be invariant under the Galois group.

Theorem 5.1.4. *(Deligne) For all primes $p \not| N, p \neq \ell$*

$$tr(\Phi_p^{-1}|W(\pi_f)) = \lambda(\pi_p), \det(\Phi_p^{-1}|W(\pi_f)) = p^{n+1}\omega(\pi_f)(p)$$

This theorem is much deeper than the previous ones. The assertion a) follows from the theory of automorphic forms on Gl_2 and b) requires some tools from algebraic geometry. We have to consider the reduction $M_N \times$ Spec$(\mathbb{F}_p)$ and to look at the reduction of the Hecke operator T_p modulo p. I will resume this discussion in Chap. V.

We conclude by giving a few applications.

A) To our modular cusp form $\Delta(z)$ we attach the Hecke L-function

$$L(\Delta, s) = \int_0^\infty \Delta(iy)y^s \frac{dy}{y} = \frac{\Gamma(s)}{(2\pi)^s} \sum_{n=1}^\infty \frac{\tau(n)}{n^s} = \frac{\Gamma(s)}{(2\pi)^s} \prod_p \frac{1}{1 - \tau(p)p^{-s} + p^{11-2s}}$$

the product expansion has been discovered by Ramanujan and has been proved by Mordell and Hecke.

Now it is in any textbook on modular forms that the transformation rule

$$\Delta(-\frac{1}{z}) = z^{12}\Delta(z)$$

implies that $L(\Delta, s)$ defines a holomorphic function in the entire s plane and satisfies the functional equation

$$L(\Delta, s) = (-1)^{12/2}L(\Delta, 12-s) = L(\Delta, 12-s).$$

This function $L(\Delta, s)$ is the prototype of an automorphic L-function. The above theorem shows that it is equal to a "motivic" L-function. We gave some vague explanations of what this possibly means: We can interpret the projective system $(\mathcal{M}/\ell^n\tilde{\mathcal{M}})_{et}$ as the $\ell-$adic realization of a motive:

$$\mathcal{M} = \mathrm{Sym}^{10}(R^1(\pi : \mathcal{E} \to S))$$

(All this is a translation of Deligne‘s reasoning into a more sophisticated language.)

It is a general hope that "motivic" L-functions $L(M, s)$ have nice properties as functions in the variable s (meromorphicity, control of the poles, functional equation). So far the only cases, in which one could prove such nice properties are cases where one could identify the "motivic" L-function to an automorphic L function. The greatest success of this strategy is Wiles‘ proof of the Shimura-Taniyama-Weil conjecture, but also the Riemann ζ-function is a motivic $L-$ function and Riemann‘s proof of the functional equation follows exactly this strategy.

B) But we also have a flow of information in the opposite direction. In 1973 Deligne proved the Weil conjectures, which in this case say that the two roots of the quadratic equation

$$x^2 - \tau(p)x + p^{11} = 0$$

have absolute value $p^{11/2}$, i.e. they have the same absolute value. This implies the famous Ramanujan- conjecture

$$\tau(p) \leq 2p^{11/2}$$

and for more than 50 years this has been a brain-teaser for mathematicians working in the field of modular forms.

End of probably removed section

The ℓ-adic Galois representation in the first non trivial case

Again we consider the module $\mathcal{M} = \mathcal{M}_{10}[-10]$. We choose a prime ℓ and for some reason let us assume $\ell > 7$. Then we can consider the cohomology groups

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}/\ell^n\tilde{\mathcal{M}})$$

and the projective limit

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}_\ell) = \lim_{\leftarrow} H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}/\ell^n\tilde{\mathcal{M}}).$$

. We know that

$$H^1_{et}(M_1 \times_{\mathrm{Spec}(\mathbb{Z})} \bar{\mathbb{Q}}, (\mathcal{M}/\ell^n\tilde{\mathcal{M}})_{et}) \xrightarrow{\sim} H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}/\ell^n\tilde{\mathcal{M}}).$$

On the etale cohomology groups we have an action of the Galois group hence we get an action

$$\rho_\ell^{(n)}: \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}(H^1(\Gamma\backslash\mathbb{H}, (\mathcal{M}/\ell^n\tilde{\mathcal{M}})_{et})). \tag{5.81}$$

From Galois theory we get a finite normal extension $K_\ell^{(n)}/\mathbb{Q}$ which is defined by $\mathrm{Gal}(\bar{\mathbb{Q}}/K_\ell^{(n)}) = \ker(\rho^{(n)})$. The representation $\rho_\ell^{(n)}$ is unramified outside ℓ, and this means that the finite extension $K_\ell^{(n)}/\mathbb{Q}$ is unramified outside ℓ.

From the fundamental exact sequence we get a diagram

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ & & H^0(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell) & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \to & H^1_c(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell) & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \to & H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell) & \to & H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell) & \to & H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell) & \to & 0 \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array} \tag{5.82}$$

the vertical and the horizontal sequence are exact sequences of Hecke$\times$ Galois modules. Here we may replace $\mathbb{Z}_\ell$ by $\mathbb{Z}/\ell^n\mathbb{Z}$. We computed these Hecke modules in section 3.3.4, the cohomology $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell)$ is free of rank 3 and $H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell)$ is free of rank one. We get the two Galois modules

$$\rho_!: \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}(H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell)), \text{ and } \rho_\partial: \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}(\mathbb{Z}_\ell^\times). \tag{5.83}$$

The ℓ-adic Tate character $\alpha_\ell: \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^\times$ is defined by the rule: For all $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and all ℓ^n-th roots of unity $\zeta \in \bar{\mathbb{Q}}$ we have $\sigma(\zeta) = \zeta^{\alpha_\ell(\sigma)}$. Then it is not difficult to see (or well known) that $\rho_\partial = \alpha_\ell^{11}$. The representation ρ_∂ is the $\ell-$ adic realisation of the Tate-motive $\mathbb{Z}(-11)$. (For a slightly more precise explanation I refer to MixMot.pdf on my home-page). On $\mathbb{Z}_\ell(-1) = H^2(\mathbb{P}^1 \times \bar{\mathbb{Q}}, \mathbb{Z}_\ell)$ the Galois group acts by the Tate-character α_ℓ

For the representation $\rho_!$ the above theorem of Deligne gives

$$\det(\mathrm{Id} - \rho(\Phi_p^{-1})t | H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell)) = 1 - \tau(p)t + p^{11}t^2 \tag{5.84}$$

We also have $\det(\rho(\sigma)) = \alpha_\ell^{11}(\sigma)$ and we can ask what is the image of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ in $\mathrm{Gl}(H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell) = \mathrm{Gl}_2(\mathbb{Z}_\ell)$. This question is discussed in [86]. If $\ell \neq 691$ then the Hecke algebra induces a splitting (Manin-Drinfeld principle)

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell) = H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell)) \oplus \mathbb{Z}_\ell \tag{5.85}$$

where T_p acts by multiplication by $p^{11}+1$ on the second summand.

Now Swinnerton-Dyer shows in [86] that for $\ell \neq 23, 691$ the image of the Galois group under $\rho_!$ is as large possible, it is the inverse image of $(\mathbb{F}_\ell^\times)^{11}$

From now on we choose $\ell = 691$ and our coefficient system $\tilde{\mathcal{M}}_{10}$. Then we get a diagram of Hecke modules

$$
\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & H^0(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z}) & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \to & H^1_c(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z}) & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \to & H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z}) & \to H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z}) & \to & H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z}) & \to & 0 & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
\tag{5.86}
$$

We learned in the probably removed section that we have an action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on this diagram and this action of the Galois group commutes with the action of the Hecke algebra. The two modules $H^0(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z}), H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z})$ are isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$ and a Hecke operator T_p acts by the eigenvalue $p^{11}+1 \mod \ell$. The module $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z}) = \mathbb{Z}/\ell\mathbb{Z} \oplus \mathbb{Z}/\ell\mathbb{Z}$ and the Hecke operator acts by the eigenvalue $\tau(p)$.

The Galois group acts on $H^0(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z})$, resp.$H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z})$ by α_ℓ^0 resp. α_ℓ^{-11}, here α_ℓ is the reduction of the Tate character $\mod \ell$. We also know that we have the inclusion of Galois modules

$$j : \mathbb{Z}/\ell\mathbb{Z}(-11) \hookrightarrow H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z}), \tag{5.87}$$

We want to understand the two Galois modules $H^1_c(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z})$ and $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z})$, There is perfect pairing with values in $\mathbb{Z}/\ell\mathbb{Z}(-11)$ between them, hence we have to study only one of them say $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z})$,

From the above considerations it follows that we have ia basis e_1, e_0, e_{-1} of this module such that a $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts by the matrix

$$\rho(\sigma) = \begin{pmatrix} \alpha_\ell(\sigma)^{-11} & u_{12}(\sigma) & u_{13}(\sigma) \\ 0 & 1 & u_{23}(\sigma) \\ 0 & 0 & \alpha_\ell(\sigma)^{-11} \end{pmatrix} \in B(\mathbb{Z}/\ell\mathbb{Z}) \tag{5.88}$$

We want to describe the image of the Galois group in $B(\mathbb{Z}/\ell\mathbb{Z})$. Let $T^{(1)}(\mathbb{Z}/\ell\mathbb{Z})$ be the torus

$$\begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix}; t \in \mathbb{Z}/\ell\mathbb{Z}^\times \tag{5.89}$$

and let $U(\mathbb{Z}/\ell\mathbb{Z})$ be the unipotent radical in $B(\mathbb{Z}/\ell\mathbb{Z})$. Then I claim

Theorem 5.1.5. *The image of the Galois group is $T^{(1)}(\mathbb{Z}/\ell\mathbb{Z}) \ltimes U(\mathbb{Z}/\ell\mathbb{Z})$*

Here are arguments why this must be the case.

The quotient $B(\mathbb{Z}/\ell\mathbb{Z})/U(\mathbb{Z}/\ell\mathbb{Z}) = T^{(1)}(\mathbb{Z}/\ell\mathbb{Z})$ and the resulting map $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to T^{(1)}(\mathbb{Z}/\ell\mathbb{Z})$ is surjective. We have to show that the restriction map $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_\ell)) \to U(\mathbb{Z}/\ell\mathbb{Z})$ is surjective. The center of $U(\mathbb{Z}/\ell\mathbb{Z})$ is the group $U_{1,3}(\mathbb{Z}/\ell\mathbb{Z})$ where $u_{1,2} = u_{2,3} = 0$. Let $V(\mathbb{Z}/\ell\mathbb{Z})$ be the quotient $U(\mathbb{Z}/\ell\mathbb{Z})/U_{1,3}(\mathbb{Z}/\ell\mathbb{Z})$. It suffices to show that $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_\ell)) \to V(\mathbb{Z}/\ell\mathbb{Z})$, is surjective because the commutators of the elements in $V(\mathbb{Z}/\ell\mathbb{Z})$ fill up the elements in $U_{1,3}((\mathbb{Z}/\ell\mathbb{Z})$. Then it becomes clear that it suffices to find a $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ with $u_{1,2}(\sigma), u_{2,3}(\sigma) \not\equiv 0 \mod \ell$ because we still have action of $T^{(1)}(\mathbb{Z}/\ell\mathbb{Z})$ by conjugation on the image of the Galois group.

Now we apply the Eichler-Shimura congruence relation which says that for any p we have

$$\rho(\Phi_p)^2 - T_p\rho(\Phi_p) + p^{11} Id = 0. \tag{5.90}$$

and if we are courageous enough to compute with 3×3 matrices we find for $\sigma = \Phi_p$

$$\begin{pmatrix} 0 & 0 & u_{12}(\Phi_p)u_{23}(\Phi_p) + (-1+p^{11})u_{13}(\Phi_p) - p^{11}t^{(p)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0. \tag{5.91}$$

Hence the right upper entry must be zero. If $p \equiv 1 \mod 691$ this means that $u_{12}(\Phi_p)u_{23}(\Phi_p) - p^{11}t^{(p)} = 0$ and this implies: If $t^{(p)} \neq 0$ then $u_{12}(\Phi_p)$ and $u_{23}(\Phi_p) \neq 0$. But we have seen that $t^{(p)} = 0$ implies the stronger congruence $\tau(p) \equiv p^{11} + 1 \mod 691^2$. (See section 3.3.4) But now the prime $p_1 = 6911$ is congruent to 1 mod 691 but $\tau(p_1)$ is not congruent to $691^{11} + 1$ modulo 691^2. Hence $u_{12}(\Phi_{p_1}) \neq 0, u_{23}(\Phi_{p_1}) \neq 0$. The claim follows.

By definition $K_\ell^{(1)}/\mathbb{Q}$ is the normal extension of $\mathbb{Q}$ such that

$$\mathrm{Gal}(K_\ell^{(1)}/\mathbb{Q}) = T^{(1)}(\mathbb{Z}/\ell\mathbb{Z}) \ltimes U(\mathbb{Z}/\ell\mathbb{Z}) := B^{(1)}(\mathbb{Z}/\ell\mathbb{Z}), \tag{5.92}$$

this extension is unramified outside ℓ. It contains the field of ℓ-th roots of unity, i.e. $\mathbb{Q}(\zeta_\ell) \subset K_\ell^{(1)}$. The Galois group $\mathrm{Gal}(K_\ell^{(1)}/\mathbb{Q}(\zeta_\ell)) = U(\mathbb{Z}/\ell\mathbb{Z})$. This group has a center $U_{13}(\mathbb{Z}/\ell\mathbb{Z}) = \mathbb{Z}/\ell\mathbb{Z}$, this is also the center of the larger group $\mathrm{Gal}(K_\ell^{(1)})/\mathbb{Q})$. We define the subfield $K_\ell^{(1,0)}$ by requiring that $\mathrm{Gal}(K_\ell^{(1,0)}/\mathbb{Q}) = \mathrm{Gal}(K_\ell^{(1)}/\mathbb{Q}(\zeta_\ell))/U_{13}(\mathbb{Z}/\ell\mathbb{Z})$. Then $K_\ell^{(1,0)}/\mathbb{Q})$ is the composite of two cyclic extensions $K_\ell^{(1,!}/\mathbb{Q}(\zeta_\ell)$ .and $K_\ell^{(1,\partial)}/\mathbb{Q}(\zeta_\ell)$. These two extensions have the faithful two dimensional representations

$$\begin{aligned} \rho_! :\ & \mathrm{Gal}(K_\ell^{(1,!)}/\mathbb{Q}) \to \mathrm{Gl}(H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z})) \\ & \sigma \mapsto \rho_!(\sigma) = \begin{pmatrix} 1 & u_{23}(\sigma) \\ 0 & \alpha_\ell(\sigma)^{-11} \end{pmatrix} \\ \rho_\partial :\ & \mathrm{Gal}(K_\ell^{(1,\partial)}/\mathbb{Q}) \to \mathrm{Gl}(H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z})/\mathbb{Z}/\ell\mathbb{Z}e_1) \\ & \sigma \mapsto \rho_\partial(\sigma) = \begin{pmatrix} \alpha_\ell(\sigma)^{-11} & u_{12}(\sigma) \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{5.93}$$

Now we invoke the theory of crystalline representations. We consider the restriction of the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}_\ell)$ to the Galois group

$\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$. This representation is crystalline and I think that this implies that as a $\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ module it has a filtration

$$\{(0)\} \subset \mathbb{Z}_\ell(0) \subset H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}_\ell) \text{ with } H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}_\ell)/\mathbb{Z}_\ell(0) = \mathbb{Z}_\ell(-11)$$

here $\mathbb{Z}_\ell(0) = H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}_\ell)^{I_\ell}$, where I_ℓ is the inertia group. Therefore we get a direct sum decomposition for the $\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ module

$$H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z}) = \mathbb{Z}/\ell\mathbb{Z}(0) \oplus \mathbb{Z}/\ell\mathbb{Z}(-11) \tag{5.94}$$

and this implies that $K_\ell^{(1,!)}/\mathbb{Q}(\zeta_\ell)$ is also unramified at ℓ hence it is an unramified extension.

This unramified extension extension has been constructed by Ribet in [74]., it has also been constructed in [46]. At the end of that paper we raise the question for a decomposition law. This means that for any prime p we want to find a rule to determine the conjugacy class of $\rho(\Phi_p), \rho_!(\Phi_p) \dots$. If $p \not\equiv 1 \mod \ell$. then the two conjugacy classes $\rho_!(\Phi_p), \rho_\partial(\Phi_p)$ are semi simple and determined by their eigenvalues. But if $p \equiv 1 \mod \ell$ then $\rho(\Phi_p)$ is unipotent and here are several possibilities for the conjugacy class.

Theorem 5.1.6. *If $p \equiv 1 \mod \ell$ and if the horizontal long exact sequence of $\mathbb{Z}[T_p]$ (See 3.97) modules splits then p splits completely either in the field $K_\ell^{(1,!)}$ or in the field $K_\ell^{(1,\partial)}$.*

If $p \equiv 1 \mod \ell$ and if the horizontal long exact sequence of $\mathbb{Z}[T_p]$ modules does not split then both fields $K_\ell^{(1,!)}/\mathbb{Q}(\zeta_\ell)$ and the field $K_\ell^{(1,\partial)}/\mathbb{Q}(\zeta_\ell)$ are inert at the primes above p.

The density of primes which satisfy $p \equiv 1 \mod 691$ and $\tau(p) \equiv p^{11} + 1 \mod 691^2$ is equal to $\frac{1}{238395}$

For the curios reader: The first such prime is $p = 3178601$. We leave it as an exercise for the reader to find out whether it splits completely in $K_\ell^{(1,!)}$ or in $K_\ell^{(1,\partial)}$. It is the 228759-th prime.

Finally we have a brief look at the action of the Galois group on $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}_\ell)$. Again we choose a basis e_1, e_0, e_{-1} the element e_1 maps to a generator in the boundary cohomology and e_0, e_{-1} form a basis of $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}_\ell)$ we assume that this basis reduces $\mod \ell$ to the basis which we denoted by the same letters. Then

$$\rho(\sigma) = \begin{pmatrix} \alpha_\ell(\sigma)^{-11} & u_{12}(\sigma) & u_{13}(\sigma) \\ 0 & a(\sigma) & b(\sigma) \\ 0 & \ell c(\sigma) & d(\sigma) \end{pmatrix} \in \mathrm{Gl}_3(\mathbb{Z}_\ell), \tag{5.95}$$

where $a(\sigma) \equiv 1 \mod \ell, d(\sigma) \equiv \alpha^{-11}(\sigma) \mod \ell$.

We claim that there is a σ with $c(\sigma) \not\equiv 0 \mod \ell$

For a prime p and the Frobenius Φ_p we get $a(\Phi_p)d(\Phi_p) - \ell b(\Phi_p)c(\Phi_p) = p^{11}$ and $\tau(p) = a(\Phi_p) + d(\Phi_p)$. Now an straightforward calculation shows that for a prime $p \equiv 1 \mod \ell$ which in addition satisfies $c(\Phi_p) \equiv 0 \mod \ell$ we must have $\tau(p) \equiv p^{11} + 1 \mod \ell^2$. But $p = 1 + 10 * 691 = 6911$ does not satisfy this congruence, hence $c(\Phi_{6911}) \not\equiv 0 \mod \ell$.

The cohomology $H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell)$ has the submodule $H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell)\oplus\mathbb{Z}_\ell e_1^\dagger$, where $e_1^\dagger=\ell f_{10}^\dagger(see(3.69))$ this submodule is determined by T_2. Therefore it is invariant under the action of the Galois group, with respect to the basis $e_1^\dagger, e_0, e_{-1}$ the Galois action is given by

$$\rho^\dagger(\sigma)=\begin{pmatrix}\alpha_\ell(\sigma)^{-11} & 0 & 0\\ 0 & a(\sigma) & b(\sigma)\\ 0 & \ell c(\sigma) & d(\sigma)\end{pmatrix}\in \mathrm{Gl}_3(\mathbb{Z}_\ell), \tag{5.96}$$

where we still have $\alpha^{11}(\sigma)=\det(\begin{pmatrix}a(\sigma) & b(\sigma)\\ \ell c(\sigma) & d(\sigma)\end{pmatrix})$. It is clear from the above considerations that the image of the Galois group is given by those matrices in $\mathrm{Gl}_3(\mathbb{Z}_\ell)$ which satisfy the conditions above.

But we want to know the image of the Galois group with respect to our basis e_1, e_0, e_{-1}. For this we write $e_1=\frac{x_0e_{-1}+e_1^\dagger}{\ell}$ and then clearly

$$\rho(\sigma)=\begin{pmatrix}\alpha_\ell(\sigma)^{-11} & c(\sigma)x_0 & \frac{(d(\sigma)-\alpha_\ell(\sigma)^{-11})}{\ell}x_0\\ 0 & a(\sigma) & b(\sigma)\\ 0 & \ell c(\sigma) & d(\sigma)\end{pmatrix}\in \mathrm{Gl}_3(\mathbb{Z}_\ell), \tag{5.97}$$

We put

$$a(x_0,\sigma)=\alpha_\ell^{11}(\sigma)(u_{12}(x_0,\sigma),u_{13}(x_0,\sigma))) \tag{5.98}$$

then $\sigma\mapsto a(x_0,\sigma)$ is a one-cocycle with values in $H^1_!(11):=H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell)\otimes\mathbb{Z}_\ell(11)$. We compute its cohomology class $[\underline{v}]\in H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q},H^1_!(11))$. We start from the exact sequence of Galois-modules

$$0\to H^1_!(11)\to\frac{1}{\ell}H^1_!(11)\to\frac{1}{\ell}H^1_!(11)/H^1_!(11)\to 0 \tag{5.99}$$

where of course $\frac{1}{\ell}H^1_!(11)/H^1_!(11)=H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes\mathbb{Z}/\ell\mathbb{Z})$. This yields a long exact sequence in Galois cohomology. The element v provides a well defined element in $\tilde{v}\in H^0(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q},H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes\mathbb{Z}/\ell\mathbb{Z})$ and clearly $\delta(\tilde{v})=\underline{v}$.

Now we can say that the image of the Galois group under ρ consists of the matrices

$$\{\begin{pmatrix}x & u_{12} & u_{13}\\ 0 & a & b\\ 0 & c & d\end{pmatrix}|ad-bc=x;\ell u_{12}=c;\ell u_{13}=d-x\}\subset\mathrm{Gl}_3(\mathbb{Z}_\ell) \tag{5.100}$$

It is the cocycle condition which makes this to a subgroup. Now it is not difficult to see that this is the group of $\mathbb{Z}_\ell-$ valued points of a smooth groups scheme $\mathcal{I}^{(1)}/\mathbb{Z}_\ell\subset\mathrm{Gl}_2/\mathbb{Z}$.

We can say that we constructed a Galois-extension $K_\ell^{(\infty)}/Q$ which is unramified outside ℓ and we have an isomorphism

$$\rho_\ell:\ \mathrm{Gal}(K_\ell^{(\infty)}/\mathbb{Q})\xrightarrow{\sim}\mathcal{I}^{(1)}(\mathbb{Z}_\ell) \tag{5.101}$$

We also consider the finite extensions $K_\ell^{(r)}(\mathbb{Z}/\ell^r\mathbb{Z}) \xrightarrow{\sim} \mathcal{I}^{(1)}(\mathbb{Z}/\ell^r\mathbb{Z})$ and for $r=1$ we have $\mathcal{I}^{(1)}(\mathbb{Z}/\ell\mathbb{Z}) = B^{(1)}(\mathbb{Z}/\ell\mathbb{Z})$.

Here we see the prototype of a very general strategy to get insight into the structure of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. We wee that that value of certain L functions at a certain special argument (in this case the Riemann $\zeta(s)$-function at $s=-11$) has influence on structure of certain cohomology groups of arithmetic groups and these Hecke modules contain some information on the structure of the Galois group, We can construct certain interesting extensions of $\mathbb{Q}$ with controlled ramification. On the other hand we see that there is also a flow of information from the Galois side back to the structure of the Hecke modules.

This connection between the congruence for the values of $\tau(p)$ and the structure of the Galois group was observed by Serre in his paper [79] and proved later by Deligne [25]. Later this relationship was exploited by Ribet in [74] and by many other authors.

Here I want to point out that there is a change of paradigm between our approach and the approach by the authors mentioned above. These authors mostly look at the space of holomorphic cusp forms or what essentially amounts to the same the inner cohomology $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n\otimes\mathbb{C})$. (Eichler-Shimura isomorphism).Then these authors get the consequences for the structure of the Galois group from the congruences.

In our approach we consider the module $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n)$ and exploit our knowledge of the denominator of the Eisenstein class.

We explain this approach in a more general situation, but we also keep an eye on the computational aspects. We start from any irregular prime ℓ and an even integer $n>0$ such that $\ell^\delta \,||\, \zeta(-1-n)$. In section 3.3.8 we introduced the Hecke-modules $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n\otimes\mathbb{Z}/\ell)\{\bar{\pi}_f^{\mathrm{Eis}}\}$ or $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n\otimes\mathbb{Z}_\ell)\{\bar{\pi}_f^{\mathrm{Eis}}\}$, and Deligne's theorem tells us, that we have an action of the Galois group on these modules. This action commutes with the action of the Hecke algebra $\mathcal{H}$. More precisely we have a finite normal extension $\mathbb{Q}_{\mathrm{Eis}}(\mathcal{M}_n\otimes\mathbb{F}_\ell)/\mathbb{Q}$ and an injective homomorphism

$$\rho_{\ell,1}\ \mathrm{Gal}(\mathbb{Q}_{\mathrm{Eis}}(\mathcal{M}_n\otimes\mathbb{F}_\ell)/\mathbb{Q}) \to \mathrm{Gl}_{\mathcal{H}}(H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n\otimes\mathbb{Z}/\ell)(\{\bar{\pi}_f^{\mathrm{Eis}}\}). \tag{5.102}$$

We can replace ℓ by higher powers ℓ^r and get representations $\rho_{\ell,r}$ and in the limit we get an injection

$$\rho_\ell:\ \mathrm{Gal}(\mathbb{Q}_{\mathrm{Eis}}(\mathcal{M}_n\otimes\mathbb{Z}_\ell)/\mathbb{Q}) \to \mathrm{Gl}_{\mathcal{H}}(H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n\otimes\mathbb{Z}_\ell)(\{\bar{\pi}_f^{\mathrm{Eis}}\}). \tag{5.103}$$

and this is the representation in the above theorem of Deligne. This theorem of Deligne also asserts that the extensions $\mathbb{Q}_{\mathrm{Eis}}(\mathcal{M}_n\otimes\mathbb{Z}_\ell) \supset \mathbb{Q}_{\mathrm{Eis}}(\mathcal{M}_n\otimes\mathbb{F}_\ell)$ are unramified outside ℓ. If we want to understand these representations we have to make some further assumptions, for instance we may assume that $m(\{\bar{\pi}_f^{\mathrm{Eis}}\}) = 1$. This is of course true in our example above, we expect it to be true most of the times but we know that this is not always true (see section 3.3.9). Assuming that $m(\{\bar{\pi}_f^{\mathrm{Eis}}\}) = 1$, then we can proceed as in our example.

We know that $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n\otimes\mathbb{Z}_\ell)(\{\bar{\pi}_f^{\mathrm{Eis}}\})$ is a free $\mathbb{Z}_\ell$ module of rank 3. We can find generators $e_1, e_0, , e_{-1}$ of this module such that

$$\begin{aligned} &a)\ \mathbb{Z}/\ell^\delta e_{-1} \text{ is the submodule in theorem 3.3.1}\\ &b) \text{ the two elements } e_{-1}, e_0 \text{ generate } H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}/^\delta_\ell)(\{\bar{\pi}_f^{\rm Eis}\}). \end{aligned} \tag{5.104}$$

With respect to this basis the Hecke operator $T_p \mod \ell^\delta$ is of the form

$$T_p = \begin{pmatrix} p^{n+1}+1 & 0 & t^{(p)} \\ 0 & p^{n+1}+1 & 0 \\ 0 & 0 & p^{n+1}+1 \end{pmatrix} \mod \ell^\delta \tag{5.105}$$

Now we want to understand the representations of the Galois group. In our argument for above example we needed the that we can find a prime $p \equiv 1 \mod \ell$ such that $\tau(p) \not\equiv p^{n+1}+1 \mod 691^2$. This can not be the right condition in the general situation because we may have $\delta > 1$ Therefore we formulate the alternative condition

$$\textit{We can find a prime } p_1 \equiv 1 \mod \ell \textit{ such that } t^{(p_1)} \not\equiv 0 \mod \ell \tag{5.106}$$

This condition is difficult to verify because we have to compute the quantity $t^{(p)}$ for a very large prime. Already in our baby example we relied on the tables for the values $\tau(p)$ provided by Mathematica, we can always verify it in principle but not in practice. We formulate the much weaker condition

$$\textit{We can find a prime } p_0 \textit{ such that } t^{(p_0)} \not\equiv 0 \mod \ell \tag{5.107}$$

This assumption is almost certainly always true, but I do not have a proof. Further down we will explain that - in theory- we can verify this assertion effectively in a given case. But we will also explain that in practice this such a verification can be achieved -for small values of n- in a few seconds on the computer.

Now we explain how we can use the Galois-module structure to verify that a given Hecke-module $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_\ell)(\{\bar{\pi}_f^{\rm Eis}\})$ satisfies (5.106). The field $\mathbb{Q}_{\rm Eis}(\mathcal{M}_n \otimes \mathbb{F}_\ell)$ has a non trivial intersection with the field $\mathbb{Q}(\zeta_\ell)$, this intersection is the fixed field under the action of $\mu_d \subset \mathbb{F}_\ell^\times$, We denote it by $\mathbb{Q}(\zeta_\ell)^{\mu_d}$. We recall that $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_\ell)(\{\bar{\pi}_f^{\rm Eis}\})$ has the generators e_1, e_0, e_{-1} such that the flag $\mathbb{F}_\ell e_{-1} \subset \mathbb{F}_\ell e_{-1} \oplus \mathbb{F}_\ell e_0$ is invariant under the action of the Galois group. Again we get the two representations $\rho_{!,\ell}, \rho_{\partial,\ell}$ of $\mathrm{Gal}(\mathbb{Q}_{\rm Eis}(\mathcal{M}_n \otimes \mathbb{F}_\ell)/\mathbb{Q})$. We restrict these representations to the subgroup $\mathrm{Gal}(\mathbb{Q}_{\rm Eis}(\mathcal{M}_n \otimes \mathbb{F}_\ell)/\mathbb{Q}(\zeta_\ell)^{\mu_d})$. These restrictions are simply homomorphisms

$$\begin{aligned} u'_{1,2} :\ & \mathrm{Gal}(\mathbb{Q}_{\rm Eis}(\mathcal{M}_n \otimes \mathbb{F}_\ell)/\mathbb{Q}(\zeta_\ell)^{\mu_d}) \to \mathbb{F}_\ell \\ u'_{2,3} :\ & \mathrm{Gal}(\mathbb{Q}_{\rm Eis}(\mathcal{M}_n \otimes \mathbb{F}_\ell)/\mathbb{Q}(\zeta_\ell)^{\mu_d}) \to \mathbb{F}_\ell \end{aligned} \tag{5.108}$$

If these homomorphisms are non zero, then they are surectves and hence they provide cyclic field extensions $K^{1,!}/\mathbb{Q}(\zeta_\ell)^{\mu_d}, K^{1,\partial}/\mathbb{Q}(\zeta_\ell)^{\mu_d}$ of degree ℓ. These extension are normal over $\mathbb{Q}$ and since we assumed that $d \neq \frac{\ell-1}{2}$ these extensions must be disjoint. Then the congruence relation (5.91) implies

Lemma 5.1.1. *We find a prime $p \equiv 1 \mod \ell$ for which $t^{(p)} \not\equiv 0 \mod \ell$ if and only if both $u'_{1,2}, u'_{2,3}$ are non zero.*

Proof. Of course if one of the homomorphisms is zero then the Eichler-Shimura congruence relation implies that $t^{(p)} \equiv 0 \mod \ell$ for all $p \neq \ell, p \equiv 1 \mod \ell$. On the other hand we see that $t^{(p)} \equiv 0 \mod \ell$ implies that always one of the two quantities $u_{1,2}(\Phi_p), u_{2,3}(\Phi_p) = 0$. Now we apply the famous Chebotareff's density theorem to the two extensions $K^{1,!}, K^{1,\partial}$. In this case it says that for $T \to \infty$ the following limit exist and is equal to

$$\lim_{T\to\infty} \frac{\#\{\, p \leq T \mid\ p \equiv 1 \mod \ell; u_{i,j}(\Phi_p) = 0\}}{\#\{p \leq X\}} = \frac{1}{\ell(\ell-1)} \tag{5.109}$$

This limit is called the *density* of the set $\{p \mid p \equiv 1 \mod \ell; u_{i,j}(\Phi_p) = 0\}$ and denoted by $d(\{p \mid p \equiv 1 \mod \ell; u_{i,j}(\Phi_p) = 0\}$. Dirichlet's theorem implies that the density of the set of primes $\{p \,|\, p \equiv 1 \mod \ell\}$ is equal to $1/(\ell-1)$. Hence we know that for $i, j = 1, 2$ or $2, 3$ and very large $T >> 0$

$$\#\{\, p \leq T \mid\ p \equiv 1 \mod \ell; u_{i,j}(\Phi_p) = 0\} \simeq \frac{1}{\ell}\#\{\, p \leq T \mid\ p \equiv 1 \mod \ell\} \tag{5.110}$$

Since we know that $\ell \geq 37$ it follows that we find a $p \equiv 1 \bmod \ell$ with $u_{1,2}(\Phi_p)$ and $u_{2,3}(\Phi_p) \neq 0$. □

Of course this does not give an effective algorithm to find the prime p. We discuss this further down.

If we now assume that (5.106) is true then we can prove a generalisation of theorem 5.1.5 for this given prime ℓ. We have to modify the statement a little bit. In general the homomorphism $\alpha_\ell^{n+1} : \mathbb{Z}_\ell^\times \to {\mathbb{F}_\ell}^\times$ will not be surjective, Hence have to replace the group $T^{(1)}(\mathbb{Z}/\ell\mathbb{Z})$ in the formulation of theorem 5.1.5 by $T^{(1)}(\mathbb{Z}/\ell\mathbb{Z})^d$ where $d = gcd(n+1, \ell-1)$. From the beginning we left out the case that $\ell - 1 | n + 1$, but from now on we also assume that $d \neq \frac{\ell-1}{2}$. Then using the same arguments as in the proof of theorem 5.1.5 we get an isomorphism

$$\mathrm{Gal}(\mathbb{Q}_{\mathrm{Eis}}(\mathcal{M}_n \otimes \mathbb{Z}/\ell)/\mathbb{Q}) \xrightarrow{\sim} \mathcal{I}^{(d)}(\mathbb{Z}/\ell) = \{\begin{pmatrix} a & u_{1,2} & u_{1,3} \\ 0 & 1 & u_{2,3} \\ 0 & 0 & a \end{pmatrix} \mid a \in ((\mathbb{Z}/\ell)^\times)^d, u_{ij} \in \mathbb{Z}/\ell\} \tag{5.111}$$

Now we will show that this result provides an indication how to verify (5.107) and then (5.106) . A direct computation shows that the function

$$t : \mathcal{I}^{(d)}(\mathbb{Z}/\ell) \to \mathbb{F}_\ell; t : \gamma \mapsto u_{1,2}u_{2,3} + (-1+a)u_{1,3} \tag{5.112}$$

is constant on the conjugacy classes in $\mathcal{I}^{(d)}$. Again we invoke the Chebotareff density theorem. For $\gamma \in \mathcal{I}^{(d)}(\mathbb{Z}/\ell)$ we denote its conjugacy class by C_γ and its centraliser by Z_γ. Then the theorem says that the density of primes p for which $\Phi_p \in C_\gamma$ is $1/\#Z_\gamma$. I leave to the reader to analyse the conjugacy classes and to check that

$$d(\{p \mid \Phi_p \in C_\gamma\ ;\ t(\gamma) \neq 0\} \simeq \ell d(\{p \mid \Phi_p \in C_\gamma\ ;\ t(\gamma) \neq 0\} \tag{5.113}$$

(This can easily be made more precise) But this suggests that the probability to find $t(\Phi_p) = t^{(p)} = 0$ is roughly $\frac{1}{\ell}$ hence very small. So we must be really very unlucky if we do not find a $t^{(p)} \neq 0$ for some very small prime $p = 2, 3,$. Since we can write algorithms which compute T_p for small values of p we should be able to verify in practice (in a given case). In the six cases $n = 10, 14, 16, 18, 20, 24$ the Hecke module $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n))$ is of rank 2 and for any $\ell|\zeta(-1-n)$ our program with Gangl yields $t(2) \not\equiv 0 \mod \ell$, hence T_2 suffices to verify (5.107) in few seconds in these cases.

In principle this procedure is effective. In [59]) the authors prove effective versions of the Chebotareff density theorem, and using these results we can show, that there is a computable constant $c_0(p)$ such that we find a $p < c_0(p)$ $t^{(p)} \neq 0$ provided (5.106) is true. But this constant is terribly large and not of practical use.

Now we want to show that there is another way to verify (5.106) in a given case, which with very high probability also works in practice. We have a closer look what happens if one of the two homomorphisms $u'_{1,2}, u'_{2,3}$ is zero. This means that one of the two representation in (5.93) factors through a conjugate of the diagonal torus, Hence we can replace the basis vector $e_1 \to e_1 + xe_0$(resp.$e_0 \to e_0 + ye_{-1}$) such that with respect to this new basis we have $u_{1,2}(\sigma) = 0$, (resp, $u_{2,3}(\sigma) = 0$) for all σ in the matrix for $\rho(\sigma)$ (see 5.88).The Hecke operator T_p with respect to either of these new basis is still given by

$$T_p = \begin{pmatrix} p^{n+1}+1 & 0 & t^{(p)} \\ 0 & p^{n+1}+1 & 0 \\ 0 & 0 & p^{n+1}+1 \end{pmatrix} \mod \ell \tag{5.114}$$

and hence the Eichler-Shimura congruence relation gives us (in both cases)

$$(-1 + p^{n+1})u_{13}(\Phi_p) = p^{n+1}t^{(p)} \tag{5.115}$$

On the other hand it is clear that under our assumption the map

$$\begin{gathered} U_{1,3}: \ \mathrm{Gal}(\mathbb{Q}_{\mathrm{Eis}}(\mathcal{M}_n \otimes \mathbb{F}_\ell)/\mathbb{Q}) \to \mathbb{F}_\ell \\ \sigma \mapsto u_{1,3}(\sigma) \end{gathered} \tag{5.116}$$

is a homomorphism. We define the subfield $L \subset \mathbb{Q}_{\mathrm{Eis}}(\mathcal{M}_n \otimes \mathbb{F}_\ell)$ by $\mathrm{Gal}(\mathbb{Q}_{\mathrm{Eis}}(\mathcal{M}_n \otimes \mathbb{F}_\ell)/L) = \ker(U_{1,3})$. Then we get an injective homomorphism

$$\bar{U}_{13}: \ \mathrm{Gal}(L/\mathbb{Q}) \hookrightarrow \mathbb{F}_\ell. \tag{5.117}$$

Hence we see that either $L = \mathbb{Q}$ or $L/\mathbb{Q}$ is a cyclic extension of order ℓ which is unramified outside ℓ. Then class field theory tells us that $L/\mathbb{Q}$ is actually the unique cyclic subfield of degree ℓ in $\mathbb{Q}(\zeta_{\ell^2})$. The Galois group $\mathrm{Gal}(L/\mathbb{Q}) = \mathbb{F}_\ell$ and for any prime $p \neq \ell$ the Frobenius is given by a number $v_\ell(p)$ in $\mathbb{F}_\ell$, and this number is defined by

$$p^{\ell-1} = 1 + \ell v_\ell(p) \mod \ell^2 \tag{5.118}$$

Hence there is a number $\lambda_{n,\ell} \in \mathbb{F}_\ell$ such that for all $p \neq \ell$

$$u_{1,3}(\Phi_p) = \lambda_{n,\ell} v_\ell(p) \tag{5.119}$$

If we now that we found a prime $p_0(\neq \ell)$ such that $t^{(p_0)} \neq 0 \mod \ell$, then we get for any other prime p the relation

$$\frac{t^{(p)}}{t^{(p_0)}} = \frac{(1-p^{-11})v_\ell(p)}{(1-p_0^{-11})v_\ell(p_0)} \tag{5.120}$$

Then we look at some other small primes and check the relation (5.120). Almost certainly we will find a small prime such that (5.120) fails and we have verified that our Hecke module is not degenerated. For small values of n, ℓ this should be rather effective. Again we need a little bit of luck. I have not yet checked the six cases $n = 10, 14, 16, 18, 20, 24$ the program with Gangl is not yet written for T_3, T_5

Clearly these results are just the beginning of a very interesting story, we are just discussing the first case of a much wider circle of problems.

For instance we can allow some ramifications, this means we pass to congruence subgroups $\Gamma \subset \mathrm{Sl}_2(\mathbb{Z})$. We have to discuss the denominator issue, in this more general context some special values of Dirichlet L-functions $L(\chi, -n-1)$ will determine (or closely related to) the denominator of Eisenstein classes.

Here some experimental aspects will become of interest. In the unramified case the primes ℓ dividing $\zeta(-1-n)$ are large, the smallest are $\ell = 37, 59, \ldots$. In section 3.3.9 we discussed the multiplicity $m(\bar\pi_f^{\mathrm{Eis}})$, we get higher multiplicity if two or even more numbers are divisible by ℓ. We saw (see [3]) that there is exactly one number n which is less than 10^5 such that $m(\bar\pi_f^{\mathrm{Eis}}) \geq 2$ But we also checked that in this case we still have weak multiplicity one. The probability that in the unramified case we will ever find a case with weak multiplicity > 1 is very small.

But if we allow ramification then we may have a much larger supply of cases where a reasonably small prime ℓ divides a value $L(\chi, -n-1)$ and where we find cases with weak multiplicity > 1. Some heuristic considerations considerations could suggest to us a probability $P(\ell) > 0$ that in a case where $\ell | L(\chi, -n-1)$ (perhaps we should consider only quadratic characters) we have weak multiplicity > 1. If we now have a finite set $\mathcal{X}$ of triples (ℓ, n, χ) with $\ell | L(\chi, -n-1), n \leq \ell - 2$. If then $\#\mathcal{X} \times P(\ell) >> 1$ then we can start to look cases where the weak multiplicity of $\bar\pi_f^{\mathrm{Eis}}$ is greater than one. It may be of interest to find out whether the number of these events match our probabilistic expectations.

The next question we can ask is: If we have higher multiplicity. what can we prove about the action of the Galois group on $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}/\ell)$? Fortunately we know a prime -namely $\ell = 547$ for which we have multiplicity 2, but we still have weak multiplicity one. For this case we made some experimental computations in section 3.3.11 and we made some predictions about the structure of the Hecke modules $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{484+\alpha(\ell-1)} \otimes \mathbb{Z}/(\ell^2))$. In this case it is an interesting question to consider that case $\alpha = 100$ and find out what the structure of the Galois module is, especially describe the image of the Galois group.

But by far the most interesting issue is to generalise this approach to larger groups. We assume that somebody has written an algorithm which computes - in a given case - the cohomology groups and the arrows in the long exact sequence. Furthermore we assume that the algorithm computes some "small" Hecke operators. For instance we can verify the conjecture about the denominator 41 in [42], basically by the same method as the one we used in chapter 3 for 691.

Then the next task will be to analyse the interplay between the Galois action on the $\ell-$adic cohomology and the Hecke action on the cohomology.

Chapter 6

Analytic methods

6.1 The representation theoretic de-Rham complex

6.1.1 Rational representations

We start from a reductive group $G/\mathbb{Q}$ for simplicity we assume that the semi simple component $G^{(1)}/\mathbb{Q}$ is quasisplit. There is a unique finite normal extension $F/\mathbb{Q}, F \subset \mathbb{C}$ such that $G^{(1)} \times_{\mathbb{Q}} F$ becomes split. If $T^{(1)}/\mathbb{Q}$ is a maximal torus which is contained in a Borel subgroup $B/\mathbb{Q}$, let $U/\mathbb{Q}$ be its unipotent radical. Then the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on $X^*(T^{(1)} \times_{\mathbb{Q}} F)$. It acts by permutations on the set of positive roots $\pi_G \subset X^*(T^{(1)} \times_{\mathbb{Q}} F)$ corresponding to $B/\mathbb{Q}$. This action factors over the quotient $\mathrm{Gal}(F/\mathbb{Q})$. Then it also acts on the set of dominant weights. Since our group is quasi split we find for any dominant weight λ an absolutely irreducible $G \times_{\mathbb{Q}} F$- module $\mathcal{M}_\lambda$.

$$r : G \times_{\mathbb{Q}} F \to \mathrm{Gl}(\mathcal{M}_\lambda).$$

This representation is characterised by the following two properties. The space of invariants under the action of U is one dimensional, i.e. ${\mathcal{M}_\lambda}^U = Fe_\lambda$ and the torus acts on this one dimensional space by the character λ, i.e.$te_\lambda = \lambda e_\lambda$. We say that e_λ is a highest weight vector. Since we assumed that $\mathbb{Q} \subset F \subset \bar{\mathbb{Q}} \subset \mathbb{C}$ we get the extension

$$r_{\mathbb{C}} : (G \times_{\mathbb{Q}} F) \times_F \mathbb{C} \to \mathrm{Gl}(\mathcal{M}_\lambda \otimes_F \mathbb{C}).$$

Given such an absolutely irreducible rational representation, we can construct two new representations. We can form the dual $\mathcal{M}^\vee_{\lambda,\mathbb{C}} = \mathrm{Hom}_{\mathbb{C}}(\mathcal{M}_\lambda, \mathbb{C})$ and the complex conjugate $\overline{\mathcal{M}}_{\mathbb{C}}$ of our module $\mathcal{M}_\lambda$. On the dual module we have the contragredient representation $r^\vee$, which is defined by $\phi(r_{\mathbb{C}}(g)(v)) = r^\vee_{\mathbb{C}}(g^{-1})(\phi)(v)$.

To get the rational representation on the conjugate module $\bar{\mathcal{M}} \otimes_F \mathbb{C}$, we recall its definition: As abelian groups we have $\mathcal{M} \otimes_F \mathbb{C} = \bar{\mathcal{M}} \otimes_F \mathbb{C}$ but the action of the scalars is conjugated, we write this as $z \cdot_c m = \bar{z}m$. Then the identity gives us an identification

$$\mathrm{End}_{\mathbb{C}}(\mathcal{M} \otimes_F \mathbb{C}) = \mathrm{End}_{\mathbb{C}}(\bar{\mathcal{M}}_\lambda \otimes_F \mathbb{C}).$$

Now we define an action $\bar{r}_{\mathbb{C}}$ on $\bar{\mathcal{M}}_\lambda \otimes_F \mathbb{C}$: For $g \in G(\mathbb{C})$ we put

$$\bar{r}_{\mathbb{C}}(g)m = r_{\mathbb{C}}(g) \cdot_c m.$$

This defines an action of the abstract group $G(\mathbb{C})$, but this is in fact obtained from a rational representation. Therefore $\mathcal{M}_{\mathbb{C}}^\vee$ and $\overline{\mathcal{M}_C}$ both are given by a highest weight.

The highest weight of $\mathcal{M}_\lambda^\vee$ is $-w_0(\lambda)$. Here w_0 is the unique element $w_0 \in W$, which sends the system of positive roots Δ^+ into the system $\Delta^- = -\Delta^+$.

The highest weight of $\bar{\mathcal{M}}_\lambda \otimes_F \mathbb{C}$ is $c(\lambda)$ where $c \in \operatorname{Gal}(\mathbb{C}/\mathbb{R}) \subset \operatorname{Gal}(F/\mathbb{Q})$ is the complex conjugation acting on $X^*(T \times_{\mathbb{Q}} F)$. So we may say: $\overline{\mathcal{M}_{\lambda C}} = \mathcal{M}_{\bar{\lambda}}$.

We will call the module $\tilde{\mathcal{M}}_\lambda$ - *conjugate-autodual* or simply *c-autodual* if

$$c(\lambda) = -w_0(\lambda) \tag{6.1}$$

If our group $G/\mathbb{Q}$ is split then c acts trivially on the character module and the condition becomes $\lambda = -w_0(\lambda)$. If now in addition the element w_0 acts by -1 on the character module, the every λ is conjugate-autodual.

In the following few sections (until 6.1.6 we will always assume that our local system (resp. the corresponding representation) are local systems in $\mathbb{C}$-vector spaces (resp. $\mathbb{C}$-vector spaces $\tilde{\mathcal{M}}_\lambda$). Therefore we will suppress the factor $\otimes\mathbb{C}$.

Now we choose an arithmetic subgroup $\Gamma \subset G(\mathbb{R})$ and we will use transcendental methods to investigate the cohomology $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda)$. $\boxed{\text{HCmod}}$

6.1.2 Harish-Chandra modules and $(\mathfrak{g}, K_\infty)$-cohomology.

We consider the group of real points $G(\mathbb{R})$, it has the Lie algebra $\mathfrak{g}$, inside this Lie algebra we have the Lie algebra $\mathfrak{k}$ of the group K_∞. We have the notion of a $(\mathfrak{g}, K_\infty)$ module: This is a $\mathbb{C}$-vector space V together with an action of $\mathfrak{g}$ and an action of the group K_∞. We have certain assumptions of consistency:

i) The action of K_∞ is differentiable, this means it induces an action of $\mathfrak{k}$, the derivative of the group action.

ii) The action of $\mathfrak{g}$ restricted to $\mathfrak{k}$ is the derivative of the action of K_∞.

iii) For $k \in K_\infty, X \in \mathfrak{g}$ and $v \in V$ we have

$$(\operatorname{Ad}(k)X)v = k(X(k^{-1}v)).$$

Inside V we have have the subspace of K_∞ finite vectors, a vector v is called K_∞ *finite* if the $\mathbb{C}$- subspace generated by all translates kv is finite dimensional, i.e. v lies in a finite dimensional K_∞ invariant subspace. The K_∞ finite vectors form a subspace $V^{(K_\infty})$ and it is obvious that $V^{(K_\infty)}$ is invariant under the action of $\mathfrak{g}$, hence it is a $(\mathfrak{g}, K_\infty)$ sub module of V. We call a $(\mathfrak{g}, K_\infty)$ module a Harish-Chandra module if $V = V^{(K_\infty)}$.

For such a $(\mathfrak{g}, K_\infty)$-module we can write down a complex

$$\operatorname{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), V) = \{0 \to V \to \operatorname{Hom}_{K_\infty}(\Lambda^1(\mathfrak{g}/\mathfrak{k}), V) \to \operatorname{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}), V) \to \dots\}$$

where the differential is given by liealgc

$$d\omega(X_0, X_1, \dots, X_p) = \sum_{i=0}^{p} (-1)^i X_i \omega(X_0, \dots, \hat{X}_i, \dots, X_p) + \sum_{0 \le i < j \le p} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, , \dots, \hat{X}_j, \dots, \tag{6.2}$$

A few comments are in order. We have inclusions

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), V) \subset \mathrm{Hom}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), V) \subset \mathrm{Hom}(\Lambda^\bullet(\mathfrak{g}), V).$$

The above differential defines the structure of a complex for the rightmost term, we have to verify that the leftmost term is a subcomplex, this is not so difficult.

We define the $(\mathfrak{g}, K_\infty)$ cohomology as the cohomology of this complex, i.e.

$$H^\bullet(\mathfrak{g}, K_\infty, V) = H^\bullet(\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), V)) \tag{6.3}$$

It is clear that the map

$$H^\bullet(\mathfrak{g}, K_\infty, V^{(K_\infty)}) \to H^\bullet(\mathfrak{g}, K_\infty, V)$$

is an isomorphism.

If we have two $(\mathfrak{g}, K_\infty)$ modules V_1, V_2 and form the algebraic tensor product $W = V_1 \otimes V_2$ the we have a natural structure of a $(\mathfrak{g}, K_\infty)$ -module on W : The group K_∞ acts via the diagonal and $U \in \mathfrak{g}$ acts by the Leibniz-rule $U(v_1 \otimes v_2) = Uv_1 \otimes v_2 + v_1 \otimes Uv_2$. If both modules are Harish-Chandra modules, then the tensor product is also a Harish-Chandra module. Of course any finite dimensional rational representation of the algebraic group also yields a Harish-Chandra module.

deRhamiso

6.1.3 The representation theoretic de-Rham isomorphism

For us the $(\mathfrak{g}, K_\infty)$ module $\mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R})$,- this is the space of functions which are $\mathcal{C}_\infty$ in the variable g_∞- is one of the most important $(\mathfrak{g}, K_\infty)$ -modules. On these functions the group $G(\mathbb{R})$ acts by translations from the right, since our functions are $\mathcal{C}_\infty$ we also get an action of the Lie algebra $\mathfrak{g}$. Hence this is also a $(\mathfrak{g}, K_\infty)$-module.

If we fix the level see that $\mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R}))$ is a $(\mathfrak{g}, K_\infty) \times \mathcal{H}_{K_f}$, the Hecke algebra acts by convolution. We choose a highest weight module $\mathcal{M}_\lambda$ and apply the previous considerations to the Harish-Chandra module

$$V = \mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R}) \otimes \mathcal{M}_\lambda.$$

Notice that we can evaluate an element $f \in \mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R})) \otimes \mathcal{M}_\lambda$ in a point $\underline{g} = (g_\infty, \underline{g}_f)$ and the result $f(\underline{g}) \in \mathcal{M}_\lambda$. The Hecke algebra acts via convolution on the first factor.

Let us assume for the moment that is torsion free. . Then we can define the *sheaf of de-Rham complexes* $\Omega^\bullet_{\Gamma\backslash X}(\mathcal{M}_\lambda)$, for any open subset $V \subset \Gamma\backslash X$ the complex of sections is the de-Rham complex $\Omega^\bullet_{\Gamma\backslash X}(V) \otimes \mathcal{M}_\lambda$ (See for instance

[39], 4.10) If V is small enough a section in $.\Omega^\bullet_{\Gamma\backslash X}(V)\otimes\mathcal{M}_\lambda$ can be written as $\omega=\sum_\nu \omega_\nu\otimes m_\nu$ where $m_1,m_2,\dots,m_k$ is a basis of $\mathcal{M}_\lambda$ and then the differential is given by $\sum_\nu d\omega_\nu\otimes m_\nu$.

If consider the complex of global sections we drop the subscript and write $\Omega^\bullet(\Gamma\backslash X)\otimes\mathcal{M}_\lambda$.

We have the following fundamental fact: $\boxed{\text{Borel}}$

Proposition 6.1.1. *We have a canonical isomorphism of complexes*

$$Hom_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),\mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R})\otimes\mathcal{M}_\lambda)\xrightarrow{\sim}\Omega^\bullet(\Gamma\backslash X,\tilde{\mathcal{M}}_\lambda\otimes\mathbb{C}),$$

this isomorphism is compatible with the action of the Hecke algebra on both sides

This is rather clear. We have the projection map

$$q_\infty: G(\mathbb{R})\to G(\mathbb{R})/K_\infty = X$$

let $x_0\in X$ be the image of the identity $e\in G(\mathbb{R})$. The differential $D_q(e)$ maps the Lie algebra $\mathfrak{g}=$ tangent space of $G(\mathbb{R})$ at e to the tangent space T_{X,x_0} at $x_0\times e_f$. This provides the identification $T_{X,x_0}\xrightarrow{\sim}\mathfrak{g}/\mathfrak{k}$.

An element $\omega\in$ $\mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}),\mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R})\otimes\mathcal{M}_\lambda)$ can be evaluated on a p-tuple $(X_0,X_1,\dots,X_{p-1})$ and the result

$$\omega(X_0,X_1,\dots,X_{p-1})\in\mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R})\otimes\mathcal{M}_\lambda.$$

We want to produce an element $\tilde\omega$ in the de-Rham complex $\Omega^\bullet(\Gamma\backslash X,\tilde{\mathcal{M}}_\lambda)$. Pick a point $x\times \underline{g}_f\in X$, we find an element $(g_\infty,)\in G(\mathbb{R})\times G(\mathbb{A}_f)$ such that $g_\infty x_0=x$. Our still to be defined form $\tilde\omega$ can be evaluated at a p-tuple $(Y_0,\dots,Y_{p-1})$ of tangent vectors in $x\times\underline{g}_f$ and the result has to be an element in $\mathcal{M}_{\mathbb{C},x}$. We find a p-tuple $(X_0,X_1,\dots,X_{p-1})$ of tangent vectors at x_0 which are mapped to $(Y_0,\dots,Y_{p-1})$ under the differential D_{g_∞} of the left translation by L_{g_∞}. We put $\boxed{\text{Armand}}$

$$\tilde\omega(Y_0,\dots,Y_{p-1})(x)=g_\infty^{-1}(\omega(X_0,\dots,X_{p-1})(g_\infty,\underline{g}_f)). \tag{6.4}$$

At this point I leave it as an exercise to the reader that this gives the isomorphism we want.([16]).) Actually we do not really need that Γ torsion free, If this is not the case then $\tilde{\mathcal{M}}_\lambda$ is only an orbilocal system and we can have to take suitable invariants under the finite stabilizers, we simply have to modify the definition of $\Omega^\bullet(\Gamma\backslash X,\tilde{\mathcal{M}}_\lambda)$ accordingly.

We recall that the de-Rham complex ([39],sect. 4.8. computes the cohomology and therefore we can rewrite the de-Rham isomorphism $\boxed{\text{BodeRh}}$

$$H^\bullet(\Gamma\backslash X,\tilde{\mathcal{M}}_\lambda)\xrightarrow{\sim}H^\bullet(\ \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),\mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R})\otimes\mathcal{M}_\lambda) \tag{6.5}$$

From now on the complex $\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),\mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R})\otimes\mathcal{M}_\lambda)$ will also be called the de-Rham complex.

By the same token we can compute the cohomology with compact supports $\boxed{\text{BodeRhcs}}$

$$H^\bullet_c(\Gamma\backslash X,\tilde{\mathcal{M}}_\lambda)\xrightarrow{\sim}H^\bullet(\ \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),\mathcal{C}_{c,\infty}(\Gamma\backslash G(\mathbb{R})\otimes\mathcal{M}_\lambda) \tag{6.6}$$

where $\mathcal{C}_{c,\infty}(\Gamma\backslash G(\mathbb{R})$ are the $\mathcal{C}_\infty$ function with compact support. These isomorphisms are also valid if we drop the assumption that Γ is torsion free..

The Poincaré duality on the cohomology is induced by the pairing on the de-Rham complexes: $\boxed{\text{PD}}$

Proposition 6.1.2. *If* $\omega_1 \in \ Hom_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R}) \otimes \tilde{\mathcal{M}})$ *is a closed form and* $\omega_2 \in \ Hom_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_{\infty,c}(\Gamma\backslash G(\mathbb{R}) \otimes \tilde{\mathcal{M}}^\vee)$ *a closed form with compact support in complementary degree then the value of the cup product pairing of the classes* $[\omega_1] \in H^p(\Gamma\backslash X, \tilde{\mathcal{M}_\lambda}), [\omega_2] \in H_c^{d-p}(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda^\vee)$ *is given by*

$$< [\omega_1] \cup [\omega_2] >= \int_{\Gamma\backslash X} < \omega_1 \wedge \omega_2 >$$

(Reference Book Vol. !)

6.1.4 Input from representation theory of real reductive groups.

Let us consider an arbitrary irreducible $(\mathfrak{g}, K_\infty)$- module V. Let $\hat{K}_\infty$ be the set of irreducible continuous representations they are finite dimensional. We also assume that for any $\vartheta \in \hat{\{K\}}_\infty$ the multiplicity of ϑ in V is finite (we say that V is admissible). Then we can extend the action of the Lie-algebra $\mathfrak{g}$ to an action of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ on V and we can restrict this action to an action of the centre $\mathfrak{Z}(\mathfrak{g})$. The structure of this centre is well known by a theorem of Harish-Chandra, it is a polynomial algebra in $r =$ rank(G) variables, here the rank is the absolute rank, i.e. the dimension of a maximal torus in $G/\mathbb{Q}$. (See Chap. 4 sect. 4)

Clearly this centre respects the decomposition into K_∞ types, since these K_∞ types come with finite multiplicity we can apply the standard argument, which proves the Lemma of Schur. Hence $\mathfrak{Z}(\mathfrak{g})$ has to act on V by scalars, we get a homomorphism $\chi_V : \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$, which is defined by

$$zv = \chi_V(z)v.$$

This homomorphism is called the *central character* of V.

A fundamental theorem of Harish-Chandra asserts that for a given central character there exist only finitely many isomorphism classes of irreducible, admissible $(\mathfrak{g}, K_\infty)$-modules with this central character.

Of course for any rational finite dimensional representation $r : G/\mathbb{Q} \to$ Gl$(\mathcal{M}_\lambda)$ we can consider $\mathcal{M}_\lambda \otimes \mathbb{C}$ as $(\mathfrak{g}, K_\infty)$-module. If $\mathcal{M}_\lambda$ is absolutely irreducible with highest weight λ (See chap. IV) then it also has a central character $\chi_\mathcal{M} = \chi_\lambda$.

Wigner's lemma: *Let* V *be an irreducible, admissible* $(\mathfrak{g}, K_\infty)$*-module, let* $\mathcal{M} = \mathcal{M}_\lambda$*, a finite dimensional, absolutely irreducible rational representation. Then* $H^\bullet(\mathfrak{g}, K_\infty, V \otimes \mathcal{M}_\mathbb{C}) = 0$ *unless we have*

$$\chi_V(z) = \chi_{\mathcal{M}^\vee}(z) = \chi_{\lambda^\vee}(z) \text{ for all } z \in \mathfrak{Z}(\mathfrak{g})$$

Since we also know that the number of isomorphism classes of irreducible, admissible $(\mathfrak{g}, K_\infty)$-modules with a given central character is finite, we can conclude that for a given absolutely irreducible rational module $\mathcal{M}_\lambda$ the number of isomorphism classes of irreducible, admissible $(\mathfrak{g}, K_\infty)$-modules V with $H^\bullet(\mathfrak{g}, K_\infty, V \otimes \mathcal{M}_\mathbb{C}) \neq 0$ is finite.

The proof of Wigner's lemma is very elegant. We have $\mathcal{M} \otimes V = \mathcal{M}^\vee \otimes V$ and hence we have $H^0(\mathfrak{g}, K_\infty, \mathcal{M} \otimes V) = \mathrm{Hom}(\mathcal{M}^\vee, V)^{(\mathfrak{g}, K_\infty)} = \mathrm{Hom}_{(\mathfrak{g}, K_\infty)}(\mathcal{M}^\vee, V)$. In [16] , Chap.I 2.4 it is shown, that the category of $\mathfrak{g}, K_\infty$ -modules has enough injective and projective elements (See [16], I. 2.5) . If I is an injective $(\mathfrak{g}, K_\infty)$-module then $\mathcal{M} \otimes I$ is also injective because for any $\mathfrak{g}, K_\infty$-module A we have $\mathrm{Hom}(A, \mathcal{M} \otimes I) = \mathrm{Hom}(\mathcal{M}^\vee, I)$. Hence an injective resolution $0 \to V \to I^0 \to I^1 \ldots$ yields an injective resolution $0 \to \mathcal{M} \to \mathcal{M} \otimes I^0 \to \mathcal{M} \otimes I^1 \ldots$ and from this we get

$$H^q(\mathfrak{g}, K_\infty, \mathcal{M} \otimes V) = \mathrm{Ext}^q_{(\mathfrak{g}, K_\infty)}(\mathcal{M}^\vee, V).$$

Any $z \in \mathfrak{Z}(\mathfrak{g})$ induces an endomorphism of $\mathcal{M}_\lambda$ and V. Since $\mathrm{Ext}^\bullet$ is functorial in both variables, we see that z induces endomorphisms z_1 (via the action on $\mathcal{M}_\lambda$) and z_2 (via the action on V) on $\mathrm{Ext}^q_{(\mathfrak{g}, K_\infty)}(\mathcal{M}^\vee, V)$. We show that $z_1 = z_2$. This is clear by definition for $\mathrm{Ext}^0_{(\mathfrak{g}, K_\infty)}(\mathcal{M}^\vee, V) = \mathrm{Hom}_{(\mathfrak{g}, K_\infty)}(\mathcal{M}^\vee, V)$: For $z \in \mathfrak{Z}(\mathfrak{g})$ and $\phi \in \mathrm{Hom}_{(\mathfrak{g}, K_\infty)}(\mathcal{M}^\vee, V), m \in \mathcal{M}_\lambda$ we have $z_1\phi(m) = \phi(zm) = z_2(\phi(m))$. To prove it for an arbitrary q we use devissage and induction. We embed V into an injective $(\mathfrak{g}, \mathbb{K})$ module I and get an exact sequence

$$0 \to V \to I \to I/V \to 0$$

and from this we get

$$\mathrm{Ext}^{q-1}_{(\mathfrak{g}, K_\infty)}(\mathcal{M}_\lambda, I/V) = \mathrm{Ext}^q_{(\mathfrak{g}, K_\infty)}(\mathcal{M}_\lambda, V) \text{ for } q > 0.$$

Now by induction we know $z_1 = z_2$ on the left hand side, so it also holds on the right hand side.

If now $\chi_V \neq \chi_{\mathcal{M}^\vee}$ then we can find a $z \in \mathfrak{Z}(\mathfrak{g})$ such that $\chi_{\mathcal{M}^\vee}(z) = 0, \chi_V(z) = 1$. This implies that $z_1 = 0$ and $z_2 = 1$ on all $\mathrm{Ext}^q_{(\mathfrak{g}, K_\infty)}((\mathfrak{g}, K_\infty)(\mathcal{M}_\lambda, V)$. Since we know that $z_1 = z_2$ we see that the identity on $\mathrm{Ext}^q_{(\mathfrak{g}, K_\infty)}(\mathcal{M}_\lambda, V)$ is equal to zero and this implies the assertion.

On the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ we have an antiautomorphism $u \mapsto^t u$ which is induced by the antiautomorphism $X \mapsto -X$ on the Lie algebra $\mathfrak{g}$. If V is an admissible $(\mathfrak{g}, K_\infty)$-module, then we can form the dual module $V^\vee$ and if we denote the pairing between $V, V^\vee$ by $< , >_V$ then

$$< Uv, \phi >_V = < v, ^t U\phi >_V \quad \text{for all } U \in \mathfrak{U}(\mathfrak{g}), v \in V, \phi \in V^\vee.$$

If V is irreducible, then it has a central character and we get

$$\chi_{V^\vee}(z) = \chi_V(^t z).$$

This applies to finite dimensional and to infinite dimensional $(\mathfrak{g}, K_\infty)$-modules.

6.1.5 Representation theoretic Hodge-theory.

We consider irreducible unitary representations $G(\mathbb{R}) \to U(H)$. We know from the work of Harish-Chandra:

1) If we fix an isomorphism class ϑ irreducible representations of K_∞ then the isotypical subspace $\dim_{\mathbb{C}} H(\vartheta) \leq \dim(\vartheta)^2$, i.e. ϑ occurs at most with multiplicity $\dim(\vartheta)$.

2) The direct sum $\sum_{\vartheta \in \hat{K}_\infty} H(\vartheta) = H^{(K_\infty)} \subset H$ is dense in H and it is an admissible irreducible Harish-Chandra -module.

We call an irreducible $(\mathfrak{g}, K_\infty)$-module unitary, if it is isomorphic to such an $H^{(K_\infty)}$.

For a given $G/\mathbb{R}$ and any rational irreducible module $\mathcal{M}_\lambda$ Vogan and Zuckerman give a finite list of certain irreducible, admissible $(\mathfrak{g}, K_\infty)-$ modules $A_{\mathfrak{q}}(\lambda)$, for which $H^\bullet(\mathfrak{g}, K_\infty, A_{\mathfrak{q}}(\lambda) \otimes \mathcal{M}_\lambda) \neq 0$ they compute these cohomology group. This list contains all unitary, irreducible $(\mathfrak{g}, K_\infty)-$modules, which have non trivial cohomology with coefficients in $\mathcal{M}_\lambda$.

For the following we refer to [16] Chap. II , §1-2 . We want to apply the methods of Hodge-theory to compute the cohomology groups $H^\bullet(\mathfrak{g}, K_\infty, V \otimes \mathcal{M}_\lambda)$ for an unitary $(\mathfrak{g}, K_\infty)$-module V. This means have a positive definite scalar product $< \, , \, >_V$ on V, for which the action of K_∞ is unitary and for $U \in \mathfrak{g}$ and $v_1, v_2 \in V$ we have $< Uv_1, v_2 >_V + < v_1, Uv_2 >_V = 0$.

We assume that $\mathcal{M}_\lambda$ is conjugate-autodual. In the next step we introduce for all p a hermitian form on $\mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda)$. To do this we construct a hermitian form on $\mathcal{M}_\lambda$.

(The following considerations are only true modulo the centre). We consider the Lie algebra and its complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$. On this complex vector space we have the complex conjugation $^- : U \mapsto \bar{U}$. We rediscover $\mathfrak{g}$ as the set of fixed points under $^-$. We also have the Cartan involution Θ which is the involution which has $\mathfrak{k}$ as its fixed point set. Then we get the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \text{ where } \mathfrak{p} \text{ is the -1 eigenspace of } \Theta.$$

The Killing form is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, we have for the Lie bracket $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. We consider the invariants under $^- \circ \Theta$, this is the Lie algebra $\mathfrak{g}_c = \mathfrak{k} \oplus \sqrt{-1} \otimes \mathfrak{p}$. On this real Lie algebra the Killing form is negative definite and $\mathfrak{g}_c$ is the Lie algebra of an algebraic group $G_c/\mathbb{R}$ whose base extension $G_c \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} G \otimes_{\mathbb{R}} \mathbb{C}$ and whose group $G_c(\mathbb{R})$ of real points is compact (this is the so called compact form of G). We still have the representation $G_c/\mathbb{R} \to \mathrm{Gl}(\mathcal{M}_\lambda)$ which is irreducible and hence we find a hermitian form $< \, , \, >_\lambda$ on $\mathcal{M}_\lambda$, which is invariant under $G_c(\mathbb{R})$ and which is unique up to a scalar.

This form satisfies the equations

$$< Um_1, m_2 >_{\mathcal{M}} + < m_1, Um_2 >_\lambda = 0 \text{ for all } m_1, m_2 \in \mathcal{M}_\lambda, U \in \mathfrak{k}$$

this is the invariance under K_∞ and

$$< Um_1, m_2 >_{\mathcal{M}} = < m_1, Um_2 >_\lambda \text{ for all } m_1, m_2 \in \mathcal{M}_\lambda, U \in \mathfrak{p}$$

this is the invariance under $\sqrt{-1} \otimes \mathfrak{p}$.

Now we define a hermitian metric on $V \otimes \mathcal{M}_\lambda$, we simply take the tensor product $< \ , \ >_V \otimes < \ , \ >_\lambda = < \ , \ >_{V\otimes\lambda}$. Finally we define the (hermitian) scalar product on $\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), V\otimes\mathcal{M}_\lambda)$. We choose and orthonormal (with respect to the Killing form) basis $E_1, E_2, \dots, E_d$ on $\mathfrak{p}$, we identify $\mathfrak{g}/\mathfrak{k} \xrightarrow{\sim} \mathfrak{p}$. Then a form $\omega \in \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), V\otimes\mathcal{M}_\lambda)$ is given by its values $\omega(E_I) \in V \otimes \mathcal{M}_\lambda$, where $I = \{i_1, i_2, \dots, i_p\}$ runs through the ordered subsets of $\{1, 2, \dots, d\}$ with p elements. For $\omega_1, \omega_2 \in \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda)$ we put

$$< \omega_1, \omega_2 > = \sum_{I, |I|=p} < \omega_1(E_I), \omega_2(E_I) >_{V\otimes\lambda} \tag{6.7}$$

Now we can define an adjoint operator

$$\delta : \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda) \to \mathrm{Hom}_{K_\infty}(\Lambda^{p-1}(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda), \tag{6.8}$$

which can be defined by a straightforward calculation. We simply write a formula for δ: For an element E_i we define $E_i^*(v \otimes m) = -E_i v \otimes m + v \otimes E_i m$. Then we can define δ by the following formula:

We have to evaluate $\delta(\omega)$ on $E_J = (E_{i_1}, \dots, E_{i_{p-1}})$ where $J = \{i_1, \dots, i_{p-1}\}$. We put

$$\delta(\omega)(E_J) = \sum_{i \notin J} (-1)^{p(i, J\cup\{i\})} E_i^* \omega_{J\cup\{i\}},$$

where $p(i, J\cup\{i\})$ denotes the position of i in the ordered set $J\cup\{i\}$. With this definition we get for a pair of forms $\omega_1 \in \mathrm{Hom}_{K_\infty}(\Lambda^{p-1}(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda)$ and $\omega_2 \in \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda)$ (See [16], II, prop. 2.3)

$$< d\omega_1, \omega_2 > = < \omega_1, \delta\omega_2 > \tag{6.9}$$

We define the Laplacian $\Delta = \delta d + d\delta$. Then we have ([16] , II ,Thm.2.5)

$$< \Delta\omega, \omega > \geq 0 \text{ and we have equality if and only if } d\omega = 0, \delta\omega = 0 \tag{6.10}$$

Inside $\mathfrak{Z}(\mathfrak{g})$ we have the Casimir operator C (See Chap. 4). An element $z \in \mathfrak{Z}(\mathfrak{g})$ acts on $V \otimes \mathcal{M}_\lambda$ by $z \otimes \mathrm{Id}$ via the action on the first factor and by the scalar $\chi_\lambda(z)$ via the action on the second factor. Then we have

Kuga's lemma : *The action of the Casimir operator and the Laplace operator on* $Hom_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda)$ *are related by the identity*

$$\Delta = C \otimes Id - \chi_\lambda(C).$$

If the $(\mathfrak{g}, K_\infty)$ *module is irreducible, then* Δ *acts by multiplication by the scalar* $\chi_V(C) - \chi_\lambda(C)$

This has the following consequence

If V *is an irreducible unitary* $\mathfrak{g}, K_\infty$*- module and if* $\mathcal{M}_\lambda$ *is an irreducible representation with highest weight* λ *then*

$$H^\bullet(\mathfrak{g}, K_\infty, V\otimes\mathcal{M}_\mathbb{C}) = \begin{cases} 0 & \text{if } \chi_V(C) - \chi_\lambda(C) \neq 0 \\ Hom_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda) & \text{if } \chi_V(C) - \chi_\lambda(C) = 0 \end{cases}.$$

This only applies for unitary $\mathfrak{g}, K_\infty$-modules, but for these it is much stronger: It says that under the assumption $\chi_V(C) = \chi_\lambda(C)$ we have $\chi_V = \chi_\lambda$ (we only have to test the Casimir operator) and it says that all the differentials in the complex are zero.

6.1.6 Input from the theory of automorphic forms

We apply this to the s90okf square integrable functions on $G(\mathbb{Q})\backslash G(\mathbb{A})/K_f$. Because of the presence of a non trivial center, we have to consider functions which transform in a certain way under the action of the center. We may assume that coefficient system $\mathcal{M}_\lambda$ has a central character and this central character defines a character ζ_λ on the maximal $\mathbb{Q}$-split torus $S \subset C$. This character can be evaluated on $S^0(\mathbb{R})$ this is the connected component of the identity of the real valued points of S. The map $z_\infty \mapsto (z_\infty, 1, \dots, 1, \dots) \in S(\mathbb{A})$ is an embedding of $S^0(\mathbb{R})$ into $G(\mathbb{A})$. It follows from [14] that the quotient $G(\mathbb{Q})S^0(\mathbb{R})\backslash G(\mathbb{A})/K_f$ has finite volume. We define the space of functions

$$\mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R})\zeta_\infty^{-1}) \tag{6.11}$$

to be the subspace of those $\mathcal{C}_\infty$ functions which satisfy $f(z_\infty \underline{g}) = \zeta_\infty^{-1}(z_\infty) f(\underline{g})$ for all $z_\infty \in S^0(\mathbb{R})$. The isogeny $d_C : C \to C'$ (see **??**) induces an isomorphism $S^0(\mathbb{R}) \xrightarrow{\sim} S'^{,0}(\mathbb{R})$, where S' is the maximal $\mathbb{Q}$ split torus in C'. Therefore we get a character $\zeta'_\infty : S'^{,0}(\mathbb{R}) \to \mathbb{R}^\times_{>0}$ and this is also a character $\zeta'_\infty : G(\mathbb{R}) \to \mathbb{R}^\times_{>0}$. Its restriction to $S^0(\mathbb{R})$ is ζ_∞. If now $f \in \mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R})\zeta_\infty^{-1})$ then

$$f(\underline{g})\zeta'_\infty(\underline{g}) \in \mathcal{C}_\infty(G(\mathbb{Q})S^0(\mathbb{R})\backslash G(\mathbb{A})/K_f) \tag{6.12}$$

We say that $f \in \mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R})\zeta_\infty^{-1})$ is square integrable if $\boxed{\text{sqint}}$

$$\int_{(G(\mathbb{Q})S^0(\mathbb{R})\backslash G(\mathbb{A})/K_f)} |f(\underline{g})\zeta'_\infty(\underline{g})|^2 d\underline{g} < \infty \tag{6.13}$$

and this allows us to define the Hilbert space $L^2(\Gamma\backslash G(\mathbb{R})\zeta_\infty^{-1})$. Since the space $(G(\mathbb{Q})S^0(\mathbb{R})\backslash G(\mathbb{A})/K_f)$ has finite volume we know that

$$\zeta'_\infty \in L^2(\Gamma\backslash G(\mathbb{R})\zeta_\infty^{-1}).$$

The group $G(\mathbb{R})$ acts on $\mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R})\zeta_\infty^{-1})$ by right translations and hence we get by differentiating an action of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ on it. We define by $\mathcal{C}^{(2)}_\infty(\Gamma\backslash G(\mathbb{R})\zeta_\infty^{-1})$ the subspace of functions f for which Uf is square integrable for all $U \in \mathfrak{U}(\mathfrak{g})$.

This allows us to define a sub complex of the de-Rham complex $\boxed{\text{Ltwo}}$

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{C}^{(2)}_\infty(\Gamma\backslash G(\mathbb{R}), \zeta_\infty^{-1}) \otimes \mathcal{M}_\lambda). \tag{6.14}$$

We will not work with this complex because its cohomology may show some bad behavior. (See remark below).

We do something less sophisticated, we simply define $H^\bullet_{(2)}(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda) \subset H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda)$ to be the image of the cohomology of the complex (6.14) in the cohomology. Hence $H^\bullet_{(2)}(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda)$ is the space of cohomology classes which can be represented by square integrable forms.

Remark: Some authors also define L^2- de-Rham complexes, using the above complex (6.14) and then they take suitable completions to get complexes of

Hilbert spaces. These complexes also give cohomology groups which run under the name of L^2-cohomology. These L^2-cohomology groups are related but not necessarily equal to our $H^\bullet_{(2)}(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda)$. They can be infinite dimensional.

The Hilbert space $L^2(\Gamma\backslash G(\mathbb{R}), \zeta_\infty^{-1})$ is a module for $G(\mathbb{R}) \times \mathcal{H}_{K_f}$ the group $G(\mathbb{R})$ acts by unitary transformations and the algebra $\mathcal{H}_{K_f}$ is selfadjoint.

Let us assume that $H = H_{\pi_\infty\times\pi_f}$ is an irreducible unitary module for $G(\mathbb{R})\times \mathcal{H} = \bigotimes_p' \mathcal{H}_p$ and assume that we have an inclusion of this $G(\mathbb{R}) \times \mathcal{H}$-module

$$j : H \hookrightarrow L^2(\Gamma\backslash G(\mathbb{R}), \zeta_\infty^{-1}).$$

It follows from the finiteness results in 6.1.5 that induces an inclusion into the space of square integrable $\mathcal{C}_\infty$ functions

$$H^{(K_\infty)} \hookrightarrow \mathcal{C}_\infty^{(2)}(\Gamma\backslash G(\mathbb{R}), \zeta_\infty^{-1})^{(K_\infty)}.$$

We consider the $(\mathfrak{g}, K_\infty)-$ cohomology of this module with coefficients in our irreducible module $\mathcal{M}_\lambda$, we assume $\chi_V(C) = \chi_\lambda(C)$. We have $H^\bullet(\mathfrak{g}, K_\infty, H \otimes \mathcal{M}_\lambda) = \mathrm{Hom}_{K_\infty}(\mathfrak{g}, K_\infty, H^{(K_\infty)} \otimes \mathcal{M}_\lambda)$ and get

$$H^\bullet(\mathfrak{g}, K_\infty, H^{(K_\infty)} \otimes \mathcal{M}_{\mathbb{C}}) \xrightarrow{j^\bullet} H^\bullet(\mathfrak{g}, K_\infty, \mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R}), \zeta_\infty^{-1})^{(K_\infty)} \otimes \mathcal{M}_\lambda).$$

This suggests that we try to "decompose" $\mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R}), \zeta_\infty^{-1})^{(K_\infty)}$ into irreducibles and then investigate the contributions of the irreducible summands to the cohomology. Essentially we follow the strategy of [Bo-Ga] and [12] but instead of working with complexes of Hilbert spaces we work with complexes of $\mathcal{C}_\infty$ forms and modify the arguments accordingly.

It has been shown by Langlands, that we have a decomposition into a discrete and a continous spectrum

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) = L^2_{\mathrm{disc}}(\Gamma\backslash G(\mathbb{R}) \oplus L^2_{\mathrm{cont}}(\Gamma\backslash G(\mathbb{R}),$$

where $L^2_{\mathrm{disc}}(\Gamma\backslash G(\mathbb{R})$ is the closure of the sum of all irreducible closed subspaces occuring in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)$ and where $L^2_{\mathrm{cont}}(\Gamma\backslash G(\mathbb{R})$ is the complement.

The discrete spectrum $L^2_{\mathrm{disc}}(\Gamma\backslash G(\mathbb{R})$ contains as a subspace the *cuspidal spectrum* $L^2_{\mathrm{cusp}}(\Gamma\backslash G(\mathbb{R}))$:

A function $f \in L^2(\Gamma\backslash G(\mathbb{R}))$ is called a *cusp form* if for all proper parabolic subgroups $P/\mathbb{Q} \subset G/\mathbb{Q}$, with unipotent radical $U_P/\mathbb{Q}$ the integral

$$\mathcal{F}^P(f)(g) = \int_{U_P(\mathbb{Q})\backslash U_P(\mathbb{A})} f(\underline{u}\underline{g})d\underline{u} = 0,$$

this means that the integral is defined for almost all $\underline{g}$ and zero for almost all $\underline{g}$. The function $\mathcal{F}^P(f)(\underline{g})$, which is an almost everywhere defined function on $P(\mathbb{Q})\backslash G(\mathbb{A})/K_f$ is called the constant Fourier coefficient of f along $P/\mathbb{Q}$. The cuspidal spectrum the intersection of all the kernels of the $\mathcal{F}^P$.

If our group is anisotropic, then it does not have any proper parabolic subgroup and in this case we have $L^2_{\mathrm{cusp}}(\Gamma\backslash G(\mathbb{R}) = L^2_{\mathrm{disc}}(\Gamma\backslash G(\mathbb{R}) = L^2(\Gamma\backslash G(\mathbb{R})$.

For any unitary $G(\mathbb{R}) \times \mathcal{H}$- module $H_\pi = H_{\pi_\infty} \otimes H_{\pi_f}$ we put

$$W_{\text{cusp}}(\pi) := \operatorname{Hom}_{G(\mathbb{R})\times\mathcal{H}}(H_\pi, L^2_{\text{cusp}}(\Gamma\backslash G(\mathbb{R})). \tag{6.15}$$

We can ignore the $\mathcal{H}$-module structure and define

$$W_{\text{cusp}}(\pi_\infty) = \operatorname{Hom}_{G(\mathbb{R})}(H_{\pi_\infty}, L^2_{\text{cusp}}(\Gamma\backslash G(\mathbb{R})).$$

Then the dimension of $W_{\text{cusp}}(\pi_\infty)$ is the multiplicity $m_{\text{cusp}}(\pi_\infty)$. It has been shown by Gelfand-Graev and Langlands that

$$m_{\text{cusp}}(\pi_\infty) = \sum_{\pi_f} \dim(W_{\pi,\text{cusp}}) < \infty.$$

We get a decomposition into isotypical subspaces

$$L^2_{\text{cusp}}(\Gamma\backslash G(\mathbb{R}) = \overline{\bigoplus_{\pi_\infty\otimes\pi_f} (L^2_{\text{cusp}}(\Gamma\backslash G(\mathbb{R})(\pi_\infty \times \pi_f)},$$

where $(L^2_{\text{cusp}}(\Gamma\backslash G(\mathbb{R})(\pi_\infty \times \pi_f)$ is the image of $W_{\pi,\text{cusp}} \otimes H_\pi$ in $L^2_{\text{cusp}}(\Gamma\backslash G(\mathbb{R})$.

The cuspidal spectrum has a complement in the discrete spectrum, this is the *residual spectrum* $L^2_{\text{res}}((\Gamma\backslash G(\mathbb{R})$. It is called residual spectrum, because the irreducible subspaces contained in it are obtained by residues of Eisenstein classes.

Again we define $W_{\text{res}}(\pi) = \operatorname{Hom}_{G(\mathbb{R})\times\mathcal{H}}(H_\pi, L^2_{\text{res}}(\Gamma\backslash G(\mathbb{R}))$, (resp. $W_{\text{res}}(\pi_\infty) = \operatorname{Hom}_{G(\mathbb{R})}(H_{\pi_\infty}, L^2_{\text{cusp}}(\Gamma\backslash G(\mathbb{R}))$, and it is a deep theorem of Langlands that $m_{\text{res}}(\pi_\infty) = \dim(W_{\text{res}}(\pi_\infty) < \infty$. Hence we get a decomposition

$$L^2_{\text{res}}(\Gamma\backslash G(\mathbb{R}) = \overline{\bigoplus_{\pi_\infty\otimes\pi_f} (L^2_{\text{res}}(\Gamma\backslash G(\mathbb{R})(\pi_\infty \times \pi_f)}.$$

If our group $G/\mathbb{Q}$ is isotropic, then the one dimensional space of constants is in the residual (discrete) spectrum but not in the cuspidal spectrum.

Langlands has given a description of the continnuos spectrum using the theory of Eisenstein series, we have a decomposition $\boxed{\text{decomp-cont}}$

$$L^2_{\text{cont}}(\Gamma\backslash G(\mathbb{R}) = \overline{\bigoplus_{\Sigma} \tilde{H}^+_P(\pi_\Sigma)}, \tag{6.16}$$

we briefly explain this decomposition following [Bo-Ga]. The Σ are so called cuspidal data, this are pairs (P, π_Σ) where P is a proper parabolic subgroup and π_Σ is a representation of $M(\mathbb{A}) = P(\mathbb{A})/U(\mathbb{A})$ occurring in the discrete spectrum $L^2_{\text{cusp}}(M(\mathbb{Q})\backslash M(\mathbb{A}))$.

Let $M^{(1)}/\mathbb{Q}$ be the semi simple part of M and recall that C/Q was the center of $G/\mathbb{Q}$. We consider the character module $Y^*(P) = \operatorname{Hom}(C \cdot M^{(1)}, \mathbb{G}_m)$. The elements $Y^*(P)\otimes\mathbb{C}$ provide homomorphisms $\gamma\otimes z : M(\mathbb{A})/C(\mathbb{A})M^{(1)}(\mathbb{A}) \to \mathbb{C}^\times$. (See (4.20)). The module $Y^*(P)\otimes\mathbb{Q}$ comes with a canonical basis which is given by the dominant fundamental weights γ_μ which are trivial on $M^{(1)}$. We define

$$\Lambda_\Sigma = Y^*(P) \otimes i\mathbb{R} = \{\sum_\mu \gamma_\mu \otimes it_\mu | t_\mu \in \mathbb{R}\}$$

this is a group of unitary characters. For $\sigma \in \Lambda_\Sigma$ we define the unitarily induced representation

$$\begin{gathered}\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\pi_\Sigma \otimes (\sigma + \rho_P) = I_P^G \pi_\Sigma \otimes \sigma \\ \{f : G(\mathbb{A}) \to L^2_{\mathrm{res}}(M(\mathbb{Q})\backslash M(\mathbb{A}))(\pi_\Sigma) | f(\underline{p}\underline{g}) = (\sigma + |\rho_P|)(\underline{p})\pi_\Sigma(\underline{p}) f(\underline{g})\}\end{gathered} \tag{6.17}$$

where of course $\underline{p} \in P(\mathbb{A}), \underline{g} \in G(\mathbb{A})$ and $\rho_P \in Y^*(P) \otimes \mathbb{Q}$ is the half sum of the roots in the unipotent radical of P. This gives us a unitary representation of $G(\mathbb{A})$. Let d_Σ be the Lebesgue measure on Λ_Σ then we can form the direct integral unitary representations

$$H_P(\pi_\Sigma) = \int_{\Lambda_\Sigma} I_P^G \pi_\Sigma \otimes \sigma \, d_\Sigma \sigma \tag{6.18}$$

The theory of Eisenstein series gives us a homomorphism of $G(\mathbb{R}) \times \mathcal{H}$ -modules

$$\mathrm{Eis}_P(\pi_\Sigma) : H_P(\pi_\Sigma) \to L^2_{\mathrm{cont}}(\Gamma\backslash G(\mathbb{R}). \tag{6.19}$$

Let us put

$$\Lambda_\Sigma^+ = \{\sum_\mu \gamma_\mu \otimes i t_\mu | t_\mu \geq 0\}$$

then the restriction

$$\mathrm{Eis}_P(\pi_\Sigma) : H_P^+(\pi_\Sigma) = \int_{\Lambda_\Sigma^+} I_P^G \pi_\Sigma \otimes \sigma \, d_\Sigma \sigma \to L^2_{\mathrm{cont}}(\Gamma\backslash G(\mathbb{R}). \tag{6.20}$$

is an isometric embedding. The image will be denoted by $\tilde{H}_P^+(\pi_\Sigma)$ these spaces are the elementary subspaces in [B-G]. Two such elementary subspaces $\tilde{H}_P^+(\pi_\Sigma), \tilde{H}_{P_1}^+(\pi_{\Sigma_1})$ are either orthogonal to each other or they are equal. We get the above decomposition if we sum over a suitable set of representatives of cuspidal data.

Now we are ready to discuss the contribution of the continuous spectrum to the cohomology. If we have a closed square integrable form

$$\omega \in \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_\infty^{(2)}(\Gamma\backslash G(\mathbb{R}) \otimes \mathcal{M}_\lambda),$$

then we can decompose it

$$\omega = \omega_{\mathrm{res}} + \omega_{\mathrm{cont}},$$

both summands are $\mathcal{C}^2_\infty$ and closed.

Proposition 6.1.3. *The cohomology class $[\omega_{\mathrm{cont}}]$ is trivial.*

Proof. This now the standard argument in Hodge theory, but this time we apply it to a continuous spectrum instead of a discrete one. We follow Borel-Casselman and prove their Lemma 5.5 (See[B-C]) in our context. We may assume that ω_∞ lies in one of the summands, i.e. $\omega_{\mathrm{cont}} = \mathrm{Eis}(\int_{\Lambda_\Sigma} \omega^\vee(\sigma) d_\Sigma \sigma)$ where $\omega^\vee(\sigma) \in \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), I_P^G \pi_\Sigma \otimes \sigma \otimes \mathcal{M}_\lambda))$ is the Fourier transform of ω_∞ in the L^2., (theorem of Plancherel). As it stands the expression $\int_{\Lambda_\Sigma} \omega^\vee(\sigma) d_\Sigma \sigma)$ does not make sense because the integrand is in L^2 and not necessarily in L^1. If we choose a symmetric positive definite quadratic form $h(\sigma) = \sum_{\nu,\mu} b_{\nu,\mu} t_\nu t_\mu$ and a positive real number τ then the function

$$h_\tau(\sigma) = (1 + \tau h(\sigma)^m)^{-1} \in L^2(\Lambda_\Sigma)$$

and then $\omega^\vee(\sigma) h_\tau(\sigma)$ is in L^1 and by definition

$$\lim_{\tau \to 0} \int_{\Lambda_\Sigma} \omega^\vee(\sigma) h_\tau(\sigma) d_\Sigma \sigma) = \int_{\Lambda_\Sigma} \omega^\vee(\sigma) d_\Sigma \sigma \tag{6.21}$$

where the convergence is in the L^2 sense. Since $\omega_\infty \in \operatorname{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), I_P^G \pi_\Sigma \otimes \sigma \otimes \mathcal{M}_\lambda)$ we get get that $\omega^\vee(\sigma)$ has the following property

For any polynomial $P(\sigma) = \sum a_{\underline{\mu}} t^{\underline{\mu}}$ in the variables t_μ and with real coefficients the section $\boxed{\text{diffmult}}$

$$\omega^\vee(\sigma) P(\sigma) \text{ is square integrable} \tag{6.22}$$

this follows from the well known rules that differentiating a function provides multiplication by the variables for the Fourier transform.

The Lemma of Kuga implies

$$\Delta(\omega^\vee(\sigma)) = (\chi_\sigma(C) - \chi_\lambda(C)) \omega^\vee(\sigma)$$

and if $\sigma = \sum \gamma_\mu \otimes it_-\mu$ the eigenvalue is

$$\chi_\sigma(C) - \chi_\lambda(C) = \sum a_{\nu,\mu} t_\nu t_\mu + \sum b_\mu t_\mu + c_{\pi_\Sigma} - c_\lambda. \tag{6.23}$$

where c_{π_Σ} is the eigenvalue of the Casimir operator of $M^{(1)}$ on π_Σ If the $t_\mu \in \mathbb{R}$ then this expression is always ≤ 0 especially we see that the quadratic form on the right hand side is negative definite. This implies that for $\sigma \in \Lambda_F$ the expression $\chi_\sigma(C) - \chi_\lambda(C)$ assumes a finite number of maximal values all of them ≤ 0 and hence

$$V_\Sigma = \{\sigma | \chi_\sigma(C) - \chi_\lambda(C) = 0\} \tag{6.24}$$

is a finite set of point. This set has measure zero, since we assumed that P was a proper parabolic subgroup. The of σ for which $H^\bullet(\mathfrak{g}, K_\infty, H_{\Lambda_\Sigma}(\sigma) \otimes \mathcal{M}_\mathbb{C}) \neq 0$ is finite. We choose a $\mathcal{C}_\infty$ function $h_\Sigma(\sigma)$ which is positive, which takes value 1 in a small neighbourhood of V_Σ, which takes values ≤ 1 in a slightly larger neighbourhood and which is zero outside this second neighbourhood. Then we write

$$\omega_\infty = \operatorname{Eis}(\int_{\Lambda_\Sigma^+} h_\Sigma(\sigma) \omega^\vee(\sigma) d_\Sigma \sigma) + \operatorname{Eis}(\int_{\Lambda_\Sigma^+} (1 - h_\Sigma(\sigma)) \omega^\vee(\sigma) d_\Sigma \sigma)$$

We have $d\omega^\vee(\sigma) = 0$ and hence we get

$$\Delta((1 - h_\Sigma(\sigma)) \omega^\vee(\sigma) = d\big((\chi_\sigma(C) - \chi_\lambda(C))(1 - h_{\Sigma(\sigma)}) \delta \omega^\vee(\sigma)\big)$$

and this implies that

$$\operatorname{Eis}(\int_{\Lambda_\Sigma^+} (1 - h_\Sigma(\sigma)) \omega^\vee(\sigma) d_\Sigma \sigma) = d \operatorname{Eis}(\int_{\Lambda_\Sigma^+} (1 - h_\Sigma(\sigma))(\chi_\sigma(C) - \chi_\lambda(C))^{-1} \delta \omega^\vee(\sigma) d_\Sigma \sigma)$$

It is clear that the integrand in the second term- $\int_{\Lambda_\Sigma^+}(1-h_\Sigma(\sigma))(\chi_\sigma(C)-\chi_\lambda(C))^{-1}\delta\omega^\vee(\sigma)$ still satisfies (6.22) and then our well known rules above imply that $\psi=$ Eis$(\int_{\Lambda_\Sigma^+}(1-h_\Sigma(\sigma))(\chi_\sigma(C)-\chi_\lambda(C))^{-1}\delta\omega^\vee(\sigma)d_\Sigma\sigma)$ is $\mathcal{C}^2_\infty$. Therefore the second term in our above formula is a boundary.

$$\omega_{\text{cont}} = \int_{\Lambda_\Sigma} h_\Sigma(\sigma)\omega(\sigma)d_\Sigma\sigma + d\psi.$$

This is true for any choice of h_Σ. Hence the scalar product $<\omega-d\psi,\omega-d\psi>$ can be made arbitrarily small. Then we claim that the cohomology class $[\omega]\in H^\bullet(\text{ Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}),\mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R})\otimes\mathcal{M}_\lambda)$ must be zero. This needs a tiny final step.

We invoke Poincaré duality: A cohomology class in $[\omega]\in H^p(\Gamma\backslash X,\tilde{\mathcal{M}}_\lambda)$ is zero if and only the value of the pairing with any class $[\omega_2]\in H_c^{d-p}(\Gamma\backslash X,\tilde{\mathcal{M}}_\lambda^\vee)$ is zero. But the (absolute) value $[\omega]\cup[\omega_2]$ of the cup product can be given by an integral (See Prop.6.1.2). Therefore it can be estimated by the norm $<\omega-d\psi,\omega-d\psi>$ (Cauchy-Schwarz inequality) and hence must be zero. $\square$

As usual we denote by $\widehat{G(\mathbb{R})}$ the unitary spectrum, for us it is simply the set of unitary irreducible representations of $G(\mathbb{R})$. Given $\tilde{\mathcal{M}}_\lambda$, we define

$$\text{Coh}_2(\lambda)=\{\pi_\infty\in\widehat{G(\mathbb{R})}|\ H^\bullet(\mathfrak{g},K_\infty,H_{\pi_\infty}\otimes\tilde{\mathcal{M}}_\lambda)\neq 0\} \tag{6.25}$$

The theorem of Harish-Chandra says that this set is finite.

Let

$$H_{\text{Coh}_2(\lambda)}=\bigoplus_{\pi:\pi_\infty\in\text{Coh}_2(\lambda)} L^2_{\text{disc}}(\Gamma\backslash G(\mathbb{R})(\pi_\infty\times\pi_f)=\bigoplus_{\pi:\pi_\infty\in\text{Coh}_2(\lambda)} H_{\pi_\infty}(\pi_f) \tag{6.26}$$

the theorem of Gelfand-Graev and Langlands assert that this is a finite sum of irreducible modules. This space decomposes again into $H^{\text{cusp}}_{\text{Coh}_2(\lambda)}\oplus H^{\text{res}}_{\text{Coh}_2(\lambda)}$

Then we get the following theorem which is due to Borel, Garland, Matsushima and Murakami $\boxed{\text{Bo-Ga-Mu}}$

Theorem 6.1.1. *a)The map*

$$H^\bullet(\mathfrak{g},K_\infty,H^{(K_\infty)}_{\text{Coh}_2(\lambda)}\otimes\mathcal{M}_\lambda)=\ Hom_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),H^{(K_\infty)}_{\text{Coh}_2(\lambda)}\otimes\mathcal{M}_\lambda)\to H^\bullet_{(2)}(\Gamma\backslash X,\tilde{\mathcal{M}}_\lambda)$$

surjective. Especially the image contains $H^\bullet_!(\Gamma\backslash X,\tilde{\mathcal{M}}_\lambda)$.

b) (Borel) The homomorphism

$$H^\bullet(\mathfrak{g},K_\infty,H^{(\text{cusp},K_\infty)}_{\text{Coh}_2(\lambda)}\otimes\mathcal{M}_\lambda)\to H^\bullet(\Gamma\backslash X,\tilde{\mathcal{M}}_\lambda)$$

is injective.

In [13] Prop.5.6, they do not consider the above space $H^\bullet_{(2)}(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda)$ we added an $\epsilon > 0$ to this proposition by claiming that this space is the image.

In general the homomorphism

$$H^\bullet(\mathfrak{g}, K_\infty, H^{\text{res}}_{\text{res}(\lambda)} \otimes \mathcal{M}_\lambda) \to H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda)$$

is not injective. We come back to this issue in the next section.

If we denote by $H^\bullet_{\text{cusp}}(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda)$ the image of the homomorphism in b), then we get a filtration of the cohomology by four subspaces $\boxed{\text{four}}$

$$H^\bullet_{\text{cusp}}(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda) \subset H^\bullet_!(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda) \subset H^\bullet_{(2)}(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda) \subset H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda). \tag{6.27}$$

We get the representation theoretic Hodge decomposition

$$\bigoplus_{\pi_\infty} W_{\text{cusp}}(\pi_\infty) \otimes H^\bullet_{\text{cusp}}(\mathfrak{g}, K_\infty, H_{\pi_\infty} \otimes \mathcal{M}_\lambda) \xrightarrow{\sim} H^\bullet_{\text{cusp}}(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda) \tag{6.28}$$

If we replace the subscript $_{\text{cusp}}$ by $_!$ the corresponding map is still surjective but may be not injective.

We want to point out that our space $H^\bullet_{(2)}(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda)$ is not the space denoted by the same symbol in the paper [12]. They define L^2 cohomology as the complex of square integrable forms, i.e. ω and $d\omega$ have to be square integrable. But then a closed form ω which is in L^2 gives the trivial class in their cohomology if we can write $\omega = d\psi$ where ψ must also be square integrable. In our definition we do not have that restriction on ψ.

The semi-simplicity of the inner cohomology

Now we assume again that our representation $\tilde{\mathcal{M}}_\lambda$ is defined over some number field F we consider it as a subfield of $\mathbb{C}$. In other word we have a representation $r : G \times F \to \text{Gl}(\mathcal{M}_\lambda)$. We have defined $H^\bullet_!(\Gamma\backslash X, \tilde{\mathcal{M}})$, this is a finite dimensional F-vector space and Theorem 3.1.1 in Chapter 3 asserts that this is a semi simple module under the Hecke algebra. The following argument shows that this is an easy consequence of our results above.

The module $H_1 \subset L^2_{\text{disc}}(\Gamma\backslash G(\mathbb{R})$ can also be decomposed into a finite direct sum of irreducible $G(\mathbb{R}) \times \mathcal{H}_{K_f}$ modules

$$H_1 = \bigoplus_{\pi_\infty \otimes \pi_f \in \hat{H}_1} (H_{\pi_\infty} \otimes H_{\pi_f})^{m_1(\pi_\infty \times \pi_f)},$$

this module is clearly semi-simple. Of course it is not a $(\mathfrak{g}, K_\infty)$-module, but we can restrict to the K_∞-finite vectors and get

$$H^\bullet(\mathfrak{g}, K_\infty, H_1^{(K_\infty)} \otimes \mathcal{M}_\lambda \otimes \mathbb{C}) = \bigoplus_{\pi_\infty \otimes \pi_f \in \hat{H}_1} (\ \text{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), H_{\pi_\infty} \otimes \mathcal{M}_{\mathbb{C}}) \otimes H_{\pi_f})^{m_1(\pi_\infty \times \pi_f)}$$

This is a decomposition of the left hand side into irreducible $\mathcal{H}_{K_f}$ modules. Now we have the surjective map

$$H^\bullet(\mathfrak{g}, K_\infty, H_1^{(K_\infty)} \otimes \mathcal{M}_\lambda \otimes \mathbb{C}) \to H^\bullet_{(2)}(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C})$$

hence it follows that $H^\bullet_{(2)}(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C}))$ is a semi simple $\mathcal{H}_{K_f}$ module and hence also $H^\bullet_!(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda)$ is a semi simple $\mathcal{H}_{K_f}$ module.

Friendship

We touch upon a question which comes up naturally in this context. Assume we have a non zero isotypical submodule $H^\bullet_!(\Gamma\backslash X, \mathcal{M}_\lambda)(\pi_f)$. Then we know that there is a unitary $(\mathfrak{g}, K_\infty)$ module H_{π_∞} with $\pi_\infty \in \mathrm{Coh}(\lambda)$ such that we can embed $H_{\pi_\infty} \times H_{\pi_f}$ into $L^2_{\mathrm{disc}}(\Gamma\backslash G(\mathbb{R})$. The interesting question is:

Given π_f, what are the possible choices for π_∞?.

We can formulate this differently. We recall that

$$W_?(\pi_\infty \otimes \pi_f) = \mathrm{Hom}_{G(\mathbb{R})\times\mathcal{H}_{K_f}}(H_{\pi_\infty} \otimes H_{\pi_f}, L^2_?(\Gamma\backslash G(\mathbb{R})$$

where ? = cusp or = (2) resp. disc then we get the surjective map

$$\bigoplus_{\pi_\infty} W_?(\pi_\infty \times \pi_f) \otimes H_{\pi_\infty} \otimes H_{\pi_f} \to H^\bullet_?(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda)(\pi_f) \qquad (6.29)$$

which is an isomorphism if ? = cusp. The friends of π_f are those π_∞ where $W_?(\pi_\infty \times \pi_f) \neq 0$.

This question may become very delicate and we will not discuss it profoundly. (As J. Arthur puts it : π_f looks around and asks "Who is my friend?") In principle we give a complete answer to these questions in the low dimensional cases discussed in section (4.1.5), i.e $G/\mathbb{R} = \mathrm{Gl}_2/\mathbb{Q}$ and $G/\mathbb{R} = R_{\mathbb{C}(\mathbb{R}}(\mathrm{Gl}_2/\mathbb{C})$.

In section (6.1.5) we mentioned the Vogan-Zuckerman classification of unitary representations with non trivial cohomology. More precisely Vogan and Zuckerman construct a family of $(\mathfrak{g}, K_\infty)$ irreducible modules $A_\mathfrak{q}(\lambda)$ for which they show $H^\bullet(\mathfrak{g}, K_\infty, A_\mathfrak{q}(\lambda) \otimes \mathcal{M}_\lambda) \neq 0$, and they compute $H^\bullet(\mathfrak{g}, K_\infty, A_\mathfrak{q}(\lambda) \otimes \mathcal{M}_\lambda)$ explicitly. Moreover they show that any irreducible unitary module V with $H^\bullet(\mathfrak{g}, K_\infty, V \otimes \mathcal{M}_\lambda) \neq 0$, is isomorphic to an $A_\mathfrak{q}(\lambda)$.

We give some very cursory description of their construction. Let $T_1^c/\mathbb{R}$ be a maximal torus in $K_\infty^{(1)}/\mathbb{R}$. Then it is clear that the centraliser $T/\mathbb{R}$ is a maximal torus in $G/\mathbb{R}$. In section 1.1.2l we introduced the one dimensional torus $\mathbb{S}^1/\mathbb{R}$ and we choose an isomorphism $i_0 : \mathbb{S}^1 \times_\mathbb{R} \mathbb{C} \xrightarrow{\sim} \mathbb{G}_m/\mathbb{C}$. We consider cocharacters $\chi : \mathbb{S}^1/\mathbb{R} \to T_1^c/\mathbb{R}$. Such a cocharacter defines a centraliser $Z_\chi \subset G/\mathbb{R}$ and a parabolic $P_\chi/\mathbb{C} \subset G \times_\mathbb{R} \mathbb{C}$, this parabolic subgroup also depends on i_0. (See section **??**) The Lie-algebra $\mathfrak{q} = \mathrm{Lie}(P_\chi/\mathbb{C})$ is the $\mathfrak{q}$ in $A_\mathfrak{q}(\lambda)$. We will denote these modules also by $A_\chi(\lambda)$, i.e. $A_\chi(\lambda) := A_\mathfrak{q}(\lambda)$ if χ and $\mathfrak{q}$ are related as above.

The second datum is a highest weight $\lambda \in X^*(T \times_\mathbb{R} \mathbb{C})$, it has to satisfy two conditions

a) The weight λ is c-autodual (see (6.1)) i.e. $c(\lambda) = -w_0\lambda)$.

b) The highest weight λ is trivial on the semi simple part $Z_\chi^{(1)}$ or what amounts to the same λ extends to a character $\lambda : P_\chi \to \mathbb{G}_m/\mathbb{C}$.

We have two extreme cases. In the first case the cocharacter χ is trivial, then the centraliser is the entire group $G \times_{\mathbb{R}} \mathbb{C}/\mathbb{C}$ and then the condition b) implies that $\lambda = 0$. This implies that $\mathcal{M}_\lambda$ is one dimensional, $\mathfrak{q} = \mathfrak{g}$ and $A_{\mathfrak{q}}(0)$ is this trivial one dimensional $(\mathfrak{g}, K_\infty)$-module.

But on the other hand for $\lambda = 0$ we do not have any constraint on the χ, i.e. we get a non trivial irreducible module $A_\chi(0)$ for any χ. But it is not known in general which of these modules are unitary.

In the second case χ is regular, this means that $Z_\chi = T$ and $P_\chi = B_\chi$ is a Borel subgroup, we have no constraint on λ. In this case the $A_{\mathfrak{q}}(\lambda) = A_{\mathfrak{b}}(\lambda)$ are the so called *tempered representations* (see [16], IV, 3.6).

The regular cocharacters $\chi \in X_*(T_1^c) \otimes \mathbb{R}$ lie in the complement of finitely many hyperplanes, hence the set $(X_*(T_1^{(c)}) \otimes \mathbb{R}))^{(0)}$ of regular characters is a finite union of connected components. It is clear from the description that the module $A_{\mathfrak{q}}(\lambda)$ does not change, if χ moves inside a connected component. Finally we have the action of the real Weyl group $W(\mathbb{R}) = N(T)(\mathbb{R})/T(\mathbb{R})$ on $X_*(T_1^c) \otimes \mathbb{R}$ and again it is clear that the isomorphism type does not change if we conjugate χ by an element in $W(\mathbb{R})$. Hence we can say that the tempered $A_{\mathfrak{p}}(\lambda)$ are parametrised by $\pi_0((X_*(T_1^{(c)}) \otimes \mathbb{R}))^{(0)})/W(\mathbb{R})$.

We have a brief look at the case that $G^{(1)}/\mathbb{R}$ has a compact maximal torus T_1^c,i.e. $T = T$. This case played an important role in the section on the Gauss-Bonnet formula. Then

$$T_1^c \times_{\mathbb{R}} \mathbb{C} \subset K_\infty^{(1)} \times_{\mathbb{R}} \mathbb{C} \subset G^{(1)} \times_{\mathbb{R}} \mathbb{C},$$

hence T_1^c is a maximal torus in both reductive groups. We have the two (absolute) Weyl groups $W_{K_\infty} = W(\mathbb{R}) = N_{K_\infty}(T)(\mathbb{C})/T(\mathbb{C})$ and $W_G = N_G(T)(\mathbb{C})/T(\mathbb{C})$ The big Weyl group W_G acts simply transitively on the set of connected components of $(X_*(T_1^{(c)}) \otimes \mathbb{R}))^{(0)}$. Hence we have $W_G = \pi_0(X_*(T_1^{(c)}) \otimes \mathbb{R}))^{(0)})$ once we choose a base point $[\chi_0] \in \pi_0(X_*(T_1^{(c)}) \otimes \mathbb{R}))^{(0)})$ and therefore we get a family

$$\{A_{w\chi_0}(\lambda)\}_{w \in W_{K_\infty} \backslash W_G}, \tag{6.30}$$

and the results of Vogan and Zuckerman assert:

These repesentations are unitary, they are pairwise non isomorphic, and they are the Harish-Chandra modules attached to the discrete series representations of $G(\mathbb{R})$.

The cohomology groups are given by

$$H^q(\mathfrak{g}, K_\infty, A_{w\chi_0}(\lambda) \otimes \mathcal{M}_\lambda) = \begin{cases} \mathbb{C} & \textit{if } q = \frac{d}{2} \\ 0 & \textit{else} \end{cases} \tag{6.31}$$

Now it is clear that for a regular highest weight λ regular the condition b) forces the cocharacter χ to be regular.

We come back to the question raised above. Assume λ is regular and we have an isotypical component $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda)(\pi_f)$. Then the possible "friends" are the $A_\chi(\lambda)$ with χ regular. Hence we get

$$H^\bullet_!(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda)(\pi_f) = \bigoplus_{w\in W_K\backslash W_G} H^{\frac{d}{2}}(\mathfrak{g}, K_\infty, A_{w\chi_0}(\lambda)\otimes \mathcal{M}_\lambda)^{m(w\chi_0\times\pi_f)} = \bigoplus_{w\in W_K\backslash W_G} \mathbb{C}^{m(w\chi_0\times\pi_f)} \tag{6.32}$$

where $m(w\chi_0\times\pi_f)$ is the multiplicity of $A_{w\chi_0}(\lambda)\times\pi_f$. in $L^2_{\mathrm{disc}}(\Gamma\backslash G(\mathbb{R})$. (In Arthurs' words : If) λ is regular then the only friends of a $\pi_f\in \mathrm{Coh}(H^\bullet_!(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda)$ are the $w\chi_0$.)

If we refrain from decomposing into isotypical subspaces then we get a simpler formula

$$H^\bullet_!(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda) = \bigoplus_{w\in W_K\backslash W_G} \mathbb{C}^{m(w\chi_0)} \tag{6.33}$$

where of course $m(w\chi_0)$ is the multiplicity of $A_{w\chi_0}(\lambda)$ in $L^2_{\mathrm{disc}}(\Gamma\backslash G(\mathbb{R})$. Actually we know that $A_{w\chi_0}(\lambda)$ must even lie in the cuspidal spectrum (see [?]). In principle we already used this fact because we tacitly used the theorem of Borel (see Thm 6.1.1, b).

6.1.7 Cuspidal vs. inner

Now we remember that in the previous sections we made the convention (See end of (6.1.1)) that our coefficient systems $\mathcal{M}_\lambda$ are $\mathbb{C}$ vector spaces. We now revoke this convention and recall that the coefficient systems $\mathcal{M}_\lambda$ should be replaced by $\mathcal{M}_\lambda\otimes_F\mathbb{C}$, where F is some number field over which $\mathcal{M}_\lambda$ is defined. Then in the above list (6.27) of four subspaces in the cohomology the second and the fourth subspace have a natural structure of F-vector spaces and they have a combinatorial definition, whereas the first and third subspace need some input from analysis in their definition. In other words if we replace $\mathcal{M}_\lambda$ in (6.27) by $\mathcal{M}_\lambda\otimes_F\mathbb{C}$ then (6.27) can be written as

$$H^\bullet_{\mathrm{cusp}}(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda\otimes_F\mathbb{C}) \subset H^\bullet_!(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda)\otimes_F\mathbb{C} \subset H^\bullet_{(2)}(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda\otimes_F\mathbb{C}) \subset H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda)\otimes_F\mathbb{C} \tag{6.34}$$

It is a very important question to understand the discrepancy between the first two steps. If λ is regular then it follow from the results of [64] that in fact

$$H^\bullet_{\mathrm{cusp}}(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda\otimes_F\mathbb{C}) = H^\bullet_!(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda)\otimes_F\mathbb{C} \tag{6.35}$$

but without the assumption λ regular this is not true for interesting reasons.

Of course we should also take the action of the Hecke algebra into account. If π_f is the isomorphism type of an absolutely irreducible Hecke module which is defined over F. Then we can consider

$$H^\bullet_{\mathrm{cusp}}(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda\otimes_F\mathbb{C})(\pi_f) \subset H^\bullet_!(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda)\otimes_F\mathbb{C}(\pi_f) \tag{6.36}$$

and compare these two modules. We will say that π_f is *strongly inner* if we have equality.

We come back to this issue in part II.

$\boxed{\text{Conseq}}$

6.1.8 Consequences

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Vanishing theorems

If V is unitary and irreducible, then we have that $\bar{V} \xrightarrow{\sim} V^\vee$ and this implies for the central character

$$\overline{\chi_V(z)} = \chi_{V^\vee}(z) \text{ for all } z \in \mathfrak{Z}(\mathfrak{g}).$$

Combining this with Wigner's lemma we can conclude

If V is an irreducible unitary $(\mathfrak{g}, K_\infty)$-module, $\mathcal{M}_\lambda$ is an irreducible rational representation, and if

$$H^\bullet(\mathfrak{g}, K_\infty, V \otimes \mathcal{M}_\lambda) \neq 0$$

then $\chi_{\mathcal{M}_\lambda^\vee}(z) = \chi_{\mathcal{M}_\lambda}({}^t z) = \chi_{\bar{\mathcal{M}}_\lambda}(z)$

In other words: For an unitary irreducible $(\mathfrak{g}, K_\infty)$-module V the cohomology with coefficients in an irreducible rational representation $\mathcal{M}$ vanishes, unless we have $\mathcal{M}_\lambda^\vee \xrightarrow{\sim} \bar{\mathcal{M}}_\lambda$, or in terms of highest weights unless $-w_0(\lambda) = c(\lambda)$. (See 3.1.1)

If we combine this with the considerations following Wigner's lemma we get

Corollary 6.1.1. *If $\mathcal{M}$ is an absolutely irreducible rational representation and if $\mathcal{M}_\lambda^\vee$ is not isomorphic to $\bar{\mathcal{M}}_\lambda$ then*

$$H^\bullet_{(2)}(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda) = 0.$$

Hence also

$$H^\bullet_!(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda) = 0.$$

We will discuss examples for this in section 6.1.8

The group $G/\mathbb{Q} = \mathbf{Sl}_2/\mathbb{Q}$

Let us consider the group $G/\mathbb{Q} = \mathrm{Sl}_2/\mathbb{Q}$. We have tautological representation $\mathrm{Sl}_2 \hookrightarrow \mathrm{Gl}(\mathbb{Q}^2) = \mathrm{Gl}(V)$ and we get all irreducible representations of we take the symmetric powers $\mathcal{M}_n = \mathrm{Sym}^n(V)$ of V. (See 2, these are the $\mathcal{M}_n[m]$ restricted to Sl_2, then the m drops out.)

In this case the Vogan-Zuckerman list is very short. It is discussed in [Slzwei] for the groups $\mathrm{Sl}_2(\mathbb{R})$ and $\mathrm{Sl}_2(\mathbb{C})$, where both groups are considered as real Lie-groups.

In the case $\mathrm{Sl}_2(\mathbb{R})$ we have the trivial module $\mathbb{C}$ and for any integer $k \geq 2$ we have two irreducible unitarizable $(\mathfrak{g}, K_\infty)$-modules $\mathcal{D}_k^\pm$ (the discrete series representations) (See [Slzwei], 4.1.5). These are the only $(\mathfrak{g}, K_\infty)$-modules which have non trivial cohomology with coefficients in a rational representation. If we now pick one of our rational representation $\mathcal{M}_n$, then the non vanishing cohomology groups are

$$H^q(\mathfrak{g}, K_\infty, \mathcal{M}_n \otimes \mathbb{C}) = \mathbb{C} \text{ for } n = 0, q = 0, 2$$
$$H^q(\mathfrak{g}, K_\infty, \mathcal{D}_k^\pm \otimes \mathcal{M}_n \otimes \mathbb{C}) = \mathbb{C} \text{ for } n = k-2, q = 1$$

The trivial $(\mathfrak{g}, K_\infty)$-module $\mathbb{C}$ occurs with multiplicity one in $L^2(\Gamma\backslash G(\mathbb{R})$ hence we get for the trivial coefficient system a contribution

$$H^\bullet(\mathfrak{g}, K_\infty, \mathbb{C}\otimes\mathcal{M}_n\otimes\mathbb{C}) = H^0(\mathfrak{g}, K_\infty, \mathbb{C})\oplus H^2(\mathfrak{g}, K_\infty, \mathbb{C}) = \mathbb{C}\oplus\mathbb{C} \to H^\bullet_{(2)}(\Gamma\backslash X, \mathbb{C}).$$

This map is injective in degree 0 and zero in degree 2.

For the modules $\mathcal{D}_k^\pm$ we have to determine the multiplicities $m^\pm(k)$ of these modules in the discrete spectrum of $L^2(\Gamma\backslash G(\mathbb{R})$. A simple argument using complex conjugation tells us $m^+(k) = m^-(k)$. Now we have the fundamental observation made by Gelfand and Graev, which links representation theory to automorphic forms:

We have an isomorphism

$$Hom_{(\mathfrak{g},K_\infty)}(\mathcal{D}_k^+, L^2_{\text{disc}}(\Gamma\backslash G(\mathbb{R})) \xrightarrow{\sim} S_k(\Gamma\backslash\mathbb{H}) =$$

space of holomorphic cusp forms of weight k and level Γ

This is also explained in [Slzwei] on the pages following 23. We explain how we get starting from a holomorphic cusp form f of weight k an inclusion

$$\Phi_f : \mathcal{D}_k^+ \hookrightarrow L^2_{\text{disc}}(\Gamma\backslash G(\mathbb{R})$$

and that this map $f \mapsto \Phi_f$ establishes the above isomorphism. This gives us the famous Eichler-Shimura isomorphism

$$S_k(\Gamma\backslash\mathbb{H}) \oplus \overline{S_k(\Gamma\backslash\mathbb{H})} \xrightarrow{\sim} H^1_!(\Gamma\backslash X, \tilde{\mathcal{M}}_{k-2}).$$

The group $G/\mathbb{Q} = R_{F/\mathbb{Q}}(\mathbf{Sl}_2/F)$.

For any finite extension $F/\mathbb{Q}$ we may consider the base restriction $G/\mathbb{Q} = R_{F/\mathbb{Q}}(Sl_2/F)$. (See Chap-II. 1.1.1). Here we want to consider the special case the $F/\mathbb{Q}$ is imaginary quadratic. In this case we have $G \otimes \mathbb{C} = \mathrm{Sl}_2 \times \mathrm{Sl}_2/\mathbb{C}$ the factors correspond to the two embeddings of F into $\mathbb{C}$. The rational irreducible representations are tensor products of irreducible representations of the two factors $\mathcal{M}_\lambda = \mathcal{M}_{k_1} \otimes \mathcal{M}_{k_2}$ where again $\mathcal{M}_k = \mathrm{Sym}^k(\mathbb{C}^2)$. These representations are defined over F.

In this case we discuss the Vogan-Zuckerman list in [Slzwei], here we want to discuss a particular aspect. We observe that

$$\mathcal{M}_\lambda^\vee \xrightarrow{\sim} \mathcal{M}_{k_1} \otimes \mathcal{M}_{k_2}, \bar{\mathcal{M}}_\lambda = \mathcal{M}_{k_2} \otimes \mathcal{M}_{k_1}$$

and hence our corollary above yields for any choice of K_f

$$H^\bullet_!(\Gamma\backslash X, \mathcal{M}) = 0 \text{ if } k_1 \neq k_2.$$

In Chapter II we discuss the special examples in low dimensions. We take $F = \mathbb{Q}[i]$ and $\Gamma = \mathrm{Sl}_2[\mathbb{Z}[\mathbf{i}]]$ this amounts to taking the standard maximal compact subgroup $K_f = \mathrm{Sl}_2[\hat{\mathcal{O}}_F]$. If now for instance $k_1 > 0$ and $k_2 = 0$, then we get $H^\bullet_!(\Gamma\backslash X, \mathcal{M}_\lambda) = 0$. Hence we have by definition $H^\bullet_!(\Gamma\backslash X, \mathcal{M}) = H^\bullet_{\mathrm{Eis}}(\Gamma\backslash X, \tilde{\mathcal{M}})$ and we have complete control over the Eisenstein- cohomology in this case. Hence we know the cohomology in this case if we apply the analytic methods.

On the other hand in Chapter 2 we have written an explicit complex of finite dimensional vector spaces, which computes the cohomology. It is not clear to me how we can read off this complex the structure of the cohomology groups.

We get another example where this phenomenon happens, if we consider the group $\mathrm{Sl}_n/\mathbb{Q}$ if $n > 2$. In Chap.2 1.2 we describe the simple roots $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$, accordingly we have the fundamental highest weights $\omega_1, \ldots, \omega_{n-1}$. The element w_0 (See 6.1.1) has the effect of reversing the order of the weights. Hence we see that for $\lambda = \sum n_i\omega_i$ we have

$$H^\bullet_!(\Gamma\backslash X, \mathcal{M}_\lambda) = 0$$

unless we have $-w_0(\lambda) = \lambda$ and this means $n_i = n_{n-1-i}$.

The algebraic K-theory of number fields

I briefly recall the definition of the K-groups of an algebraic number field $F/\mathbb{Q}$. We consider the group $\mathrm{Gl}_n(\mathcal{O}_F)$, it has a classifying space BG_n. We can pass to the limit $\lim\limits_{n\to\infty} \mathrm{Gl}_n(\mathcal{O}_F) = \mathrm{Gl}(\mathcal{O}_F) = G$ and let BG its classifying space. Quillen invented a procedure to modify this space to another space BG^+, whose fundamental group is now abelian, but which has the same homology and cohomology as BG. Then he defines the algebraic K-groups as

$$K_i(\mathcal{O}_F) = \pi_i(\mathrm{BG}^+).$$

The space is an H-space, this means that we have a multiplication $m : \mathrm{BG}^+ \times \mathrm{BG}^+ \to \mathrm{BG}^+$ which has a two sided identity element. Then we get a homomorphism $m^\bullet : H^\bullet(\mathrm{BG}^+, \mathbb{Z}) \to H^\bullet(\mathrm{BG}^+ \times \mathrm{BG}^+, \mathbb{Z})$ and if we tensorize by $\mathbb{Q}$ and apply the Künneth-formula then we get the structure of a Hopf algebra on the Cohomology

$$m^\bullet : H^\bullet(\mathrm{BG}^+, \mathbb{Q}) \to H^\bullet(\mathrm{BG}^+, \mathbb{Q}) \otimes H^\bullet(\mathrm{BG}^+, \mathbb{Q})$$

Then a theorem of Milnor asserts that the rational homotopy groups

$$\pi_i(\mathrm{BG}^+) \otimes \mathbb{Q} = \ \mathrm{prim}(H^i(\mathrm{BG}, \mathbb{Q}),$$

where prim are the primitive elements, i.e. those elements $x \in H^i(\mathrm{BG}, \mathbb{Q})$ for which

I sketch a second application. We discuss the group $G = R_{F/\mathbb{Q}}(\mathrm{Gl}_n/F)$, where $F/\mathbb{Q}$ is an algebraic number field. the coefficient system $\tilde{\mathcal{M}}_\lambda = \mathbb{C}$ is trivial. In this case Borel, Garland and Hsiang have shown hat in low degrees $q \leq n/4$

$$H^q(\Gamma\backslash X, \mathbb{C}) = H^q_{(2)}\Gamma\backslash X, \mathbb{C}).$$

On the other hand it follows from the Vogan-Zuckerman classification ([89], that the only irreducible unitary $(\mathfrak{g}, K_\infty)$ modules V, for which $H^q(\mathfrak{g}, K_\infty, V) \neq 0$ and $q \leq n/4$ are one dimensional.

Hence we see that in low degrees

$$H^q(\mathfrak{g}, K_\infty, \mathbb{C}) \to H^q(\Gamma\backslash X, \mathbb{C})$$

is an isomorphism (Injectivity requires some additional reasoning.)

On the other hand we have $H^q(\mathfrak{g}, K_\infty, \mathbb{C}) = \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathbb{C})$ and obviously this last complex is isomorphic to the complex $\Omega^\bullet(X)^{G(\mathbb{R})}$ of $G(\mathbb{R})$-invariant forms on the symmetric space $G(\mathbb{R})/K_\infty$. Our field has different embeddings $\tau : F \hookrightarrow \mathbb{C}$, the real embeddings factor through $\mathbb{R}$, they form the set S_∞^{real} and the pairs of may conjugate embeddings into $\mathbb{C}$ form the set S_∞^{comp} Then

$$X = \prod_{v \in S_\infty^{\mathrm{real}}} \mathrm{Sl}_n(\mathbb{R})/SO(n) \times \prod_{S_\infty^{\mathrm{comp}}} \mathrm{Sl}_n(\mathbb{C})/SU(n).$$

Now the complex $\Omega^\bullet(X)^{G(\mathbb{R})}$ of invariant differential forms (all differentials are zero) does not change if we replace the group

$$G(\mathbb{R}) = \prod_{v \in S_\infty^{\mathrm{real}}} \mathrm{Sl}_n(\mathbb{R}) \times \prod_{S_\infty^{\mathrm{comp}}} \mathrm{Sl}_n(\mathbb{C})$$

by its compact form $G_c(\mathbb{R})$ and then we get the complex of invariant forms on the compact twin of our symmetric space

$$X_c = \prod_{v \in S_\infty^{\mathrm{real}}} SU_n(\mathbb{R})/SO(n) \times \prod_{S_\infty^{\mathrm{comp}}} (SU(n) \times SU(n))/SU(n),$$

but then

$$\Omega(X_c)^{G_c(\mathbb{R})} = H^\bullet(X_c, \mathbb{C}).$$

The cohomology of the topological spaces like the one on the right hand side has been computed by Borel in the early days of his career. **Referenz**

If we let n tend to infinity, we can consider the limit of these cohomology groups, then the limit becomes a Hopf algebra and we can consider the primitive elements

At this point we encounter an interesting problem. We have the three subspaces (See end of 3.2)

$$H^\bullet_{\mathrm{cusp}}(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C}) \subset H^\bullet_!(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda) \otimes \mathbb{C} \subset H^\bullet_{(2)}(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C}) \subset H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda) \otimes \mathbb{C},$$

note the positions of the tensor symbol $\otimes$. The first and the third space are only defined after we tensorize the coefficient system by $\mathbb{C}$, whereas the second and the fourth cohomology groups by definition F vector spaces tensorized by $\mathbb{C}$.

Now the question is whether the first and the third space also have a natural F -vector space structure. Of course we get a positive answer, if the Manin-Drinfeld principle holds. All the vector spaces are of course modules under the Hecke algebra and we and we can look at their spectra

$$\begin{array}{cc} \Sigma(H^\bullet_{\mathrm{cusp}}(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C})) = \Sigma_{\mathrm{cusp}} & \Sigma(H^\bullet_!(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C})) = \Sigma_! \\ \Sigma(H^\bullet_{(2)}(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C})) = \Sigma_{(2)} & \Sigma(H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C})) = \Sigma^{\cdot} \end{array}$$

If now for instance $\Sigma_{\rm cusp}\cap(\Sigma_!\setminus\Sigma_{\rm cusp})=\emptyset$ then we can define $H^\bullet_{\rm cusp}(\Gamma\backslash X,\tilde{\mathcal{M}}_\lambda)\subset H^\bullet_!(\Gamma\backslash X,\tilde{\mathcal{M}}_\lambda)$ as the subspace which is the sum of the isotypical components in $\Sigma_{\rm cusp}$.

If this is the case we say that the cuspidal cohomology is *intrinsically definable* and we get a canonical decomposition

$$H^\bullet_!(\Gamma\backslash X,\tilde{\mathcal{M}}_\lambda)=H^\bullet_{\rm cusp}(\Gamma\backslash X,\tilde{\mathcal{M}}_\lambda)\oplus H^\bullet_{!,\rm noncusp}(\Gamma\backslash X,\tilde{\mathcal{M}}_\lambda).$$

The classical Manin-Drinfeld principle refers to the two spectra $\Sigma_!\subset\Sigma$, if it is true in this case we get a decomposition

$$H^\bullet(\Gamma\backslash X,\tilde{\mathcal{M}}_\lambda)=H^\bullet_!(\Gamma\backslash X,\tilde{\mathcal{M}}_\lambda)\oplus H^\bullet_{\rm Eis}(\Gamma\backslash X,\tilde{\mathcal{M}}_\lambda)$$

the canonical complement is called the Eisenstein cohomology. (See Chap. 2 2.2.3 and Chap 3 section 5.)

6.1.9 Extroduction

We return to our fundamental exact sequence $\boxed{\text{fil0}}$

$$0\to H^\bullet_!(\Gamma\backslash X,\tilde{\mathcal{M}})\to H^\bullet(\Gamma\backslash X,\tilde{\mathcal{M}})\xrightarrow{r}H^\bullet(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X),\tilde{\mathcal{M}})\to\ldots \tag{6.37}$$

We know that $H^\bullet_!(\Gamma\backslash X,\tilde{\mathcal{M}})\subset H^\bullet_{(2)}(\Gamma\backslash X,\tilde{\mathcal{M}})$ and we gained some understanding of the latter space using analytic methods, we have seen that classes in $[\omega]\in H^\bullet_{(2)}(\Gamma\backslash X,\tilde{\mathcal{M}}\otimes\mathbb{C})$ can be represented by harmonic forms ω.

We also want to understand the cohomology of the boundary and we want to describe the image of the restriction map r. This leads us to very difficult questions, again we have to use analytic tools (Langlands theory of Eisenstein series), we will only very superficially touch this subject in this first part of this book.

If we want to compute $H^\bullet(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X),\tilde{\mathcal{M}})=H^\bullet(\partial(\Gamma\backslash\bar{X}),\tilde{\mathcal{M}})$ we invoke the spectral sequences (2.63, 2.64), their $E_2^{p,q}$ term is is the cohomology of the complex $\boxed{\text{Epqone}}$

$$\to:\bigoplus_{[P]:d(P)=p+1}H^q(\Gamma_P\backslash X,\tilde{\mathcal{M}})\xrightarrow{d_1^{p,q}}\bigoplus_{[Q]:d(Q)=p+2}H^q(\Gamma_Q\backslash X,\tilde{\mathcal{M}})\to. \tag{6.38}$$

We can go one step further and employ the spectral sequence (2.52) and decompose invoke Kostant's theorem to decompose the cohomology of the fiber into weight spaces, we explain this briefly in the following subsection.

$\boxed{\text{growth}}$

The cohomology of unipotent groups

We drop the subscript $_P$, we know that the group scheme $U/\mathbb{Q}$ is a unipotent group scheme, let $A=A(U)$ be its affine algebra (see section 1.1.1).. Then $U/\mathbb{Q}$ has a filtration by subschemes $U_0=\{e\}\subset U_1\subset U_2\subset\ldots U_{m-1}\subset U_m$ such that

$U_i/U_{i-1} \xrightarrow{\sim} \mathbb{G}_a$. The subgroup $\Gamma_U \subset U(\mathbb{Q})$ is Zariski dense, more precisely we know the following: If $\Gamma_i = U_i(\mathbb{Q}) \cap \Gamma$ then $\Gamma_i/\Gamma_{i-1} \xrightarrow{\sim} \mathbb{Z} \subset U_i/U_{i-1}(\mathbb{R}) \xrightarrow{\sim} \mathbb{R}$.

We consider the category of $U/\mathbb{Q}$ modules $\mathrm{Mod_U}$ (see 1.1.1.) Then it is clear that the functor $\mathcal{M} \to \mathcal{M}^U$ is equal to $\mathcal{M} \to \mathcal{M}^{\Gamma_U}$. (Our $\mathbb{Z}-$ module $\mathcal{M}$ above is now a $\mathbb{Q}-$ vector space, i.e. we consider coefficient systems with rational coefficients.)

We choose the action of U on A by left translations on A. It follows from Frobenius reciprocity that the $U/\mathbb{Q}$ module A is an injective module in Mod_U (See ???) This implies that we get an injective resolution of the $U/\mathbb{Q}$ -module $\mathbb{Q}$ by

$$0 \to \mathbb{Q} \to A \to (A/\mathbb{Q}) \otimes A \to \dots = 0 \to \mathbb{Q} \to I^0 \to I^1 \to \tag{6.39}$$

and hence

$$H^q(U, \mathcal{M}) = H^q(\Gamma_U \backslash U(\mathbb{R}), \mathcal{M}) = H^q(0 \to (I^1 \otimes \mathcal{M})^U \to (I^2 \otimes \mathcal{M})^U \to \dots) = H^q((I^\bullet \otimes \mathcal{M})^U) \tag{6.40}$$

Since $U/\mathbb{Q}$ is the unipotent radical of the parabolic group $P/\mathbb{Q}$, the parabolic group $P/\mathbb{Q}$ acts via the adjoint action on the modules $I^m \otimes \mathcal{M}$ This action respects the submodules $(I^m \otimes \mathcal{M})^U$ and $U/\mathbb{Q}$ acts trivially on $(I^m \otimes \mathcal{M})^U$, this implies that the modules $I^m \otimes \mathcal{M})^U$ are $M/\mathbb{Q} = (P/U)/\mathbb{Q}$ modules. The group $M/\mathbb{Q}$ is reductive and we know that the category of $M/\mathbb{Q}$ modules is semi simple (???). This implies that we can decompose

$$(I^\bullet \otimes \mathcal{M})^U = \mathbb{H}^\bullet(U, \mathcal{M}) \oplus ACI(I^\bullet)^U \tag{6.41}$$

where the first summand is a complex of $M/\mathbb{Q}$ -modules in which all the differentials are zero and the second is an acyclic complex of $M/\mathbb{Q}$- modules. Hence

$$H^\bullet(U, \mathcal{M}) = H^\bullet(\Gamma_U, \mathcal{M}) \xrightarrow{\sim} \mathbb{H}^\bullet(U, \mathcal{M}) \tag{6.42}$$

We get a "smaller" resolution from the (algebraic) de-Rham complex of differential forms. On the smooth affine scheme $U/\mathbb{Q}$ we have the sheaves of differential forms $\Omega^p_U = \Lambda^p \Omega^1_U$ ([40] ,7.5) and we have the de-Rham complex

$$\Omega(U)^\bullet = 0 \to \mathbb{Q} \to A \to \Omega^1(U) \to \Omega^2(U) \to \dots \tag{6.43}$$

where $\Omega^p(U) = \Omega^p_U(U)$ is the module of global sections and $A = \Omega^0(U)$. These modules of differentials are free A modules, hence they are injective. Since our unipotent group scheme $U/\mathbb{Q}$ is isomorphic to the affine space $\mathbb{A}^d$ (as affine scheme) we see easily that this complex is exact, hence it provides an acyclic resolution. As before we get the cohomology by taking the complex $(\Omega^p(U) \otimes \mathcal{M})^U$ of invariants under the action of $U/\mathbb{Q}$. Since an $U/\mathbb{Q}$- invariant differential form with values in $\mathcal{M}$ is determined by its value at the identity e the complex of invariants under $U/\mathbb{Q}$ becomes

$$0 \to \mathcal{M} \to \operatorname{Hom}(\mathfrak{u}, \mathcal{M}) \to \operatorname{Hom}(\Lambda^2\mathfrak{u}, \mathcal{M}) \to \dots = 0 \to \operatorname{Hom}(\Lambda^\bullet\mathfrak{u}, \mathcal{M}) \tag{6.44}$$

and the cohomology of this complex is the cohomology $H^\bullet(\mathfrak{u}, \mathcal{M})$. We still have the action of $P/\mathbb{Q}$ on $\mathfrak{u}$ by the adjoint action, hence we get an action of P on $\operatorname{Hom}(\Lambda^\bullet\mathfrak{u}, \mathcal{M})$ and we have

Theorem 6.1.2. *(van Est [?])*

$$H^\bullet(\mathfrak{u},\mathcal{M}) \xrightarrow{\sim} \mathbb{H}^\bullet(\mathfrak{u},\mathcal{M}) = (\mathrm{Hom}(\Lambda^\bullet\mathfrak{u},\mathcal{M}))^U,$$

Proof. later □

A famous theorem of Kostant yields a description of the $M/\mathbb{Q}$ module $(\mathrm{Hom}(\Lambda^\bullet\mathfrak{u},\mathcal{M}))^U$, Let $\lambda \in X^*(T)$ be the highest weight of $\mathcal{M}$, i.e. we have $\mathcal{M} = \mathcal{M}_\lambda$. The set

$$W^P = \{w \in W \mid \alpha \in \Delta_M^+,\ w^{-1}(\alpha) \in \Delta^+\} \tag{6.45}$$

is the set of Kostant representatives for $W^M\backslash W$. For any $w \in W^P$ we define the element

$$\omega_w = \Lambda_{\alpha\in\Delta_U; w^{-1}\alpha<0}\ u_\alpha^\vee \otimes e_{w\lambda} \tag{6.46}$$

It is clear that this element lies in $(\mathrm{Hom}(\Lambda^\bullet\mathfrak{u},\mathcal{M}_\lambda))^U$ and hence it is the highest weight vector of an irreducible M-module

$$\mathbb{H}^\bullet(\mathfrak{u},\mathcal{M}_\lambda)_{w\cdot\lambda} \subset \mathbb{H}^\bullet(\mathfrak{u},\mathcal{M}_\lambda)$$

where $w\cdot\lambda = w(\lambda+\rho)-\rho$

Now the theorem theorem of Kostant says

Theorem 6.1.3.

$$\mathbb{H}^\bullet(\mathfrak{u},\mathcal{M}) = \bigoplus_{w\in W^P} \mathbb{H}^\bullet(\mathfrak{u},\mathcal{M})_{w\cdot\lambda} = \bigoplus_{w\in W^P} \mathbb{H}^{l(w)}(\mathfrak{u},\mathcal{M})_{w\cdot\lambda}$$

where we have to be aware that the summand $\mathbb{H}^\bullet(\mathfrak{u},\mathcal{M})_{w\cdot\lambda}$ sits in degree $l(w)$

Proof. Rather clear after the preparation. □

Since the differentials in the complex $\mathbb{H}^\bullet(\mathfrak{u},\mathcal{M}_\lambda)$ are zero, the spectral sequence degenerates and we get

$\boxed{\text{cohboundstrat}}$

$$H^n(\Gamma_P\backslash X^P(\underline{c}_{\pi'}),\tilde{\mathcal{M}}) \xrightarrow{\sim} \bigoplus_{w\in W^P} H^{n-l(w)}(\Gamma_M\backslash X^M(r(\underline{c}_{\pi'})),\mathbb{H}^\bullet\widetilde{(\mathfrak{u},\mathcal{M})}_{w\cdot\lambda}), \tag{6.47}$$

this is the decomposition of the cohomology of the boundary stratum $\partial_P(\Gamma\backslash X)$ into weight spaces. We insert this decomposition into the spectral sequence and get a more precise description of the $E_1^{p,q}$ page (6.38):

$$\begin{gathered}\xrightarrow{d_1^{p-1,q}} \bigoplus_{[P]:d(P)=p+1} \bigoplus_{w\in W^P} H^{q-l(w)}(\Gamma_{M_P}\backslash X^{M_P},\mathbb{H}^{l(w)}\widetilde{(\mathfrak{u}_P,\mathcal{M})}_{w\cdot\lambda}) \xrightarrow{d_1^{p,q}} \\ \xrightarrow{d_1^{p,q}} \bigoplus_{[Q]:d(Q)=p+2} \bigoplus_{v\in W^Q} H^{q-l(v)}(\Gamma_{M_Q}\backslash X^{M_Q},\mathbb{H}^{l(v)}\widetilde{(\mathfrak{u}_Q,\mathcal{M})}_{v\cdot\lambda}) \xrightarrow{d_1^{p+1,q}}\end{gathered} \tag{6.48}$$

and it becomes clear that the computation of the cohomology of this complex, i.e. the computation of the $E_2^{p,q}$, and the differentials will become a very delicate issue, The computation of the higher pages of the spectral sequence will be even more difficult.

But I am convinced that a thorough study of the deeper pages and of the map r will be very rewarding. We get interesting application to number theory. We can prove rationality results for special values of L-functions (divided by a well chosen period.) In certain cases the value of these normalised $L-$ values $L^{\mathrm{ar}}(\pi_f,\nu)=\frac{L\pi_f,\nu)}{\Omega_{\epsilon(\nu)}}$ carry some relevant arithmetic information. It may tell us something about the structure of some cohomology groups as Hecke-modules (for instance the denominator of Eisenstein classes) and as a consequence something about the structure of the Galois group.

We have executed this program in some very specific cases in this book. We discussed the rationality of normalised special $L-$ values in a special situation at the end of chapter 4 (see (4.219)). For some more general cases I refer to [45],[47]. In these papers the authors always work on the $E_2^{0,\bullet}$ page of the spectral sequence.

In the note [**?**] I consider the group $\mathrm{Sl}_4(\mathbb{Z})$ and carry out some speculative computation which indicate that $d_2^{\bullet,\bullet}\neq 0$. Hence we get $E_2^{\bullet,\bullet}\neq E_3^{\bullet,\bullet}$, we also get a rationality relation between special values of the Riemann ζ- function which of course will be well known.

The other application is intensively studied in the chapters 3-5 for the case $\mathrm{Sl}_2(\mathbb{Z})$. We also consider the case $\mathrm{Sl}_2(\mathbb{Z}[\mathbf{i}])$ where the answer is less complete and some experimental computations should be done. I also refer to the papers [42] and [44], in both papers I produce situations where we expect

The prime ℓ divides normalised L value $\Longrightarrow$ ℓ divides the denominator of an Eisenstein class.

But it requires new ideas to prove such an assertions, it seems that at this moment we should make some numerical experiments.

Finis operis

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The following items can be obtained from my home page

`http://www.math.uni-bonn.de/people/harder`

[MixMot-2015.pdf] Modular Construction of Mixed Motives

[SecOps.pdf] Secondary Operations on the Cohomology of Harish-Chandra Modules

Index