Cohomology of Arithmetic Groups

by

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The following is a first version of Chapter VI (probably the last one) of a book on the cohomology of arithmetic groups. The intention of the book is to give a fairly elementary introduction into the subject and to show that it yields interesting applications to number theory. This last chapter VI is really the heart of the matter. I refer to the previous chapters, which exist in a preliminary version (in german) and are available at the MPI or my office in Be1.

The main result in Chapter VI is Theorem I. This Theorem is an elementary statement on the structure of the cohomology as a module for the Hecke-algebra. Its proof in 6.2 and 6.3 is also elementary but rather long

The really difficult and by no means elementary part of this chapter VI is section 6.1, where I try to explain how Theorem I yields insight into the structure of cyclotomic fields(the theorem of Herbrand-Ribet).

I also add the introduction to the book, the reader may get some impression of the concept of the whole book. Any comments will be welcome.

Introduction: This book is meant to be an introduction into the cohomology of arithmetic groups. This is certainly a subject of interest in its own right, but my main goal will be to illustrate the arithmetical applications of this theory. I will discuss the application to the theory of special values of L-functions and the theorem of Herbrand-Ribet (See Chap V, [Ri], Chap VI, Theorem II).

On the other hand the subject is also of interest for differential geometers and topologists, since the arithmetic groups provide so many interesting examples of Riemannian manifolds.

My intention is to write an elementary introduction. The text should be readable by graduate students. This is not easy, since the subject requires a considerable background: One has to know some homological algebra (cohomology and homology of groups, spectral sequences, cohomology of sheaves), the theory of Lie groups, the structure theory of semisimple algebraic groups, symmetric spaces, arithmetic groups, reduction theory for arithmetic groups. At some point the theory of automorphic forms enters the stage, we have to understand the theory of representations of semi-simple Lie groups and their cohomology. Finally when we apply all this to number theory (in Chap. V and VI) one has to know a certain amount of algebraic geometry ($\ell-$adic cohomology, Shimura varieties (in the classical case of elliptic modular functions)) and some number theory(classfield theory, $L-$functions and their special values).

I will try to explain as much as possible of the general background. This should be possible, because already the simplest examples namely the Lie groups $Sl_2(\mathbb{R})$ and $Sl_2(\mathbb{C})$ and their arithmetic subgroups $Sl_2(\mathbb{Z})$ and $Sl_2(\mathbb{Z}[\sqrt{-1}])$ are very interesting and provide deep applications to number theory. For these special groups the results needed from the structure theory of semisimple groups, the theory of symmetric spaces and reduction theory are easy to explain. I will therefore always try to discuss a lot of things for our special examples and then to refer to the literature for the general case.

I want to some words about the general framework.

Arithmetic groups are subgroups of Lie groups. They are defined by arithmetic data. The classical example is the group $Sl_2(\mathbb{Z})$ sitting in the real Lie group $Sl_2(\mathbb{R})$ or the group $Sl_2(\mathbb{Z}[\sqrt{-1}])$ as a subgroup of $Sl_2(\mathbb{C})$, which has to be viewed as real Lie group (See ..). Of course we may also consider $Sl_n\mathbb{Z} \subset Sl_n\mathbb{R}$ as an arithmetic group. We get a slightly more sophisticated example, if we start from a quadratic form, say

$$f(x_1, x_2, \ldots, x_n) = -x_1^2 + x_2^2 + \ldots + x_n^2$$

the orthogonal group $O(f)$ is a linear algebraic group defined over the field $\mathbb{Q}$ of rational numbers, the group of its real points is the group $O(n,1) = O(f)(\mathbb{R})$ and the group of integral matrices preserving this form is an arithmetic subgroup $\Gamma \subset O(f)(\mathbb{R})$

The starting point will be an arithmetic group $\Gamma \subset G_\infty$, where G_∞ is a real Lie group. This group is always the group of real points of an algebraic group over $\mathbb{Q}$ or a subgroup of finite index in it. To this group G_∞ one associates a symmetric space $X = G_\infty/K_\infty$, where K_∞ is a maximal compact subgroup of G_∞, this space is diffeomorphic to $\mathbb{R}^d$. The next datum we give ourselves is a Γ-module $\mathcal{M}$ from which we construct a sheaf $\tilde{\mathcal{M}}$ on the quotient space $\Gamma\backslash X$. This sheaf will be what topologists call a local coefficient system, if Γ acts without fixed points on X. We are interested in the cohomology groups

$$H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}).$$

Under certain conditions we have an action of a big algebra of operators on these cohomology groups, this is the so called Hecke algebra $\mathbf{H}$, it originates from the structure of the arithmetic group Γ (Γ has many subgroups of finite index, which allow the passage to coverings of $\Gamma\backslash X$ and we have maps going back and forth). It is the structure of the cohomology groups $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})$ as a module under this algebra $\mathbf{H}$, which we want to study, these modules contain relevant arithmetic information.

Now I give an overview on the Chapters of the book.

In chapter I we discuss some basic concepts from homological algebra, especially we introduce to the homology and cohomology of groups, we recall some facts from the cohomology of sheaves and give a brief introduction into the theory of spectral sequences.

Chapter II introduces to the theory of linear algebraic groups, to the theory of semi simple algebraic groups and the corresponding Lie groups of their real points. We give some examples and we say something about the associated symmetric spaces. We consider the action of arithmetic groups on these symmetric spaces, and discuss some classical examples in detail. This is the content of reduction theory. As a result of this we introduce the Borel-Serre compactification $\Gamma\backslash\bar{X}$ of $\Gamma\backslash X$, which will be discussed in detail for our examples. After this we take up the considerations of chapter I and define and discuss the cohomology groups of arithmetic groups with coefficients in some Γ-modules $\mathcal{M}$. We shall see that these cohomology groups are related (and under some conditions even equal) to the cohomology groups of the sheaves $\tilde{\mathcal{M}}$ on $\Gamma\backslash X$. Another topic in this chapter is the discussion of the homology groups, their relation to the cohomology with compact supports and the Poincaré duality. We will also explain the relations between the cohomology with compact supports the ordinary cohomology and the cohomology of the boundary of the Borel-Serre compactification. Finally we introduce the Hecke operators on the cohomology. We discuss these operators in detail for our special examples, and we prove some classically well known relations for them in our context. In these classical cases we also compute the cohomology of the boundary as a module over the Hecke algebra **H**

At the end of this chapter we give some explicit procedures, which allow an explicit computation of these cohomology groups in some special cases. It may be of some interest to develop such computational techniques sinces this allows to carry out numerical experiments (See .. and ...). We shall also indicate that this apparently very explicit procedure for the computation of the cohomology does not give any insight into the structure of the cohomology as a module under the Hecke algebra. This chapter II is still very elementary.

In Chapter III we develop the analytic tools for the computation of the cohomology. Here we have to assume that the Γ-module $\mathcal{M}$ is a $\mathbb{C}$-vector space and is actually obtained from a rational representation of the underlying algebraic group. In this case one may interprete the sheaf $\tilde{\mathcal{M}}$ as the sheaf of locally constant sections in a flat bundle, and this implies that the cohomology is computable from the de-Rham-complex associated to this flat bundle. We could even go one step further and introduce a Laplace operator so that we get some kind of Hodge-theory and we can express the cohomology in terms of harmonic forms. Here we encounter serious difficulties since the quotient space $\Gamma\backslash X$ is not compact. But we will proceed in a different way. Instead of doing analysis on $\Gamma\backslash X$ we work on $\mathcal{C}_\infty(\Gamma\backslash G_\infty)$. This space is a module under the group G_∞, which acts by right translations, but we rather consider it as a module under the Lie algebra $\mathfrak{g}$ of G_∞ on which also the group K_∞ acts, it is a $(\mathfrak{g}, K)$-module.

Since our module $\mathcal{M}$ comes from a rational representation of the underlying group G, we may replace the de-Rham-complex by another complex

$$H^\bullet(\mathfrak{g}, K, \mathcal{C}_\infty(\Gamma\backslash G_\infty) \otimes \mathcal{M}),$$

this complex computes the so called $(\mathfrak{g}, K)$-cohomology. The general principle will be to "decompose" the $(\mathfrak{g}, K)$-module $\mathcal{C}_\infty$ into irreducible submodules and therefore to compute the cohomology as the sum of the contributions of the individual submodules. This is a group theoretic version of the classical approach by Hodge-theory. Here we have to overcome two difficulties. The first one is that the quotient $\Gamma\backslash G_\infty$ is not compact and hence the above decomposition does not make sense, the second is that we have to understand the irreducible $(\mathfrak{g}, K)$-modules and their cohomology. The first problem is of analytical nature, we will give some indication how this can be solved by passing to certain subspaces of the cohomology the so called cuspidal and the discrete part of the cohomology. We shall state some general results, which are mainly due to A. Borel and H. Garland. We shall shall also state some general results concerning the second problem. The general result in this chapter is a partial generalization of the theorem of Eichler-Shimura, it desribes the cuspidal part of the cohomology in terms of irreducible representations occuring in the space of cusp forms and contains some information on the discrete cohomology, which is slightly weaker. We shall also give some indications how it can be proved.

In the next chapter IV we resume the discussion of the previous chapter but we restrict our attention to the specific groups $Sl_2(\mathbb{R})$ and $Sl_2(\mathbb{C})$ and their arithmetic subgroups. At first we give a rather detailed discussion of their representation-theory (i.e. the theory of representations of the corresponding $(\mathfrak{g}, K)$-modules) and we compute also the $(\mathfrak{g}, K)$-cohomology of the most important $(\mathfrak{g}, K)$-modules, this is the second ingredient in the theorem of Eichler -Shimura. But in this special case we give also a complete solution for the analytical difficulties, so that in this case we get a very precise formulation of the Eichler-Shimura theorem, together with a rather complete proof.

In the following chapter V we discuss the Eisenstein-cohomology. The theorem of Eichler-Shimura desribes only a certain part of the cohomology , the Eisenstein -cohomology is meant to fill the gap, it is complementary to the cuspidal cohomology. These Eisenstein classes are obtained by an infinite summation process, which sometimes does not converge and is made convergent by analytic continuation. We shall discuss in detail the cases of the special groups $Sl_2(\mathbb{R})$ and $Sl_2(\mathbb{C})$ (the second case is not yet in the manuscript). Here we will be able to explain an arithmetic application of our theory. Recall that we have to start from a rational representation of the underlying algebraic group $G/\mathbb{Q}$ and this representation is defined over $\mathbb{Q}$ or at least over some number field. Hence we actually get a Γ-module $\mathcal{M}$ which is a $\mathbb{Q}$- vector space, and hence we may study the cohomology $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})$ which then is a $\mathbb{Q}$-vector space. The Eisenstein classes are a priori defined by transcendental means, so they define a subspace in $H^\bullet(\mathfrak{g}, K, \tilde{\mathcal{M}})_\mathbb{C}$. But we have still the action of the Hecke-algebra $\mathbf{H}$, and this acts on the $\mathbb{Q}$-vector space $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})$, and using the so called Manin-Drinfeld argument we can characterize the space of Eisenstein-classes as an isotypic piece in the cohomology, hence it is defined over $\mathbb{Q}$. We shall indicate that we can evaluate the now rational Eisenstein-classes on certain homology-classes, which are also defined over $\mathbb{Q}$, hence the result is a rational number. On the other hand we can-using the trancendental definition of the Eisenstein class-express the result of this evaluation in terms of special values of $L-$functions. This yields rationality results for special values of $L-$functions (see [Ha] and [Ha -Sch]). This gives us the first arithmetic informations of our theory.

In Chapter VI we discuss the arithmetic properties of the Eisenstein-classes. in the previous chapter we have seen, that the Eisenstein-classes are rational classes despite of the fact, that they are obtained by an infinite summation. Now we will discuss the extremely special case where $\Gamma = Sl_2(\mathbb{Z})$ and our Γ-module is

$$\mathcal{M}_n/p\,\mathcal{M}_n = \{\sum a_\nu X^\nu Y^{n-\nu} |\ a_\nu \in \mathbb{Z}[\frac{1}{6}]\}.$$

We also introduce the dual module

$$\mathcal{M}_n/p\,\mathcal{M}_n^\vee = \mathrm{Hom}(\mathcal{M}, \mathbb{Z}[\frac{1}{6}]).$$

We then ask whether the Eisenstein-class is actually an integral class, this means whether it is contained in $H^1(\Gamma\backslash X), \mathcal{M}_n/p\,\mathcal{M}_n^\vee)$. The answer is no in general, the Eisenstein-class has a denominator, which is apart from powers of 2 and 3 exactly the numerator of the number

$$\zeta(1-(n+2)) = \pm\frac{B_{n+2}}{n+2}.$$

(See Chap. VI, Theorem I) This result is obtained by testing the Eisenstein-classes on certain homology classes, the so called modular symbols, which have been introduced in chapter II. This result generalizes results of Haberland [Hab] and my student [Wg]. I will indicate that this result has arithmetic implications in the direction of the theorem of Herbrand -Ribet. We cannot prove this theorem here since we need some other techniques from arithmetic algebraic geometry to complete the proof. We shall also discuss some congruence relations between Eisenstein classes of different weights, which arise from congruence relations on the level of sheaves. These congruence relations between the sheaves have also been exploited by Hida and R. Taylor

Finally I want to discuss some possible generalizations of all this and some open interesting problems. During the whole book I always tried to keep the door open for such generalizations. I presented the cohomology of arithmetic groups in such a way that we have the necessary tools to extend our results. This may have had the effect, that the presentation of the results in the classical case of $Sl_2(\mathbb{Z})$ looks to complicated, but I hope it will pay later on.

Some of these generalisations are discussed in [HS].

I want to explain a few notations, that have been introduced in earlier chapters and may be not so clear.

H is the usual upper half plane and $\tilde{H}$ is the Borel-Serre completion of it: To each point $r \in \mathbb{P}^1(\mathbb{Q})$ we add a line $H_{r,\infty}$ which we interprete as the set of real Borel-subgroups in opposition to the Borel-subgroup corresponding to r. The group Γ is $Sl_2(\mathbb{Z})$, if $\mathcal{M}$ is a Γ module, then $\tilde{\mathcal{M}}$ we be the corresponding sheaf on $\Gamma\backslash H$ or on $\Gamma\backslash\tilde{H}$. The inclusion $\Gamma\backslash H \hookrightarrow \Gamma\backslash\tilde{H}$ induces an isomorphism on cohomology. The boundary of the Borel-Serre compactification $\Gamma\backslash\tilde{H}$ is denoted by $\partial(\Gamma\backslash\tilde{H})$

Chapter VI

The arithmetic properties of Eisenstein classes

6.1: **The main result and its arithmetic consequences**.

We apply our previous results to a very specific situation. Our arithmetic group will be the group $\Gamma = SL(2,\mathbb{Z})$, it acts on the upper half plane H. We put $R = \mathbb{Z}[\frac{1}{6}]$ and we consider the following two R-modules

$$\mathcal{M}_n/p\,\mathcal{M}_n = \Big\{\sum\nolimits_{\nu=0}^{n} a_\nu X^\nu Y^{n-\nu} | a_\nu \in R\Big\}$$

and

$$\mathcal{M}_n^\vee = \Big\{\sum\nolimits_{\nu=0}^{n} a_\nu \binom{n}{\nu} X^\nu Y^{n-\nu} | a_\nu \in R\Big\}.$$

The group Γ acts on these modules and if

$$e_\nu = X^{\frac{n}{2}+\nu} Y^{\frac{n}{2}-\nu}, e_\nu^\vee = \binom{n}{\frac{n}{2}+\nu} e_\nu,$$

we have a Γ-invariant pairing defined by $< e_\nu, e_{-\mu}^\vee >= \delta_{\nu,\mu}$ (See 5.6). We study the cohomomology of the sheaves $\tilde{\mathcal{M}}_n$ and $\tilde{\mathcal{M}}_n^\vee$ on $\Gamma\backslash\tilde{H}$. I recall that n should be even, if we want these sheaves to be different from zero. Of course the two modules become equal if we tensorize by $\mathbb{Q}$, the result is denoted by $\mathcal{M}_{n,\mathbb{Q}}$. For any finitely generated R-module M we denote by M_{int} the quotient of M by its torsion submodule. We consider the following diagram

$$\begin{array}{ccc} H^1(\Gamma\backslash\tilde{H}, \tilde{\mathcal{M}}_n^\vee) & \xrightarrow{r} & H^1(\partial(\Gamma\backslash\tilde{H}), \tilde{\mathcal{M}}_n^\vee) \\ \downarrow & & \downarrow \\ H^1(\Gamma\backslash\tilde{H}, \tilde{\mathcal{M}}_n^\vee)_{int} & \xrightarrow{r} & H^1(\partial(\Gamma\backslash\tilde{H}), \tilde{\mathcal{M}}_n^\vee)_{int} \, . \\ \cap & & \cap \\ H^1(\Gamma\backslash\tilde{H}, \tilde{\mathcal{M}}_{n,\mathbb{Q}}) & \xrightarrow{r} & H^1(\partial(\Gamma\backslash\tilde{H}), \tilde{\mathcal{M}}_{n,\mathbb{Q}}) \end{array}$$

We have seen in the preceeding Chapter (5.7) that

$$H^1(\partial(\Gamma\backslash\tilde{H}), \tilde{\mathcal{M}}_n^\vee) = R\omega_n \oplus torsion$$

where for all primes p the Hecke operator T_p acts nilpotently on $torsion \otimes \mathbb{Z}_{(p)}$ and satisfies

$$T_p\omega_n = (p^{n+1}+1)\omega_n.$$

The image of the class ω_n in $H^1(\partial(\Gamma\backslash\tilde{H}), \tilde{\mathcal{M}}_{n,\mathbb{Q}})$ has a canonical lifting to a class $Eis_n \in H^1(\Gamma\backslash\tilde{H}, \tilde{\mathcal{M}}_{n,\mathbb{Q}})$ which is characterized by the two properties

$$r(Eis_n) = \omega_n \text{ and } T_p(Eis_n) = (p^{n+1}+1)Eis_n$$

If we intersect the subspace $\mathbb{Q} \cdot Eis_n$ with the R-module $H^1_{int}(\Gamma\backslash\tilde{H}, \tilde{\mathcal{M}}_n^\vee)$ we get a primitive submodule

$$\mathbb{Q} \cdot Eis_n \cap H^1(\Gamma\backslash\tilde{H}, \tilde{\mathcal{M}}_n^\vee)_{int} = R \cdot eis_n,$$

where eis_n is unique up to an element in R^*. My student Wang showed in his dissertation that the Hecke operator T_p acts also nilpotently on the torsion of $H^1(\Gamma\backslash\tilde{H}, \tilde{\mathcal{M}}_n^\vee) \otimes \mathbb{Z}_{(p)}$ and therefore we have a canonical lifting of eis_n to a class in $H^1(\Gamma\backslash\tilde{H}, \tilde{\mathcal{M}}_n^\vee)$ which we also call eis_n. If we restrict the class eis_n to the cohomology of the boundary, the we find

$$r(eis_n) = a(n) \cdot \omega_n,$$

where the number $a(n)$ is unique up to an element in R^*. This number can be interpreted as the *denominator* of the Eisensteinclass Eis_n, I am interested in its prime factorisation. For any prime $p > 3$ we define $\delta_p(n)$ to be the highest power of p dividing $a(n)$ i.e. in the usual notation we have

$$p^{\delta_p(n)} || a(n) \text{ or } \delta_p(n) = ord_p(a(n)).$$

Then we have the following

Theorem I: *For $p > 3$ we have*

$$\delta_p(n) = ord_p(\zeta(1-(n+2)))$$

In his dissertation [Wg] my student Wang proved the weaker result

$$p|a(n) \iff p|\zeta(1-(n+2)),$$

some of his ideas enter in the proof of the above theorem. The proof will be given in 6.2-6.3.

I want to discuss the arithmetic applications of this theorem. To do this I have to explain the connections to étale cohomology. The fundamental point is:

For any natural number m the cohomology groups

$$H^1(\Gamma\backslash\tilde{H}, \tilde{\mathcal{M}}_n^\vee \otimes \mathbb{Z}/p^m\mathbb{Z})$$

are not only modules for the Hecke-algebra but we have also an action of the Galois-group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on these modules, which commutes with the action of the the Hecke-algebra.

This gives us a structure of a Hecke×Galois-module on these cohomology groups. We may pass to the projective limit over all m, then

$$\lim_{\leftarrow}(H^1(\Gamma\backslash\tilde{H}, \tilde{\mathcal{M}}_n^\vee \otimes \mathbb{Z}/p^m\mathbb{Z}))$$

is also a Hecke×Galois–module and the famous theorem of Eichler-Shimura provides us some information on the structure of this module in terms of automorphic forms (See [De] and 6.1.1). We shall exploit the following principle:

The denominator of the Eisenstein-class (i.e. the number $\delta_p(n)$) has some influence on the structure of this Hecke× Galois-module and this forces the module to tell us something on the arithmetic of the cyclotomic field $\mathbb{Q}(\zeta_p)$ (The theorem of Herbrand-Ribet)

I will explain in 6.1.2.2 how the denominator influences the structure of the module. First I will try to give some idea how one gets this action of the Galois-group.

The starting point is that $\Gamma\backslash H$ is actually the set of complex points of a quasiprojective algebraic variety $\mathcal{S}/\mathbb{Q}$ and the above sheaves can be interpreted as sheaves for the étale site. This yields the action of the Galois-group on $H^1(\Gamma\backslash\tilde{H}, \tilde{\mathcal{M}}_n^\vee \otimes \mathbb{Z}_p)$. This will be explained briefly in the next section, a real understanding of these things requires considerable knowledge some results in arithmetic algebraic geometry (étale cohomology , large parts of the article of Deligne-Rapoport on the modular interpretation of the curve

$\Gamma\backslash H$ and the theory of p-adic representations of the Galois-group in the sense of the theory of Fontaine-Messing-Faltings.)

6.1.1 *The interpretation of $\tilde{\mathcal{M}}_n^\vee/N\tilde{\mathcal{M}}_n^\vee$ as étale sheaves:* To begin we define an action of $Gl_2(\mathbb{Z}/N\mathbb{Z})$ on $\tilde{\mathcal{M}}_n^\vee/N\tilde{\mathcal{M}}_n^\vee$ by the rule $\sigma(P(X,Y)) = P(aX+cY, bX+dY)det(\sigma)^{-n}$. We observe that we may twist this action by a power of the determinant, we just multiply the result of the above action by $det(\sigma)^\nu$ for some fixed ν the resulting $Gl_2(\mathbb{Z}/N\mathbb{Z})$-module is called $\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee[\nu]$. I have to explain some algebraic geometry, especially some implications of the results of Deligne - Rapoport [De-Ra].

The quotient $\Gamma\backslash H$ is actually the Riemann-sphere minus the point at infinity. It has, considered as an algebraic variety, a canonical model over $\mathbb{Q}$, it is the projective line $\mathbb{P}^1/\mathbb{Q}$ minus the point at infinity, I want to call this $\mathcal{S}/\mathbb{Q}$. We have the canonical embedding

$$j : \mathcal{S} \longrightarrow \mathbb{P}^1.$$

To any natural number N exists a curve $\mathcal{S}_N$ defined over $\mathbb{Q}$ and a map

$$\pi_N : \mathcal{S}_N \longrightarrow \mathcal{S},$$

which is an étale covering outside the points $0, 1 \in \mathbb{P}^1(\mathbb{Q})$ and the covering group is $Gl_2(\mathbb{Z}/N\mathbb{Z})/(\pm Id)$. (On the transcendental level this covering is in principle obtained by passing to the congruence subgroup

$$\Gamma(N) = \{\gamma \in \Gamma | \gamma \equiv Id \mathrm{mod}(N)\},$$

but it is slightly more complicated than that, the quotient $\Gamma(N)\backslash H$ yields only one connected component of $\mathcal{S}_N(\mathbb{C})$.) Let $\mathcal{S}'$ be the complement of the two points $0, 1$. By construction our sheaf $\tilde{\mathcal{M}}_n^\vee/N\tilde{\mathcal{M}}_n^\vee$ becomes trivial on $\mathcal{S}_N(\mathbb{C})$ if we pull it back by the map π_N. Hence we may also consider it as the trivial sheaf for the étale site on $\mathcal{S}_N$. We restrict it to the open subscheme $\mathcal{S}_N\backslash\pi^{-1}(0,1)$. The group $Gl_2(\mathbb{Z}/N\mathbb{Z})$ acts on this sheaf and since this is the fundamental group of the covering, it defines a sheaf on $\mathcal{S}'$. This is a standard procedure for the construction of sheaves. We have the embedding

$$i : \mathcal{S}' \longrightarrow \mathcal{S}$$

and the direct image $i_*(\tilde{\mathcal{M}}_n^\vee/N\tilde{\mathcal{M}}_n^\vee)$ is an étale sheaf on $\mathcal{S}$ which we also denote by $\tilde{\mathcal{M}}_n^\vee/N\tilde{\mathcal{M}}_n^\vee$ (The functor i_* is exact since we assume $6 \not| N$). Now we have the following fundamental facts:

(i) *We have an action of the Hecke algebra on the étale cohomology groups*

$$H^1_{\text{ét}}(\mathcal{S} \times_\mathbb{Q} \overline{\mathbb{Q}}, \mathcal{M}_n^\vee/N\mathcal{M}_n^\vee),$$

which commutes with the action of the Galois-group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$) .

(ii) *We have a comparison isomorphism*

$$\Phi : H^1_{\text{ét}}(\mathcal{S} \times_\mathbb{Q} \overline{\mathbb{Q}}, \mathcal{M}_n^\vee/N\mathcal{M}_n^\vee) \longrightarrow H^1(\Gamma\backslash H, \mathcal{M}_n^\vee/N\mathcal{M}_n^\vee),$$

which commutes with the action of the Hecke algebra on both sides.

(Of course we can perform the same construction also with the dual sheaf $\mathcal{M}_n/p\mathcal{M}_n$).

We define as usual $\hat{\mathbb{Z}} = \underleftarrow{\lim}(\mathbb{Z}/N\mathbb{Z})$ and define $\hat{R} = \hat{\mathbb{Z}}[\frac{1}{6}]$ then we have of course

$$\underleftarrow{\lim}(\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee) = \mathcal{M}_n^\vee \otimes \hat{R} = \hat{\mathcal{M}}_n^\vee$$

and this is again a Γ- module, which defines a sheaf $\tilde{\hat{\mathcal{M}}}_n^\vee$ on $\Gamma\backslash\tilde{H}$ and since we have no cohomology in degree two one checks easily that

$$H^1(\Gamma\backslash\tilde{H}, \tilde{\hat{\mathcal{M}}}_n^\vee) = \underleftarrow{\lim}(H^1(\Gamma\backslash\tilde{H}, \mathcal{M}_n^\vee/N\mathcal{M}_n^\vee)) = H^1(\Gamma\backslash\tilde{H}, \tilde{\mathcal{M}}_n^\vee) \otimes \hat{R}.$$

(One has to use the fact that the map $\mathbb{Z} \to \hat{R}$ is faithfully flat.) In the context of the étale cohomology we do not consider the cohomology with coefficients in the sheaf $\tilde{\hat{\mathcal{M}}}_n^\vee$, this does not give a good result, instead we define

$$H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\hat{\mathcal{M}}}_n^\vee) := \lim_{\leftarrow}(H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{M}_n^\vee/N\mathcal{M}_n^\vee)).$$

Using the comparison isomorphism and the above assertions concerning the transcendental cohomology we get a comparison

$$\Phi : H^1(\Gamma\backslash\tilde{H}, \tilde{\hat{\mathcal{M}}}_n^\vee) \to H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\hat{\mathcal{M}}}_n^\vee),$$

which of course commutes with the action of the Hecke-algebra. This comparison-isomorphism gives us the structure of a Galois-module to the transcendental cohomology if we extend the coefficients from $\tilde{\mathcal{M}}_n^\vee$ to $\tilde{\mathcal{M}}_n^\vee \otimes \hat{R}$.

Now we fix a prime p. It is known that $\hat{R} = \prod_{p,p\neq 2,3} \mathbb{Z}_p$ and choosing the above prime means that we project to one component in the product. It also amounts to the following: In the above construction we tacitly assumed that we took the projective limit with respect to the set of all integers N and the ordering was given by divisibility. Now we perform the same constructions as above but we restrict N to the set of powers of p. Then we put $\tilde{\mathcal{M}}_{n,p}^\vee = \tilde{\mathcal{M}}_n^\vee \otimes \mathbb{Z}_p$ and the above comparison gives us an isomorphism

$$\Phi : H^1(\Gamma\backslash\tilde{H}, \tilde{\mathcal{M}}_{n,p}^\vee) \to H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{n,p}^\vee),$$

where the right hand side is defined by a projective limit. If we restrict to the prime p we have the additional bonus that the representation of the Galois-group is unramified outside of the prime p. (This is not at all obvious and depends on the fact, that our scheme $\mathcal{S}$ has a smooth model over Spec($\mathbb{Z}$).

We also have the same construction for the cohomology with compact supports. If we want to define it in the context of étale cohomology we recall that we have the compactification $j : \mathcal{S} \hookrightarrow \mathbb{P}^1$. We extend the étale sheaf $\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee$ by zero to a sheaf $j_!(\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee)$ on $\mathbb{P}^1$.([SGA 4 1/2],) Then we have a comparison isomorphism

$$\Phi_c : H^1_c(\Gamma\backslash\tilde{H}, \mathcal{M}_n^\vee/N\mathcal{M}_n^\vee) \to H^1_{\text{ét}}(\mathbb{P}^1 \times_{\mathbb{Q}} \overline{\mathbb{Q}}, j_!(\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee)),$$

and we put

$$H^1_{\text{ét}}(\mathbb{P}^1 \times_{\mathbb{Q}} \overline{\mathbb{Q}}, j_!(\tilde{\hat{\mathcal{M}}}_n^\vee)) := \lim_{\leftarrow}(H^1_{\text{ét}}(\mathbb{P}^1 \times_{\mathbb{Q}} \overline{\mathbb{Q}}, j_!(\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee)).$$

We define always $H^1_!$ to be the image of H^1_c in H^1, then we have

$$H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\hat{\mathcal{M}}}_n^\vee) = \lim_{\leftarrow}(H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{M}_n^\vee/N\mathcal{M}_n^\vee))$$

and this is again a Hecke$\times$Galois-module. Again we restrict N to the powers of our given prime p, we get an exact sequence

$$0 \to H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{n,p}^\vee) \to H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{n,p}^\vee) \to H^1(\partial(\Gamma\backslash\tilde{H}), \tilde{\mathcal{M}}_{n,p}^\vee) \to 0, \qquad (Seq)$$

so far the last term is only defined in the transcendental context, but since the first two terms have a Galois-module structure, it inherits also such a structure.

We still may go one step further and tensorize the sheaves (or the cohomology groups) with $\mathbb{Q}_p$ then we get $\mathbb{Q}_p$ vector spaces together with an action of the Hecke-algebra and the Galois-group. We apply the results in V 5.9 and get :

If we go to a suitable finite algebraic extension E_ϖ of $\mathbb{Q}_p$ then we get a decomposition in isotypical spaces under the Hecke-algebra

$$H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{n,p}^\vee) \otimes E_\varpi = \bigoplus_\pi H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{n,p}^\vee)(\pi) \oplus E_\varpi \cdot Eis_n,$$

where the isotypical E_ϖ- vector spaces $H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{n,p}^\vee)(\pi)$ are two-dimensional Galois -modules and where $E_\varpi \cdot Eis_n$ is a one dimensional Galois-module .

At this point I can state the famous and fundamental result of Eichler- Shimura and Deligne (See[De]): *The action of the Galois-group* $\mathrm{Gal}(\bar{Q}/\mathbb{Q})$ *on* $H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^{\vee}_{n,p})(\pi)$ *is unramified outside* p *and for any prime* ℓ *different from* p *the trace and the determinant of the inverse Frobenius* Φ_ℓ^{-1} *are given by*

$$tr(\Phi_\ell^{-1}) = T_\ell(\pi) \text{ and } det(\Phi_\ell^{-1}) = \ell^{n+1}$$

This determines these modules as modules for the Galois-group. We will determine the structure of $E_\varpi \cdot Eis_n$ as a module for the Galois-group in the next section. Its structure is much simpler.

If $\mathcal{O}_\varpi$ is the ring of integers then $(H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^{\vee}_{n,p}) \otimes \mathcal{O}_\varpi)_{int}$ is a free $\mathcal{O}_\varpi$-module, the above decomposition will in general not introduce a decomposition of this module. But if we choose an ordering of the summands in the decomposition over E_ϖ, we get a filtration on $(H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^{\vee}_{n,p}) \otimes \mathcal{O}_\varpi)_{int}$. The successive quotients of this filtration are lattices in the corresponding vector spaces $H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^{\vee}_{n,p})(\pi)$ or $E_\varpi \cdot Eis_n$. They are Hecke$\times$Galois-modules.

6.1.2 *The Galois-module* $\mathbb{Z}_p \cdot eis_n$: We want to determine the Galois-module $\mathbb{Z}_p \cdot eis_n$. To do this we have to investigate the Galois-module- structure of $H^0(\partial(\Gamma\backslash\tilde{H}), \tilde{\mathcal{M}}^{\vee}_{n,p})$ and $H^1(\partial(\Gamma\backslash\tilde{H}), \tilde{\mathcal{M}}^{\vee}_{n,p})$. To be more precise we have to introduce a Galois-module structure on these cohomology-groups which fits into our diagram, and then we have to compute it.

To state our result we have to introduce the Tate-module $\mathbb{Z}_p(1)$. The group of p^m-th roots of unity

$$\mu_{p^m} = \{\zeta \in \bar{\mathbb{Q}} \mid \zeta^{p^m} = 1\}$$

is (non canonically) isomorphic to the module $\mathbb{Z}/p^m\mathbb{Z}$ and the Galois-group acts on this module by a homomorphism

$$\alpha : \mathrm{Gal}(\bar{Q}/\mathbb{Q}) \to (\mathbb{Z}/p^m\mathbb{Z})^*,$$

which is defined by the rule $\sigma(\zeta) = \zeta^{\alpha(\sigma)}$. If we pass to the projective limit over all m we get $\varprojlim(\mu_{p^m}) \approx \mathbb{Z}_p$ and the Galois-group acts on this limit by this limit of the above α's, this is a character

$$\alpha : \mathrm{Gal}(\bar{Q}/\mathbb{Q}) \to \mathbb{Z}_p^*$$

which is the so called Tate-character. We denote the module $\mathbb{Z}_p$ with the above action of the Galois-group on it by $\mathbb{Z}_p(1)$. We define $\mathbb{Z}_p(\nu)$ to be the Galois-module $\mathbb{Z}_p$ with the action $\sigma(x) = \alpha(\sigma)^\nu x$ for $\sigma \in \mathrm{Gal}(\bar{Q}/\mathbb{Q}), x \in \mathbb{Z}_p$. I assert

Proposition 6.1.2.1:*The Galois-module* $\mathbb{Z}_p \cdot eis_n$ *is isomorphic to* $\mathbb{Z}_p(-n-1)$.

This is by no means obvious, I will try to give an outline of the proof, I do not know whether I should advise the reader to skip it.

We constructed the étale sheaves $\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee$ on $\mathcal{S}$ and we have two way to extend it to a sheaf on the compactification: We discussed already the extension $j_!(\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee)$ whose stalk at infinity is zero, this is an exact functor. We may also take the direct image $j_*(\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee)$ on the compactification $\mathbb{P}^1$. Then we have to take into account that this direct image functor is not exact, hence we have to consider the derived functors $R^\bullet j_*(\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee)$. We get a spectral sequence

$$H^\bullet_{\text{ét}}(\mathbb{P}^1 \times_{\mathbb{Q}} \bar{\mathbb{Q}}, R^\bullet j_*(\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee)) \Longrightarrow H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{M}_n^\vee/N\mathcal{M}_n^\vee),$$

to state this in modern terms, we may compute the cohomology of our sheaf on the open piece also as the cohomology of a complex of sheaves on the compactification. This yields us the exact sequence

$$H^1_{c,\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{M}_n^\vee/N\mathcal{M}_n^\vee) \to H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{M}_n^\vee/N\mathcal{M}_n^\vee) \to H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, R^\bullet j_*(\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee)/j_!(\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee)) \to$$

the quotient in the argument of the last term is a complex of sheaves which is concentrated in the point at infinity. Hence we may consider it as a complex of $\mathbb{Z}/N\mathbb{Z}$- modules on which we have an action of the Galois-group, simply because an étale sheaf on Spec($\mathbb{Q}$) is simply a module for the Galois-group. Then the H^1 of this complex of sheaves is simply the stalk $R^1j_*(\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee)_\infty$. Under the present conditions we can pass to the projective limit and we still get an exact sequence

$$H^1_{c,\text{ét}}(\mathcal{S}\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{M}_n^\vee\otimes\hat{R})\to H^1_{\text{ét}}(\mathcal{S}\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{M}_n^\vee\otimes\hat{R})\to R^1j_*(\mathcal{M}_n^\vee\otimes\hat{R})_\infty,$$

we shall see that the last term is actually equal to $H^1_{\text{ét}}(\partial(\Gamma\backslash\tilde{H}),\tilde{\mathcal{M}}_n^\vee\otimes\hat{R})$ hence we gain an action of the Galois group on the cohomology of the boundary after we tensorize by $\hat{R}$. We also know that the cohomology of the boundary splits off a canonical direct summand $\hat{R}\cdot Eis_n$ which is also a Galois-module, this is the one we want to understand. (The reader should observe that in the previous chapters the cohomology of the boundary was computed from the Borel-Serre compactification, this is an object that has nothing to do with algebraic geometry).

To get the structure of these Galois-modules we remind ourselves of what would we do in the transcendental context. We take a little disc D_∞ around the point ∞ in $\mathbb{P}^1(\mathbb{C})$, the intersection of D_∞ with $\Gamma\backslash\tilde{H}=\mathcal{S}(\mathbb{C})$ is the punctured disc $\dot{D}_\infty$, we may restrict our sheaf $\mathcal{M}_n^\vee$ to $\dot{D}_\infty$. We have the embedding

$$j:D_\infty\to\dot{D}_\infty$$

and we want to compute the derived functors $R^\bullet j_*(\mathcal{M}_n^\vee)$. We recall that our sheaves $\mathcal{M}_n^\vee$ where defined through an action of the group Γ, but it is clear that the restriction of the sheaf to the punctured disc is obtained from the action of the fundamental group $\pi_1(\dot{D}_\infty)=\Gamma_\infty=\mathbb{Z}$ on $\mathcal{M}_n^\vee$. Since we are only interested in the free part we may replace the sheaf $\tilde{\mathcal{M}}_n^\vee$ by $\tilde{\mathcal{M}}_n$. We have an emdedding

$$\mathcal{M}_n/p\,\mathcal{M}_n\hookrightarrow\mathcal{M}_{n+1},$$

which is given by $X^\nu Y^{n-\nu}\to X^{\nu+1}Y^{n-\nu}$ and which commutes with the action of $\Gamma_\infty=\pi_1(\dot{D}_\infty)$. Hence we have exact sequences

$$0\to\tilde{\mathcal{M}}_n\to\widetilde{\mathcal{M}_{n+1}}\to R\cdot\widetilde{Y^{n+1}}\to 0$$

of sheaves on $\dot{D}_\infty$. It is easy to see that the boundary operator of the long exact sequence in cohomology provides an isomorphism modulo torsion

$$H^0(\dot{D}_\infty,R\cdot Y^{n+1})\to H^1(\dot{D}_\infty,\tilde{\mathcal{M}}_n).$$

This gives an alternative method to compute the cohomology of the boundary. The point is that this can be imitated in the arithmetic context and then we will be able to read off the Galois-module structure. Let u be the uniformizing element at ∞ we replace the disc by the spectrum of the power series ring

$$D\hat{}_\infty=\mathrm{Spec}(\mathbb{Q}[[u]]),$$

and $\dot{D}\hat{}_\infty=D\hat{}_\infty\backslash\{\infty\}$.

For any integer N we define $D\hat{}_N=\mathrm{Spec}(\mathbb{Q}[\zeta_N][[v]])$, where ζ_N is a primitive N-th root of unity and $v^N=u$. We have a map $D\hat{}_N\to D\hat{}_\infty$, which becomes étale if we remove the point ∞. The Galois group of this étale covering is isomorphic to the group of matrices

$$B_N=\{\sigma=\begin{pmatrix}a&b\\0&1\end{pmatrix}|a\in(\mathbb{Z}/N\mathbb{Z})^*,b\in\mathbb{Z}/N\mathbb{Z}\}\subset Gl_2(\mathbb{Z}/N\mathbb{Z}).$$

This group acts on our module $\mathcal{M}_n/p\,\mathcal{M}_n/N\mathcal{M}_n/p\,\mathcal{M}_n$ and by the same procedure that gave us the sheaves $\tilde{\mathcal{M}}_n$ on $\mathcal{S}$ we get the restriction of these sheaves to$\dot{D}_\infty\hat{}$ if we restrict the group action to B_N. Hence we have the exact sequences of sheaves on$\dot{D}_\infty\hat{}$ as before (remember the twist in the definition of $\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee$ as $Gl_2(\mathbb{Z}/N\mathbb{Z})$-module)

$$0 \to \widetilde{\mathcal{M}_n/N\mathcal{M}_n} \to \widetilde{\mathcal{M}_{n+1}/N\mathcal{M}_{n+1}} \to (\mathbb{Z}/N\mathbb{Z})Y^{n+1} \to 0.$$

This yields a coboundary map

$$j_*(R\cdot Y^{n+1}) \to R^1 j_*(\widetilde{\mathcal{M}_n/N\mathcal{M}_n})$$

which becomes an isomorphism modulo torsion if we pass to the projective limit over N. This implies that the Galois group acts on $\hat{R}\cdot Eis_n$ in the same way as it acts on the left hand side. But it is clear that the element $\sigma \in B_N$ acts on $(R/NR)Y^{n+1}$ by multiplication by a^{-n-1}. (This explains the strange twist we introduced, when we defined the $Gl_2(\mathbb{Z}/N\mathbb{Z})$, it has the effect that the Galois-group acts trivially on $R^0 j_*(\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee))$

If we pass to the limit over $N = p^k$, then we see that

$$R^j_*(\tilde{\mathcal{M}}_n^\vee) = \lim_{\leftarrow}(R^1(\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee)) = \mathbb{Z}_p(-n-1) + torsion,$$

and this is the assertion of our proposition.

6.1.2.2: $\mathbb{Z}_p$*-Hecke modules*: At this point I want to explain some very simple principles concerning the structure of modules under the Hecke algebra:

Here I want to look at the Hecke-algebra $\mathbf{H}$ as the polynomial algebra over $\mathbb{Z}$ generated by the indeterminates T_ℓ one of them for each prime ℓ. We fix a prime p and we consider finitely generated $\mathbb{Z}_p$-modules X on which $\mathbf{H}$ acts. I want to make the following assumption

The Hecke-operator T_p *acts nilpotently on* X_{tors}

If I want to make I category out of these objects I should require that cokernels of maps have this property too. I claim that each such module has a canonical decomposition

$$X = X_{\mathrm{nil}} \oplus X_{\mathrm{ord}},$$

so that T_p acts topologically nilpotent on X_{nil} (i.e we have $T_p^m(X_{\mathrm{nil}}) \subset pX_{\mathrm{nil}}$ for some m) and T_p induces an isomorphism on X_{ord}. The module X_{ord} is called the ordinary part of X it is torsion free. If we apply this construction to X_{int} we get $X_{int,ord} = X_{\mathrm{ord}}$.

This is indeed very elementary. We consider the vector space $X \otimes \bar{\mathbb{Q}}_p$ and decompose it into generalized eigenspaces under the Hecke algebra. This means that we have a finite set $Spec(X)$ of homomorphisms

$$\lambda : \mathbf{H} \to \bar{\mathbb{Q}}_p$$

such that we get a decomposition into generalized eigenspaces

$$X \otimes \bar{\mathbb{Q}}_p = \bigoplus_{\lambda \in Spec(X)} Z_\lambda,$$

where $Z_\lambda = (\xi \in X \otimes \bar{\mathbb{Q}}_p | (T_\ell - \lambda(T_\ell))^N \xi = 0)$ for a suitably large number N. Since X is a finitely generated $\mathbb{Z}_p$-module the values $\lambda(T_\ell)$ will be integers in $\bar{\mathbb{Q}}_p$, we decompose $Spec(X) = Spec(X)_{tnilp} \cup Spec(X)_{ord}$ according to whether $\lambda(T_p)$ is in the maximal ideal or it is a unit. Then we get a decomposition

$$X \otimes \bar{\mathbb{Q}}_p = \bigoplus_{\lambda \in Spec(X)_{tnilp}} Z_\lambda \oplus \bigoplus_{\lambda \in Spec(X)_{ord}} Z_\lambda = Z_{tnilp} \oplus Z_{ord}.$$

The two summands are invariant under the action of $\mathcal{G}al_{\text{pur}}$ and therefore this decomposition descends to a decomposition over $\mathbb{Q}_p$:

$$X \otimes \mathbb{Q}_p = Y_{tnilp} \oplus Y_{ord}$$

and we define

$$X_{int,tnilp} := Y_{tnilp} \cap X_{int} \quad X_{int,ord} = Z_{ord} \cap X_{int}.$$

Now one has to prove that

$$X_{int} = X_{int,tnilp} \oplus X_{int,ord},$$

it is clear that the left hand side contains the direct sum on the right hand side. I leave this as an exercise to the reader.

This proves the claim for X_{int}, it follows from our general assumption on the torsion that we have a section from X_{ord} in X_{int} back to X. The following assertions are now obvious

(i) X_{ord} is a free $\mathbb{Z}_p$-module, its rank is equal to the sum of the dimensions of the spaces Z_λ if λ runs over $Spec(X)_{ord}$

(ii) We get a decomposition

$$X \otimes \mathbb{Z}/p = X_{\text{nil}} \otimes \mathbb{Z}/p \oplus X_{\text{ord}} \otimes \mathbb{Z}/p$$

where the first summand is the generalized eigenspace to the eigenvalue 0 for T_p and where T_p induces an isomorphism on the second summand.

(iii) The functor $ord : X \to X_{\text{ord}}$ would be an exact functor if we had made a category out of these modules in the above sense. (This is not true for *int*).

We define the Eisenstein-part of the spectrum: Let $(\bar{\pi})$ be the maximal ideal of the ring of intgers in $\bar{\mathbb{Q}}_p$, we define

$$Spec_{Eis}(X) = (\lambda \in Spec(X)_{ord} | \lambda(T_\ell) \equiv \ell^{n+1} + 1 \text{mod}(\bar{\pi}) \text{ for all } \ell).$$

The same reasoning as before yields that the space

$$(X \otimes \bar{\mathbb{Q}}_p)_{Eis} := \bigoplus_{\lambda \in Spec_{Eis}(X)} Z_\lambda$$

descends to a subspace in $X_{ord} \otimes \mathbb{Q}_p$ and we have a decomposition

$$X_{ord} \otimes \mathbb{Q}_p = (X_{\text{ord}} \otimes \mathbb{Q}_p)_{nonEis} \oplus (X_{\text{ord}} \otimes \mathbb{Q}_p)_{Eis}.$$

Again it is also clear, that intersecting this direct sum decomposition with X_{ord} gives us

$$X_{\text{ord}} = X_{ord,nonEis} \oplus \text{Eis}\, X$$

and alltogether

$$X = X_{\text{nil}} \oplus X_{ord,nonEis} \oplus \text{Eis}\, X.$$

The following facts are obvious

(iv) Any endomorphism of X, which commutes with the action of the Hecke algebra leaves this decomposition invariant

(v) Rank$(\text{Eis}\, X)$ equals the sum of the dimensions of the Z_λ with $\lambda \in Spec_{Eis}(X)$.

(vi) $\text{Eis}\, X \otimes \mathbb{Z}/p$ is the submodule of $X \otimes \mathbb{Z}/p$ on which all the operators $T_\ell - (\ell^{n+1} + 1)$ act nilpotently.

We apply this to our exact sequence (Seq) and we restrict it to the Eisenstein-part, this yields

$$0 \to H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{Eis} \to H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{Eis} \to H^1(\partial(\Gamma\backslash\tilde{H}), \tilde{\mathcal{M}}^\vee_{n,p}) \to 0,$$

of course the third term is already in the Eisenstein-part.

Now we discuss the influence of the denominator of the Eisenstein-class on the structure of the cohomology as Hecke$\times$Galois-module. As before we write the denominator as $p^{\delta_p(n)}$, by construction we get an exact sequence

$$0 \longrightarrow H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{Eis} \oplus \mathbb{Z}_p \cdot eis_n \longrightarrow H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{Eis} \longrightarrow \mathbb{Z}/p^{\delta_p(n)} \longrightarrow 0.$$

tensorizing this sequence with $\mathbb{Z}/p^{\delta_p(n)}$gives us an exact sequence

$$H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{Eis} \otimes \mathbb{Z}/p^{\delta_p(n)} \oplus (\mathbb{Z}/p^{\delta_p(n)}) \cdot eis_n \to H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{Eis} \otimes \mathbb{Z}/p^{\delta_p(n)} \to \mathbb{Z}/p^{\delta_p(n)} \to 0.$$

The kernel of the last arrow is $H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{Eis}$ and hence we get a surjective map

$$H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{Eis} \otimes \mathbb{Z}/p^{\delta_p(n)} \oplus (\mathbb{Z}/p^{\delta_p(n)}) \cdot eis_n \to H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{Eis} \otimes \mathbb{Z}/p^{\delta_p(n)}.$$

This map is of course of the form (Id, Ψ), where Ψ is a map

$$\Psi : (\mathbb{Z}/p^{\delta_p(n)}) \cdot eis_n \to H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{Eis} \otimes \mathbb{Z}/p^{\delta_p(n)}.$$

I claim

Lemma:*This map is injective and commutes with the action of the Hecke-operators and the Galois-group.*

Proof: The injectivity follows from the fact that $\mathbb{Z}_p \cdot eis_n$ is a primitive submodule hence it is a direct summand (as a $\mathbb{Z}_p$-module) and therefore $\mathbb{Z}/p^{\delta_p(n)} \cdot eis_n$ injects into $H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{Eis} \otimes \mathbb{Z}/p^{\delta_p(n)}$. The rest is clear.

6.1.3. *The arithmetic consequences*: Now we are ready to discuss the influence of the denominator of the Eisenstein-class on the arithmetic of cyclotomic fields. I recall the decomposition of the cohomology-groups into eigenspaces for the Hecke algebra (See 6.1.1). If we use our results from the previous section then we find that $\text{rank} H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{\text{Eis}}$ is 2 times the number of $\pi \in Spec(H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p}))$ which satisfy

$$\pi(T_\ell) \equiv \ell^{n+1} + 1 \bmod p \text{ for all } \ell$$

Therefore we know that $\text{rank}(H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{\text{Eis}}) \geq 2$ if $\delta_p(n) > 0$. This is a classical assertion on congruences between Eisenstein- series and cusp forms, statements of this kind occur in the work if Doi, Hida, Koike and Ribet.

A pair $(p, n+2)$ is called *irregular* if $\delta_p(n) > 0$. I propose to call $(p, n+2)$ *tamely irregular* if the rank of $H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{\text{Eis}}$ is equal to 2. I call it*wildly irregular* if it is greater than 2. We shall see in the next section that the type of irregularity of $(p, n+2)$ depends only on $(n+2)\bmod(p-1)$.

In general the Galois group $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ or in our situation $\text{Gal}(E_\varpi/\mathbb{Q}_p)$ acts on $Spec(H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p}))$, but if the pair is tame the the single element in the Eisenstein-part of the spectrum is of course defined over $\mathbb{Q}_p$.

I want to formulate the theorem of Herbrand-Ribet, actually I will state a stronger version of it in case of a tamely irregular prime. One will see from our arguments how one gets the theorem of Herbrand-Ribet without this restriction. It seems to me that our result in this special situation is even stronger than the consequences one gets from the Main-conjecture of Mazur-Wiles. In the following discussion we abbreviate $\delta_p(n) = \delta$

Theorem II: *If $p^\delta | \zeta(1-(n+2))$ and if the pair $(p, n+2)$ is tame then the cyclotomic field $\mathbb{Q}(\zeta_{p^\delta})$ of p^δ-th roots of unity has a cyclic extension $L/\mathbb{Q}(\zeta_{p^\delta})$ of degree p^δ, which is normal over $\mathbb{Q}$, everywhere unramified and so that the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\delta}/\mathbb{Q}) = (\mathbb{Z}/p^\delta\mathbb{Z})^*$ acts on $\mathrm{Gal}(L/\mathbb{Q}(\zeta_{p^\delta}))$ by multiplication by x^{-n-1}.*

If $\delta = 1$ this is the part of Ribet in the theorem of Herbrand-Ribet if we drop the assumption of tameness.

Proof: Let us define $X = H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^{\vee}_{n,p})_{\mathrm{Eis}}$ Since we assume that the pair $(p, n+2)$ is tame this is a free $\mathbb{Z}_p$- module of rank 2. The $\mathbb{Z}/p^\delta$-module $X \otimes \mathbb{Z}/p^\delta$ is also free of rank 2 and it contains $\mathbb{Z}/p^\delta \cdot eis_n$ as a free, rank one Galois-submodule, we get a short exact sequence

$$0 \to eis_n \to X \otimes \mathbb{Z}/p^\delta \to \mathbb{Z}/p^\delta \to 0.$$

The Galois group $\mathrm{Gal}(\bar{Q}/\mathbb{Q})$ acts by $\alpha^{-n-1}\mathrm{mod}(\mathbb{Z}/p^\delta)$ on the submodule and trivially on the quotient, the first assertion follows from proposition 6.1.2.1 the second from the formula for the determinant of the Frobenius in the theorem of Eichler-Shimura-Deligne quoted above.(This comes from a simple duality argument). We restrict the action to the Galois group $\mathrm{Gal}(\mathbb{Q}/\mathbb{Q}(\zeta_{p^\delta}))$, let $L^\sharp/\mathbb{Q}(\zeta_{p^\delta})$ be the smallest extension over which this module becomes trivial. Hence we have an action of $\mathrm{Gal}(L^\sharp/\mathbb{Q}(\zeta_{p^\delta}))$ on $X \otimes \mathbb{Z}/p^\delta$.

I quote a result from the theory of p-adic representations. We choose a prime in $\bar{\mathbb{Q}}$ this provides an embedding of the local Galois group $\mathcal{G}al_{\mathrm{pur}} \subset \mathrm{Gal}(\bar{Q}/\mathbb{Q})$. We have some information concerning the action of this local group on a Hecke eigenspace $H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^{\vee}_{n,p})(\pi)$ if π is ordinary, i.e. $\pi(T_p)$ is a unit in $\mathcal{O}_\varpi$. In this case this result asserts, that we have an exact sequence of $\mathcal{G}al_{\mathrm{pur}}$-modules

$$0 \to E_\varpi(0) \to H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^{\vee}_{n,p})(\pi) \to E_\varpi(-n-1) \to 0.$$

Such a sequence induces on any $\mathcal{G}al_{\mathrm{pur}}$ invariant $\mathcal{O}_\varpi$-lattice in $H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^{\vee}_{n,p})(\pi)$ a filtration, where the submodule and the quotient have to be replaced by $\mathcal{O}_\varpi(0)$ and $\mathcal{O}_\varpi(-n-1)$.

We apply this to our element π that makes up the Eisenstein-part of the spectrum. Then the local theorem implies that we have a canonical splitting of the $\mathcal{G}al_{\mathrm{pur}}$- module

$$X \otimes \mathbb{Z}/p^\delta = \mathbb{Z}/p^\delta(-n-1) \oplus \mathbb{Z}/p^\delta(0),$$

because the local filtration goes in the opposite direction.

Now we proceed along similar lines as Ribet (See [Ri2]). We choose a basis e, f to our lattice X such that these two basis vectors give this splitting if we reduce modulo p^δ and of course e should be the one that reduces to eis_n. (This depends of course on the choice of $\mathcal{G}al_{\mathrm{pur}} \subset \mathrm{Gal}(\bar{Q}/\mathbb{Q})$). Then we have for $\sigma \in \mathrm{Gal}(\bar{Q}/\mathbb{Q})$ that

$$\sigma(e) = a(\sigma)e + c(\sigma)f$$
$$\sigma(f) = b(\sigma)e + d(\sigma)f$$

and we have that modulo p^δ the matrix reduces to

$$\begin{pmatrix} \alpha(\sigma)^{-n-1} & c(\sigma) \\ 0 & 1 \end{pmatrix}.$$

Let us assume that we find an element σ in the Galois group for which $c(\sigma) \not\equiv 0 \mathrm{mod} p$. We know that the local Galois group $\mathcal{G}al_{\mathrm{pur}}$ gives us all the elements

$$\begin{pmatrix} \alpha(\tau)^{-n-1} & 0 \\ 0 & 1 \end{pmatrix}$$

if we reduce modulo p^δ and therefore we find also an element σ in the global Galois group which has the matrix

$$\begin{pmatrix} 1 & c(\sigma) \\ 0 & 1 \end{pmatrix}$$

and $c(\sigma) \not\equiv 0 \mathrm{mod} p$. This element generates a cyclic subgroup of order p^δ in $\mathrm{Gal}(L^\sharp/\mathbb{Q}(\zeta_{p^\delta}))$, which is invariant under the action of the action of $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\delta})/\mathbb{Q})$. The Galois group $\mathrm{Gal}(L^\sharp/\mathbb{Q}(\zeta_{p^\delta}))$ has exponent p^δ so it is a $\mathbb{Z}/(p^\delta)$ module and the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\delta})/\mathbb{Q}) = (\mathbb{Z}/(p^\delta))^*$ acts on this module by multiplication by x^{-n-1}. Moreover it is clear that $L^\sharp/\mathbb{Q}(\zeta_{p^\delta})$ is unramified everywhere, because the p-adic Galois modules are certainly unramified outside p and at p we have the above reduction of the action of $\mathcal{G}al_{\mathrm{pur}}$ to the diagonal group. So we are through if the above assumption is true. But if it is not true then we modify our lattice to $\mathbb{Z}_p \cdot e \oplus \mathbb{Z}_p \cdot (f/p)$. This new lattice will be invariant under the action of the Galois group and with respect to this basis we have to make the following changes in our matrix above:

$$c(\sigma) \to c(\sigma)/p$$

$$d(\sigma) \to d(\sigma) \cdot p$$

To this lattice we can apply the same arguments as before, actually we could now even compute mod $p^{\delta+1}$. So we will eventually construct the desired extension unless we have $c(\sigma) = 0$ for all $\sigma \in \mathrm{Gal}(\bar{Q}/\mathbb{Q})$. But this is not possible, because the representation of the Galois-group on $X \otimes \mathbb{Q}_p$ is irreducilble ([Ri1], Thm.2.3), hence the theorem is proved.

Remarks: 6.1.4: If we drop the assumption that the pair is tamely irregular, then we may have several π in $Spec(H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{\mathrm{Eis}})$. If we order them they induce a filtration on our lattice $X \otimes \mathcal{O}_\varpi = H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{\mathrm{Eis}} \otimes \mathcal{O}_\varpi$. Let e be the ramification index of E_ϖ over $\mathbb{Q}_p$ and $\Delta = e \cdot \delta$. The same arguments as before show that we have still an injective map $\Psi : \mathcal{O}_\varpi/(\varpi^\delta) \cdot eis_n \to X \otimes \mathcal{O}_\varpi/(\varpi^\delta)$. This induces a non trivial map into one step of this filtration. To this step we can obviously apply the same reasoning as before (with a little modification). Therefore we see that we can prove the theorem of Herbrand-Ribet without the assumption that p is tame.

6.1.5: After the discussion of the proof of theorem I we shall see in next section that we give indeed a modular construction of the minus part of the Hilbert classfield of the field $\mathbb{Q}(\zeta_p)$ under the assumption that the prime is tamely irregular.

6.1.6: Our method here differs in one point from the methods used by Ribet and Mazur-Wiles. In some sense they use the de-Rham-realization of the cohomology and exploit informations they get from there to get results on the p-adic representations. They read off from the q-expansion of the classical Eisenstein-series, that it has a denominator as a de-Rham cohomology class. From there they get some information on the Jordan-Hölder series of the $\mathbb{Z}_p$ -integral cusp forms as a module under the Hecke-algebra. Then they invoke the theorem of Eichler-Shimura and Deligne to get some information on the Galois-module structure of the steps in the filtration. Here we work directly with the p-adic realization, which to me seems to be more natural and opens some perspectives for generalisations (See [HS] and 6.4). Our point of view is that of Haberland in [Hab], but he falls short proving Theorem I because he restricts the weights (See 6.4.1 and [Hab], 5.2 Satz 3)

6.2. **Modular Symbols:** The patient reader may have forgotten that we still have to prove Theorem I.

The main idea of the proof is quite simple. We construct certain homology classes which generate $H_1(\Gamma\backslash\tilde{H}, \underset{\sim}{\mathcal{M}}_n)$ and we evaluate the Eisenstein classe on these homology classes. The result of this evaluation gives us a collection of rational numbers. The smallest common denominator of these rational numbers will be the denominator of the Eisenstein class. Here we need the theory of singular homology with coefficients as it is explained in Kapitel E 1(Chapter II in the final version).

6.2.1. The homology classes are the so called modular symbols (with coefficients). At the present moment $\Gamma \subset SL_2(\mathbb{Z})$ may be an arbitrary congruence subgroup. To any two points $r, s \in \mathbb{P}^1(\mathbb{Q})$ we have the geodesic circle $C_{r,s}$ in the upper half plane which joins these two points. We recall the discussion of the Borel-Serre compactification in V, 5.1. To any point $r \in \mathbb{P}^1(\mathbb{Q})$ we added the line $H_{r,\infty}$ of real Borel subgroups in opposition to r and we defined

$$\tilde{H} = H \cup \bigcup_{r\in\mathbb{P}^1(\mathbb{Q})} H_{r,\infty}.$$

It is clear form the definition that $C_{r,s}$ closes up to an intervall in $\tilde{H}$. It hits $H_{r,\infty}$ in the points s and $H_{s,\infty}$ in the point r. Let us denote the point s on $H_{r,\infty}$ by $\{s\}_r$. For $m \in \mathcal{M}_n$ we introduce the 1-chain

$$C_{r,s} \otimes m \in C_1(\Gamma\backslash\tilde{H}, \underset{\sim}{\mathcal{M}}_n).$$

Its boundary is

$$\partial(C_{r,s} \otimes m) = \{s\}_r \otimes m - \{r\}_s \otimes m.$$

(We orient the chain by going from r to s.) This chain is actually a relative 1-cycle, i. e. an element of

$$Z_1(\Gamma\backslash\tilde{H}, \partial(\Gamma\backslash\tilde{H}), \underset{\sim}{\mathcal{M}}_n)$$

and one can prove that these cycles generate the first relative homology group

$$H_1(\Gamma\backslash\tilde{H}), \underset{\sim}{\mathcal{M}}_n)$$

(Diplomarbeit Gebertz, Bonn 198.). From now on we assume that $\Gamma = SL_2(\mathbb{Z})$ and we consider the special 1-chains

$$C_{\infty,0} \otimes e_\nu \qquad\qquad \nu = -\frac{n}{2}, \dots, \frac{n}{2}.$$

We compute its boundary, since

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

interchanges 0 and ∞ in $\mathbb{P}^1(\mathbb{Q})$, we obtain

$$\partial(C_{\infty,0} \otimes e_\nu) = \{0\}_\infty \otimes e_\nu - \{\infty\}_0 \otimes e_\nu = \{0\}_\infty \otimes (e_\nu - w\, e_\nu)$$

(We may visualize these 1-chains on the quotient $\Gamma\backslash\tilde{H}$. We have the path from ∞ to i on the imaginary axis. If we "tensor" it by e_ν we get a 1-chain $[\infty, i] \otimes e_\nu$. Now we make a turn and go back on the same path but we "tensor" by $w\, e_\nu$. This 1-chain has the boundary

$$\{0\}_\infty \otimes (e_\nu - w\, e_\nu) + \{i\} \otimes (e_\nu - w\, e_\nu).$$

But since i is a fixed point for w the second term is zero.)

We abbreviate

$$\mathcal{Z}_\nu^{(f)} = C_{\infty,0} \otimes e_\nu,$$

and denote by $[\mathcal{Z}_\nu^{(f)}]$ the corresponding relative homology class.

Our Eisenstein classes Eis_n are cohomology classes hence they cannot be evaluated on the relative homology classes $[\mathcal{Z}_\nu^{(f)}]$. But we can try to bound the boundary $\partial\mathcal{Z}_\nu^{(f)} \in C_0(\partial(\Gamma\backslash\tilde{H}), \underset{\sim}{\mathcal{M}}_n)$ by a 1-chain in $C_1(\partial(\Gamma\backslash\tilde{H}), \underset{\sim}{\mathcal{M}}_n)$, i.e. we can try to write

$$\partial\mathcal{Z}_\nu^{(f)} = \partial\mathcal{Z}_\nu^{\infty}$$

with some $\mathcal{Z}_\nu^{\infty} \in C_1(\partial(\Gamma\backslash\tilde{H}), \underset{\sim}{\mathcal{M}}_n)$. If we can do this then

$$\mathcal{Z}_\nu^{(f)} - \mathcal{Z}_\nu^{\infty} \in Z_1(\Gamma\backslash\tilde{H}, \underset{\sim}{\mathcal{M}}_n)$$

would define an absolute homology class. This does not always work, we have to use the Hecke-operators to modify this construction.

The evaluation of the Eisenstein class on classes of this type will provide us the information on the denominator.

6.2.2. We want to discuss the problem to find $\mathcal{Z}_\nu^{\infty}$. The group

$$U_{\infty,\infty} = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\} = \mathbb{R}$$

acts by definition simply transitively on $H_{\infty,\infty}$. We identify $\mathbb{R} = H_{\infty,\infty}$ by this action and taking the base point $0 \in \mathbb{P}^1(\mathbb{R})$. A point $x \in \mathbb{R}$ when viewed as a point on $H_{\infty,\infty}$ is denoted by $\{x\}_\infty$.

If we want to bound $\{0\}_\infty \otimes e_\nu$, we try to write

$$\{0\}_\infty \otimes e_\nu = \partial([0,1]_\infty \otimes P)$$

where $P \in \mathcal{M}_n$ and this means we have to write

$$e_\nu = (1-T)P(X,Y) = P(X,Y) - P(X, X+Y),$$

where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ as in 5.7. If we pass to rational coefficients, we can find such a P if and only if $\nu \neq -\frac{n}{2}$, i.e. $e_\nu \neq Y^n$. Since we want to bound $\{0\}_\infty(e_\nu - w\, e_\nu) = \{0\}_\infty \otimes (e_\nu - (-1)^{\frac{n}{2}+\nu} e_{-\nu})$, we always require $\nu \neq \pm\frac{n}{2}$. In this case we know that we may bound by a $P \in \mathcal{M}_{n,\mathbb{Q}}$. But we need something better since we want to get integral homology classes. In this case we only know that $\{0\}_\infty \otimes (e_\nu - w\, e_\nu)$ defines a torsion element in $H_0(\partial(\Gamma\backslash\tilde{H}), \mathcal{M}_n)$. At this point we have to use the Hecke operators to annihilate these classes. To be more precise we localize at the prime p (the one for which we want to prove Theorem I), and this means we tensorize our modules with $\mathbb{Z}_{(p)}$, the local ring at p. For simplicity we denote the localized modules and sheaves still by $\mathcal{M}_n, \tilde{\mathcal{M}}_n, \ldots$. Then we have seen in 5.7. that for $\nu \neq \pm\frac{n}{2}$ the zero cycles $\{0\} \otimes e_{\pm\nu}$ define classes which will be annihilated by a power T_p^m of the Hecke operator at p. Hence we may write

$$T_p^m(\{0\} \otimes (e_\nu - w\, e_\nu)) = \partial\mathcal{Z}_\nu^{\infty}$$

with $\mathcal{Z}_\nu^{\infty} \in C_1(\partial(\Gamma\backslash\tilde{H}), \mathcal{M}_n)$ and

$$T_p^m\ \mathcal{Z}_\nu^{(f)}\ -\ \mathcal{Z}_\nu^{\infty}$$

will be a 1-cycle which defines a homology class in $H_1(\Gamma\backslash\tilde{H}, \underset{\sim}{\mathcal{M}}_n)$: Hence we may evaluate

$$\langle Eis_n, T_p^m\ \mathcal{Z}_\nu^{(f)}\ -\ \mathcal{Z}_\nu^\infty\rangle.$$

The result is a rational number. We also take into account that the 1-chain $\mathcal{Z}_\nu^\infty$ is not unique. It may be modified by a cycle $C \in Z_1(\partial(\Gamma\backslash\tilde{H}), \underset{\sim}{\mathcal{M}}_n)$. This changes the evaluation above by

$$\langle Eis_n, [C]\rangle,$$

and this is an integer (in $\mathbb{Z}_{(p)}$) since Eis_n restricted to the boundary is an integral class. Hence we should consider

$$\langle Eis_n, T_p^m \mathcal{Z}_\nu^{(f)} - \mathcal{Z}_\nu^\infty\rangle \in \mathbb{Q}/\mathbb{Z}_{(p)}.$$

We have the following

Lemma 6.2.2.1: *The p-part of the denominator of the Eisenstein class Eis_n is equal to the maximal p-denominator of the numbers*

$$\langle Eis_n, T_p^m\ \mathcal{Z}_\nu^{(f)} - \mathcal{Z}_\nu^\infty\rangle.$$

Proof: It is not difficult to see that the $[\mathcal{Z}_\nu^{(f)}]$ generate the relative homology $H_1(\Gamma\backslash\tilde{H}, \partial(\Gamma\backslash\tilde{H}), \underset{\sim}{\mathcal{M}}_n)$ (Diplomarbeit, A. Geberts or exercise). Since we obviously have $\mathcal{Z}_\nu^{(f)} = \pm\mathcal{Z}_{-\nu}^{(f)}$ it suffices to look at $\nu \geq 0$.

We recall that we are looking for the p-denominator of the Eisenstein class Eis_n, i.e. we want to find the smallest $\delta \geq 0$ such that $p^\delta Eis_n$ becomes integral. It follows from Poincaré-duality that we can find a homology class $\xi \in H_1(\Gamma\backslash\tilde{H}, \underset{\sim}{\mathcal{M}}_n)$ such that

$$\langle p^\delta Eis_n, \xi\rangle = 1$$

and hence

$$\langle Eis_n, \xi\rangle = \frac{1}{p^\delta}.$$

We send ξ into the relative homology and we may write its image ξ' in the form

$$\xi' \ = \sum_{\nu=0}^{\frac{n}{2}-1}\ r_\nu\ [\mathcal{Z}_\nu^{(f)}] \qquad\qquad e_\nu \in \mathbb{Z}_{(p)}.$$

(The class $\mathcal{Z}_{\frac{n}{2}}^{(f)}$ cannot occur since otherwise the boundary of ξ' in $H_0(\partial(\Gamma\backslash\tilde{H}), \underset{\sim}{\mathcal{M}}_n)$ would not be zero).

We apply T_p^m. We have from our adjunction formula

$$\begin{aligned}\langle Eis_n, T_p^m\xi\rangle = \langle T_p^m\ Eis_n, \xi\rangle &= (p^{n+1}+1)\cdot\langle Eis_n, \xi\rangle \\ = \frac{(p^{n+1}+1)^m}{p^\delta}\end{aligned},$$

and hence $T_p^m\xi$ gives us the same denominator.

Now

$$T_p^m\xi' = \sum r_\nu T_p^m \mathcal{Z}_\nu^{(f)}.$$

This class lifts back to a class in the absolute homology by our previous construction, and we have that

$$\sum r_\nu(T_p^m \mathcal{Z}_\nu^{(f)} - \mathcal{Z}_\nu^\infty) \quad \text{and} \quad \xi$$

differ by an element that comes from $H_1(\partial(\Gamma\backslash\tilde{H}), \underset{\sim}{\mathcal{M}}_n)$. On this difference the Eisenstein class Eis_n takes an integral value and now the Lemma is clear.

I propose to call a 1-cycle, which is a sum of chains of the type $C_{r,s} \otimes m$ and chains in the boundary a *modular symbol*.

6.2.3 Now we discuss the evaluation of the Eisenstein class on these cycles.

We will represent the Eisenstein class by a suitable 1-form. We will modify our cycle into another one which is homologous to it, and whose support is in $\Gamma\backslash H$. Then we may evaluate by integration and eventually we push the cycle back into its original position.

We have a closer look at the cycle

$$(T_p^m)\mathcal{Z}_\nu^{(f)} - \mathcal{Z}_\nu^\infty = \mathcal{Z}_\nu.$$

We shall see that the chain $(T_p^m)\mathcal{Z}_\nu^{(f)}$ will be a combination of chains of the form $C_{\infty,r} \otimes x_r$ with $x_r \in \mathcal{M}_n$. In the neighborhood of $\Gamma_\infty\backslash H_{\infty,\infty}$ in $\Gamma\backslash\tilde{H}$. Such a cycle is supported by two (or one) vertical lines, so that we get the following picture for $(T_p^m)\mathcal{Z}_\nu^{(f)}$ at infinity

$\Gamma_\infty\backslash H_{\infty,\infty}$

The chain $\mathcal{Z}_\nu^\infty$ is supported in $\Gamma_\infty\backslash H_{\infty,\infty}$. It is clear that we get a cycle homologous to $T_p^m\mathcal{Z}_\nu^{(f)} - \mathcal{Z}^{(\infty)}$ if we chop of the cylinder on a finite level far enough out. Then the boundary will be

$$\Gamma_\infty\backslash\{z \mid \mathrm{Im}(z) = y\}$$

and our modified cycle will be

$$\mathcal{Z}_\nu[y] = T_p^m\mathcal{Z}_\nu^{(f)}[y] - \mathcal{Z}_\nu^\infty[y] \in Z_1(\Gamma\backslash H, \underset{\sim}{\mathcal{M}}_n),$$

where $\mathcal{Z}_\nu^\infty[y]$ is the chain corresponding to $\mathcal{Z}_\nu^\infty$ in $\Gamma_\infty\backslash\{z \mid \mathrm{Im}(z) = y\}$.

If we represent the class Eis_n by a 1-form $\widetilde{Eis}_n$, then we have

$$\langle Eis_n, \mathcal{Z}_\nu\rangle = \langle Eis_n, \mathcal{Z}_\nu[y]\rangle = \int\limits_{\mathcal{Z}_\nu[y]} \widetilde{Eis}_n = \int\limits_{(T_p)^m\mathcal{Z}_\nu^{(f)}[y]} \widetilde{Eis}_n - \int\limits_{\mathcal{Z}_\nu^\infty[y]} \widetilde{Eis}_n.$$

Proposition 6.2.3.1 :*We can choose the representing form $\widetilde{Eis}_n$ such that*

$$T_p\,\widetilde{Eis}_n = (p^{n+1}+1)\,\widetilde{Eis}_n$$

and such that the limits

$$\lim_{y\to\infty} \int\limits_{(T_p^m)\mathcal{Z}_\nu^{(f)}[y]} \widetilde{Eis_n} \qquad \text{and} \qquad \lim_{y\to\infty} \int\limits_{\mathcal{Z}_\nu^{\infty}[y]} \widetilde{E}is_n$$

exist. Then we get

$$\langle Eis_n, \mathcal{Z}_\nu \rangle \;=\; (p^{n+1}+1)^m \int\limits_{\mathcal{Z}_\nu^{(f)}} \widetilde{E}is_n - \lim_{y\to\infty} \int\limits_{\mathcal{Z}_\nu^{\infty}[y]} \widetilde{E}is_n.$$

Proof: We recall the construction of the Eisenstein classes. Given n we have the character $\chi_\infty = \chi = (n+1, n)$ to which we attach the $(\mathfrak{g}, K)$-module I_{χ_∞}. The Eisenstein intertwining operator gives us an embedding

$$Eis : I_{\chi_\infty} \hookrightarrow \mathcal{A}(SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})).$$

We have seen that the Hecke operators act by scalars on the module $Eis(\chi_\infty)$ and using 5.5.4 we see that they act with eigenvalues $\{p^{n+1}+1\}_p$ on

$$\mathrm{Hom}_K(\Lambda^1(\mathfrak{g}/\mathfrak{k}), I_{\chi_\infty} \otimes \mathcal{M}_{n,\mathbb{C}}).$$

We have seen in 4.3.3 that

$$\mathrm{Hom}_K(\Lambda^1(\mathfrak{g}/\mathfrak{k}), I_{\chi_\infty} \otimes \mathcal{M}_{n,\mathbb{C}}) = \mathrm{Hom}_{K^T}(\Lambda^1(\mathfrak{a}/\mathfrak{k}^T \oplus \mathfrak{u}), \mathbb{C}_{\chi_\infty + s} \otimes \mathcal{M}_{n,\mathbb{C}})$$

where of course $\mathfrak{k}^T = 0$ and $K^T = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\}$. We have the generator $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ of $\mathtt{a}$ and $E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ of $\mathfrak{u}$ and therefore we can define the element

$$\tilde{\omega}_n : \begin{cases} H & \longrightarrow 0 \\ E_+ & \longrightarrow 1 \otimes Y^n \end{cases}$$

of $\mathrm{Hom}_{K^T}(\Lambda^1(\mathfrak{a} \oplus \mathfrak{u}), \mathbb{C}_{\chi_\infty + \delta} \otimes \mathcal{M}_{n,\mathbb{C}})$ and

$$\widetilde{E}is_n = Eis(\omega_n) \in \mathrm{Hom}_K(\Lambda^1(\mathfrak{g}/\mathfrak{k}), \mathcal{A}(\Omega_2(\mathbb{Z})\backslash SL_2(\mathbb{R})^\bullet \otimes \mathcal{M}_{n,\mathbb{C}}).$$

If we restrict the Eisenstein class to a neighvorhood of the boundary $\Gamma_\infty \backslash \overline{H}_{\infty,\infty}$, i.e. to

$$\Gamma_\infty \setminus \left\{ z \mid \mathrm{Im}(z) > y_0 \right\} \subset \Gamma \backslash H$$

then it is dominated by its constant term. We have seen in 5.4.5 that this constant term is

$$\tilde{\omega}_n \in \mathrm{Hom}_K(\Lambda^1(\mathfrak{g}/\mathfrak{k}), I_\infty \otimes \mathcal{M}_{n,\mathbb{C}})) \subset \mathrm{Hom}_K(\Lambda^1(\mathrm{g}/\mathrm{k}), \mathcal{C}_\infty(\Gamma_\infty \backslash SL_2(\mathbb{R})) \otimes \mathcal{M}_{n,\mathbb{C}}).$$

But $\tilde{\omega}_n$ vanisches on the tangent vectors of the chains $C_{\infty,r} \otimes m$ in a neighborhood of $\Gamma_\infty \backslash \overline{H}_{\infty,\infty}$ and therefore we see that

$$\widetilde{E}is_n(\text{Tangentvectors of } T_p^m \mathcal{Z}_\nu^{(f)} \text{ of length } 1)$$

goes rapidly to zero, if we approach $\Gamma_\infty \backslash H_{\infty,\infty}$ or if $y \to \infty$. (Faster than $0(y^{-N})$ for any N.) This implies that the limit

$$\lim_{y\to\infty} \int\limits_{\mathcal{Z}_\nu^{(f)}[y]} \widetilde{E}is_n$$

exists. (Of course this depends on the choice of our form $\tilde{\omega}_n$. There are several choices possible which gice the same cohomology class on $\Gamma_\infty\backslash H$. If we had taken the classical holomorphic Eisenstein series, we would have some trouble.) But now it is also clear that

$$\lim_{y\to\infty} \int\limits_{\mathcal{Z}_\nu^\infty[y]} \widetilde{Eis}_n$$

converges, and if we want to complete the limit, we can substitute $\widetilde{Eis}_n$ by its constant term. The proposition is proved.

I want to discuss the computation of the limit. The following considerations are purely technical and consist of bookkeeping of the definitions.

We recall the construction of the compactification. We have

$$\begin{array}{ccc} \Gamma_\infty\backslash H & \hookrightarrow & \Gamma_\infty\backslash\tilde{H} \\ \| & & \| \\ S^1\times\mathbb{R}_{>0} & \hookrightarrow & S^1\times(\mathbb{R}_{>0}\cup\{\infty\}). \end{array}$$

A 1-chain in $C_1(\Gamma_\infty\backslash H_{\infty,\infty},\underset{\sim}{\mathcal{M}}_n)$ is a linear combination of expressions

$$c=[a,b]\otimes P$$

where $P(X,Y)\in\mathcal{M}_n$ and where we identified $H_{\infty,\infty}\simeq\mathbb{R}$ (see 6.2). If we push the 1-chain to a finite level, we get

$$c_y[a,b]=[a+iy,b+iy]\otimes P.$$

We want a formula for

$$\lim_{y\to\infty} \int\limits_{c_y[a,b]} \widetilde{Eis}_n.$$

In Kapitel E5 we gave the rule how to compute such integrals. We lift the map

$$\begin{array}{cccc} \sigma & : & [0,1] & \longrightarrow & H \\ \sigma & : & t & \longrightarrow & iy_0+a+t(b-a) \end{array}$$

to a map into the group. For simplicity we change the group from $SL_2(\mathbb{R})$ to $GL_2(\mathbb{R})^+$, and we choose

$$\begin{array}{ccccc} \tilde{\sigma} & : & [0,1] & \longrightarrow & GL_2(\mathbb{R}) \\ \tilde{\sigma} & : & t & \longrightarrow & \begin{pmatrix} y_0 & a+t(b-a) \\ 0 & 1 \end{pmatrix}. \end{array}$$

Then in the notations of Kapitel E, p. 23 we have

$$\sigma^*(t)(\frac{\partial}{\partial t})=\frac{b-a}{y}\,E_+,$$

and the formula at the bottom of that page yields

$$\int\limits_{c_y[a,b]} \widetilde{Eis}_n=\frac{b-a}{y}\int\limits_0^1 \langle\widetilde{\mathrm{Eis}}_n(\tilde{\sigma}(t))(E_*),\tilde{\sigma}(t)^{-1}m\rangle\,dt.$$

Since the Eisenstein series is dominated by its constant term, we may substitute $\tilde{\omega}_n$ for $\widetilde{Eis}_n$ when we pass to the limit $y \to \infty$. By definition we have

$$\tilde{\omega}_n(\tilde{\sigma}(t))(E_+) = y_0^{n+1}\tilde{\omega}_n(t)(E_+) = y_0^{n+1}Y^n.$$

We have

$$\tilde{\sigma}(t)^{-1}P(X,Y) = \tilde{\sigma}(t)^{-1}\sum a_\nu X^\nu Y^{n-\nu} = \sum a_\nu y_0^{-\nu} X^\nu \Big(\frac{-a-t(b-a)}{y_0}X+Y\Big)^{n-\nu}.$$

Since we have to pair with Y^n we only have to consider the coefficient of X^n and get for the integral

$$\int\limits_{c_y[a,b]} \tilde{\omega}_n = (b-a)\int\limits_0^1 \sum_{v=0}^{n} a_\nu(-a-t(b-a))^{u-\nu}dt = \int\limits_{-b}^{-a}(\sum a_\nu n^{n-\nu})du.$$

Hence we find the following formula for the integration of the Eisensteinclass against a 1-chain $[a,b]\otimes P(X,Y) \in C_1(\partial(\Gamma_\infty\backslash H_{\infty,\infty}), \underset{\sim}{\mathcal{M}}_n)$. We substitute $1,u$ for X,Y into $P(X,Y)$ and get

$$\lim_{y\to\infty}\int\limits_{c_y[a,b]} \widetilde{Eis}_n = \int\limits_{-b}^{-a} P(1,u)\;du. \tag{$**$}$$

6.2.4 We are now ready to carry out the evaluation. Before starting I will state the result. For any ν with $-\frac{n}{2} < \nu < \frac{n}{2}$ we define the following numbers mod$\mathbb{Z}_{(p)}$

$$b_{n,\nu} = \begin{cases} 0 & \text{if} \quad p-1 \nmid \frac{n}{2}+\nu+1 \\[1ex] \dfrac{1}{p\cdot(\frac{\frac{n}{2}+\nu+1}{p-1})} & \text{if} \quad p-1 \mid \frac{n}{2}+\nu+1. \end{cases}$$

Then we shall prove

$$\langle Eis_n, (T_p)^m \mathcal{Z}_\nu^{(f)} - \mathcal{Z}_\nu^\infty\rangle = (p^{n+1}+1)^m\cdot\frac{\zeta(1-(\frac{n}{2}+\nu+1))\zeta(1-(\frac{n}{2}-\nu+1))}{\zeta(1-(n+2))} + b_{n,\nu} + b_{n,-\nu} \bmod \mathbb{Z}_{(p)},$$

where of course $\zeta(\;)$ is the ordinary Riemann-zeta function.

It is clear that this has to be done in two steps. We will prove that

$$\int\limits_{\mathcal{Z}_\nu^{(f)}} \widetilde{Eis}_n = \frac{\zeta(1-(\frac{n}{2}+\nu+1))\cdot\zeta(1-(\frac{n}{2}-\nu+1))}{\zeta(1-(n+2))}$$

and

$$\int\limits_{\mathcal{Z}_\nu^\infty} \widetilde{Eis}_n = -b_{n,\nu} - b_{n,-\nu} \bmod \mathbb{Z}_{(p)}.$$

Let us look at the boundary terms first. We consider $\mathcal{M}_{n,\mathbb{Q}}$ as $GL_2(\mathbb{Q})$-module with the action

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} P(X,Y) = P(aX+cY, bX+dY)$$

i.e. without any twist by a power of the determinant. Then we have the advantage that $M_2(\mathbb{Z})$ acts on $\mathcal{M}_n$.

Let us consider the zero-cycle $\{0\}_\infty \otimes e_\nu$. For m large enough we have to bound

$$(T_p)^m(\{0\}_\infty \otimes e_\nu)$$

by a 1-chain (remember that we localized at the prime p). The general formula for the Hecke operator is

$$T_p(\{r\}_\infty \otimes m) = \sum_{j \neq \mathrm{mod} p} \left\{ \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} r \right\}_\infty \otimes \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} m + \{pr\}_\infty \otimes \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} m.$$

One checks easily that it decomposes into a sum of two operators

$$U_p : \{r\} \otimes m \longrightarrow \sum_{j \mathrm{mod} p} \left\{ \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} r \right\}_\infty \otimes \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} m$$

and

$$V_p : \{r\} \otimes m \longrightarrow \{pr\} \otimes \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} m.$$

We have

$$\begin{aligned} V_p U_p \{r\}_\infty \otimes m &= V_p \left(\sum \left\{ \tfrac{r+j}{p} \right\}_\infty \otimes \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} m \right) = \\ &= \sum \{r+j\}_\infty \otimes \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \cdot p^n m = p^{n+1} \cdot \{r\} \otimes m \end{aligned}$$

.

Hence if we evaluate

$$(U_p + V_p)^m \{0\}_\infty \otimes e_\nu,$$

and we take into account that

$$V_p : \{0\}_\infty \otimes e_\nu \to p^{\frac{n}{2}+\nu} \{0\}_\infty \otimes e_\nu,$$

we find that this is equal to

$$U_p^m \{0\}_\infty \otimes e_\nu + c_1 U_p^{m-1} \{0\}_\infty \otimes e_\nu + c_2 U_p^{m-2} \{0\}_\infty \otimes e_\nu \ldots$$

where

$$\begin{aligned} c_1 &\equiv p^{\frac{n}{2}+\nu} \mathrm{mod}\, p^n \\ c_2 &\equiv p^{(\frac{n}{2}+\nu)^2} \mathrm{mod}\, p^n \\ &\vdots \end{aligned} .$$

(This must be so since only those products of U's and$V$'s which have the V's at their right end can give an contribution which is non zero mod p.)

Now we have for all m

$$U_p^m(\{0\} \otimes e_\nu) = \sum_{j_1, \ldots, j_m = 0}^{p-1} \left\{ \frac{j_1 + j_2\, p + \ldots + j_m\, p^{m-1}}{p^m} \right\}_\infty \otimes \begin{pmatrix} 1 & j_m \\ 0 & p \end{pmatrix} \cdots \begin{pmatrix} 1 & j_1 \\ 0 & p \end{pmatrix} e_\nu.$$

We put

$$\underline{j} = j_1 + j_2\, p \ldots j_m\, p^{m-1},$$

then the sum becomes

$$\sum_{j_1, \ldots, j_m = 0}^{p-1} \left\{ \frac{\underline{j}}{p^m} \right\}_\infty \otimes X^{\frac{n}{2}+\nu} (\underline{j} X + p^m Y)^{\frac{n}{2}-\nu}.$$

Let us look at the individual terms in the sum. I claim that for large m we can find a $P \in \mathcal{M}_n$ such that

$$X^{\frac{n}{2}+\nu}(\underline{j}X + p^m Y)^{\frac{n}{2}-\nu} = \partial P = P(X,Y) - P(X,-X+Y).$$

This is so because the term Y^n does not occur, X^n can be written in that form and all other terms are highly divisible by p. We substitute $X = 1$ and $Y = u$. Then we get

$$(\underline{j} + p^m u)^{\frac{n}{2}-\nu} = \partial P(1,u) = P(1,u) - P(1,u-1).$$

We write

$$P(1,u) = a_0 + a_1 u + \dots$$

with $a_i \in \mathbb{Z}_{(p)}$, the constant coefficient a_0 is arbitrary. Now we have written the zero-cycle

$$\{\underline{j}/p^m\}_\infty \otimes X^{\frac{n}{2}+\nu}(\underline{j}X + p^m Y)^{\frac{n}{2}-\nu}$$

a boundary of the 1-cycle

$$[\underline{j}/p^m, \underline{j}/p^m + 1]_\infty \otimes P,$$

so it yields according to our formula (∗∗) the contribution

$$\int\limits_{-\underline{j}/p^m-1}^{-\underline{j}/p^m} P(1,u)\,du = \tilde{P}_1(-\underline{j}/p^m) - \tilde{P}_1(-\underline{j}/p^m - 1)$$

where $\tilde{P}_1$ is a primitive polynomial for the polynomial $P(1,u)$. We normalize it to $\tilde{P}_1(0) = 0$, hence

$$\tilde{P}_1(u) = a_0\, u + \frac{a_1}{2}\, u^2 + \frac{a_3}{3}\, u^3 \dots .$$

The $\tilde{Q}_1(u) = \tilde{P}_1(u) - \tilde{P}_1(u-1)$ is a primitive polynomial for

$$Q(1,u) = (\underline{j} + p^m u)^{\frac{n}{2}-\nu}$$

and therefore we get for our above integral

$$\int\limits_{-\underline{j}/p^m-1}^{-\underline{j}/p^m} P_1(u)\,du = \tilde{Q}_1(-\underline{j}/p^m).$$

We do not yet know the constant term of $\tilde{Q}_1$ (it is not arbitrary) but in any case we have

$$\tilde{Q}_1(u) = \frac{1}{(\frac{n}{2}-\nu+1)p^m} \cdot (\underline{j} + p^m u)^{\frac{n}{2}-\nu+1} + \tilde{Q}_1(0) - \frac{\underline{j}^{\frac{n}{2}-\nu+1}}{(\frac{n}{2}-\nu+1)p^m}$$

and hence

$$\int\limits_{-\underline{j}/p^{m-1}}^{-\underline{j}/p^m} P_1(u)\,du = \tilde{Q}_1(0) - \frac{\underline{j}^{\frac{n}{2}-\nu+1}}{p^m(\frac{n}{2}-\nu+1)} = -\tilde{P}_1(-1) - \frac{\underline{j}^{\frac{n}{2}-\nu+1}}{p^m(\frac{n}{2}-\nu+1)}.$$

Let us consider

$$-\tilde{P}_1(-1) = +a_0 - \frac{a_1}{2} + \frac{a_2}{3} \dots .$$

Since we have

$$(\underline{j} + p^m u)^{\frac{n}{2}-\nu} = P_1(u) - P_1(u-1),$$

we have

$$P_1(u) = \underline{j}u + P_1^*(u),$$

where the coefficients of $P_1^*(u)$ are highly divisible by p. Hence we see (since $p > 2$) that $-P_1(-1) \in \mathbb{Z}_{(p)}$ and the contribution of an individual term in the sum giving $U_p^m(\{0\}_\infty \otimes e_\nu)$ is

$$-\frac{\underline{j}^{\frac{n}{2}-\nu+1}}{(\frac{n}{2}-\nu+1)p^m} \bmod \mathbb{Z}_{(p)}.$$

We consider the expression

$$S_m(\mu) = \frac{1}{p^m \mu} \sum_{j=0}^{p^m-1} j^\mu \bmod \mathbb{Z}_{(p)}.$$

Since $\underline{j}$ runs over the integers mod p^m, we have to consider sums ($\mu = 1, 2, \ldots$)

$$S_m(\mu) = \frac{1}{p^m \mu} \sum_{j=0}^{p^m-1} j^\mu,$$

and look at their value as an element in $\mathbb{Q}/\mathbb{Z}_{(p)}$. One checks that the values become stationary if m is sufficiently large, and we get the same value if we sum over any system of representatives. I claim that

$$S_m(\mu) \equiv 0 \bmod \mathbb{Z}_{(p)} \qquad \text{if} \qquad p-1 \nmid \mu$$

and

$$S_m(\mu) \equiv \frac{-1}{p \cdot \frac{p-1}{\mu}} \qquad \text{if} \qquad p-1 \mid \mu$$

provided that m is sufficiently large. I leave this as an excercise for the reader. Now we have to sum up the contributions form the $p^{(\frac{n}{2}+\nu)\alpha} U_p^{m-\alpha}\{0\}_\infty \otimes e_\nu$. They are zero mod $\mathbb{Z}_{(p)}$ if $\alpha > 0$ because the factor in front cancels the deniminator, hence we have proved the desired formula for the contribution of the infinite part $\mathcal{Z}_\nu^\infty$.

To evaluate the finite part we recall that the Eisensteinclass is a form with values in $\mathcal{M}_{n,\mathbb{C}}^\vee$ and the chains take values in $\mathcal{M}_{n,\mathbb{C}}$. In the previous considerations we let $\alpha \in GL_2(\mathbb{Q})$ act on $\mathcal{M}_{n,\mathbb{Q}}$ by

$$\alpha P(X,Y) = P(aX + cY, bX + dY),$$

and hence it should act on $P^\vee \in \mathcal{M}_{n,\mathbb{C}}^\vee$ by

$$\alpha P^\vee(X,Y) = P(aX + cY, bX + dY)\det(\alpha)^{-n}.$$

But it is clear that the value of the integral does not change if we twist both actions by powers of the determinant which are inverse to each other. Hence we twist the action on $\mathcal{M}_{n,\mathbb{Q}}$ by the factor $\det(\alpha)^{-\frac{n}{2}}$. This has the effect that the center of GL_2 acts trivially on $\mathcal{M}_{n,\mathbb{Q}}$ and $\mathcal{M}_{n,\mathbb{Q}}^\vee$. Another effect is that we can work on the group PGL_2 instead of SL_2, which has some advantages.

We want to compute the integral

$$\int_{\mathcal{Z}_\nu^{(f)}} \widetilde{Eis}_n.$$

and of course we should follow the rules from Chapter E. We lift the chain into the group and define

$$\begin{array}{lcll} \tilde\sigma & : & \mathbb{R}^*_{>0} & \longrightarrow & PGL_2^+(\mathbb{R}) \\ \tilde\sigma & : & t & \longrightarrow & \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \end{array} ,$$

where we consider $\mathbb{R}^*_{>0} \subset [0,\infty] = [0,1]$. Then the support of our chain $\mathcal{Z}_\nu^{(f)}$ is given by the map

$$\sigma : t \longrightarrow \tilde\sigma(t) i = ti.$$

The differential D_σ of σ maps the tangent vector $t \cdot \frac{\partial}{\partial t}$ at a point t_0 to the tangent vector $y \cdot \frac{\partial}{\partial y}$ at $t_0 i = y_0$ and hence

$$D^{-1}_{L_{\tilde\sigma(t_0)}} \circ D_\sigma(t_0) \cdot (t \frac{\partial}{\partial y}) = \frac{\partial}{\partial y} \mid_i .$$

We have the canonical identification of $\mathfrak{g}/\mathfrak{k}$ to the tangent space T_i^H and under this identification the vector

$$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}/\mathfrak{k}$$

corresponds to $\frac{\partial}{\partial y} \mid_i$. We have to compute the integral

$$\int_{\mathcal{Z}_\nu^{(f)}} \widetilde{Eis}_n = \int_0^\infty \langle \tilde\sigma(t) \widetilde{Eis}_n(\tilde\sigma(t))(h_1), e_\nu \rangle \frac{dt}{t} =$$

$$= \lim_{y \to \infty} \int_{1/y}^{y} \langle \tilde\sigma(t) \widetilde{Eis}_n(\tilde\sigma(t))(h_1), e_\nu \rangle \frac{dt}{t}.$$

We want to transform this integral into an integral over an adelic variable. We consider the torus

$$T_1 = \left\{ \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \right\} \subset GL_2$$

which is identified to the multiplicative group by the map

$$\chi : x \longrightarrow \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}.$$

This yields identifications

$$\begin{array}{lcll} \chi_{\mathbb{A}} & : & I_{\mathbb{Q}} & \xrightarrow{\sim} & T_1(\mathbb{A}) \\ & & I_{\mathbb{Q}}/\mathbb{Q}^* & \longrightarrow & T_1(\mathbb{A})/T_1(\mathbb{Q}) . \end{array}$$

If we divide $T_1(\mathbb{A})$ by the maximal compact subgroup $K_{f,0}^{T_1} = \prod_p \mathbb{Z}_p^*$, then we get

$$T_1(\mathbb{A})/T(\mathbb{Q}) K_{f,0}^{T_1} \simeq \mathbb{R}^*_{>0}.$$

This identification is induced by the map

$$t_\infty \longrightarrow \left(\begin{pmatrix} t_\infty & 0 \\ 0 & 1 \end{pmatrix}, 1, \dots, 1, \dots \right).$$

We recall the adelic construction of the Eisenstein series in 5.8. To the character

$$\chi \; : \; B(\mathbb{A}) \longrightarrow \mathbb{C}^*$$
$$\chi \; : \; \begin{pmatrix} \underline{t}_1 & \underline{u} \\ 0 & \underline{t}_2 \end{pmatrix} \longrightarrow |\underline{t}_1|^{\frac{n+1}{2}} \cdot |\underline{t}_2|^{\frac{-n+1}{2}}$$

we have the induced $(\mathfrak{g}, K) \times G(\mathbb{A}_f)$-modul $\underline{I}_\chi$ which we map by the Eisenstein intertwining operator

$$Eis \; : \; \underline{I}_\chi \longrightarrow \mathcal{C}_\infty(GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}))$$
$$Eis \; : \; \Psi(\underline{g}) \longrightarrow \sum_{\gamma \in B(\mathbb{Q})\backslash GL_2(\mathbb{Q})} \Psi(\gamma \underline{g})$$

into the space of automorphic forms. (It does not make a difference if we work on PGL_2 or on GL_2.) We evaluate the Eisenstein operator at the element

$$\Psi_0 = \Psi_\infty \otimes \Psi_f^0$$

where Ψ_f^0 is the standard spherical function and where $\Psi_\infty(g_\infty) = \tilde{\omega}_n(h_1)(g_\infty)$ it is an element in $\underline{I}_\chi \otimes \mathcal{M}_{n,\mathbb{C}}^\vee$. Then our integral

$$\int_{\mathbb{R}^*_{>0}} \langle \tilde{\sigma}_\infty(t_\infty) Eis(\Psi_0)(\tilde{\sigma}_\infty(t_\infty)), e_\nu \rangle \, \frac{dt_\infty}{t_\infty}.$$

We can move the element $\tilde{\sigma}_\infty(t_\infty) = \begin{pmatrix} t_\infty & 0 \\ 0 & 1 \end{pmatrix}$ to the other side, since e_ν is a weight vector for the torus we get by defintion of the action that this is equal to

$$\int_{\mathbb{R}^*_{>0}} t_\infty^{-\nu} \langle Eis(\Psi_0)(\tilde{\sigma}_\infty(t_\infty)), e_\nu \rangle \, \frac{dt_\infty}{t_\infty}.$$

This is now equal to the integral

$$\int_{I_\mathbb{Q}/\mathbb{Q}^*} |\underline{t}|^{-\nu} \langle Eis(\Psi_0)(\chi_\mathbb{A}(\underline{t})), e_\nu \rangle \, \frac{d\underline{t}}{|\underline{t}|}.$$

Here $\frac{d\underline{t}}{|\underline{t}|}$ is an invariant measure on $I_\mathbb{Q}$. It is the product of local measures $\frac{dt_\infty}{t_\infty}$ at infinity and $\mu_p = c_p \frac{dt_p}{|t_p|}$ at the finite places where at the finite place the volume of $T_1(\mathbb{Z}_p) = \mathbb{Z}_p^*$ is normalized to one.

We recall the definition of the Eisenstein class and get for that integral

$$\int_{I_\mathbb{Q}/\mathbb{Q}^*} |\underline{t}|^{-\nu} \sum_{\gamma \in B(\mathbb{Q})\backslash GL_2(\mathbb{Q})} \langle \Psi_0(\gamma \cdot \chi_\mathbb{A}(\underline{t})), e_\nu \rangle \, \frac{d\underline{t}}{|\underline{t}|}.$$

The torus $T_1(\mathbb{Q})$ acts on $B(\mathbb{Q})\backslash GL_2(\mathbb{Q}) = \mathbb{P}^1(\mathbb{Q})$ with three orbits. We have

$$GL_2(\mathbb{Q}) = B(\mathbb{Q}) \cup B(\mathbb{Q}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cup B(\mathbb{Q}) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} T_1(\mathbb{Q}).$$

The first two points corespond to fixed points, on the third the torus $T_1(\mathbb{Q})$ acts simply transitively. We get

$$\sum_{\gamma \in B(\mathbb{Q})\backslash GL_2(\mathbb{Q})} \langle \Psi_0(\chi_\mathbb{A}(\underline{t})), e_\nu \rangle = \langle \Psi_0(\chi_\mathbb{A}(\underline{t})), e_\nu \rangle + \langle \Psi_0(w \cdot \chi_\mathbb{A}(\underline{t}) w^{-1} w), e_\nu \rangle$$
$$+ \sum_{a \in T_1(\mathbb{Q})} \langle \Psi_0 \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} a \cdot \chi_\mathbb{A}(\underline{t}) \right), e_\nu \rangle.$$

The first two terms are zero. For this one has to recall the definition of $\tilde{\omega}_n$ and one has to take into account that $\nu \neq \pm\frac{n}{2}$. So we are left with

$$\int\limits_{T_1(\mathbb{A})/T_1(\mathbb{Q})} |\underline{t}|^{-\nu} \left(\sum_{a \in T_1(\mathbb{Q})} \langle \Psi_0 \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} a \cdot \chi_{\mathbb{A}}(\underline{t}) \right), e_\nu \rangle \right) \frac{d\underline{t}}{|\underline{t}|}.$$

Of course we want to write this as

$$\int\limits_{T_1(\mathbb{A})} |\underline{t}|^{-\nu} \langle \Psi_0 \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} a \cdot \chi_{\mathbb{A}}(\underline{t}) \right), e_\nu \rangle \frac{d\underline{t}}{|\underline{t}|}.$$

It is clear that we are allowed to do so if we can prove that the integral over $T_1(\mathbb{A})$ is absolutely convergent. But this will be clear from the following computation.

The point is of course that now the integral can be written as a product over local integrals. We have that the above integral is equal to

$$\prod_v \int\limits_{T_1(\mathbb{Q}_v)} |t_v|^{-\nu} \langle \Psi_{o,v} \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} t_v \right), e_\nu \rangle \frac{dt_v}{|t_v|}.$$

(We will see in the process of the computation that the local integrals are absolutely convergent and that the infinite product is so too.)

I studied integrals of this kind in my papers [Ha1] and [Ha2]. I more or less copy the corresponding computations here. At a finite place p we have to evaluate

$$\int\limits_{T_1(\mathbb{Q}_p)} |t_p|^{-\nu} \cdot \Psi_{0,p} \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} t_p & 0 \\ 0 & 1 \end{pmatrix} \right) \frac{dt_p}{|t_p|} = \int\limits_{T_1(\mathbb{Q}_p)} |t_p|^{-\nu} \cdot \Psi_{0,p} \left(\begin{pmatrix} t_p & 0 \\ t_p - 1 & 1 \end{pmatrix} \right) \frac{dt_p}{|t_p|}.$$

We write $t_p = p^\mu \cdot \epsilon$, where μ runs over the integers and ϵ over the units.

We apply the Iwasawa decomposition and get

$$\begin{pmatrix} \epsilon p^\mu & 0 \\ \epsilon p^\mu - 1 & 1 \end{pmatrix} = \begin{cases} \begin{pmatrix} p^\mu & * \\ 0 & 1 \end{pmatrix} k_p & \text{if } \mu \geq 0 \\ \begin{pmatrix} 1 & * \\ 0 & p^\mu \end{pmatrix} k_p & \text{if } \mu < 0 \end{cases}$$

and hence we have

$$\Psi_{0,p} \left(\begin{pmatrix} \epsilon p^\mu & 0 \\ \epsilon p^\mu - 1 & 1 \end{pmatrix} \right) = \begin{cases} p^{-\mu(\frac{n}{2}+1)} & \text{if } \mu \geq 0 \\ p^{\mu(\frac{n}{2}+1)} & \text{if } \mu < 0 \end{cases}.$$

(Note that we have unitary induction here).

Since we have normalized the measure to give volume one on the units, we find

$$\int\limits_{T_1(\mathbb{Q}_p)} |t_p|^{-\nu} \cdot \Psi_{0,p} \left(\begin{pmatrix} t_p & 0 \\ t_p - 1 & 1 \end{pmatrix} \right) \frac{dt_p}{|t_p|} = \sum_{\mu=0}^{\infty} p^{\mu(\nu - \frac{n}{2} - 1)} + \sum_{\mu=1}^{\infty} p^{-\mu(\nu + \frac{n}{2} + 1)} =$$

$$\frac{1}{1 - p^{-(\frac{n}{2}+1-\nu)}} + \frac{p^{-\frac{n}{2}-1-\nu}}{1 - p^{-(\frac{n}{2}+1+\nu)}} = \frac{1 - p^{-n-2}}{(1 - p^{-(\frac{n}{2}+1-\nu)}) \cdot (1 - p^{-(\frac{n}{2}+1+\nu)})}$$

These local integrals are certainly absolutely convergent for all finite places. If we multiply them together, the infinite product is also convergent and gives the value

$$\frac{\zeta(\frac{n}{2}+1+\nu)\zeta(\frac{n}{2}+1-\nu)}{\zeta(n+2)}.$$

Again we made use of the assumption that $\nu \neq \pm\frac{n}{2}$.

I investigate the integral at the infinite place. We have to compute

$$\int\limits_{T_1(\mathbb{R})} |t|^{-\nu} \langle \tilde{\omega}_n(h_1)\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right), e_\nu\rangle \,\frac{dt}{|t|}.$$

We have to unravel the definition of $\tilde{\omega}_n$ which is by definition an element in

$$\mathrm{Hom}_K(\Lambda^1(\mathfrak{g}/\mathfrak{k}), I_{\chi_\infty} \otimes \mathcal{M}^\vee_{n,\mathbb{C}}) = \mathrm{Hom}_{K^T}(\Lambda^1(\mathfrak{t}\oplus\mathfrak{u}), \mathbb{C}_{\chi_\infty+\delta_\infty} \otimes \mathcal{M}^\vee_{n,\mathbb{C}})$$

(see 4.3.3) and for $g \in PGL_2(\mathbb{R})$ we have

$$\tilde{\omega}_n(h_1)(g) = \tilde{\omega}_n(h_1)(bk) = b^{\chi_\infty+\delta_\infty}\tilde{\omega}_n(h_1)(k)$$

and

$$\tilde{\omega}_n(h_1)(k) = k\,\tilde{\omega}_n(\mathrm{ad}(k)^{-1}h_1)(1) = k\,\tilde{\omega}_n(a(k)h_1 + b(k)E_+)$$

and $\tilde{\omega}_n(a(k)h_1 + b(k)E_+)(1) = b(k)Y^n$ by definition. We write

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ t & 1 \end{pmatrix} = b(t)\cdot k(t) = \begin{pmatrix} \frac{t}{\sqrt{1+t^2}} & * \\ 0 & \sqrt{1+t^2} \end{pmatrix}\cdot\begin{pmatrix} \frac{1}{\sqrt{1+t^2}} & \frac{-t}{\sqrt{1+t^2}} \\ \frac{t}{\sqrt{1+t^2}} & \frac{1}{\sqrt{1+t^2}} \end{pmatrix}$$

and get

$$\tilde{\omega}_n(h_1)\left(\begin{pmatrix} t & 0 \\ t & 1 \end{pmatrix}\right) = \frac{|t|^{\frac{n}{2}+1}}{(1+t^2)^{\frac{n}{2}+1}}\tilde{\omega}_n(h_1)(k(t)) =$$

$$\frac{|t|^{\frac{n}{2}+1}\cdot t}{(1+t^2)^{\frac{n}{2}+2}}\; k(t)\tilde{\omega}_n(\ldots h_1 + \frac{t}{1+t^2}E_+)(1) \;=\; \frac{|t|^{\frac{n}{2}+1}}{(1+t^2)^{\frac{n}{2}+1}}\; k(t)Y^n.$$

and we get the integral

$$\int\limits_{\mathbb{R}^*} \frac{|t|^{\frac{n}{2}+1-\nu}\cdot t}{(1+t^2)^{\frac{n}{2}+2}}\,\langle Y^n, k(t)^{-1}e_\nu\rangle\,\frac{dt}{|t|} \;=\; \int\limits_{\mathbb{R}^*} \frac{|t|^{\frac{n}{2}+1-\nu}\cdot t^{\frac{n}{2}-\nu+1}}{(1+t^2)^{n+2}}\,\frac{dt}{|t|}.$$

This integral vanishes if $\frac{n}{2}+1-\nu \equiv 1 \bmod 2$. If $\frac{n}{2}+1-\nu$ is even, we get the following value

$$2\int\limits_0^\infty \frac{t^{n+2-2\nu}}{(1+t^2)^{n+2}}\,\frac{dt}{t}.$$

We substitute $1+t^2 = \frac{1}{w}$, then $t = (\frac{1-w}{w})^{\frac{1}{2}}$ and $\frac{dt}{t} = -\frac{1}{2}\,\frac{1}{w(1-w)}$, and we get as value for our integral

$$\int\limits_0^1 (1-w)^{\frac{n}{2}+1-\nu-1}w^{\frac{n}{2}+1^{\nu}-1}dw = \frac{\Gamma(\frac{n}{2}+\nu+1)\Gamma(\frac{n}{2}-\nu+1)}{\Gamma(n+2)}.$$

(By the way, the integral is absolutely convergent!).

If we multiply everything together, we find

$$\int_{\mathcal{Z}_\nu^{(f)}} \widetilde{Eis}_n = \frac{\Gamma(\frac{n}{2}+\nu+1)\cdot\Gamma(\frac{n}{2}-\nu+1)}{\Gamma(n+2)} \cdot \frac{\zeta(\frac{n}{2}+\nu+1)\zeta(\frac{n}{2}-\nu+1)}{\zeta(n+2)}.$$

If we exploit the functional equation for the Riemann ζ-function which tells us

$$\zeta(2k) \;=\; \frac{2^{2k-1}\pi^{2k}}{(2k-1)!}\,(-1)^k\cdot\zeta(1-2k),$$

we find

$$\int_{\mathcal{Z}_\nu^{(f)}} \widetilde{Eis}_n \;=\; \frac{\zeta(1-(\frac{n}{2}+\nu+1))\cdot\zeta(1-(\frac{n}{2}-\nu+1))}{\zeta(1-(n+2))}$$

which now together with the formula for the boundary term gives us the formula

$$\langle Eis_n, T_p^m\mathcal{Z}_\nu^{(f)}-\mathcal{Z}_\nu^\infty\rangle = (p^{n+1}+1)^m\,\frac{\zeta(1-(\frac{n}{2}+\nu+1))\zeta(1-(\frac{n}{2}-\nu+1))}{\zeta(1-(n+2))} + b_{n,\nu}+b_{n,-\nu} \bmod \mathbb{Z}_{(p)}.$$

I want to point out that this formula is true for all values ν which satisfy $-\frac{n}{2}<\nu<\frac{n}{2}$, we do not need a parity condition since the values of the ζ-function vanish if $1-(\frac{n}{2}\pm\nu+1)$ is odd.

Using Lemma 6.2.2.1 we can easily determine the denominator of the Eisenstein class. We look what happens for the different values of ν. We assume $p^{\delta(n)}\|\zeta(1-(n+2))$. We have three cases:

I. Case: $p-1\nmid(\frac{n}{2}-\nu+1)(\frac{n}{2}+\nu+1)$.

In this case the $b_{n,\pm\nu}$ and $\zeta(1-(\frac{n}{2}\pm\nu+1))$ are integral at p. Hence we find the denominator of Eis_n evaluated at $\mathcal{Z}_\nu$ is

$$p^{\delta(n)-\mathrm{ord}_p\zeta(1-(\frac{n}{2}-\nu+1))\cdot\zeta(1-(\frac{n}{2}+\nu+1))}.$$

II. Case: $p-1\mid\frac{n}{2}-\nu+1$ and $p-1\nmid\frac{n}{2}+\nu+1$.

Let us assume $p^{\alpha-1}(p-1)\|\frac{n}{2}-\nu+1$. Then p^α is the demoninator of $\zeta(1-(\frac{n}{2}-\nu+1))$. The Kummer congruences yield

$$\zeta(1-(n+2))\equiv\zeta(1-(\frac{n}{2}+\nu+1)) \bmod p^\alpha.$$

We write

$$\zeta(1-(\frac{n}{2}+\nu+1)) = \zeta(1-(n+2))+p^\alpha\cdot Z(\nu,n)$$

with $Z(\nu,n)\in\mathbb{Z}_{(p)}$ and substitute this into the first part of our formula. We get

$$\langle Eis_n,\mathcal{Z}_\nu\rangle \equiv \zeta(1-(\frac{n}{2}-\nu+1))\cdot\left(1+p^\alpha\frac{Z(\nu,n)}{\zeta(1-(n+2))}\right)+b_{n,\nu}+b_{n,-\nu}.$$

The classical Clausen-von Standt congruences tell us that

$$\zeta(1-(\frac{n}{2}-\nu+1)) = \frac{-1}{p(\frac{\frac{n}{2}-\nu+1}{p-1})}+\nu.$$

With $\nu\in\mathbb{Z}_{(p)}$ this gives us and hence

$$\langle Eis_n,\mathcal{Z}_\nu\rangle \equiv \frac{-1}{p(\frac{\frac{n}{2}-\nu+1}{p-1})}+\nu-\frac{p^\alpha}{p\cdot(\frac{\frac{n}{2}-\nu+1}{p-1})}\frac{Z(\nu,n)}{\zeta(1-(n+2))}+\nu p^\alpha\cdot\frac{Z(\nu,n)}{\zeta(1-(n+2))}+b_{n,\nu}+b_{n,\nu}.$$

The first term cancels against $b_{n,-\nu}$, in the first factor of the third term the p^α cancels. Therfore it is this term which determines the denominator: it is

$$p^{\mathrm{ord}_p(\zeta(1-(n+2))-\mathrm{ord}_p Z(\nu,n)}$$

III. Case: $p-1 \mid \frac{n}{2}-\nu+1$ and $p-1 \mid \frac{n}{2}+\nu+1$.

In this case the numerator of $\zeta(1-(n+2))$ is a unit at p and the whole expression is obviously integral.

We are now ready to prove Theorem I in the case $n > p$. We will discuss the case $n < p$ in the next section.

Obviously it suffices to prove that

$$\min_{\nu} \quad \mathrm{ord}_p(\zeta(1-(\tfrac{n}{2}-\nu+1))\cdot\zeta(1-(\tfrac{n}{2}+\nu+1)) = 0$$

if ν runs over the integers $\nu = 0$ to $\frac{n}{2}-1$ for which $\frac{n}{2}-\nu+1 \equiv 0 \bmod 2$ and $p-1 \nmid \frac{n}{2} \pm \nu + 1$. We reduce the arguments $(\frac{n}{2}-\nu+1) \bmod p-1$, i.e. we write

$$\frac{n}{2}-\nu+1 \equiv m(-\nu) \bmod p-1$$

$$2 \le m(-\nu) < p-1.$$

If ν runs over the given set of numbers, the number $m(\nu)$ (or $m(-\nu)$) runs over the even residue classes mod $(p-1)$, and we always have $m(\nu)+m(-\nu) \equiv n+2 \bmod (p-1)$. We have the Kummer congruences

$$\zeta(1-(\frac{n}{2}+\nu+1)) \equiv \zeta(1-m(\nu)) \bmod p.$$

If now for all even numbers k, k' with $k, k' = 2, 4, \dots, p-3$ and $k+k' \equiv n+2 \bmod (p-1)$ we had

$$p \mid \zeta(1-k)\zeta(1-k')$$

then at least $[\frac{p}{4}]$ of the numbers $\zeta(1-k)$ for $k = 2, \dots, p-3$ would be divisible by p and this contradicts the following inequality of Carlitz on the irregularity index:

This inequality asserts that the irregularity index $i(p)$ which is the number of even $k's$ between 2 and $p-3$ for which $p \mid \zeta(1-k)$ is less then

$$i(p) \le \frac{p+3}{4} - \log_p(2) \cdot \frac{p-1}{4}$$

(see [ ],...). Hence we have proved Theorem I in the case $p > n$.

To get the result in general we have to investigate congruences between coefficient systems. This goes back to Ash-Stevens. It has been used by Wang in his dissertation and is apparently also related to the work of Hida and Taylor.

6.3. Congruences between coefficient systems: We only consider the simplest case. Let n_0 be even $0 < n_0 \le p-3$. We put $n = n_0 + (p-1)$. We look at our module $\mathcal{M}_n$ as a module over $R = \mathbb{Z}[\frac{1}{6}]$. It has the basis

$$\{X^\nu Y^{n-\nu}\}_{\nu=0,\dots,n}.$$

The dual module $\mathcal{M}_n^\vee$ is contained in $\mathcal{M}_n$ after localizing at p. It has a basis

$$X^n, X^{n-1}Y, \dots, X^pY^{n-p}, pX^{p-1}Y^{n-(p-1)}, \dots, pX^{n-(p-1)}Y^{p-1}, X^{n-p}Y^p, \dots, Y^n.$$

We have $p\mathcal{M}_n \subset \mathcal{M}_n^\vee \subset \mathcal{M}_n$, hence we find the submodul

$$\mathcal{X}_n = \mathcal{M}_n^\vee / p\mathcal{M}_n \subset \mathcal{M}_n / p\mathcal{M}_n,$$

as a $\mathbb{F}_p$-basis of this module is given by the monomials

$$X^n, X^{n-1}Y, \ldots, X^p Y^{n-p}, X^{n-p}Y^p, \ldots, Y^n.$$

If we consider the action of the torus

$$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \qquad\qquad t \in \mathbb{F}_p^*$$

on $\mathcal{X}_n$ we find that the monomials are eigenvectors with weights

$$t^n, t^{n-1}, \ldots, t^p, t^{n-p}, \ldots, t, 1.$$

It is clear that the weights t^n and 1 occur with multiplicity one. The other weights come in pairs

$$\begin{array}{lcl} t^{n-1} & \text{and} & t^{n-p} = t^{n-1-(p-1)} \\ \vdots & \vdots & \vdots \\ t^p & \text{and} & t \end{array} .$$

We consider the $SL_2(\mathbb{F}_p)$ submodule generated by X^n. It certainly contains Y^n, and we have

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} X^n = X^u + \sum_{\nu=1}^{n-p} u^\nu W_\nu + u^n X^u$$

where $W_\nu = \alpha_\nu X^{n-\nu} Y^\nu + \beta_\nu X^{n-\nu-(p-1)} Y^{\nu+(p-1)}$ with some $\alpha_\nu, \beta_\nu \in \mathbb{F}_p^*$. I claim

Proposition 6.3.1: *The elements X^n, Y^n and the W_ν generate a submodule of $\mathcal{M}_n/p\mathcal{M}_n$ which is isomorphic to $\mathcal{M}_{n_0}/p\mathcal{M}_{n_0}$.*

Proof: This is rather obvious. One has to observe that the subspace in question is the space generated by the polynomials

$$(\alpha X + \beta Y)^n \qquad\qquad \alpha, \beta \in \mathbb{F}_p$$

which is certainly invariant. Then it is clear that the map Φ sending

$$X^{n_0} \longrightarrow X^n, Y^{n_0} \longrightarrow Y$$

extends to an isomorphism from $\mathcal{M}_{n_0}/p\mathcal{M}_{n_0}$ to that subspace. Hence we get the exact sequence

$$0 \longrightarrow \mathcal{M}_{n_0}/p\mathcal{M}_{n_0} \xrightarrow{\phi} \mathcal{M}_n/p\mathcal{M}_n \longrightarrow \mathcal{Y}_n \longrightarrow 0$$

of $GL_2(\mathbb{F}_p)$ and hence Γ-modules. This induces an exact sequence in cohomology

$$\longrightarrow H^1(\Gamma\backslash H, \mathcal{M}_{n_0}/p\mathcal{M}_{n_0}) \longrightarrow H^1(\Gamma\backslash H, \mathcal{M}_n/p\mathcal{M}_n) \longrightarrow H^1(\Gamma\backslash H, \mathcal{Y}_n) \longrightarrow .$$

Proposition 6.3.2: *The action of the Hecke-Algebra on $H^\bullet(\Gamma\backslash H, \mathcal{M}_m)$ induces an action on the groups $H^\bullet(\Gamma\backslash H, \mathcal{M}_m/p\mathcal{M}_m)$ and extends to an action on the long exact sequence. The Hecke operator T_p acts nilpotently on $H^\bullet(\Gamma\backslash H, \mathcal{Y}_n)$.*

Proof: We have to comment briefly on the functoriality properties of Hecke operators. If $\mathcal{M},\mathcal{N}$ are any two Γ-modules and if $\varphi:\mathcal{M}\to\mathcal{N}$ is a Γ-module homomorphism, then we may ask wether this map φ induces a map on cohomology which commutes with the action of T_p (or T_ℓ).

We recall the discussion in 5.5. To get a Hecke Operator we needed an element $\alpha\in G(\mathbb{Q})$ and an element

$$u_\alpha\in \mathrm{Hom}_{\Gamma(\alpha)}(\mathcal{M}^{(\alpha)},\mathcal{M}).$$

Of course it is clear that φ induces $\Gamma(\alpha)$ homomorphisms

$$\begin{array}{ccccc}\varphi & : & \mathcal{M}^{(\alpha)} & \longrightarrow & \mathcal{N}^{(\alpha)} \\ \varphi & : & \mathcal{M} & \longrightarrow & \mathcal{N}\quad ,\end{array}$$

so what we need is that the elements

$$u_\alpha\in \mathrm{Hom}_{\Gamma(\alpha)}(\mathcal{M}^{(\alpha)},\mathcal{M}), v_\alpha\in \mathrm{Hom}_{\Gamma(\alpha)}(\mathcal{N}^{(\alpha)},\mathcal{N})$$

give rise to a commutative diagram

$$\begin{array}{ccc}\mathcal{M}^{(\alpha)} & \xrightarrow{\quad\varphi\quad} & \mathcal{N}^{(\alpha)} \\ \big\downarrow u_\alpha & & \big\downarrow v_\alpha \\ \mathcal{M} & \xrightarrow{\quad\varphi\quad} & \mathcal{N}\quad .\end{array}$$

It is clear that such a choice of the u_α,v_α gives rise to Hecke operators which commute with φ. If we are in the case discussed in 5.5.0 it is obvious that the natural choice of u_α,v_α gives such a diagram, hence we do not have problems with the operators T_ℓ where $\ell\neq p$.

If $\ell=p$, we recall the construction in 5.6. It is clear that the choice of u_α there for $\alpha=\begin{pmatrix}p&0\\0&1\end{pmatrix}$ does on

$$\mathcal{M}_{n_0}/p\mathcal{M}_{n_0}\qquad\qquad \mathcal{M}_n/p\mathcal{M}_n$$

exactly the same thing. It maps all $X^{n-\nu}Y^\nu$ with $\nu>0$ to zero and map

$$\begin{array}{llll}u_\alpha: X^{n_0} & \longrightarrow\ Y^{n_0} & \text{on} & \mathcal{M}_{n_0}/p\mathcal{M}_{n_0} \\ u_\alpha: X^{n} & \longrightarrow\ Y^{n} & \text{on} & \mathcal{M}_{n}/p\mathcal{M}_{n}.\end{array}$$

Hence we get indeed such a diagram as above and moreover we see that this operator induces obviously zero on Y_n, hence also on the cohomology (So we proved actually something much stronger.).

It is clear that we also get an exact sequence in the opposite direction

$$0\longrightarrow \mathcal{Y}_n^\vee\longrightarrow \mathcal{M}_n^\vee/p\mathcal{M}_n^\vee\longrightarrow \mathcal{M}_{n_0}^\vee/p\mathcal{M}_{n_0}^\vee\longrightarrow 0,$$

where we have of course the same assertions concerning the Hecke operators.

We defined the classes $eis_{n_0}\in H^1(\Gamma\backslash H,\tilde{\mathcal{M}}_{n_0}^\vee)$ and $eis_n\in H^1(\Gamma\backslash H,\tilde{\mathcal{M}}_n^\vee)$. If we reduce them mod p, we get one-dimensional subspaces

$$\begin{array}{lcl}\langle\overline{eis_{n_0}}\rangle & \subset & H^1(\Gamma\backslash H,\mathcal{M}_{n_0}\widetilde{/p}\mathcal{M}_{n_0}) \\ \langle\overline{eis_{n}}\rangle & \subset & H^1(\Gamma\backslash H,\mathcal{M}_{n}\widetilde{/p}\mathcal{M}_{n}).\end{array}$$

We prove the following

Theorem III: *The map $\Phi^\vee$ sends $\langle\bar{e}is_n\rangle$ to $\langle\bar{e}is_{n_0}\rangle$.*

Before I come to the proof of this theorem, I will show that it has Theorem I for $n_0 < p-1$ is a corollary. We know that we can write

$$\begin{aligned} eis_{n_0} &= a(n_0)\ Eis_{n_0} \\ eis_n &= a(n)\ Eis_n\ . \end{aligned}$$

It follows from Kummer's congruences that $p \mid \zeta(1-(n_0+2))$ if and only if $p \mid \zeta(1-(n+2))$. Let us assume that this is the case. We know already that $p \mid a(n)$. This implies that eis_n is zero if we restrict it to the boundary. But then it is clear that eis_{n_o} is also zero after restriction to the boundary and hence we have $p(a(n_0)$. But then our evaluation formula for the Eisenstein class on the modular symbols tells us that there must be a number ν with $-\frac{n_0}{2} < \nu < \frac{n_0}{2}$, $\frac{n_0}{2} + \nu + 1 \equiv 0 \mathrm{mod}\ 2$ such that

$$p \nmid \zeta(1-(\frac{n_0}{2}+\nu+1))\zeta(1-(\frac{n_0}{2}-\nu+1))$$

and the rest is clear.

We now come to the proof of the theorem 6.3.3. We do it by testing the classes on the modular symbols.

We start from the cycles $\mathcal{Z}_{0,\nu} \in H^1(\Gamma\backslash H, \tilde{\mathcal{M}}_{n_0})$ which generate this homology group. Since we have $n_0 < p$ they are rather simple. If we have

$$e_{0,\nu} = X^{\frac{n_0}{2}+\nu} \cdot Y^{\frac{n_0}{2}-\nu},$$

we may write

$$e_{0,\nu} - w\, e_{0,\nu} = P_{0,\nu}(X,Y) - P_{0,\nu}(X, X+Y)$$

with $P_{0,\nu} \in \mathcal{M}_{n_0} \otimes \mathbb{Z}_{(p)}$. Therefore we do not need to apply the Hecke operator to define $\mathcal{Z}_{0,\nu}$. We simply take

$$\mathcal{Z}_{0,\nu} = \mathcal{Z}_{0,\nu}^{(f)} - \mathcal{Z}_{0,\nu}^{(\infty)} = \mathcal{Z}_{0,\nu}^{(f)} - [0,1]_\infty \otimes P_{0,\nu}.$$

Let $\bar{\mathcal{Z}}_{0,\nu}$ be the reduction mod p of this 1-cycle. The Map Φ sends it to

$$\Phi(\bar{\mathcal{Z}}_{0,\nu}) \in Z_1(\Gamma\backslash H, \underset{\sim}{\mathcal{M}_n/p\,\mathcal{M}_n}).$$

All we need to prove is that there exists a constant $\alpha \neq 0$ in $\mathbb{F}_p^*$ such that

$$\alpha\langle \bar{e}is_{n_0}, \bar{\mathcal{Z}}_{o,\nu}\rangle = \langle \Phi^\vee(\bar{e}is), \bar{\mathcal{Z}}_{0,\nu}\rangle,$$

and since the second term is equal to

$$\langle \bar{e}is_n, \Phi(\bar{\mathcal{Z}}_{0,\nu})\rangle,$$

we need to prove

$$\alpha\langle e\bar{i}s_{n_0}, \bar{\mathcal{Z}}_{o,\nu}\rangle = \langle \bar{e}is_n, \Phi(\bar{\mathcal{Z}}_{0,\nu})\rangle.$$

For the term on the left hand side we have our formula. Since it is simply equal to

$$\alpha\langle eis_{n_0}, \mathcal{Z}_{o,\nu}\rangle \bmod p.$$

For the second term we can do the same, but there is still a difficulty. We have to interprete $\Phi(\bar{\mathcal{Z}}_{0,\nu})$ as the reduction mod p of a modular symbol.

We define

$$\bar{g}_\nu = \Phi(\bar{e}_{0,\nu}) = \xi_\nu\ \bar{e}_{\nu+\frac{p-1}{2}} + \eta_\nu\ \bar{e}_{\nu-\frac{p-1}{2}}$$

and we lift these elements to

$$g_\nu = \tilde{\xi}_\nu \; e_{\nu+\frac{p-1}{2}} + \tilde{\eta}_\nu \; e_{\nu-\frac{p-1}{2}},$$

i.e. we lift the coefficients. (The numbers $\xi_\nu \eta_\nu$ coincide with the α, β up to a shift in the indices.) We wrote

$$e_{0,\nu} - w\, e_{0,\nu} = \partial P_{0,\nu}(X,Y) = P_{0,\nu}(X,Y) - P_{0,\nu}(X, X+Y)$$

and of course

$$\partial \Phi(\bar{P}_{0,\nu}) = \bar{g}_\nu - w\, \bar{g}_\nu.$$

We write

$$P_{o,\nu} = \sum_{\mu=0}^{n_0} a_\mu X^{n_0-\mu} Y^\mu,$$

and we define

$$P_\nu^{\#} = \sum a_\mu (\tilde{\alpha}_\mu X^{n-\mu} Y^\mu + \tilde{\beta}_\mu X^{n-(p-1)-\mu} Y^{p-1+\mu})$$

where $\tilde{\alpha}_\mu, \tilde{\beta}_\mu$ are lifts of the α_μ, β_μ so we have $\tilde{\alpha}_\mu + \tilde{\beta}_\mu \equiv 1 \bmod p$ and $\tilde{\beta}_0 = 0$, $\tilde{\alpha}_{n_0} = 0$. We have

$$\partial P_\nu^{\#} = g_\nu - w\, g_\nu + p\, h_\nu.$$

We can write

$$X^{n+1-p} Y^{p-1} - X^{p-1} Y^{n+1-p} = \partial R,$$

and it is clear that we $\gamma_\nu \in \mathbb{Z}_{(p)}$ and $Q_\nu \in \mathcal{M}_n \otimes \mathbb{Z}_{(p)}$ such that

$$p\, h_\nu = p\, \partial Q_\nu + \gamma_\nu\, \partial R.$$

We modify $P_\nu^{\#}$ into $P_\nu = P_\nu^{\#} - p\, Q_\nu$, then P_ν is a lifting of $\Phi(\bar{P}_{o,\nu})$ and

$$\partial P_\nu = g_\nu - w\, g_\nu + \gamma_\nu \partial R.$$

Hence it is clear that the cycle $\Phi(\bar{\mathcal{Z}}_{0,\nu})$ is reduction mod p of the following modular symbol

$$\mathcal{Z}_\nu = \tilde{\xi}_\nu \; \mathcal{Z}^{(f)}_{\nu+\frac{p-1}{2}} + \tilde{\eta}_\nu \; \mathcal{Z}^{(f)}_{\nu-\frac{p-1}{2}} + p\, \gamma_\nu \; \mathcal{Z}^{(f)}_{\frac{n}{2}+1-p} - [0,1]_\infty \otimes P_\nu$$

and

$$\langle \bar{eis}_n, \Phi(\bar{\mathcal{Z}}_{0,\nu}) \rangle = \langle eis_n, \tilde{\mathcal{Z}}_\nu \rangle \bmod p.$$

We know already that we may choose eis_n as

$$eis_n = \zeta(1-(n+2)) Eis_n$$

and we define

$$eis'_{n_0} = \zeta(1-(n_0+2)) \cdot Eis_{n_0}.$$

Then $eis'_{n_0} \in \mathbb{Z}_{(p)}\; eis_{n_0}$. We prove the congruence

$$\langle eis_n, \mathcal{Z}_\nu \rangle \equiv \langle eis'_{n_0}, \mathcal{Z}_{0,\nu} \rangle \bmod p.$$

The contribution from the finite part of the modular symbols is

$$\zeta(1-(\frac{n_0}{2}+\nu+1)) \cdot \zeta(1-(\frac{n_0}{2}-\nu+1))$$

for $\mathcal{Z}_{0,\nu}$ and

$$\tilde{\xi}_\nu\,\zeta(1-(\frac{n_0}{2}+\nu+1+p-1))\cdot\zeta(1-(\frac{n_0}{2}-\nu+1))+\tilde{\eta}_\nu\,\zeta(1-(\frac{n_0}{2}+\nu+1))\cdot\zeta(1-(\frac{n_0}{2}-\nu+1+p-1))$$

+ contribution of $\mathcal{Z}^{(f)}_{\frac{n}{2}+1-p}$. The last contribution is zero because $\frac{n}{2}+\frac{n}{2}+1-p+1$ is odd and the ζ-value vanisches. Then it is clear that the contributions of the finite part are congruent. This follows from Kummer's congruences and the fact that $\tilde{\xi}_\nu+\tilde{\eta}_\nu\equiv 1 \bmod p$.

We have to compare the contribution sof

$$-[0,1]_\infty\otimes P_{0,\nu}\qquad\text{and}\qquad-[0,1]_\infty\otimes P_\nu.$$

This means we have to compare the values

$$\zeta(1-(n_0+2))\int_0^1 P_{0,\nu}(u)\,du=a_0+\frac{a_1}{2}+\ldots\frac{a_{n_0}}{n_0+1}$$

and

$$\zeta(1-(n+2))\int_0^1 P_\nu(u)\,du$$

modulo the prime p. We have to recall our requirements on P_ν. First of all we required that P_ν is a lifting of $\Phi(\bar{P}_{0,\nu})$ hence

$$P_\nu(u)=a_0+a_1(\tilde{\alpha}_1u+\tilde{\beta}_1u^p)+\ldots+a_{n_0}u^n+p\,H_\nu.$$

We see that for the evaluation of the second integral we need only the coefficients of $P_\nu \bmod p$ except for the coefficient of u^{p-1}. Since we assumed $n_0<p+1$ we write

$$P_\nu(u)=a_0+a_1(\tilde{\alpha}_1u+\tilde{\beta}_1u^p)+\ldots+a_{n_0}u^n+\gamma\,p\,u^{p-1}+p\,H_\nu^{\#}$$

where the coefficient of u^{p-1} in $H_\nu^{\#}$ is zero $\bmod\,p^2$.

Now we get for the second integral $\bmod\,p$

$$a_o+a_1\left(\frac{\tilde{\alpha}_1}{2}+\tilde{\beta}_1\right)+a_2\left(\frac{\tilde{\alpha}_2}{3}+\frac{\tilde{\beta}_2}{2}\right)+\ldots+\frac{a_{n_0}}{n_0}+\gamma.$$

We see that there is no problem if $\delta_p(n_0)$ and $\delta_p(n)>0$, since both contributions are zero. So we are left with the case that $p\nmid\zeta(1-(n_0+2))$ and $p\nmid\zeta(1-(n+2))$. (In this case I could refer to the Dissertation of Xiandong Wang,... .)

The reader should be a little bit puzzled. How can this expression be equal to the first one? We have to prove that

$$a_1\tilde{\beta}_1+a_2\frac{\tilde{\beta}_2}{2}+\ldots+\frac{a_{n_0}}{n_0}+\gamma=a_1\frac{\tilde{\beta}_1}{2}+a_2\frac{\tilde{\beta}_2}{3}+\ldots+\frac{a_{n_0}}{n_0+1}.$$

We still have the requirement that

$$\partial\,P_\nu=g_\nu-w\,g_\nu+\gamma_\nu\cdot(X^{n+1-p}Y^{p-1}-X^{p-1}Y^{n+1-p}),$$

if we substitute $1,u$ for X,Y then

$$P_\nu(u)-P_\nu(1+u)=\tilde{\alpha}_\nu u^\nu+\tilde{\beta}_\nu u^{\nu+p-1}+(-1)^\nu(\tilde{\alpha}_\nu u^{n-\nu}+\tilde{\beta}_\nu u^{n_0-\nu})+\gamma_\nu(u^{p-1}-u^{n_0})$$

where ν of course still runs from $-\frac{n_0}{2}+1$ to $\frac{n_0}{2}-1$. The only thing that counts is that the coefficient of u^{p-2} on the right hand side is zero.

We have

$$\partial\, P_\nu = P_\nu(u) - P_\nu(-1+u) = \partial(a_0 + a_1(\tilde{\alpha}, u+\tilde{\beta}_1, u^p) + \ldots + a_{n_0}u^n + \gamma\, p\, u^{p-1}) + p\cdot \partial\, H^{\#}_\nu .$$

It is clear that the contribution of $p\,\partial\, H^{\#}_\nu$ to the coefficient u^{p-2} is zero mod p^2. On the other hand

$$\partial\, u^{p-1} = u^{p-1} - (n+u)^{p-1} = u^{p-1} - (u^{p-1} + (p-1)u^{p-2} + \ldots) = -(p-1)u^{p-2}$$

and

$$\partial\, u^{p+\mu} = u^{p+\mu} - (u^{p+\mu} + \ldots + \binom{p+\mu}{\mu+2}u^{p-2} + \ldots) = \ldots - \binom{p+\mu}{\mu+2}u^{p-2}\ldots.$$

One sees that

$$\binom{p+\mu}{\mu+2} \;=\; p\cdot\frac{(p+\mu)\ldots(p+1)\cdot(p-1)}{(\mu+2)!}$$

and the value of this mod p^2 is

$$-p\,\frac{1}{(\mu+1)(\mu+2)}.$$

Hence the vanishing of the coefficient of u^{p-2} gives us the relation

$$a_1\frac{\tilde{\beta}_1}{1\cdot 2} + a_2\frac{\tilde{\beta}_2}{2\cdot 3} + \ldots + a_{n_0}\cdot\frac{1}{n_0(n_0+1)} + \gamma \equiv 0 \bmod p$$

and this is what we wanted.

Since the $\bar{\mathcal{Z}}_{0,\nu}$ generate the homology $H_1(\Gamma\backslash H, \tilde{\mathcal{M}}_{n_0}/p\,\tilde{\mathcal{M}}_{n_0})$ we have proved that $\Phi^\vee$ sends eis_n to eis'_{n_0}. By construction we have $eis'_{n_0} = u\cdot eis_{n_0}$ with $u\in\mathbb{Z}_{(p)}$. We still have to prove that u is a unit. This is clear from the exact sequence

$$\longrightarrow H^1(\Gamma\backslash H, Y_n^\vee) \longrightarrow H^1(\Gamma\backslash H, \tilde{\mathcal{M}}_n^\vee/p\,\tilde{\mathcal{M}}_n^\vee)\xrightarrow{\Phi^\vee} H^1(\Gamma\backslash H, \mathcal{M}_{n_0}/p\,\mathcal{M}_{n_0}) \longrightarrow .$$

The map cannot send $\bar{eis}_n$ to zero since the Hecke operator T_p acts nilpotently on the term on the left.

6.1.2 *The Galois-module* $\mathbb{Z}_p\cdot eis_n$: We want to determine the Galois-module $\mathbb{Z}_p\cdot eis_n$. To do this we have to investigate the Galois-module- structure of $H^0(\partial(\Gamma\backslash\tilde{H}), \tilde{\mathcal{M}}_{n,p}^\vee)$ and $H^1(\partial(\Gamma\backslash\tilde{H}), \tilde{\mathcal{M}}_{n,p}^\vee)$. To be more precise we have to introduce a Galois-module structure on these cohomology-groups which fits into our diagram, and then we have to compute it.

To state our result we have to introduce the Tate-module $\mathbb{Z}_p(1)$. The group of p^m-th roots of unity

$$\mu_{p^m} = \{\zeta\in\bar{\mathbb{Q}} \mid \zeta^{p^m} = 1\}$$

is (non canonically) isomorphic to the module $\mathbb{Z}/p^m\mathbb{Z}$ and the Galois-group acts on this module by a homomorphism

$$\alpha : \mathrm{Gal}(\bar{Q}/\mathbb{Q}) \to (\mathbb{Z}/p^m\mathbb{Z})^*,$$

which is defined by the rule $\sigma(\zeta) = \zeta^{\alpha(\sigma)}$. If we pass to the projective limit over all m we get $\varprojlim(\mu_{p^m}) \approx \mathbb{Z}_p$ and the Galois-group acts on this limit by this limit of the above α's, this is a character

$$\alpha : \mathrm{Gal}(\bar{Q}/\mathbb{Q}) \to \mathbb{Z}_p^*$$

which is the so called Tate-character. We denote the module $\mathbb{Z}_p$ with the above action of the Galois-group on it by $\mathbb{Z}_p(1)$. We define $\mathbb{Z}_p(\nu)$ to be the Galois-module $\mathbb{Z}_p$ with the action $\sigma(x) = \alpha(\sigma)^\nu x$ for $\sigma \in \mathrm{Gal}(\bar{Q}/\mathbb{Q}), x \in \mathbb{Z}_p$. I assert

Proposition 6.1.2.1:*The Galois-module $\mathbb{Z}_p \cdot eis_n$ is isomorphic to $\mathbb{Z}_p(-n-1)$.*

This is by no means obvious, I will try to give an outline of the proof, I do not know whether I should advise the reader to skip it.

We constructed the étale sheaves $\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee$ on $\mathcal{S}$ and we have two way to extend it to a sheaf on the compactification: We discussed already the extension $j_!(\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee)$ whose stalk at infinity is zero, this is an exact functor. We may also take the direct image $j_*(\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee)$ on the compactification $\mathbb{P}^1$. Then we have to take into account that this direct image functor is not exact, hence we have to consider the derived functors $R^\bullet j_*(\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee)$. We get a spectral sequence

$$H^\bullet_{\text{ét}}(\mathbb{P}^1 \times_{\mathbb{Q}} \bar{\mathbb{Q}}, R^\bullet j_*(\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee)) \Longrightarrow H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{M}_n^\vee/N\mathcal{M}_n^\vee),$$

to state this in modern terms, we may compute the cohomology of our sheaf on the open piece also as the cohomology of a complex of sheaves on the compactification. This yields us the exact sequence

$$H^1_{c,\text{ét}}(\mathcal{S}\times_{\mathbb{Q}}\bar{\mathbb{Q}}, \mathcal{M}_n^\vee/N\mathcal{M}_n^\vee) \to H^1_{\text{ét}}(\mathcal{S}\times_{\mathbb{Q}}\bar{\mathbb{Q}}, \mathcal{M}_n^\vee/N\mathcal{M}_n^\vee) \to H^1_{\text{ét}}(\mathcal{S}\times_{\mathbb{Q}}\bar{\mathbb{Q}}, R^\bullet j_*(\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee)/j_!(\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee)) \to$$

the quotient in the argument of the last term is a complex of sheaves which is concentrated in the point at infinity. Hence we may consider it as a complex of $\mathbb{Z}/N\mathbb{Z}$- modules on which we have an action of the Galois-group, simply because an étale sheaf on $\mathrm{Spec}(\mathbb{Q})$ is simply a module for the Galois-group. Then the H^1 of this complex of sheaves is simply the stalk $R^1 j_*(\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee)_\infty$. Under the present conditions we can pass to the projective limit and we still get an exact sequence

$$H^1_{c,\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{M}_n^\vee \otimes \hat{R}) \to H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{M}_n^\vee \otimes \hat{R}) \to R^1 j_*(\mathcal{M}_n^\vee \otimes \hat{R})_\infty,$$

we shall see that the last term is actually equal to $H^1_{\text{ét}}(\partial(\Gamma\backslash\tilde{H}), \tilde{\mathcal{M}}_n^\vee \otimes \hat{R})$ hence we gain an action of the Galois group on the cohomology of the boundary after we tensorize by $\hat{R}$. We also know that the cohomology of the boundary splits off a canonical direct summand $\hat{R} \cdot Eis_n$ which is also a Galois-module, this is the one we want to understand. (The reader should observe that in the previous chapters the cohomology of the boundary was computed from the Borel-Serre compactification, this is an object that has nothing to do with algebraic geometry).

To get the structure of these Galois-modules we remind ourselves of what would we do in the transcendental context. We take a little disc D_∞ around the point ∞ in $\mathbb{P}^1(\mathbb{C})$, the intersection of D_∞ with $\Gamma\backslash\tilde{H} = \mathcal{S}(\mathbb{C})$ is the punctured disc $\dot{D}_\infty$, we may restrict our sheaf $\mathcal{M}_n^\vee$ to D_∞. We have the embedding

$$j : D_\infty \to \dot{D}_\infty$$

and we want to compute the derived functors $R^\bullet j_*(\mathcal{M}_n^\vee)$. We recall that our sheaves $\mathcal{M}_n^\vee$ where defined through an action of the group Γ, but it is clear that the restriction of the sheaf to the punctured disc is obtained from the action of the fundamental group $\pi_1(\dot{D}_\infty) = \Gamma_\infty = \mathbb{Z}$ on $\mathcal{M}_n^\vee$. Since we are only interested in the free part we may replace the sheaf $\tilde{\mathcal{M}}_n^\vee$ by $\tilde{\mathcal{M}}_n$. We have an emdedding

$$\mathcal{M}_n/p\mathcal{M}_n \hookrightarrow \mathcal{M}_{n+1},$$

which is given by $X^\nu Y^{n-\nu} \to X^{\nu+1}Y^{n-\nu}$ and which commutes with the action of $\Gamma_\infty = \pi_1(\dot{D}_\infty)$. Hence we have exact sequences

$$0 \to \tilde{\mathcal{M}}_n \to \widetilde{\mathcal{M}_{n+1}} \to R \cdot \widetilde{Y^{n+1}} \to 0$$

of sheaves on $\dot{D}_\infty$. It is easy to see that the boundary operator of the long exact sequence in cohomology provides an isomorphism modulo torsion

$$H^0(\dot{D}_\infty, R\cdot Y^{n+1}) \to H^1(\dot{D}_\infty, \tilde{\mathcal{M}}_n).$$

This gives an alternative method to compute the cohomology of the boundary. The point is that this can be imitated in the arithmetic context and then we will be able to read off the Galois-module structure. Let u be the uniformizing element at ∞ we replace the disc by the spectrum of the power series ring

$$D\hat{}_\infty = \mathrm{Spec}(\mathbb{Q}[[u]]),$$

and $\dot{D}\hat{}_\infty = D\hat{}_\infty \backslash \{\infty\}$.

For any integer N we define $D\hat{}_N = \mathrm{Spec}(\mathbb{Q}[\zeta_N][[v]])$, where ζ_N is a primitive N-th root of unity and $v^N = u$. We have a map $D\hat{}_N \to D\hat{}_\infty$, which becomes étale if we remove the point ∞. The Galois group of this étale covering is isomorphic to the group of matrices

$$B_N = \{\sigma = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} | a \in (\mathbb{Z}/N\mathbb{Z})^*, b \in \mathbb{Z}/N\mathbb{Z}\} \subset Gl_2(\mathbb{Z}/N\mathbb{Z}).$$

This group acts on our module $\mathcal{M}_n/p\,\mathcal{M}_n/N\mathcal{M}_n/p\,\mathcal{M}_n$ and by the same procedure that gave us the sheaves $\tilde{\mathcal{M}}_n$ on $\mathcal{S}$ we get the restriction of these sheaves to $\dot{D}\hat{}_\infty$ if we restrict the group action to B_N. Hence we have the exact sequences of sheaves on $\dot{D}\hat{}_\infty$ as before (remember the twist in the definition of $\mathcal{M}_n^\vee/N\mathcal{M}_n^\vee$ as $Gl_2(\mathbb{Z}/N\mathbb{Z})$-module)

$$0 \to \widetilde{\mathcal{M}_n/N\mathcal{M}_n} \to \widetilde{\mathcal{M}_{n+1}/N\mathcal{M}_{n+1}} \to (\mathbb{Z}/N\mathbb{Z})Y^{n+1} \to 0.$$

This yields a coboundary map

$$j_*(R\cdot Y^{n+1}) \to R^1 j_*(\widetilde{\mathcal{M}_n/N\mathcal{M}_n})$$

which becomes an isomorphism modulo torsion if we pass to the projective limit over N. This implies that the Galois group acts on $\hat{R}\cdot Eis_n$ in the same way as it acts on the left hand side. But it is clear that the element $\sigma \in B_N$ acts on $(R/NR)Y^{n+1}$ by multiplication by a^{-n-1}. (This explains the strange twist we introduced, when we defined the $Gl_2(\mathbb{Z}/N\mathbb{Z})$, it has the effect that the Galois-group acts trivially on $R^0 j_*(\mathcal{M})$)

If we pass to the limit over $N = p^k$, then we see that

$$R^j_*(\tilde{\mathcal{M}}_n^\vee) = \lim_{\leftarrow}(R^1(\mathcal{M})) = \mathbb{Z}_p(-n-1) + torsion,$$

and this is the assertion of our proposition.

6.1.2.2: *$\mathbb{Z}_p$-Hecke modules*: At this point I want to explain some very simple principles concerning the structure of modules under the Hecke algebra:

Here I want to look at the Hecke-algebra $\mathbf{H}$ as the polynomial algebra over $\mathbb{Z}$ generated by the indeterminates T_ℓ one of them for each prime ℓ. We fix a prime p and we consider finitely generated $\mathbb{Z}_p$-modules X on which $\mathbf{H}$ acts. I want to make the following assumption

The Hecke-operator T_p acts nilpotently on X_{tors}

If I want to make I category out of these objects I should require that cokernels of maps have this property too. I claim that each such module has a canonical decomposition

$$X = X_{\mathrm{nil}} \oplus X_{\mathrm{ord}},$$

so that T_p acts topologically nilpotent on $X_{\rm nil}$ (i.e we have $T_p^m(X_{\rm nil}) \subset pX_{\rm nil}$ for some m) and T_p induces an isomorphism on $X_{\rm ord}$. The module $X_{\rm ord}$ is called the ordinary part of X it is torsion free. If we apply this construction to X_{int} we get $X_{int,ord} = X_{\rm ord}$.

This is indeed very elementary. We consider the vector space $X \otimes \bar{\mathbb{Q}}_p$ and decompose it into generalized eigenspaces under the Hecke algebra. This means that we have a finite set $Spec(X)$ of homomorphisms

$$\lambda : \mathbf{H} \to \bar{\mathbb{Q}}_p$$

such that we get a decomposition into generalized eigenspaces

$$X \otimes \bar{\mathbb{Q}}_p = \bigoplus_{\lambda \in Spec(X)} Z_\lambda,$$

where $Z_\lambda = (\xi \in X \otimes \bar{\mathbb{Q}}_p | (T_\ell - \lambda(T_\ell))^N \xi = 0)$ for a suitably large number N. Since X is a finitely generated $\mathbb{Z}_p$-module the values $\lambda(T_\ell)$ will be integers in $\bar{\mathbb{Q}}_p$, we decompose $Spec(X) = Spec(X)_{tnilp} \cup Spec(X)_{ord}$ according to whether $\lambda(T_p)$ is in the maximal ideal or it is a unit. Then we get a decomposition

$$X \otimes \bar{\mathbb{Q}}_p = \bigoplus_{\lambda \in Spec(X)_{tnilp}} Z_\lambda \oplus \bigoplus_{\lambda \in Spec(X)_{ord}} Z_\lambda = Z_{tnilp} \oplus Z_{\rm ord}.$$

The two summands are invariant under the action of $\mathcal{G}al_{\rm pur}$ and therefore this decomposition descends to a decomposition over $\mathbb{Q}_p$:

$$X \otimes \mathbb{Q}_p = Y_{tnilp} \oplus Y_{\rm ord}$$

and we define

$$X_{int,tnilp} := Y_{tnilp} \cap X_{int} \quad X_{int,ord} = \mathrm{ord}\, Z \cap X_{int}.$$

Now one has to prove that

$$X_{int} = X_{int,tnilp} \oplus X_{int,ord},$$

it is clear that the left hand side contains the direct sum on the right hand side. I leave this as an exercise to the reader.

This proves the claim for X_{int}, it follows from our general assumption on the torsion that we have a section from $\mathrm{ord}\, X$ in X_{int} back to X. The following assertions are now obvious

(i) $\mathrm{ord}\, X$ is a free $\mathbb{Z}_p$-module, its rank is equal to the sum of the dimensions of the spaces Z_λ if λ runs over $Spec(X)_{ord}$

(ii) We get a decomposition

$$X \otimes \mathbb{Z}/p = X_{\rm nil} \otimes \mathbb{Z}/p \oplus X_{\rm ord} \otimes \mathbb{Z}/p$$

where the first summand is the generalized eigenspace to the eigenvalue 0 for T_p and where T_p induces an isomorphism on the second summand.

(iii) The functor $ord : X \to X_{\rm ord}$ would be an exact functor if we had made a category out of these modules in the above sense. (This is not true for int).

We define the Eisenstein-part of the spectrum: Let $(\bar{\pi})$ be the maximal ideal of the ring of intgers in $\bar{\mathbb{Q}}_p$, we define

$$Spec_{Eis}(X) = (\lambda \in Spec(X)_{ord} | \lambda(T_\ell) \equiv \ell^{n+1} + 1 \mathrm{mod}(\bar{\pi}) \text{ for all } \ell).$$

The same reasoning as before yields that the space

$$(X \otimes \bar{\mathbb{Q}}_p)_{Eis} := \bigoplus_{\lambda \in Spec_{Eis}(X)} Z_\lambda$$

descends to a subspace in $X_{ord} \otimes \mathbb{Q}_p$ and we have a decomposition

$$X_{ord} \otimes \mathbb{Q}_p = (X_{\text{ord}} \otimes \mathbb{Q}_p)_{nonEis} \oplus (X_{\text{ord}} \otimes \mathbb{Q}_p)_{Eis}.$$

Again it is also clear, that intersecting this direct sum decomposition with X_{ord} gives us

$$X_{\text{ord}} = X_{ord,nonEis} \oplus \text{Eis}\, X$$

and alltogether

$$X = X_{\text{nil}} \oplus X_{ord,nonEis} \oplus \text{Eis}\, X.$$

The following facts are obvious

(iv) Any endomorphism of X, which commutes with the action of the Hecke algebra leaves this decomposition invariant

(v) Rank(Eis X) equals the sum of the dimensions of the Z_λ with $\lambda \in Spec_{Eis}(X)$.

(vi) Eis $X \otimes \mathbb{Z}/p$ is the submodule of $X \otimes \mathbb{Z}/p$ on which all the operators $T_\ell - (\ell^{n+1} + 1)$ act nilpotently.

We apply this to our exact sequence (Seq) and we restrict it to the Eisenstein-part, this yields

$$0 \to H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{Eis} \to H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{Eis} \to H^1(\partial(\Gamma \backslash \tilde{H}), \tilde{\mathcal{M}}^\vee_{n,p}) \to 0,$$

of course the third term is already in the Eisenstein-part.

Now we discuss the influence of the denominator of the Eisenstein-class on the structure of the cohomology as Hecke$\times$Galois-module. As before we write the denominator as $p^{\delta_p(n)}$, by construction we get an exact sequence

$$0 \longrightarrow H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{Eis} \oplus \mathbb{Z}_p \cdot eis_n \longrightarrow H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{Eis} \longrightarrow \mathbb{Z}/p^{\delta_p(n)} \longrightarrow 0.$$

tensorizing this sequence with $\mathbb{Z}/p^{\delta_p(n)}$gives us an exact sequence

$$H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{Eis} \otimes \mathbb{Z}/p^{\delta_p(n)} \oplus (\mathbb{Z}/p^{\delta_p(n)}) \cdot eis_n \to H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{Eis} \otimes \mathbb{Z}/p^{\delta_p(n)} \to \mathbb{Z}/p^{\delta_p(n)} \to 0.$$

The kernel of the last arrow is $H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{Eis}$ and hence we get a surjective map

$$H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{Eis} \otimes \mathbb{Z}/p^{\delta_p(n)} \oplus (\mathbb{Z}/p^{\delta_p(n)}) \cdot eis_n \to H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{Eis} \otimes \mathbb{Z}/p^{\delta_p(n)}.$$

This map is of course of the form (Id, Ψ), where Ψ is a map

$$\Psi : (\mathbb{Z}/p^{\delta_p(n)}) \cdot eis_n \to H^1_!(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^\vee_{n,p})_{Eis} \otimes \mathbb{Z}/p^{\delta_p(n)}.$$

I claim

Lemma:*This map is injective and commutes with the action of the Hecke-operators and the Galois-group.*

Proof: The injectivity follows from the fact that $\mathbb{Z}_p \cdot eis_n$ is a primitive submodule hence it is a direct summand (as a $\mathbb{Z}_p$-module) and therefore $\mathbb{Z}/p^{\delta_p(n)} \cdot eis_n$ injects into $H^1_{\text{ét}}(\mathcal{S} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^{\vee}_{n,p})_{Eis} \otimes \mathbb{Z}/p^{\delta_p(n)}$. The rest is clear.

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