A short note which owes its existence to some discussions with J. Bergstroem, C. Faber, G. van der Geer , A. Mellit and J. Schwermer

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We consider the group $G/\operatorname{Spec}(\mathbb{Z}) = GSp_3/\operatorname{Spec}(\mathbb{Z})$ with Dynkin-Diagram

$$\alpha_1 \quad - \quad \alpha_2 \quad <= \quad \alpha_3$$

and choose a coefficient system

$$\mathcal{M}_\lambda = \mathcal{M}_{n_1\gamma_1+n_2\gamma_2+n_3\gamma_3}.$$

which provides a sheaf $\tilde{\mathcal{M}}_\lambda$ on $S^G_{K_f} = Sp_3(\mathbb{Z})\backslash\mathbb{H}_3$. We consider the cohomology of the boundary, we select the boundary stratum corresponding to the parabolic subgroup P given by $\alpha_1 \quad - \quad \times \quad <= \quad \alpha_3$, the semi simple part is $M = PSl_2 \times Sp_1$. The first factor has to be viewed as the linear factor and corresponds to α_1, the other factor is the hermitian factor.

We look at the Kostant representatives, we have two of length 4 and two of length 3

$$\begin{matrix} w_1 = s_2s_1s_3s_2 & w_2 = s_2s_3s_2s_1 \\ v_1 = s_2s_1s_3 & v_2 = s_2s_3s_2 \end{matrix}.$$

If Θ_P is the longest Kostant representatives then $w_1 = \Theta_P v_1, w_2 = \Theta_P v_2$ and we get

$$w_1(\lambda+\rho)-\rho = (2+n_2+2n_3)\gamma^M_{\alpha_1} + (2+n_1+n_2+n_3)\gamma^M_{\alpha_3} + 1/2(-6-n_2)\gamma_{\alpha_2}$$

$$v_1(\lambda+\rho)-\rho = (2+n_2+2n_3)\gamma^M_{\alpha_1} + (2+n_1+n_2+n_3)\gamma^M_{\alpha_3} + 1/2(-4+n_2)\gamma_{\alpha_2}$$

$$w_2(\lambda+\rho)-\rho = (4+n_1+2n_2+2n_3)\gamma^M_{\alpha_1} + n_3\gamma^M_{\alpha_3} + 1/2(-6-n_1)\gamma_{\alpha_2}$$

$$v_2(\lambda+\rho)-\rho = (4+n_1+2n_2+2n_3)\gamma^M_{\alpha_1} + n_3\gamma^M_{\alpha_3} + 1/2(-4+n_1)\gamma_{\alpha_2}$$

$$\Theta_P(\lambda+\rho)-\rho = n_1\gamma^M_{\alpha_1} + n_3\gamma^M_{\alpha_3} + \frac{1}{2}(-5-\frac{n_1}{2}-n_2-n_3)\gamma_{\alpha_2}$$

We consider the two cases of length 4. We denote the two coefficients at $\gamma^M_{\alpha_1}$(resp.)$\gamma^M_{\alpha_3}$ by d_1(resp.)d_3. Then the cohomology $H^2(S^M_{K^M_f}, \mathcal{M}(w\cdot\lambda))$ is given by the Künneth-formula, the factors are given by holomorphic modular forms of weight $d_1+2 = k_1, d_3+2 = k_3$. Since we work on level 1 these weights must be even.

We give ourselves these numbers d_1, d_3 and ask for the solutions in integers $n_1, n_2, n_3 \geq 0$ (or > 0 if we restrict to the regular case). Of course we have one degree of freedom and it is clear the interesting variable is the one in the coefficient of γ_2, i.e. n_2 in the case $w = w_1$ and n_1 in the case $w = w_2$.

I want the following computation to be on one page!

In the first case we have the two equations

$$n_2 + 2n_3 = d_1 - 2$$
$$n_1 + n_2 + n_3 = d_3 - 2$$

hence

$$n_3 = \frac{d_1 - 2 - n_2}{2}; \;\; n_3 \geq 0, \; n_2 \leq d_1 - 2$$

Substituting this into the second equation yields

$$n_1 + \frac{n_2}{2} = d_3 - 1 - \frac{d_1}{2}$$

and therefore

$$n_2 \in \{\min(d_1 - 2, 2d_3 - d_1 - 2), \ldots, 2, 0\}, \; n_2 \equiv 0 \mod 2 \qquad (Case1)$$

and for the coefficient in front of γ_{α_2} we get

$$\{-3, \ldots, -2 - \frac{\min(d_1, 2d_3 - d_1)}{2}\}$$

This string of integers is not empty if and only if the minimum is > 0. Especially we need

$$d_1 \geq 2 \text{ and } d_3 - \frac{d_1}{2} \geq 1. \qquad (uneqI)$$

If we are in the second case we have the two equations

$$n_1 + 2n_2 + 2n_3 = d_1 - 4$$
$$n_3 = d_3$$

and this implies

$$n_1 \in \{d_1 - 4 - 2d_3, \ldots, 2, 0\} \qquad (Case2)$$

and we need the inequality

$$d_1 \geq 2d_3 + 4 \qquad (uneqII)$$

Then we see that the factor in front of γ_{α_2} runs through the interval

$$\{-3, \ldots, -1 - \frac{d_1}{2} + d_3\}$$

We consider the Eisenstein cohomology in degree 6

$$\begin{array}{ccc} H^6(S^G_{K_f}, \mathcal{M}_\lambda) & \xrightarrow{r} & H^6(\partial(S^G_{K_f}), \tilde{\mathcal{M}}_\lambda) \\ & & \uparrow i_P \\ & & H^2_!(S^M_{K^M_f}, \tilde{\mathcal{M}}(w\cdot\lambda)) \end{array}$$

where w is one of the two elements of length 4 (which one depends on λ) and where i_P is an inclusion. An eigenspace in

$$H^2_!(S^M_{K^M_f}, \tilde{\mathcal{M}}(w\cdot\lambda))(\tau\times\sigma) \subset (H^2_!(S^M_{K^M_f}, \tilde{\mathcal{M}}(w\cdot\lambda))$$

is essentially given by a pair f resp.g of holomorphic cusp forms of weight k_1 resp. k_3.

We have to look at the Eisenstein cohomology given by this pair of forms and we have to study the factor in front of the second term in the constant term. According to Grbac-Schwermer it is of the following form: We have to lift σ to $H = Gl_3$ by the symmetric square lift, we get a cuspidal cohomology class $\Pi_f = \mathrm{Sym}^2(\sigma_f)$, i.e we get an isotypical module in the inner (cuspidal) cohomology

$$H^2_!(S^H_{K^H_f}, \tilde{\mathcal{N}}_\mu)(\Pi_f) \subset H^2_!(S^H_{K^H_f}, \tilde{\mathcal{N}}_\mu),$$

where $\mathcal{N}_\mu$ is the module with highest weight $\mu = d_3(\gamma^H_\alpha + \gamma^H_\beta)$.

Then the factor in front of the second term is

$$\frac{L(\tau\times\Pi, s-1)L(\tau,\Lambda^2, 2s-1)}{L(\tau\times\Pi, s)L(\tau,\Lambda^2,2s)},$$

where $L(\tau\times\Pi, s)$ is the Rankin-Selberg L function of $\tau\times\Pi$ and since $r=2$ (in the notation of Grbac-Schwermer) $L(\tau,\Lambda^2,s) = \zeta(s)$ the Riemann $\zeta-$ function. This expression must be evaluated at a certain specific argument $s_w \in \mathbb{Z}$ which depends on our data. It depends of course on certain conventions on the definition of the L-function, but there are some very natural conventions which we will make here. These conventions are different from the conventions in the automorphic literature.

Now we come to the speculative part. I refer to my articles on my home page "Eiscoh-rank.." , "p-adic-fin" and "Mixedmot-Intro." The construction provides a mixed Tate-motive $M(\tau_f\times\Pi_f)$ this motive has a Betti-de-Rham extension class

$$[M(\tau_f\times\Pi_f)]_{\text{B-dRh}} \in \mathrm{Ext}^1_{\text{B-dRh}}(\mathbb{Z}(-m-1), \mathbb{Z}(0)) \in \mathbb{R}$$

where

$$m = n_2 \text{ in case 1 and } m = n_1 \text{ in case 2}$$

depending on the case in which we are. Hence we conclude $s_w = \frac{m}{2}+1$.

The image of the group of mixed Tate motives in $\mathrm{Ext}^1_{\text{B-dRh}}(\mathbb{Z}(-m-1, \mathbb{Z}(0))$ is the $\mathbb{Q}$ vector space generated by $\zeta'(-m)$ (provided this mixed Tate-motive is not "exotic"). Up to some transcendental periods this extension class is given by

$$\frac{1}{\Omega(\tau_f\times\Pi_f)^{\epsilon(m/2)}}\frac{L(\tau\times\Pi,\frac{m}{2})}{L(\tau\times\Pi,\frac{m}{2}+1)}\zeta'(-m).$$

Hence the number number in front should be rational. For a prime ℓ which satisfies some certain assumptions (large or ordinary for $\tau \times \sigma$) we hope that the following implication must be true:

The Eisenstein section

$$\mathrm{Eis} : H^2_!(S^M_{K^M_f}, \tilde{\mathcal{M}}(w \cdot \lambda))(\tau \times \sigma) \to H^6(S^G_{K_f}, \mathcal{M}_\lambda) \text{ is integral at } \ell \Longrightarrow$$

$$\frac{1}{\Omega(\tau_f \times \Pi_f)^{\epsilon(m/2)}} \frac{L(\tau \times \Pi, \frac{m}{2})}{L(\tau \times \Pi, \frac{m}{2}+1)} \text{ is integral at } \ell$$

If the Eisenstein section is not integral then we must have congruences.

This conjectural statement can be formulated without any assumption on the existence of an abelian category of mixed motives.

At the end I want to discuss briefly how we can prove the the rationality of

$$\frac{1}{\Omega(\tau_f \times \Pi_f)^{\epsilon(m/2)}} \frac{L(\tau \times \Pi, \frac{m}{2})}{L(\tau \times \Pi, \frac{m}{2}+1)} \qquad (rational)$$

for $m/2$ in the range given by the labels $(Case1), (Case2)$ on p.2. The L-function $L(\tau \times \Pi, s)$ is a Rankin-Selberg L function on $Gl_2 \times Gl_3$. For these Rankin-Selberg L-functions rationality results for special values are known. Here I discuss the method which will be used in my forthcoming paper with Raghuram.

We change horses and start from the group $G = Sl_5/\operatorname{Spec}(\mathbb{Z})$. We choose the standard Borel subgroup B of upper triangular matrices. The group $M = Sl(Gl_2 \times Gl_3)$ is the Levi-quotient of a maximal parabolic subgroup containing B, and B induces a Borel subgroup $\bar{B}$. We consider highest weights $\mu = d_1\gamma^M_{\alpha_1} + d_3(\gamma^M_{\alpha_3} + \gamma^M_{\alpha_4}) + a\gamma_{\alpha_2}$ for M, we assume $d_1 \equiv 0 \mod 2$ and consider the cohomology

$$H^2_!(S^M_{K^M_f}, \mathcal{N}_\mu).$$

We have to ask ourselves whether these cohomology groups occurs in the boundary cohomology

$$H^5(\partial_P(S^G_{K_f}), \mathcal{M}_\lambda),$$

for a suitable coefficient system $\mathcal{M}_\lambda$. This means we have to find an element λ and an element w of length 3 in W^P such that

$$w(\lambda + \rho) - \rho = \mu = d_1\gamma^M_{\alpha_1} + d_3(\gamma^M_{\alpha_3} + \gamma^M_{\alpha_4}) + a\gamma_{\alpha_2}$$

It is an easy calculation that there are just 2 elements of length 3 namely

$$w_1 = s_2s_3s_4 \text{ and } w_2 = s_2s_3s_1.$$

A straightforward calculation shows that

$$w_1(\lambda+\rho)-\rho = (3+m_1+m_2+m_3+m_4)\gamma^M_{\alpha_1}+m_2\gamma^M_{\alpha_3}+m_3\gamma^M_{\alpha_4}+\frac{1}{16}(-15+3m_1+m_2-m_3-3m_4)\gamma_{\alpha_2}$$

Of course this implies $m_2 = m_3$ and then

$$w_1(\lambda+\rho)-\rho = (3+m_1+2m_2+m_4)\gamma^M_{\alpha_1} + m_2(\gamma^M_{\alpha_3}+\gamma^M_{\alpha_4}) + \frac{1}{2}(-5+m_1-m_4)\gamma_{\alpha_2}.$$

We have to solve the following equation in positive integers

$$\begin{gathered} 3+m_1+2m_2+m_4 = d_1 \\ m_2 = d_3 \end{gathered}$$

This is simple, we know that d_1 must be even, we get

$$m_1 + m_4 = d_1 - 2d_3 - 3$$

and hence we must have

$$d_1 - 2d_3 - 3 \geq 0.$$

Taking into account that d_1 is even this is the inequality $(uneqII)$. Then we get from positive to negative

$$m_1 - m_4 \in \{d_1 - 2d_3 - 3, d_1 - 2d_3 - 5, \ldots, -(d_1 - 2d_3 - 3)\}.$$

Therefore we get for the coefficient in front of γ_{α_2}

$$w_1(\lambda+\rho)-\rho = \cdots + (\{-4+\frac{d_1}{2} - d_3, \ldots, -1 + d_3 - \frac{d_1}{2}\})\gamma_{\alpha_2}$$

and here we see that the right half of this interval (in direction to the negative numbers) is exactly the interval we found in $(Case2)$ on p. 2

For w_2 we get the constraint $m_1 + m_2 = m_3 + m_4$ and we eliminate m_4. We get

$$w_2(\lambda+\rho)-\rho = (1+m_2+m_3)\gamma^M_{\alpha_1} + (1+m_1+m_2)(\gamma^M_{\alpha_3}+\gamma^M_{\alpha_4}) + \frac{1}{2}(-5+m_2-m_3)\gamma_{\alpha_2}$$

$$\begin{gathered} 1+m_2+m_3 = d_1 \\ 1+m_1+m_2 = d_3 \\ m_1+m_2 \geq m_3 \end{gathered}$$

Subtracting the first from the second and subtracting the first from 2 times the second yields

$$\begin{gathered} m_1 - m_3 = d_3 - d_1 \\ 1+2m_1+m_2-m_3 = 1+m_1+m_2-m_3+m_1 = 1+m_4+m_1 = 2d_3 - d_1. \end{gathered}$$

From this we get $2d_3 - d_1 - 1 \geq 0$ and $m_1 \geq d_3 - d_1$ (the second inequality is only relevant if $d_3 \geq d_1$)

$$m_2 - m_3 = 2d_3 - d_1 - 1 - 2m_1$$

This implies

$$\frac{m_2-m_3}{2}\in\{\min(\frac{d_3}{2}-d_1-\frac{1}{2},d_1-\frac{1}{2}),\min(\frac{d_3}{2}-d_1-\frac{1}{2},d_1-\frac{1}{2})-1,\dots,-(\min(\frac{d_3}{2}-d_1-\frac{1}{2},d_1-\frac{1}{2}))\}$$

This yields for the coefficient in front of γ_{α_2} the values

$$\{-3+\min(\frac{d_3}{2}-d_1-\frac{1}{2},d_1--\frac{1}{2}),\dots,-2-\min(\frac{d_3}{2}-d_1,d_1)\}$$

and again the right half is exactly the string in $(Case1)$ on p.2.

This makes it clear that the values in $(rational)$ are indeed rational for all possible value of m (provided Raghuram and I did not make a mistake.)

The cohomological or motivic L-function

I have to say a few words concerning the definition of the L-functions which appear on p. 3. I am not really satisfied by the following exposition, eventually these considerations deserve a more lucid exposition.

If $E/\mathbb{Q}$ is an elliptic curve, then we can speak of $H^1(E)$ as a pure motive of weight 1, we define its L-function as usual by

$$L(E,s)=\prod_{p\notin S}\frac{1}{(1-\alpha_p p^{-s})(1-\beta_p p^{-s})}L_S(\pi,s)$$

We know that to our E we find an irreducible representation

$$\pi=\pi_\infty\otimes_p\pi_p$$

in $L^2_0(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)$ which "corresponds" to E. For all $p\notin S$ the local representation π_p is in the unitary principal series and given by a unitary character $\lambda_p: T(\mathbb{Q}_p)\to S^1\subset\mathbb{C}^\times$. This gives us two numbers

$$\lambda(\begin{pmatrix}p&0\\0&1\end{pmatrix})=\alpha'_p,\lambda(\begin{pmatrix}1&0\\0&p\end{pmatrix})=\beta'_p$$

and "corresponds" means that

$$\alpha_p=\sqrt{p}\alpha'_p,\beta_p=\sqrt{p}\beta'_p.$$

The L function attached to π is given by (we take the tautological representation of the dual group)

$$L(\pi,s)=\prod_{p\notin S}\frac{1}{(1-\alpha'_p p^{-s})(1-\beta'_p p^{-s})}L_S(\pi,s)$$

Hence we see that $L(E,s+\frac{1}{2})=L(\pi,s)$, hence the L-functions differ by a shift in the argument s.

I propose to call $L(\pi,s)$ the "unitary" or "automorphic" L-function and $L(E,s)$ the "motivic" $L-$ function. But we could also call it the "cohomological" $L-$

function. To our elliptic curve, or our modular form π we find an isotypical subspace in

$$H^1_!(S^{Gl_2}_{K_f}, \mathcal{O}_F)(\pi_f) \subset H^1_!(S^{Gl_2}_{K_f}, \mathcal{O}_F)(\pi_f)$$

in the cohomology. On this subspace the Hecke operator T_p acts by the eigenvalue $a_p = \alpha_p + \beta_p$ and the diagonal Hecke operator acts by the eigenvalue p. Then we see that we can express the local factors

$$L_p(E, s) = \frac{1}{1 - a_p p^{-s} + pp^{-2s}}$$

where now the factors a_p, p are eigenvalues.

You find a related consideration in my paper "Eis-coh-rank.." in 2.1.4. for modular forms of arbitrary weight.

We work with the following principle: Our motive $H^1(E)$ is is a member in a string of motives namely

$$\{H^1(E) \otimes \mathbb{Z}(n)\}_{n \in \mathbb{Z}}$$

But it is clear that $H^1(E) = H^1(E) \otimes \mathbb{Z}(0)$ is a distinguished member in this family: If we start from the negative side and go to the positive side, then it is the first member where the eigenvalues of the inverse Frobenius Φ_p are always (for all choices of $E/\mathbb{Q}$) integers. If we go one step further the eigenvalues become divisible by p. We could also express this in terms of Hodge numbers.

I claim that here is a general principle. If we have a reductive group $G/\mathbb{Q}$ (let us assume that it is split) and if we have an isotypical subspace in the integral cohomology

$$H^1_!(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_F)(\pi_f) \subset H^1_!(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_F)$$

then we can attach L functions to it:

$$L^{\text{coh}}(\pi_f, r, s) = \prod_{p \notin S} \frac{1}{1 - a_p^{(1)}(\pi_p)p^{-s} + a_p^{(2)}(\pi_p)p^{-2s} \dots}$$

where the $a_p^{(i)}$ are eigenvalues of suitable Hecke operators $T(t_p, u_{t_p})$ multiplied by a power $p^{m(i)}$ with $m(i) \geq 0$ and there is at least one i for which $m(i) = 0$.

This L-function is invariant under "Tate-twist": If $\gamma : G \to G_m$ is a character then we can twist $\mathcal{M}_\lambda \to \mathcal{M}_\lambda \otimes \mathbb{Z}\gamma$, this is nothing else than adding γ to λ, i.e. replacng λ by $\lambda + \gamma$. Then we can twist π_f to $\pi_f \otimes |\gamma|_f^{-1}$ and this module occurs in $H^1_!(S^G_{K_f}, \tilde{\mathcal{M}}_{\lambda+\gamma} \otimes \mathcal{O}_F)$. Then the L function does not change, i.e. we get

$$L^{\text{coh}}(\pi_f, r, s) = L^{\text{coh}}(\pi_f \otimes |\ \ |_f^{-1}, r, s).$$

This L function $L^{\text{coh}}(\pi_f, r, s)$ may be expressed in terms of an unitary $L-$ function. To define this unitary L-function we observe that π_f is the finite part of a representation π which after a suitable twist by some unique $|\gamma|^b$ occurs in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. Then we can define the automorphic (or unitary) L function

$$L(\pi|\gamma|^b, r, s) = L^{\text{aut}}(\pi_f, r, s)$$

and we get a relation

$$L^{\text{coh}}(\pi_f, r, s + d(r, \lambda)) = L^{\text{aut}}(\pi_f, r, s),$$

where $d(\lambda)$ is a positive integer or half integer which only depends on the semi simple part of λ.

I insert a section from a text which arouse from the discussions with Raghuram:

Automorphic L-function versus Cohomological (Motivic ?) L-function

To our irreducible automorphic representation $\pi = \Phi(H_{\pi_\infty} \otimes H_{\pi_f})$ we can attach L-functions. To do this we follow Langlands: Our representation is a tensor product of local representations, i.e. $\pi = \pi_\infty \otimes \otimes'_p \pi_p$. Since we assume that we are in the unramified case, we know that at the finite places the local components are unramified principal series. This means that we can (temporarily) think of H_{π_p} as a one dimensional $\mathbb{C}$ vector space which is generated by a spherical function $\phi_\mu : Gl_n(\mathbb{Q}_p) \to \mathbb{C}$. Here

$$\mu : T(\mathbb{Q}_p)/T(\mathbb{Z}_p) \to \mathbb{C}^\times$$

is a character, which is unique modulo the action of the Weyl group. We encode it by a n-tuple of complex numbers $(z_1, \ldots, z_n)$ which are defined by (the p below is placed at the i-th spot on the diagonal)

$$\mu : \begin{pmatrix} 1 & 0 & 0 & & \ldots & 0 \\ 0 & \ddots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & p & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & & \ddots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \mapsto z_i$$

This character μ (the Satake parameter of π_p) can be viewed as an element in the Langlands dual group, i.e $\mu \in G^\vee(\mathbb{C}) = Gl_n(\mathbb{C})$. We choose an irreducible representation $r : G^\vee(\mathbb{C}) \to Gl(V)$ and define the local Euler factor at p by

$$L_p(\pi, r, s) = \frac{1}{\det(\mathrm{Id} - r(\mu)p^{-s}|V)}$$

and the global automorphic L-function as

$$L(\pi, r, s) = L_\infty(\pi_\infty, r, s) \prod_p L_p(\pi, r, s).$$

At the moment we do not discuss the Euler-factor at the infinite place. From now on we only discuss the case that r is the tautological representation of $Gl_n(\mathbb{C})$. In this special case the local Euler factor becomes

$$L_p(\pi, r, s) = L_p(\pi, s) = \frac{1}{\det(\mathrm{Id} - r(\mu)p^{-s}|V)} = \prod_i \frac{1}{1 - z_i p^{-s}} =$$

$$\frac{1}{1 - \sigma_1(\underline{z})p^{-s} + \sigma_2(\underline{z})p^{-2s} - \ldots \sigma_n(\underline{z})p^{-ns}}.$$

where $\sigma_\nu(\underline{z})$ is the ν-th elementary symmetric function in the variables $\underline{z} = (z_1, \ldots, z_n)$.

We want to rewrite the local Euler-factor in terms of the values of $\pi_p : \mathcal{H}_p \to \mathcal{O}_F$ on certain specific Hecke operators.

These specific Hecke operators are provided by the characteristic functions of the double cosets

$$t_p^{(\nu)} = Gl_n(\mathbb{Z}_p) \begin{pmatrix} p & 0 & 0 & & \dots & 0 \\ 0 & p & \dots & \dots & \dots & \dots \\ 0 & 0 & p & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & & \ddots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} Gl_n(\mathbb{Z}_p),$$

where the first ν entries on the diagonal are p-s and the last $n-\nu$ entries are 1-s. A well known computation shows that

$$\pi_p(t_p^{(\nu)}) = p^{\frac{(n-\nu)\nu}{2}} \sigma_\nu(\underline{z})$$

and hence

$$L_p(\pi, s) = \frac{1}{1 - p^{-\frac{(n-1)}{2}} \pi_p(t_p^{(1)}) p^{-s} + p^{-\frac{(n-2)2}{2}} \pi_p(t_p^{(2)}) p^{-2s} - \dots \pm \pi_p(t_p^{(n)}) p^{-ns}}.$$

This looks much clumsier than our defining expression for the Euler factor, but wait!

We recall that we can define the normalized operators $T(t_p, u_{t_p})$ which act on the integral cohomology $H^\bullet(S^G_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}})$, (see [book], Chap. II and III.) If we send the integral cohomology to the rational cohomology then the action of the Hecke operator t_p on $H^\bullet(S^G_{K_f}, \mathcal{M}_\lambda)$ is given by convolution and this operator does not preserve module of integral classes. To get an operator acting on the integral cohomology we have to multiply t_p by a power $p^{c(t_p,\lambda)}$, this is the power of p which yields the canonical choice u_{t_p}. Then we defined the endomorphisms

$$T(t_p, u_{t_p}) : H^\bullet(S^G_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}}) \to H^\bullet(S^G_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}}).$$

If we look at the effect of $T(t_p, u_{t_p})$ on the rational cohomology, then we get

$$T(t_p, u_{t_p}) = p^{c(t_p,\lambda)} t_p$$

In our case it is easy to determine the powers $p^{c(t_p^{(\nu)},\lambda)}$. We gave the recipe how to determine the exponents in [book], Chap. II. For the moment we do not need the selfduality of λ.

We decompose $\mathcal{M}_{\lambda,\mathbb{Z}}$ into weight spaces

$$\mathcal{M}_{\lambda,\mathbb{Z}} = \mathbb{Z}e_\lambda \oplus \dots \oplus \dots \mathbb{Z}e_{w_0\lambda}$$

here w_0 is the longest element in the Weyl group, on the highest weight λ it has the effect

$$w_0(\sum a_i\gamma_i + d\det) = -\sum_i a_{n-i}\gamma_i + d\det$$

The weight γ_i has value $p^{\min(i,\nu)-\frac{\nu}{n}}$ on $t_p^{(\nu)}$ and hence we see that $t_p^{(\nu)}$ multiplies the lowest weight vector $e_{w_0\lambda}$ by

$$p^{-(\sum_{i=1}^{n-1} a_{n-i}(\min(i,\nu)-\frac{i\nu}{n})+d\nu} = p^{-\sum_{i=1}^{n-1} a_{n-i}\min(i,\nu)+((\sum_{i=1}^{n-1}\frac{ia_{n-i}}{n})+d)\nu)} = p^{C(\lambda,\nu)}.$$

Observe that $\sum_{i=1}^{n-1}\frac{ia_{n-i}}{n}+d=\delta(\lambda)$ is an integer.

If we multiply $t_p^{(\nu)}$ by the inverse of this number then the resulting operator is the identity on the lowest weight vector $e_{w_0\lambda}$ and all other weight vectors e_η are multiplied by a power $p^{m(\eta)}$ where $m(\eta)\geq 0$. This means that $p^{-C(\lambda,\nu)}t_p^{(\nu)}$ is the canonical choice for $u_{t_p^{(\nu)}}$, i.e. on the rational cohomology we have

$$T(t_p^{(\nu)},u_{t_p^{(\nu)}}) = p^{-C(\lambda,\nu)}t_p^{(\nu)}.$$

Now we assume again that λ is selfdual. Then

$$\delta(\lambda) = d+\frac{1}{2}(\sum_{i=1}^{n-1}\frac{ia_i}{n}+\sum_{i=1}^{n-1}\frac{(n-i)a_i}{n}) = d+\frac{1}{2}(\sum_{i=1}^{n-1}a_i).$$

If we introduce the number $t(\lambda)=\sum_{i=1}^{n-1}a_i$ then we find for $\nu=1$

$$C(\lambda,1) = -\frac{1}{2}t(\lambda)+d.$$

Hence we see that the expression in the denominator of the local Euler factor starts with

$$1-p^{-s-\frac{n-1}{2}-\frac{1}{2}t(\lambda)+d}\pi_p(T(t_p^{(1)},u_{t_p^{(1)}}))\dots$$

We make the substitution $-s-\frac{n-1}{2}-\frac{1}{2}t(\lambda)+d\to -s$ and introduce the numbers

$$\Delta(\lambda^{(1)},\nu) = \nu(\frac{n-1}{2}+C(\lambda,1))-(\frac{\nu(n-\nu)}{2}+C(\lambda,\nu)) = \frac{\nu^2-\nu}{2}+\nu C(\lambda,1)-C(\lambda,\nu).$$

It is clear that the $\Delta(\lambda^{(1)},\nu)$ are non negative integers and they only depend on $(n,\lambda^{(1)})$.

This allows us to define the cohomological local Euler factor attached to an isotypical module π_f in $H^\bullet(S_{K_f}^G,\mathcal{M}_\lambda)$:

$$L_p^{\mathrm{coh}}(\pi_p,s) = \frac{1}{1-\pi_p(T(t_p^{(1)},u_{t_p^{(1)}}))p^{-s}+p^{\Delta(\lambda^{(1)},2)}\pi_p(T(t_p^{(2)},u_{t_p^{(2)}}))p^{-2s}-\dots}.$$

The global cohomological L-function will be

$$L^{\mathrm{coh}}(\pi_f,s) = \prod_p L_p^{\mathrm{coh}}(\pi_p,s).$$

This still may look clumsy in comparison with the definition of the automorphic L-function. But it has the advantage that the local Euler factor is expressed in terms of the eigenvalues of Hecke operators operating on the integral cohomology. The $\pi_p(t_p^{(\nu)}, u_{t_p^{(\nu)}})$ are algebraic integers and therefore the coefficients of $p^{-\nu s}$ are also algebraic integers. We recall that these eigenvalues do not change if we modify the coefficient system by twisting it by a power of the determinant, i.e replacing λ by $\lambda + d\det$. The twist is compensated by the modification of the $u_{t_p^{(\nu)}}$. Hence we can say that this cohomological L function is attached to the string $\{\pi_f \otimes |\ \ |^m\}_{m\in\mathbb{Z}}$ of Hecke modules (See 2.2.)

But now we hope that we are also able to relate this cohomological L-function to the motivic L-function. The Langlands philosophy predicts that we can find a motive $M(\pi_f)$ which has coefficients in F and is of rank n over F, which has the following properties:

For any prime ℓ and any $\mathfrak{l}|\ell$ in F we have the ℓ adic realization $M(\pi_f)_\mathfrak{l}$, this is an $F_\mathfrak{l}$ -vector space of rank n and we have an action of the Galois group

$$\rho_\mathfrak{l} : \ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}(M(\pi_f)_\mathfrak{l}).$$

This Galois-module is unramified at primes different from ℓ and for any prime p different from ℓ we have

$$\det(\mathrm{Id} - \Phi_p^{-1} p^{-s} | M(\pi_f)_\mathfrak{l})^{-1} = L_p^{\mathrm{coh}}(\pi_p, s).$$

In other words the cohomological L-function is equal to the motivic L-function:

$$L_p^{\mathrm{coh}}(\pi_f, s) = L(M(\pi_f).s)$$

We can say that $M(\pi_f) \xrightarrow{\sim} F^n$, the Hecke operators act as scalars via the given homomorphism $\pi_f : \mathcal{H} \to \mathcal{O}_F$. Then the Betti-realization of this motive is $M(\pi_f)_B \xrightarrow{\sim} F^n$ together with action of the complex conjugation F_∞. We also have the de-Rham realization $M(\pi_f)_{d-Rh} \xrightarrow{\sim} F^n$. On the de-Rham realization we have descending filtration $M(\pi_f)_{d-Rh} = F^0(M(\pi_f)_{d-Rh}) \supset F^1(M(\pi_f)_{d-Rh}) \supset F^2(M(\pi_f)_{d-Rh}) \cdots \supset \{(0)\} = F^\bullet(M(\pi_f)_{d-Rh}))$ by F -sub spaces. We have for the weight

$$w(M(\pi_f)_{d-Rh}) = n - 1 + t(\lambda),$$

the non trivial subquotients are of rank one, i.e. the non zero Hodge numbers are equal to one. The non trivial jumps occur at the indices $\{1+a_1, 2+a_1+a_2, \dots, n-1+a_1+\dots+a_{n-1}\}$, i.e. the cleaned up filtration is

$$M(\pi_f)_{d-Rh} \supset F^{1+a_1}(M(\pi_f)_{d-Rh}) \supset F^{2+a_1+a_2}(M(\pi_f)_{d-Rh}) \dots$$

$$\supset F^{n-1+t(\lambda)}(M(\pi_f)_{d-Rh}) \supset \{(0)\}.$$

The quadruple $(M(\pi_f)_B, F_\infty, M(\pi_f)_{d-Rh}, F^\bullet(M(\pi_f)_{d-Rh}))$ together with the comparison isomorphisms $I_\sigma : M(\pi_f)_B \times_{F,\sigma} \mathbb{C} \xrightarrow{\sim} M(\pi_f)_{d-Rh}, \sigma : F \hookrightarrow \mathbb{R}$, is called the Betti-de-Rham realization $M(\pi_f)_{B-dRh}$ of our motive.

At this point it seems to be convenient to introduce the number

$$m(\lambda) = \begin{cases} \frac{n}{2} - 1 & if \text{ n } is \quad even \\ \frac{n-1}{2} & if \text{n } is \quad odd \end{cases}$$

We also introduce the numbers $b_1 = 1 + a_1, \dots, b_\nu = \nu + a_1 + \cdots + a_\nu, \dots$.

If n is even then we have the two filtration steps around the middle

$$\supset F^{b_{m(\lambda)}}(M(\pi_f)_{d-Rh}) \supset F^{b_{m(\lambda)+1}}((M(\pi_f)_{d-Rh}) \supset$$

If n is odd we have a non zero Hodge number $(b_{m(\lambda)}, b_{m(\lambda)})$ which is of type (p,p). On the quotient $F^{b_{m(\lambda)}}(M(\pi_f)_{B-dRh})/F^{b_{m(\lambda)}+1}(M(\pi_f)_{B-dRh})$ the involution F_∞ act by a sign $(-1)^{\epsilon(M(\pi_f)_{B-dRh})}$ where $\epsilon = \epsilon(M(\pi_f)_{B-dRh} = 0, 1$.

We can form the completed L-function

$$\Lambda(M(\pi_f), s) = L_\infty(M(\pi_f)_{B-dRh}, s)L(M(\pi_f), s)$$

where the Euler-factor is determined by $M = M(\pi_f)_{B-dRh}$. The rules for this factor at infinity yield for n even

$$L_\infty(M, s) = \frac{\Gamma(s)}{(2\pi)^s}\frac{\Gamma(s-b_1)}{(2\pi)^{s-d_1}}\frac{\Gamma(s-b_2)}{(2\pi)^{s-b_2}}\cdots\frac{\Gamma(s-b_{m(\lambda)})}{(2\pi)^{s-b_{m(\lambda)}}}$$

and for n odd

$$L_\infty(M, s) = \frac{\Gamma(s)}{(2\pi)^s}\frac{\Gamma(s-b_1)}{(2\pi)^{s-b_1}}\cdots\frac{\Gamma(s-b_{m(\lambda)-1})}{(2\pi)^{s-b_{m(\lambda)-1}}}\frac{\Gamma(\frac{s-b_{m(\lambda)}+\epsilon}{2})}{\pi^{\frac{s-b_{m(\lambda)}+\epsilon}{2}}}$$

With this choice of the factor $L_\infty(M(\pi_f)_{B-dRh}, s)$ it follows from the theory of automorphic forms that the motivic L-function satisfies the expected functional equation

$$\Lambda(M(\pi_f), s) = \varepsilon(M(\pi_f))\Lambda(M(\pi_f), w(M(\pi_f)_{d-Rh})+1-s) = \varepsilon(M(\pi_f))\Lambda(M(\pi_f), n+t(\lambda)-s).$$

Of course we can forget the motive $M(\pi_f)$, everything can be defined in terms of π_f and the coefficient system (which in turn should be defined by π_f.)

Here the insertion ends.

Can we produce examples?

We return to the ratios of L-values on p.3. The L-functions which occur in these expressions are actually the "automorphic" or "unitary" L functions. But I think that I have strong reasons that we should express them in terms of the "cohomological" L-function. In the case discussed in "Eis-coh..." the arguments of evaluation are exactly the critical points of the Scholl-motive $M(f)$ attached to the automorphic form and this is equal to the cohomological L-function.

In the special case which we consider we started from two modular forms f, g of weights k_1, k_3 respectively. For both of them we have the Scholl-motive $M(f), M(g)$ and the two dimensional ℓ-adic Galois-representations

$$\rho(\tau) : \ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to Gl(M(f))_\ell), \rho(\sigma) : \ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to Gl(M(f))_\ell),$$

and we have for the Frobenii:

$$\rho(\tau)(\Phi_p^{-1}) \simeq \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix}, \ \alpha_p + \beta_p = a_p, \alpha_p\beta_p = p^{k_1-1} = p^{d_1+1}$$

$$\rho(\sigma)(\Phi_p^{-1}) \simeq \begin{pmatrix} \gamma_p & 0 \\ 0 & \delta_p \end{pmatrix}, \ \gamma_p + \delta_p = c_p, \gamma_p\delta_p = p^{k_3-1} = p^{d_3+1}$$

where a_p resp. c_p is the $p-th$ Fourier coefficient of f resp. g.

We take the symmetric square of $\rho(\sigma)$ and get

$$\rho(\text{ Sym}^2(\sigma)) : \text{ Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to Gl_3(\mathbb{Z}_\ell)$$

(here we assume that f, g have coefficients in $\mathbb{Z}$.) Then

$$\rho(\text{ Sym}^2(\sigma))(\Phi_p^{-1}) \simeq \begin{pmatrix} \gamma_p^2 & 0 & 0 \\ 0 & p^{d_3+1} & 0 \\ 0 & 0 & \delta_p^2 \end{pmatrix}$$

Then we can write the finite part of the L-function as

$$L^{\text{coh}}(\tau \times \Pi, s) = \prod_p \frac{1}{\det\big(\text{Id} - \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix} \otimes \begin{pmatrix} \gamma_p^2 & 0 & 0 \\ 0 & p^{d_3+1} & 0 \\ 0 & 0 & \delta_p^2 \end{pmatrix} p^{-s}\big)}$$

Here it becomes clear that this is the motivic L-function of the motive $M(\tau \times \Pi)$. Here the representation r of the dual group is the tensor product of the two tautological representations.

The local Euler-factor is of degree 6 it can be expressed in terms of the eigenvalues a_p, c_p and is given by

$$\big[\big(1 + \big(-a_pc_p^2 + 2a_pp^{-1+h}\big)\,p^{-s} + (a_p^2p^{-2+2h} + c_p^4p^{-1+k} - 4c_p^2p^{-2+h+k} + 2p^{-3+2h+k})p^{-2s} + (a_pc_p^2p^{-3+2h+k} + 2a_pp^{-4+3h+k})p^{-3s} + p^{-6+2(2h+k)}p^{-4s}\big)\big) * (1 - a_pp^{h-1}p^{-s} + p^{k+2h-3}p^{-2s})\big]^{-1}$$

Our motives $M(f), M(g)$ have Hodge types $\{(d_1{+}1, 0), (0, d_1{+}1), (d_3{+}1, 0), (0, d_3{+}1)\}$ and therefore we get for the Hodge type of $M(\tau \times \Pi)$

$$\{(d_1{+}2d_3{+}3, 0), (d_1{+}d_3{+}2, d_3{+}1), (d_1{+}1, 2d_3{+}2), (2d_3{+}2, d_1{+}1), (d_3{+}1, d_1{+}d_3{+}2), (0, d_1{+}2d_3{+}3)\}$$

it is pure of weight $d_1 + 2d_3 + 3$.

We reorder these Hodge type according to the size of the second component and get

$$\{(w, 0), (w-a, a), (w-b, b), (b, w-b), (a, w-a), (0, w)\},$$

where now $0 \leq a \leq b \leq \frac{w}{2}$.

From the the Hodge type or from representation-theoretic considerations we get a Γ factor at infinity which is (if I am not mistaken)

$$L_\infty(\tau \times \Pi, s) = \frac{\Gamma(s)\Gamma(s-a)\Gamma(s-b)}{(2\pi)^{3s}}$$

Again we put

$$\Lambda^{\mathrm{coh}}(\tau\times\Pi,s)=L_\infty(\tau\times\Pi,s)L^{\mathrm{coh}}(\tau\times\Pi,s).$$

This function satisfies a functional equation:

$$\Lambda^{\mathrm{coh}}(\tau\times\Pi,s)=\Lambda^{\mathrm{coh}}(\tau\times\Pi,w+1-s)$$

Once we accept this functional equation then we have fast algorithms to compute the values $\Lambda^{\mathrm{coh}}(\tau\times\Pi,s_0)$ at given argument s_0 up to very high precision.

(For classical modular forms f of weight k we have the following formula

$$\Lambda(f,s)=\sum_{n=1}^{\infty}\big((\frac{1}{2\pi})^s\frac{a_n}{n^s}\Gamma(s,2\pi nA)+(-1)^{\frac{k}{2}}(\frac{1}{2\pi})^{k-s}\frac{a_n}{n^{k-s}}\Gamma(s,2\pi n/A)\big)$$

where $\Gamma(s,2\pi nA)$ is the incomplete Γ function and where A is a strictly positive real number. The right hand side is independent of A (this gives a good test that the functional equation is really correct) and $A=1$ is the best choice. The sum is rapidly converging, because the incomplete Γ goes rapidly to zero.)

I remember that Don Zagier once mentioned that we always have such a formula to compute values of L-functions, once we can guess the functional equation and this formula can be used to confirm the guess.
This has been done by Tim Dokchitser in his Note "Computing special values of motivic L-functions. Experiment. Math. 13 (2004), no. 2, 137–149. "

Finally we discuss the special values. We have the above list of Hodge types, recall that the Hodge types lists those pairs (p,q) with $p+q=w=d_1+2d_3+2$ for which $h^{p,q}(M)\neq 0$. The Deligne conjecture predicts that we have to look at pairs (p_c,q_c) for which $p_c+q_c=w, p_c>q_c$ for which $h^{p_c,q_c}\neq 0$ and for which $h^{\nu,w-\nu}=0$ for all $q_c<\nu<p_c$. This is the critical interval $M_{\mathrm{crit}}=[(p_c,q_c),(q_c,p_c)]$ of our motive. One should look at it as an interval on the line $p+q=w$.
We look at our Hodge types

$$\{(d_1{+}2d_3{+}3,0),(d_1{+}d_3{+}2,d_3{+}1),(d_1{+}1,2d_3{+}2),(2d_3{+}2,d_1{+}1),(d_3{+}1,d_1{+}d_3{+}2),(0,d_1{+}2d_3{+}3)\}$$

We have to find the interval we have to distinguish cases. The first case is

a)

$$d_1<2d_3+1$$

Now we have two possibilities for the critical interval, it is either
a1)

$$[(2d_3+2,d_1+1),(d_1+1,2d_3+2)]$$

a2)

$$[(d_1 + d_3 + 2, d_3 + 1), (d_3 + 1, d_1 + d_3 + 2)]$$

depending on which one is smaller.

The second case is
b)

$$d_1 > 2d_3 + 1$$

In this case the critical interval is clearly

$$[(d_1 + 1, 2d_3 + 2), (2d_3 + 2, d_1 + 1)],$$

In the paper with Raghuram we will prove that we can define a period $\Omega(\tau_f \times \Pi_f)$ which under our assumptions (f, g have coefficients in $\mathbb{Q}$) is unique up to a sign such that

$$\Omega(\tau_f \times \Pi_f)^{\epsilon(a)} \frac{\Lambda^{\text{coh}}(\tau \times \Pi, a)}{\Lambda^{\text{coh}}(\tau \times \Pi, a+1)} \in \mathbb{Q} \text{ provided } p_c \geq a+1 \text{ , } a \geq q_c + 1$$

From our data $[p_c, q_c]$ and the value of a we can reconstruct the coefficient system λ.

"Large" primes occuring in the denominator of these rational number should produce congruence between eigenvalues of Hecke operators on Siegel modular forms of genus three and certain expressions in eigenvalues on pairs of modular forms of genus one.

The computation of the period is somewhat delicate. In principle we follow the recipe given in my papers

" Arithmetic Aspects of Rank one Eisenstein Cohomology"

"Interpolating coefficient systems and p-ordinary cohomology of arithmetic groups"

for which (recently updated versions) exist on my home-page. But it is not clear from the abstract definition how - given explicit data, i.e. f, g -we can really compute a number with high precision which gives us the value of the period.

There is a way out. Recall that we compute ratios of special values $a, a+1$ where a runs through an interval $[p_c - 1, q_c + 1]$ of integers, this interval can be quite long. So we simply choose our period such that for $a_0 = p_c - 1$

$$\Omega^*(\tau_f \times \Pi_f)^{\epsilon(a_0)} \frac{\Lambda^{\text{coh}}(\tau \times \Pi, a_0)}{\Lambda^{\text{coh}}(\tau \times \Pi, a_0 + 1)} = 1.$$

The correct period differs from this one by a rational number, which will have some prime factors $\{p_1, p_2, \ldots, p_r\}$ in it. Now we can start to verify the above rationality assertion for all a and we can compute these ratios as rational numbers.

Recall that we are interested in arguments a for which our ratio of L-values divided by the "correct" period has a "large" prime p in its factorization (in the denominator). Now it would be really bad luck, if this prime p would be (always) member of $\{p_1, p_2, \ldots, p_r\}$.

Hence if we find large primes p in the denominator of the ratios

$$\Omega^*(\tau_f\times\Pi_f)^{\epsilon(a)}\frac{\Lambda^{\rm coh}(\tau\times\Pi,a)}{\Lambda^{\rm coh}(\tau\times\Pi,a+1)}$$

for some values of a then we can look for congruences $\mod p$ between different kinds of Siegel modular forms.

How do these congruences look like?

We go back to the very general case that $G/\operatorname{Spec}(\mathbb{Z})$ is a Chevalley scheme and let $P\subset G$ be a maximal parabolic subgroup, here we assume that it is conjugate to its opposite. We assume that $T/\operatorname{Spec}(\mathbb{Z})$ is a maximal split torus and $T\subset B\subset P$. Let $\pi=\{\alpha_1,\alpha_2,\dots,\alpha_r\}$ be the set of simple positive roots, let $\{\gamma_1,\gamma_2,\dots,\gamma_r\}$ be the set of dominant fundamental weights. We have

$$2\frac{<\gamma_i,\alpha_j>}{<\alpha_i,\alpha_i>}=\delta_{ij},$$

the dominant weights are elements in $X^*(T)\otimes\mathbb{Q}$. We also consider the cocharacters $\{\chi_1,\chi_2,\dots,\chi_r\}\in X_*(T)\otimes\mathbb{Q}$, which form the dual basis to the roots. If we identify $X_*(T)\otimes\mathbb{Q}=X^*(T)\otimes\mathbb{Q}$ via the canonical quadratic form, then $\chi_i=\frac{2\gamma_i}{<\alpha_i,\alpha_i>}$. (The canonical quadratic form is normalized: $<\alpha_3,\alpha_3>=2$.)

Now we consider the cuspidal (inner ?) cohomology of the boundary stratum attached to P, we refer to "Arithmetic Aspects...". We pick a $\tilde{w}\in W^P$ and consider an isotypical subspace

$$H_!^{\bullet-l(\tilde{w})}(S^M,\mathcal{M}(\tilde{w}\cdot\lambda))(\sigma_f))\subset H^\bullet(\partial_P(S),\mathcal{M}).$$

Actually we should take an induced module on the left hand side, but let us assume that we only look at unramified cohomology, i.e. $K_f=G(\hat{\mathbb{Z}})$. Then induction simply means that we restrict the action of $\mathcal{H}^M$ to the action of $\mathcal{H}^G$ on $H_!^{\bullet-l(w)}(S^M,\mathcal{M}(w\cdot\lambda))$. We want to derive a formula for a "cohomological" Hecke operator in $\mathcal{H}^G$ as a sum over "cohomological" Hecke operator in $\mathcal{H}^M$.

The algebra of Hecke operators is generated by local algebras $\mathcal{H}_p^G$ and these local algebras commute (under our assumption that everything is unramified, they are even commutative).

We fix a prime p. To get Hecke operators we start from cocharacters $\chi=\sum m_i\chi_i:G_m\to T$, where the $m_i\in\mathbb{Z}$. This provides an element $\chi(p)\in T(\mathbb{Q}_p)$, and hence a double coset $K_p\chi(p)K_p$ whose characteristic function is denoted by T_χ. By convolution this defines an operator (also denoted by T_χ) on the cohomology with rational coefficients

$$T_\chi:H^\bullet(S^G,\mathcal{M}_\lambda\otimes\mathbb{Q})\to H^\bullet(S^G,\mathcal{M}_\lambda\otimes\mathbb{Q}).$$

We have defined the modified operators, which act on the cohomology with integral coefficients

$$T_\chi^{\rm coh}\,"="\,p^{<\chi,\lambda>}T_\chi:H^\bullet(S^G,\mathcal{M}_\lambda)\to H^\bullet(S^G,\mathcal{M}_\lambda).$$

We have a formula for the action of T_χ on the unramified spherical functions. We consider unramified characters $\nu_p : T(\mathbb{Q}_p) \to \mathbb{C}^\times$. Since $T(\mathbb{Q}_p) = X_*(T) \otimes \mathbb{Q}_p^\times$ we have for the module of unramified characters

$$\mathrm{Hom}_{un}(T(\mathbb{Q}_p), \mathbb{C}^\times) = \mathrm{Hom}(X_*(T), \mathbb{C}^\times) = X^*(T) \otimes \mathbb{C}^\times$$

If we pick a $\chi \in X_*(T)$ and a $\nu_p = \sum \gamma \otimes \omega_\gamma$ then

$$\nu_p(\chi(p)) = \prod \omega_\gamma^{<\chi,\gamma>} =< \chi, \nu_p >_p .$$

We have the embedding $X^*(T) \hookrightarrow \mathrm{Hom}_{un}(T(\mathbb{Q}_p), \mathbb{C}^\times)$ which is given by $\gamma \mapsto |\gamma|_p = (x \mapsto |\gamma(x)|_p)$. I want to distinguish carefully between the algebraic character and its absolute value.

Especially we have the half sum of positive roots $\rho_B^G \in X^*(T) \otimes \mathbb{Q}$ and the resulting character $|\rho_B^G|_p$.

We define the spherical function

$$\psi_{\nu_p}(g) = \psi_{\nu_p}(bk) = \nu_p(b)$$

and this will be an eigenfunction for the convolution with a Hecke operator

$$T_\chi * \psi_{\nu_p} = T_\chi^\vee(\nu_p)\psi_{\nu_p}.$$

We write a formula for $T_\chi^\vee(\nu_p)$ for the case that $\chi = \chi_i$ is one of our basis cocharacters χ_i. We look at the orbit of χ_i under the Weyl group, let W_i be the stabilizer of χ_i in W, then

$$T_\chi^\vee(\nu_p) = p^{<\chi_i,\rho_B^G>} \sum_{W/W_i} < w\chi_i, \nu_p - |\rho_B^G|_p >_p + \delta(\chi_i),$$

where $\delta(\chi_i)$ is a multiple of the identity. It is zero if for all positive roots α we have $< \chi_i, \alpha > \in \{0, 1\}$, i.e. the coefficient of the root α_i in any positive root is always ≤ 1.

If we now have an isotypical submodule $H_!^\bullet(S^G, \mathcal{M}_\lambda)(\pi_f), \pi_f = \otimes_p \pi_p$, and $\pi_p = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \nu_p$ then our above formula says

$$T_{\chi_i}^{\mathrm{coh}}(\pi_f) = p^{<\chi_i,\lambda>+<\chi_i,\rho_B^G>}(\sum_{W/W_i} < w\chi_i, \nu_p - |\rho_B^G|_p >_p) + p^{<\chi_i,\lambda>}\delta(\chi_i).$$

Now we ask for a formula for the Hecke operator on T_χ^{coh} on an isotypical piece $H_!^{\bullet - l(\tilde{w})}(S^M, \mathcal{M}(\tilde{w} \cdot \lambda))(\sigma_f))$ in the cohomology of some boundary stratum.

One of these simple computations, which I can never reproduce at any later occasion , shows

$$T_{\chi_i}^{G,\mathrm{coh}}(\mathrm{Ind}(\sigma_p)) = p^{<\chi_i, \lambda + \rho_B^G>} \sum_{w \in W^P/W_i} p^{-<w\chi_i, \tilde{w}(\lambda+\rho_B^G)>} T_{w\chi}^{M,\mathrm{coh}}(\sigma_p) + p^{<\chi_i,\lambda>}\delta(\chi_i) =$$

$$\sum_{w \in W^P/W_i} p^{<\chi_i, (\lambda+\rho_B^G) - w^{-1}\tilde{w}(\lambda+\rho_B^G)>} T_{w\chi}^{M,\mathrm{coh}}(\sigma_p) + p^{<\chi_i,\lambda>}\delta(\chi_i)$$

The factor in front is equal to one if $w=\tilde{w}$ and otherwise the exponent is a strictly positive number. Hence we get

$$T^{G,\mathrm{coh}}_{\chi_i}(\mathrm{Ind}(\sigma_p)) = T^{M,\mathrm{coh}}_{\tilde{w}\chi_i}(\sigma_p) + \qquad \text{Hecke-ind}$$

$$\sum_{w\in W^P/W_i, w\neq\tilde{w}} p^{<\chi_i,(\lambda+\rho_B^G)-w^{-1}\tilde{w}(\lambda+\rho_B^G)>} T^{M,\mathrm{coh}}_{w\chi_i}(\sigma_p) + p^{<\chi_i,\lambda>}\delta(\chi_i).$$

Let us call the first summand on the right hand side the "main" term. We observe that for $w\neq\tilde{w}$ the exponent $<\chi_i,(\lambda+\rho_B^G)-w^{-1}\tilde{w}(\lambda+\rho_B^G)>>\ 0$ and if λ is regular this is also true for $<\chi_i,\lambda>$. This tells us that the eigenvalue $T^{M,\mathrm{coh}}_{\chi_i}(\mathrm{Ind}(\sigma_p))$ is a p-adic unit if and only if $T^{M,\mathrm{coh}}_{\tilde{w}\chi}(\sigma_p)$ is a p-adic unit, provided λ is regular or $\delta(\chi_i)=0$.

(For the special case $G=GSp_2/\mathrm{Spec}(\mathbb{Z})$ and P the Siegel parabolic this yields the formulae in 3.1.2.1 in "Eisenstein Kohomologie...". The formula for $T_{p,\beta}$ is wrong, I overlooked the term $p^{<\chi_i,\lambda>}\delta(\chi_i)$. This was discovered by Gerard, the congruences for the second Hecke operator became wrong.)

Now we can formulate how the general form of a Ramunujan-type congruence should look like.

We start from an isotypical subspace $H^\bullet(S^M,\mathcal{M}(w\cdot\lambda_R))(\sigma_f)$ where $R=\mathbb{Z}[1/N]$ where N is a suitable integer. Let $I_{\sigma_f}\subset\mathcal{H}^M_R$ be the annihilator of σ_f. Then the quotient $\mathcal{H}^M_R/I_{\sigma_f}=R(\sigma_f)$ is an order in an algebraic number field $\mathbb{Q}(\sigma_f)$. We consider the second constant term of the Eisenstein series evaluated at $s_w=0$ and assume that it is of the form

$$a(\sigma_f)\mathrm{Mot}(\sigma_f)$$

where $a(\sigma_f)\in\mathbb{Q}(\sigma)$ and where $\mathrm{Mot}(\sigma_f)$ has some kind of an interpretation as an element in some $\mathrm{Ext}^1_{\mathcal{MM}}$. Now we assume that a "large" prime $\mathfrak{l}\subset R(\sigma_f)$ divides the denominator of $a(\sigma_f)$. We assume that σ_ℓ is ordinary at $\mathfrak{l}$, i.e. $T^{M,\mathrm{coh}}_{\chi_i}(\sigma_\ell)\notin\mathfrak{l}$ for all i (some i_0 ?).

Then we can hope for an isotypical component Π_f for the Hecke algebra $\mathcal{H}^G_R$ in the cohomology $H^\bullet(S^G,\mathcal{M}_\lambda)(\Pi_f)$, we consider the order $\mathcal{H}^G_R/I_{\Pi_f}=R(\Pi_f)$, we expect to find a prime $\mathfrak{l}_1\subset R(\Pi_f)$ and an isomorphism between the completions

$$\Phi: R(\Pi_f)_{\mathfrak{l}_1}\xrightarrow{\sim} R(\sigma_f)_{\mathfrak{l}}$$

such that for all primes p

$$\Phi(T^G_{\chi_i}(\Pi_p)) \equiv T^{G,\mathrm{coh}}_{\chi_i}(\mathrm{Ind}(\sigma_p)) \mod \mathfrak{l}.$$

We consider the case where our modular forms f,g have rational coefficients, i.e. are of weight $12,16,18,20,22,26$ this means that the values for d_1,d_3 are $10,14,16,18,20,24$. Following a notation in representation theory we put

$$w\cdot\lambda = w(\lambda+\rho)-\rho = d_1(w\cdot\lambda)\gamma^M_{\alpha_1}+d_3(w\cdot\lambda)\gamma^M_{\alpha_3}+1/2(-6-n_2)\gamma_{\alpha_2}.$$

Given d_1, d_3 a value a in the upper half of the above range, we solve the equations

$$d_1(w_1 \cdot \lambda) = d_1,\ d_3(w_1 \cdot \lambda) = d_3,\ \frac{n_2}{2} + \frac{d_1}{2} + d_3 + 2 = a \qquad (case1)$$

$$d_1(w_2 \cdot \lambda) = d_1,\ d_3(w_2 \cdot \lambda) = d_3,\ \frac{n_1}{2} + \frac{d_1}{2} + d_3 + 3 = a \qquad (case2)$$

We introduce the number

$$\mathbf{w} = d_1 + 2d_3 + 3$$

and observe that $\frac{d_1}{2} + d_3 + 2 = \frac{\mathbf{w}+1}{2}$ is the reflection point of the functional equation. We rewrite our equations a little bit. In $(case1)$

$$\begin{gathered} k_1 - 4 = d_1 - 2 = n_2 + 2n_3 \\ k_3 - 4 = d_3 - 2 = n_1 + n_2 + n_3 \\ 2a - \mathbf{w} - 1 = n_2 \end{gathered}$$

and in (case2)

$$\begin{gathered} k_1 - 6 = d_1 - 4 = n_1 + 2n_2 + 2n_3 \\ k_3 - 4 = d_3 - 2 = n_3 \\ 2a - \mathbf{w} - 3 = n_1 \end{gathered}$$

As it turns out that for our restricted choice of f, g we never have solutions in $(case2)$.

This gives us a unique highest weight $\lambda = \lambda(d_1, d_3, a)$ and a space of holomorphic modular cusp forms $S_{n_1,n_2,4+n_3}$ in which we should look for a cusp form satisfying congruences.

I want to give the precise form for the expected congruences. We choose the Hecke operator T_{χ_3}, this is the operator whose eigenvalues are the traces of the Frobenius, it has also the property that $< \chi_3, \alpha > \in \{-1, 0, 1\}$ for all roots α, and if we identify $X_*(T)_{\mathbb{Q}} = X_*(T)_{\mathbb{Q}}$ then $\chi_3 = \gamma_3$.

The Weyl group W is the semidirect product of S_3 and $(\mathbb{Z}/2\mathbb{Z})^3$ and is of order 48. The stabilizer W_3 of χ_3 is the subgroup S_3, this is the Weyl group of A_2. We have to study the double cosets

$$W_M \backslash W / W_3 = W^P / W_3.$$

The quotient W/W_3 has cardinality 8, on this quotient we have the action of W_M, this is the group generated by the reflections s_1, s_3 and hence is of order 4. It is clear that we have two orbits of length 2 and one orbit of length 4. Hence the sum in (Hecke-ind) has three terms.

The orbit of length 4 gives us the "main" term in our formula (Hecke-ind) and $T^{M,\mathrm{coh}}_{\tilde{w}\chi_i}(\sigma_p) = a_p(f)a_p(g)$, where of course the two factors are the eigenvalues of f, g respectively.

The two other orbits correspond to the Kostant representatives $e = (\mathrm{Id}, \Theta_P$, they are fixed by s_1, hence the W_M orbits are given by $\{(e, s_3), (\Theta_P, s_3\Theta_P)\}$. This means that for choice of w we have $T^{M,\mathrm{coh}}_{w^{-1}\tilde{w}\chi_i}(\sigma_p) = a_p(g)$, it remains to compute the factor in front. For $w = e$ or $w = \Theta_P$ this factor is

$$p^{<(\mathrm{Id}-\tilde{w}^{-1}w)\chi_3,\lambda+\rho>}$$

Our element $\tilde{w}$ is one of the two Kostant representatives w_1, w_2 on p. 1. Then $\tilde{w}^{-1}\Theta_P$ is equal to the the corresponding elements v_1, v_2. We get

$$\begin{array}{cc} <(\mathrm{Id}-w_1^{-1})\chi_3,\lambda+\rho>=n_2+n_3+2 & <(\mathrm{Id}-v_1^{-1})\chi_3,\lambda+\rho>=n_3+1 \\ <(\mathrm{Id}-w_2^{-1})\chi_3,\lambda+\rho>=n_1+n_2+n_3+3 & <(\mathrm{Id}-v_2^{-1})\chi_3,\lambda+\rho>=n_2+n_3+2 \end{array}$$

Hence we expect:

We choose triple d_1, d_3, a and a pair of eigenforms f, g with weight $d_1+2=k_1, d_3+2=k_3$. Let λ solve the appropriate equations $(case1), (case2)$. If a prime $\mathfrak{l}$ divides the **denominator** *of*

$$\Omega(\tau_f\times\Pi_f)^{\epsilon(a)}\frac{\Lambda^{\mathrm{coh}}(\tau\times\Pi,a)}{\Lambda^{\mathrm{coh}}(\tau\times\Pi,a+1)}$$

then we find an isotopical subspace $H^6_!(S^G,\mathcal{M}_\lambda)(\tilde{\Pi}_f)$ and a congruence

$$T^G_{\chi_3}(\tilde{\Pi}_p)\equiv a_p(g)(p^{n_3+1}+a_p(f)+p^{n_2+n_3+2})\mod \mathfrak{l}$$

in $(case1)$ and

$$T^G_{\chi_3}(\tilde{\Pi}_p)\equiv a_p(g)(p^{n_2+n_3+2}+a_p(f)+p^{n_1+n_2+n_3+3})\mod \mathfrak{l}$$

in $(case2)$

We compare to TABLE 1. in [BFG]: We have

$$(k_1,k_3)=(m_2,m_1)$$

and

$$r_1=n_2+n_3+2, r_2=n_3+1 \text{ in } (case1),$$

$$r_1=n_1+n_2+n_3+3, r_2=n_2+n_3+2 \text{ in } (case2).$$

Recall that we are interested in the special value $a+1$, we can say in (case1)

$$a+1=\frac{n_2+1}{2}+\frac{\mathbf{w}}{2}+1=\frac{r_1-r_2+\mathbf{w}}{2}+1$$

and in $(case2)$

$$a+1=\frac{n_2+1}{2}+\frac{\mathbf{w}}{2}+1=\frac{r_1-r_2+\mathbf{w}}{2}+1$$

Now I checked against TABLE1 in [BFG] and Anton's tables and the data match perfectly. We even see some "small" primes providing congruences. We see a 17^2 occuring in the case f of weight 12 and g of weight 18. We observe that both forms are ordinary at 17.

Remark: In our special case the expression for $T^{G,\mathrm{coh}}_{\chi_i}(\mathrm{Ind}(\sigma_p))$ is a sum of three terms, the term in the middle $a_p(g)a_p(f)$ has weight $\frac{d_1+d_2}{2}+1$ the first term has a lower weight the third term has a higher weight. The difference of the weights of the first and third term is up to a shift our evaluation point a. This means: The closer these two weights get the closer a comes to the center of the L function.

Congruences for the Klingen parabolic subgroup in the case $g = 2$.

We assume $g = 2$ and we consider the Klingen parabolic subgroup Q of $G = PSp_2/\operatorname{Spec}(\mathbb{Z})$, we have $\pi = \{\alpha_1, \alpha_2\}$, where α_1 is the short root, and α_2 is long. The Klingen parabolic subgroup is defined by the cocharacter χ_2, its unipotent radical is not commutative and contains the roots $\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2$.

We choose an irreducible module for G it has a highest weight $\lambda = n_1\gamma_1 + n_2\gamma_2$, then $n_2 \equiv 0 \mod 2$. We pick $w = s_1 s_2$, and consider the cohomology $H^1_!(S^M, H^2(\mathfrak{u}_Q, \mathcal{M}(w \cdot \lambda))$. We have

$$w \cdot \lambda = w(\lambda + \rho) - \rho = (1 + n_1 + n_2)\gamma_2^M + (-3 - n_2)\gamma_1.$$

Hence we see that $k = n_2 + n_1 + 3$ must be even.

Let us consider an isotypical submodule $H^1(S^M, \mathcal{M}(w \cdot \lambda))(\sigma_f)$. Such an eigenspace corresponds to a modular form of weight k. Then the second constant term is of the form

$$\frac{L^{\text{coh}}(\operatorname{Sym}^2(f), k + n_2)}{\Omega(\sigma_f) L^{\text{coh}}(\operatorname{Sym}^2(f), k + n_2 + 1)}.$$

We will see that this has a motivic interpretation. In the case of the Siegel parabolic subgroup the critical values in question where also in a string of integers, basically because the motive in question had Hodge numbers $(k-1, 0), (0, k-1)$. But in the case of the Klingen parabolic subgroup the non zero Hodge numbers of the motive $\operatorname{Sym}^2(f)$ will be $((2(k-1), 0), (k-1, k-1), (0, 2(k-1))$ and the $(k-1, k-1)$ piece puts a parity condition on the critical values.

More precisely we can find periods $\Omega_{even}(\sigma_f), \Omega_{odd}(\sigma_f)$ such that

$$\frac{1}{\Omega_{odd}(\sigma_f)} L^{\text{coh}}(\operatorname{Sym}^2(f), k + n_2) \in \mathbb{Q} \text{ for } n_2 \in \{-1, -3, \dots, -k+1\}$$

$$\frac{1}{\Omega_{even}(\sigma_f)} L^{\text{coh}}(\operatorname{Sym}^2(f), k + n_2 + 1) \in \mathbb{Q} \text{ for } n_2 \in \{-1, 1, 3, \dots, k-3\}$$

In this case the Hecke operator would again be T_{χ_2}, the group W_M has two orbits of length 2 on W/W_2 and we get for the Hecke eigenvalue on the induced module is $a_p(f)(1 + p^{n_2+1})$.

Hence we can look for congruences of the form

$$T_{\chi_2}(\tilde{\Pi}_p) \equiv a_p(f)(1 + p^{n_2+1}) \mod \mathfrak{l}$$

if $\mathfrak{l}$ *divides*

$$\frac{1}{\Omega_{even}(\sigma_f)} L^{\text{coh}}(\operatorname{Sym}^2(f), k + n_2 + 1)$$

and $1 \leq n_2 \leq k - 3 = n_1, n_2 \equiv 1 \mod 2$, *where* $\tilde{\Pi}_f$ *is an eigenform in* S_{n_1, n_2+3}.

The data fit perfectly with TABLE 3. in [BFG] and Anton's tables.

We have the special case $n_2 = k - 3$, and therefore $n_1 = 0$. Our space of Siegel modular forms is a space of scalar valued modular forms of weight k. In this case the congruences have been proved by Mizumoto in Math. Ann. 275, 149-161 (1986). He uses the de-Rham realization of the cohomology, he looks at the Fourier expansion of the Eisenstein series.