$d_2^{r-3,\bullet}: E_2^{r-3,\bullet} \to E_2^{r-1,\bullet-1}$

G. Harder

October 31, 2023

Let $\mathcal{G}/\mathbb{Z}$ be any semi simple (simply connected) group scheme (Chevalley scheme). We put $X = \mathcal{G}(\mathbb{R}))/K_\infty$ and $\Gamma = \mathcal{G}(\mathbb{Z})$. (You may think of $\mathcal{G} = \mathrm{Sl}_n$) Let T, B a split maximal torus and $B \supset T$ a Borel. Let Δ^+ be the set of positive roots and $\pi = \{\alpha_1, \dots, \alpha_i, \dots, \alpha_r\} \subset X^*(T)$ be the set of simple roots and $\{\dots, \gamma_i, \dots\}$ be the set of fundamental weights. If $I \subset \pi$ then P_I will be the parabolic subgroup whose Levi M_I has simple roots $\pi \setminus I$, the number $d(P_I) := \#I - 1$ is its rank minus 1. We will denote the locally symmetric space $\Gamma\backslash X = S$ and accordingly S^{M_I} will be the locally symmetric space defined by M_I. The Borel-Serre boundary is denoted by $\Gamma\backslash X$ it is the union of boundary strata $\partial_I(\Gamma\backslash X)$. These boundary strata are open in their closure. The closure $\overline{\partial_I(\Gamma\backslash X)}$ of a stratum is the union of the strata $\partial_{I'}(\Gamma\backslash X)$ with $I' \supset I$. If $P = P_I$ then $\partial_P(\Gamma\backslash X) = \partial_I(\Gamma\backslash X)$. Let $\lambda = \sum n_i\gamma_i$ be a highest weight and $\mathcal{M}_\lambda$ the resulting highest weight representation. Let $\tilde{\mathcal{M}}$ be the resulting sheaf on $\Gamma\backslash X$.

We consider the cohomology and the fundamental long exact sequence

$$\to H^q_c(\Gamma\backslash X, \tilde{\mathcal{M}}) \to H^q(\Gamma\backslash X, \tilde{\mathcal{M}}) \xrightarrow{r_\partial} H^q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) \to H^{q+1}(\dots) \tag{1}$$

We also introduce the "inner cohomology"

$$H^q_!(\Gamma\backslash X, \tilde{\mathcal{M}}) := \ker(r_\partial) = \mathrm{Im}(i_c).$$

The first -still vague problem- is

Describe the image of the restriction map r_∂

But this does not make sense, how can we describe a subspace in a vector space which we do not know. Hence we have to understand the cohomology

$$H^q(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X), \tilde{\mathcal{M}}) = H^q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}),$$

here $\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X)$ is the tubular neighbourhood described in the book.

Now homological algebra provides some non trivial tools, we have a spectral sequence that converges to the cohomology, the $E_1^{\bullet,\bullet}$ term is

$$E_1^{p,q} = \bigoplus_{P: d(P) = p+1} H^q(\partial_P(\Gamma\backslash X), \tilde{\mathcal{M}}) \tag{2}$$

and the differentials $d_1^{p,q}: E_1^{p,q} \to E_1^{p+1,q}$ are obtained from the restriction maps to the next lower stratum. We define $E_2^{p,q}: \ker(d_1^{p,q})/\mathrm{Im}(d_1^{p-1,q})$, and the we can define $d_2^{p,q} :: E_2^{p,q} \to: E_2^{p+2,q-1}$.

This spectral sequence converges, hence we get a descending filtration

$$(0) \subset \mathcal{F}^{r-1}(H^q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) \subset \dots \mathcal{F}^1(H^q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) \subset \mathcal{F}^0(H^q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) = H^q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) \tag{3}$$

and the quotients

$$\mathcal{F}^p H^q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}})/\mathcal{F}^{p+1}H^q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) = E_r^{p,q-p} = E_\infty^{p,q-p} \tag{4}$$

We also have Kostant's theorem

$$H^q(\partial_P(\Gamma\backslash X), \tilde{\mathcal{M}}) = \bigoplus_{w \in WP} H^{q-l(e)}(\Gamma_M\backslash X^M, \widetilde{H^{l(w)}(\mathfrak{u}, \mathcal{M})}(w \cdot \mathcal{M}))$$

and hence

$$E_1^{p,q} = \bigoplus_{P:d(P)=p+1} \bigoplus_{w \in W^P} H^{q-l(e)}(\Gamma_M\backslash X^M, \widetilde{H^{l(w)}(\mathfrak{u}, \mathcal{M})}(w \cdot \mathcal{M})). \tag{5}$$

We have the so called edge homomorphisms

$$H^q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) \to E_\infty^{0,q} \subset E_1^{0,q}\,; E_1^{r-1,q} \to E_\infty^{r-1,q} \hookrightarrow H^{q+r-1}(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) \tag{6}$$

We also have a homological spectral sequence with $E^1_{\bullet,\bullet}$ -term

$$E^1_{p,q} = \bigoplus_{P:d(P)=p+1} H_q(\partial_P(\Gamma\backslash X, \tilde{\mathcal{M}}) \text{ and differentials } d^1_{p,q} : E^1_{p,q} \to E^1_{p-1,q}. \tag{7}$$

the higher differentials are maps

$$d^\nu_{p,q} : E^\nu_{p,q} \to E^\nu_{p-\nu,q+\nu-1}. \tag{8}$$

This spectral sequence induces an ascending filtration

$$(0) \subset \mathcal{G}_0(H_q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) \subset \dots \mathcal{G}_{r-2}(H_q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) \subset H_q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) \tag{9}$$

which converges to $H_\bullet(\partial(\Gamma\backslash X), \tilde{\mathcal{M}})$. For this filtration we know that

$$\mathcal{G}_p(H_q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}})/\mathcal{G}_{p-1}(H_q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) = E^\infty_{p,q-p} \tag{10}$$

We have the Poincare duality pairing

$$H^\bullet(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) \times H^{d-1-\bullet}(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}^\vee) \xrightarrow{\cup} \mathbb{Q} \tag{11}$$

which also yields and identification $PD : H^{d-1-\bullet}(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}^\vee) \xrightarrow{\sim} H_\bullet(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}^\vee)$. where $< PD(\xi), \eta) >= \xi \cup \eta$ and $<\,,\,>$ is the obvious pairing between homology and cohomology.

Now I claim that a careful analysis of the spectral sequences yields that for the pairing

$$\mathcal{G}_p(H_q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}^\vee) \cup \mathcal{F}^{p+1}(H^q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) = 0 \tag{12}$$

and hence the Poincare duality pairing induces a non degenerate pairing

$$\mathcal{G}_p(H_q(\partial(\Gamma\backslash X),\tilde{\mathcal{M}}^\vee)/\mathcal{G}_{p-1}(H_q(\partial(\Gamma\backslash X),\tilde{\mathcal{M}}^\vee)\times \mathcal{F}^pH^q(\partial(\Gamma\backslash X),\tilde{\mathcal{M}})/\mathcal{F}^{p+1}(H^q(\partial(\Gamma\backslash X),\tilde{\mathcal{M}})\to\mathbb{Q}$$

$$<\ ,\ >_\infty\colon E^\infty_{p,q-p}\times E_\infty^{p,q-p}\to\mathbb{Q} \tag{13}$$

These modules are now subquotients of $E^1_{p,q-p}$ and $E_1^{p,q-p}$ and the pairing $<\ ,\ >_\infty$ is induced by the natural pairing between these modules

Now we can formulate a slightly more precise question.

I think we should always try to compute the cohomology $H^\bullet(\,\Gamma\backslash X,\tilde{\mathcal{M}}), H^\bullet(\partial(\Gamma\backslash X),\tilde{\mathcal{M}})$ and the map r at the same time. Assume that we computed the $E_1^{p,q}$ page of the spectral sequence, i.e we solved our problem for the Levi-quotients M, then we have to compute the higher pages in the spectral sequence. The computation of the terms $E_2^{p,q}$ requires that we have solved our problem for the reductive quotients of the parabolic subgroups. The computation of the differentials $d_2^{p,q}: E_2^{p,q}\to E_2^{p+2,q-1}$ may become delicate, eventually we have to compute the terms $E_\infty^{p,q}$.

Once this problem is solved we try to construct -starting from cohomology classes $\omega\in H^\bullet(\partial(\Gamma\backslash X),\tilde{\mathcal{M}})$ - Eisenstein cohomology classes $\mathrm{Eis}(\omega)]\in H^\bullet(\,\Gamma\backslash X,\tilde{\mathcal{M}}\otimes\mathbb{C})$*. The construction of these classes requires that we have to write some infinite series and we have to discuss some -sometimes delicate- issues of convergence. We go to the transcendental level and hence we extend scalars to $\mathbb{C}$. These Eisenstein series may have poles we have to take residues. Then we try to show that the images $r_\partial(\mathrm{Eis}(\omega))$ of these classes span a maximal isotropic subspace (with respect to above pairing) in the cohomology of the boundary.*

This problem has been successfully tackled by J. Bajpai, L. Guan and various other authors in some low rank cases. We come back to this later.

If we use Eisenstein series to produce classes in the image of r_∂ or to compute the differentials in the spectral sequence we have to evaluate infinite series. Hence we expect that certain transcendental quantities will show up. But on the other hand we know that all the vector spaces and all arrows are defined over $\mathbb{Q}$ there must be some elgebraicity or rationality relations between these transcendental quantities.

This principle is exploited in our book with Raghuram. There the underlying group is $\mathrm{Gl}_n/\mathbb{Z}$ we consider the first column of the spectral sequence $E_1^{0,\bullet}$. We know that the edge homomorphism $H^\bullet(\partial(\Gamma\backslash X),\tilde{\mathcal{M}})\xrightarrow{eg^0}E_\infty^{0,\bullet}$ is surjective. We define a certain subspace $E^{0,\bullet}_{1,!!}\subset E^{0,\bullet}_\infty$ of so called strongly inner classes. To do this we use the action of the Hecke algebra and and strong multiplicity one.We construct a canonical section $s: E^{0,\bullet}_{1,!!}\to H^\bullet(\partial(\Gamma\backslash X),\tilde{\mathcal{M}})$. The image $s(E^{0,\bullet}_{1,!!})$ is a direct summand, more precisely we have

$$H^\bullet(\partial(\Gamma\backslash X),\tilde{\mathcal{M}})=s(E^{0,\bullet}_{1,!!})\oplus s(E^{0,\bullet}_{1,!!})^\perp \tag{14}$$

where the second summand is the orthogonal complement with respect to the above pairing. Now we construct Eisenstein classes $\mathrm{Eis}(\omega)\in H^\bullet(\Gamma\backslash X,\tilde{\mathcal{M}}\otimes\mathbb{C})$ such that the images $r_\partial($ $\mathrm{Eis}(\omega)$ span a maximal isotropic subspace in $(E^{0,\bullet}_{1,!!})$.

The position of this subspace is determined by special values of Rankin-Selberg L functions. Since this subspace is defined over $\mathbb{Q}$ we get our rationality result for these special values.

One purpose of this note is to show that we should look at the entire spectral sequence and not only at the cuspidal part of the $E_1^{0,\bullet}$. We pick a highest weight λ let $\mathcal{M}_\lambda$ be the highest weight module. Instead of looking at the first column we consider the last column in the spectral sequence

$$E_1^{r-1,\bullet} = \bigoplus_q E_1^{r-1,q} = \bigoplus_q \bigoplus_{w\in W: l(w)=q} H^0(S^T, \mathbb{Q}(w\cdot\lambda)) = \bigoplus_q \bigoplus_{w\in W:l(w)=q} \mathbb{Q}e_{w\cdot\lambda} \tag{15}$$

where $e_{w\cdot\lambda}$ is a generator in degree $l(w)$ and non zero if and only if $w\cdot\lambda|T(\mathbb{Z})=0$. (The symmetric space S^T is essentially a point, divided by $T(\mathbb{Z})=\{\pm 1\}^r$.)

We want to compute $E_2^{r-1,\bullet}$ and consider the differential

$$d_1^{r-2,q}: E_1^{r-2,q} \to E_1^{r-1,q}$$

$$\bigoplus_{I:\#I=r-1} \bigoplus_{w_I\in W^{P_I}} H^{q-l(w_I)}(\Gamma\backslash X^{M_I}, \mathcal{M}_{w_I\cdot\lambda}) \to \bigoplus_{w\in W} \mathbb{Q}(w\cdot\lambda)(=\mathbb{Q}e_w) \tag{16}$$

the image respects the direct sum decomposition. We need to know for which $w\in W$ the space $\mathbb{Q}e_{w\cdot\lambda}$ is not in the image of $d_1^{r-2,q}$. Let us call this set $\mathcal{S}[\lambda]\subset W$.

We understand the Eisenstein cohomology of $M_I^{(1)}(=\mathrm{Sl}_2)$, and this tells us what the image

$$d_I: H^{q-l(w_I)}(\Gamma\backslash X^{M_I}, \mathbb{Q}(w_I\cdot\lambda)) \to \mathbb{Q}e_{w_I}\oplus\mathbb{Q}e_{s_iw_I} \tag{17}$$

will be. (Here $\pi\setminus I=\{\alpha_i\}$ and s_i is the reflection). Only one of the terms on the right hand side can be non zero (because $w\cdot\lambda|w_0\cdot\lambda|T(\mathbb{Z})=1$).) Then it is clear that

The element $w\in\mathcal{S}[\lambda]$ if and only for all $I\subset\pi; \#I=r-1, \{\alpha_i\}=\pi\setminus I$ the following holds:

If $w\in W^{P_I}$ then $\dim(\mathcal{M}_{w\cdot\lambda})>1$; if $w\notin W^{P_I}$ then $\dim(\mathcal{M}_{s_iw\cdot\lambda})=1$

The set $\mathcal{S}[\lambda]$ should be computed with the help of a computer. If λ is regular then it is clear that $\mathcal{S}[\lambda]$ consists just of one element namely the identity Id. We get in this case $E_2^{r-1,\bullet}=\mathbb{Q}(e_{\mathrm{Id}})$ and this injects into $H^{r-1}(\partial(\Gamma\backslash X),\mathcal{M}_\lambda)$. Hence we get, provided $|w_0\cdot\lambda|T(\mathbb{Z})=1$, non trivial cohomology $H^r(\partial(\Gamma\backslash X),\mathcal{M}_\lambda)\neq 0$.

But for non regular λ this set may become more complicated. In any case we find

$$E_2^{r-1,q} = E_1^{r-1,q}/\mathrm{Im}(d_1^{r-2,q}) = \bigoplus_{w\in\mathcal{S}[\lambda], l(w)=q} \mathbb{Q}(w\cdot\lambda) \tag{18}$$

and this says that the computation of $E_2^{r-1,q}$ is a Weyl group issue. Now we want to compute $E_3^{r-1,\bullet}$, this means that we have to compute the image of the differential

$$d_2^{r-3,\bullet}: E_2^{r-3,\bullet} \to E_2^{r-1,\bullet-1}. \tag{19}$$

This is a delicate issue. We know that $E_2^{r-3,\bullet}$ is a quotient of the kernel of the differential $E_1^{r-3,\bullet} \xrightarrow{d_1^{r-3,\bullet}} E_1^{r-2,\bullet}$. We can compute this kernel from Weyl group combinatoric and our knowledge of the restriction maps

$$H^\bullet(S^{M_I}, w\cdot\lambda) \to H^\bullet(\partial(S^{M_I}), w\cdot\lambda), \tag{20}$$

here I runs over the subsets with $\#I = r-2$ or in other words M_I is of rank 2. For these groups the above general problem is solved (Schwermer, Bajpai, Moya, Horozow, H., ...). We compute the map

$$\ker(E_1^{r-3,\bullet} \to E_1^{r-2,\bullet}) \xrightarrow{d_2^{r-3,\bullet}} E_1^{r,\bullet-1} \tag{21}$$

Now we have to use transcendental tools, we extend our coefficient system by $\mathbb{C}$ and compute the map

$$\ker(E_1^{r-3,\bullet}\otimes\mathbb{C} \xrightarrow{d_1^{r-3,\bullet}} E_1^{r-2,\bullet}\otimes\mathbb{C}) \xrightarrow{d_2^{r-3,\bullet}} E_1^{r,\bullet-1}\otimes\mathbb{C} \tag{22}$$

.

We recall that the spectral sequence is obtained from the cohomology of the double de-Rham complex. We represent a cohomology class $[\omega]$ in $\ker(d_1^{r-3,\bullet})$ by an array of closed differential forms

$$\tilde\omega = \{.....,\omega_I,..\} \in \prod_{I,\#I=r-2} \Omega^\bullet(\partial_I(\Gamma\backslash X), \mathcal{M}_\lambda\otimes\mathbb{C}) \tag{23}$$

and send this to by the horizontal boundary map to

$$d_0\tilde\omega = \{...,\omega_J,...\} \in \prod_{J,\#J=r-1} \Omega^\bullet(\partial_J(\Gamma\backslash X), \mathcal{M}_\lambda\otimes\mathbb{C}). \tag{24}$$

Here we know that we get the ω_J by taking a sum over constant Fourier coefficients

$$\mathcal{F}^{M_I,M_J} : \omega_I(.) \mapsto \int_{U_{I,J}} \omega_I(u_{I,J}\cdot)du_{I,J}$$

We know that the ω_J are representing the trivial class, hence we can write $\omega_J = d\psi_J$ where now

$$\tilde\psi = \{...,\psi_J,...\} \in \prod_{J,\#J=r-1} \Omega^{\bullet-1}(\partial_J(\Gamma\backslash X), \mathcal{M}_\lambda\otimes\mathbb{C}). \tag{25}$$

and this array of differentials is now by the horizontal boundary mapped to a differential form

$$\psi \in \Omega^{\bullet-1}(\partial_\pi(\Gamma\backslash X), \mathcal{M}_\lambda). \tag{26}$$

This is a closed form and hence it defines a cohomology class $[\psi] \in H^{\bullet-1}(\partial_J(\Gamma\backslash X), \mathcal{M}_\lambda\otimes\mathbb{C})$. This class is the image of the class $[\omega]$ under the map $d_2^{r-3,\bullet}$, i.e.

$$d_2^{r-3,\bullet}([\tilde\omega]) = [\psi] \tag{27}$$

I think the first situation where something really interesting happens is $\Gamma = \mathrm{Sl}_4(\mathbb{Z})$, we consider the trivial representation, i.e. $\lambda = 0$. Now we have to compute. My computation (partially supported by Mathematica and with the help of Taiwang Deng) gave the following result: We write the elements of the Weyl group W as products of reflections at simple roots i.e. $w = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \dots s_{\alpha_{i_k}} = \{i_1, i_2, \dots, i_k\}$. Then $\mathcal{S}[0] = \{\{1,3\}, \{1,2,3\}, (\{3,2,1\}, \{1,2,3,1,2,1\}\}$ and hence

$$E_2^{2,2} = \mathbb{Q}(\{1,3\} \cdot 0), E_2^{2,3} = \mathbb{Q}(\{1,2,3\} \cdot 0) \oplus \mathbb{Q}(\{3,2,1\} \cdot 0), E_2^{2,6} = \mathbb{Q}(\{1,2,3,1,2,1\} \cdot 0) \tag{28}$$

The $E_2^{\bullet,\bullet}$ page looks as follows

$$\begin{array}{ccc} E_2^{0,0} = \mathbb{Q} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E_2^{2,,2} = \mathbb{Q} \\ E_2^{0,3} = \mathbb{Q}^2 & 0 & E_2^{2,3} = \mathbb{Q}^2 \\ E_2^{0,4} = \mathbb{Q} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E_2^{2,6} = \mathbb{Q} \end{array} \tag{29}$$

and I am 99,9 % sure that the two arrows $d_2^{0,3} : E_2^{0,3} \to E_2^{2,,2}$ and $d_2^{0,4} : E_2^{0,4} \to E_2^{2,3}$ are non zero. Let us assume that this is true. Then the $E_3^{\bullet,\bullet}$ page looks as follows

$$\begin{array}{ccc} E_3^{0,0} = \mathbb{Q} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ E_3^{0,3} = \mathbb{Q} & 0 & E_3^{2,3} = \mathbb{Q} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E_3^{2,6} = \mathbb{Q} \end{array} \tag{30}$$

Hence

$$\begin{array}{c} H^q(\partial(\Gamma\backslash X), \mathbb{Q}) \xrightarrow{\sim} E_3^{0,q} \text{ for } q = 0,3 \\ \text{and} \\ E_3^{2,q-2} \xrightarrow{\sim} H^q(\partial(\Gamma\backslash X), \mathbb{Q}) \text{ for } q = 5,8 \end{array} \tag{31}$$

and all the other cohomology groups vanish.

We get a short exact sequence

$$0 \to E_3^{2,\bullet-2} \xrightarrow{eg^2} H^\bullet(\partial(\Gamma\backslash X), \mathbb{Q}) \xrightarrow{eg^0} E_3^{0,\bullet} \to 0, \tag{32}$$

where the $eg^\bullet$ are the so called edge homomorphisms. We have computed the cohomology off the boundary.

The map $d_2^{0,4}$ sends $x \mapsto (\alpha x, \beta x)$, we expect that at least one of the two numbers is non zero and the quotient $c = \alpha/\beta$ is a rational number (or infinity) which can be expressed in terms of special values of the ζ function. For this we use the computation of Langlands for the constant term. We come back to this later.

I have not yet computed this expression for the number c .

We still have to compute the image of r_∂. It must be a maximal isotropic subspace and hence we have the two possibilities

$$r_\partial(H^\bullet(\Gamma\backslash X,\mathbb{Q}) = H^0(\Gamma\backslash X,\mathbb{Q})\oplus H^3(\Gamma\backslash X,\mathbb{Q}) \text{ or } H^0(\Gamma\backslash X,\mathbb{Q})\oplus H^5(\Gamma\backslash X,\mathbb{Q}).$$

At this moment it is not clear to me which of the two cases happens.

If we are able to write down some Eisenstein classes $\mathrm{Eis}(\omega)$ in degree 0 and 3 which restrict non trivially to the boundary cohomology then the first case happens and we have solved our general problem in this case.

If not then we return to our fundamental exact sequence. We have to go to the transcendental level. We apply Thm. 8.1.1 from the book, we know that $H^\bullet_!(\Gamma\backslash X,\mathbb{R})\subset H^\bullet_{(2)}(\Gamma\backslash X,\mathbb{R})$ and the $H^\bullet_{(2)}$ cohomology is generated by the images of

$$H^\bullet(\mathfrak{g},K_\infty,H_{\pi_\infty})\to H^\bullet_{(2)}(\Gamma\backslash X,\mathbb{R}) \tag{33}$$

where $H_{\pi_\infty}\subset L^2_{\mathrm{disc}}(\Gamma\backslash \mathrm{Sl}_4(\mathbb{R}))$ runs over the irreducible subspaces. Since we do not allow ramification the only possibility is $\pi_\infty=\mathbf{1}_\infty=\mathbb{C}$).)

We have an action of the Hecke algebra on the fundamental exact sequence it also acts on the $E_1^{\bullet,\bullet}$ terms in the spectral sequence and this action is compatible with the differentials. Hence it also acts on the higher pages.

This allows us to define subspaces in our cohomology groups. Let $\mathbf{1}_f$ be the one dimensional Hecke module given by the constant functions, let $\mathcal{I}_1$ the ideal of Hecke operators annihilating $\mathbf{1}_f$. Then we define $H^\bullet(\Gamma\backslash X,\mathbb{Q})[\mathbf{1}_f]$ to be the submodule annihilated by a suitable high power of $\mathcal{I}_1$. Then the above computations show that for $\Gamma=\mathrm{Sl}_3(\mathbb{Z}),\Gamma=\mathrm{Sl}_4(\mathbb{Z})$

$$H^\bullet(\partial(\Gamma\backslash X),\mathbb{Q}) = H^\bullet(\partial(\Gamma\backslash X),\mathbb{Q})(\mathbf{1}_f) \tag{34}$$

(The modules $\mathbb{Q}(w\cdot 0)$ are isomorphic to $\mathbf{1}_f$.).

Now I also believe that this also holds for the "global" cohomology, i.e.

$$H^\bullet(\Gamma\backslash X,\mathbb{Q}) = H^\bullet(\Gamma\backslash X,\mathbb{Q})(\mathbf{1}_f). \tag{35}$$

These considerations can be extended to the case of any semi- simple simply connected group scheme $G/\mathbb{Z}$. Let us put $\Gamma=G(\mathbb{Z})$. Any dominant highest weight λ that satisfies $\lambda|w_0\cdot\lambda|\ T(\mathbb{Z})=1$ defines a one dimensional Hecke module $\lambda_{\mathcal{H}}$ and we can define $H^\bullet(\Gamma\backslash X,\mathbb{Q})[\lambda_{\mathcal{H}}]$. Now we consider the objects

$$r: H^\bullet(\Gamma\backslash X,\mathbb{Q})(\lambda_{\mathcal{H}}) \xrightarrow{r_\partial} H^\bullet(\partial(\Gamma\backslash X),\mathbb{Q})(\lambda_{\mathcal{H}}) \tag{36}$$

and all the pages $(E_s^{\bullet,\bullet},d_s^{\bullet,\bullet})(\lambda_{\mathcal{H}})$.

I think that it is of great interest to write an algorithm which computes all these modules and arrows in a given case. For instance we can compute $\mathcal{S}[\lambda]$ and hence $E_2^{r-1,\bullet}$. But it is clear that this can not be achieved with bare hands.

If λ is regular we know that we have an exact sequence (w_0 the longest element in the Weyl group)

$$0 \to \mathbb{Q}(\lambda) \to H^\bullet(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}_\lambda)(\lambda_{\mathcal{H}}) \to \mathbb{Q}(|w_0 \cdot \lambda)| \to 0 \tag{37}$$

where the two extremal terms sit in degree $r-1, l(w_0)$ and $H^{l(w_0)}(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda) \xrightarrow{\sim} \mathbb{Q}(w_= \cdot \lambda)$ respectively. This becomes much more complicated if λ is not regular, the most complicated case is $\lambda = 0$.

We recall some known facts from the theory of Eisenstein series.

Since we assumed $\lambda|_{T(\mathbb{Z})} = 1$ we can define the character $|\lambda| : T(\mathbb{Q})\backslash T(\mathbb{A})/T(\hat{\mathbb{Z}}) \to \mathbb{R}^\times_{>0}$. We consider the induced $G(\mathbb{R})\times$-Hecke module

$$\mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}|w_0 \cdot \lambda|\,|\rho|^s = \{f : G(\mathbb{A}) \to \mathbb{C}\,|f(\underline{b}\underline{g}\underline{k}_f) = |w_0 \cdot \lambda|\,|\rho|^s(\underline{b})f(\underline{g}) \tag{38}$$

for all $\underline{b} \in B(\mathbb{A}), \underline{g} \in G(\mathbb{A}), \underline{k}_f \in G(\hat{\mathbb{Z}})$. Here $|\rho|^s = \prod |\gamma_i|^{s_i}, s_i \in \mathbb{C}$. Since we do not allow any ramification we know that

$$\mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}|w_0 \cdot \lambda|\,|\rho|^s = \mathrm{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})}|w_0 \cdot \lambda|\,|\rho|^s \tag{39}$$

Following Langlands we define the Eisenstein intertwining operator

$$\mathrm{Eis}(\lambda, s) : \mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}|w_0 \cdot \lambda|\,|\rho|^s \to \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/G(\hat{\mathbb{Z}})) = \mathcal{C}_\infty(G(\mathbb{Z})\backslash G(\mathbb{R}))$$

$$\mathrm{Eis}(\lambda, s) : \{\underline{g} \mapsto f(\underline{g})\} \mapsto \{\underline{g} \mapsto \sum_{a \in B(\mathbb{Q})\backslash G(\mathbb{Q})} f(a\underline{g})\} \tag{40}$$

We know that this yields a meromorphic function in the variable s, its poles can be computed from the constant term

$$\mathcal{F}^B(f)(\underline{g}) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} f(\underline{u}\underline{g})d\underline{u} = \sum_{w \in W} c(w, \lambda + s)T^{\mathrm{loc}}(w, s)(f). \tag{41}$$

We explain the term on the right hand side, we begin with $T^{\mathrm{loc}}(w, s)$. The operator $T^{\mathrm{loc}}(w, s) = \prod_v T_v^{\mathrm{loc}}(w, s)$ is an intertwining operator which is a product of local operators $T_v^{\mathrm{loc}}(w,) :\ \mathrm{Ind}_{B(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}|w_0 \cdot \lambda|\,|\rho|_v^s \to\ \mathrm{Ind}_{B(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}|ww_0 \cdot \lambda|\,|\rho|_v^s$. These local operators are normalized such that they map the spherical function to the spherical function.

We know that any isomorphism type θ of an irreducible representation of the maximal compact subgroup K_∞ occurs with finite multiplicity in $\mathrm{Ind}_{B(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}|ww_0 \cdot \lambda|\,|\rho|_v^s$ and we define the Harish-Chandra module

$$\mathrm{Ind}_{B(\mathbb{R})}^{G(\mathbb{R}),(0)}|ww_0 \cdot \lambda|\,|\rho|_v^s := \bigoplus_{\theta \in \hat{K_\infty}} \mathrm{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})}|ww_0 \cdot \lambda|\,|\rho|_v^s(\theta) \tag{42}$$

For a given θ we can identify the isotypical components $\mathrm{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})}|ww_0 \cdot \lambda|\,|\rho|_v^s(\theta)$ to the fixed space $\mathcal{C}_\infty(T(\mathbb{Z})\backslash K_\infty)(\theta)$, provided we only consider those

(λ, w) for which $ww_0 \cdot \lambda|_{T(\mathbb{Z})} = 1$. Hence we see that we can interpret the isotypical components of the intertwining operator

$$\mathrm{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})}|w_0 \cdot \lambda| \ |\rho|_v^s(\theta) \overset{T(w,\theta,s)}{\longrightarrow} \ \mathrm{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})}|ww_0 \cdot \lambda| \ |\rho|_v^s(\theta) \tag{43}$$

simply as endomorphisms of a fixed finite dimensional vector space.

It is well known - or very easy to see -that the operator $T(w,\theta,s)$ is a polynomial in the variables s and hence holomorphic at $s = 0$.

We define the K_∞ invariant subspace $I_B^G[w,i] \subset \mathrm{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})}|w_0 \cdot \lambda| \ |\rho|_v^s(\theta)$ it consist of those elements f for which $T(w,\theta,s)(f)$ vanishes if $s_i = 0$, i.e. $T(w,\theta,s)(f) = s_i g$.

Now we consider the complexes

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \ \mathrm{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})}|ww_0 \cdot \lambda| \ |\rho|_v^s \otimes \tilde{\mathcal{M}}_\lambda) \tag{44}$$

Again it is clear that these complexes are acyclic if $s = \sum s_i\gamma$ is generic, i.e. all $s_i \neq 0$. But we are are interested in what happens at $s = 0$. It is clear that the differentials are holomorphic at $s = 0$, hence can evaluate at $s = 0$ and compute the cohomology

$$H^\bullet(\ \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \ \mathrm{Ind}_{B(\mathbb{R})}^{G(\mathbb{R}),(0)}|ww_0 \cdot \lambda| \otimes \tilde{\mathcal{M}}_\lambda) \tag{45}$$

here we use the formulas of Delorme.

In my book with Raghuram I introduce a $\mathbb{Q}-$ structure on everything. I explain that the group K_∞ is the group of real points of an algebraic group scheme $\mathcal{K}/\mathbb{Q}$, let $\mathcal{A}(\mathcal{K})$ be its affine algebra. Hence we can view its Lie-algebra $\mathfrak{k}$ as a vector space over $\mathbb{Q}$. The θ isotypical subspace $\mathcal{C}_\infty(T(\mathbb{Z})\backslash K_\infty)(\theta)$, is canonically isomorphic to $\mathcal{A}(T(\mathbb{Z})\backslash\mathcal{K})(\theta)\otimes\mathbb{C}$. Hence we have the Harish-Chandra modules over $\mathbb{Q}$

$$\mathcal{I}_B^G ww_0 \cdot \lambda = \bigoplus_\theta \mathcal{A}(T(\mathbb{Z})\backslash\mathcal{K})(\theta) \tag{46}$$

and we get the complex of $\mathbb{Q}$-vector spaces and its cohomology

$$H^\bullet(\ \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{I}_B^G ww_0 \cdot \lambda \otimes \mathcal{M}_\lambda) \tag{47}$$

Of course the intertwining operators $T^{\mathrm{loc}}(w,0)$ are also defined over $\mathbb{Q}$. This ends the discussion of $T^{\mathrm{loc}}(w,s)$, we study the factor $c(w,\lambda+s)$. This is a "transcendental" contribution. This factor has been computed by Langlands. Let

$$\Delta^{+,w} := \{\alpha \in \Delta^+ | \ w^{-1}\alpha \notin \Delta^+\} \tag{48}$$

and for any positive root let α let $\chi_\alpha : \mathbb{G}_m \to T$ be the cocharacter attached to α then

$$c(w,\lambda+s) = \prod_{\alpha\in\Delta^{+,w}} \frac{\xi(<\chi_\alpha, \lambda+\rho+s>)}{\xi(<\chi_\alpha, \lambda+\rho+s>+1)} \tag{49}$$

here $\xi(s) = \frac{\Gamma(s/2)}{\pi^{s/2}}\zeta(s)$ is the completed Riemann ζ-function.

We observe that always $< \chi_\alpha, \lambda + \rho > \geq 1$ and we have equality if and only if $\alpha = \alpha_i$ is a simple root and if $< \alpha_i, \lambda >= 0$ Hence we see that

The function $s \mapsto \dfrac{\xi(< \chi_\alpha, \lambda + \rho + s >)}{\xi(< \chi_\alpha, \lambda + \rho + s > +1)}$ *is regular at* $s = 0$ *unless* $\alpha = \alpha_i$ *is a simple root and* $< \alpha_i, \lambda >= 0$ *In this last case we have a simple pole at* $s_i = 0$.

Now we "evaluate" the Eisenstein intertwining operator at $s = 0$, of course we have to explain what we mean by that. We start from an element

$$\omega \in \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), \ \mathrm{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})}|w_0 \cdot \lambda| \, |\rho|_v^s(\theta),$$

we pick a p-tuple of linear independent vectors in $\mathfrak{g}/\mathfrak{k}$ and we want to attach a value to $\mathrm{Eis}(\lambda, s)(\omega(X_1, X_2 \dots, X_p))$ at $s = 0$. We look at the individual terms in the constant term (41). We define the $S_w \subset \{\alpha_i, \dots, \alpha_i, \dots, \alpha_r\}$ to be the set of those indices i for which $\alpha_i \in \Delta^{+,w}$ and $\omega(X_1, \dots, X_p) \in I_B^G[w, i]$. This guarantees that $c(w, \lambda + s)T^{\mathrm{loc}}(w, s)$ is holomorphic at $s_i = 0$. This set does not depend on the choice of the X_ν. Then it is clear that $\prod_{i, i \notin S_w} s_i c(w, \lambda + s)T^{\mathrm{loc}}(w, s)$ is regular at $s = 0$. If S is the union of all S_w then we define the Eisenstein differential form

$$\mathrm{Res}_{s=0}^S \ \mathrm{Eis}(\lambda, s)(\omega_s) = (\prod_{|not \in S} s_i) \ \mathrm{Eis}(\lambda, s)(\omega_s)|_{s=0}$$

This gives us a supply of differential forms and it is not difficult to see that there is a Harish Chandra module $\mathrm{Eis}^*(\lambda, 0) \subset \mathcal{C}_\infty(G(\mathbb{Z}) \backslash G(\mathbb{R}))$ of finite length such that $\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \ \mathrm{Eis}^*(\lambda, 0) \otimes \tilde{\mathcal{M}}_\lambda)$ contains all these differential forms.

Now I assume that the map

$$H^\bullet(\ \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \ \mathrm{Eis}^*(\lambda, 0) \otimes \tilde{\mathcal{M}}_\lambda)) \to H^\bullet(\Gamma \backslash X, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C})(\lambda) \qquad (50)$$

is surjective, I think that this follows from Frankes theorem.

I think that now we have the tools to compute the differentials in the spectral sequence. We recall the procedure on p.5 . We can represent the cohomology classes (See p.5, 23) by Eisenstein cohomology classes as above. Then we can compute the closed differential forms ω_J in (24) they are zero in cohomology: Since we know them explicitly we can compute ψ. This can also applied to higher differentials.

This tells us that any differential $d_\nu^{p,q} : E_\nu^{p,q} \otimes \mathbb{C} \to E_\nu^{p-\nu+1, q+\nu} \otimes \mathbb{C}$ can be given by a matrix

$$C(\lambda) = \begin{pmatrix} c_{1,1} & & c_{1,t} \\ & c_{a,b} & \\ c_{u,1} & & c_{u,t} \end{pmatrix} \qquad (51)$$

and the coefficients $c_{a,b}$ are linear combinations with rational coefficients of "transcendental" numbers of the form $c(w, \lambda, 0)$.

Since this matrix is equivalent to a matrix given by a linear map between two rational vector spaces this implies certain relations between these transcendental quantities. Since these quantities are monomials in values of the Riemann ζ function at strictly positive integer arguments ($\zeta(1)=1$) we get some relations amoung these numbers. But I think that people believe that all such relations follow from $\zeta(2a)\cdot\zeta(2b)=u\cdot\zeta(2(a+b)), u\in\mathbb{Q}$ and that there are no non tautological relations involving ζ values at odd positive integers.

Nevertheless it is an interesting question to compute the last column $E_\infty^{r-1,\bullet}$ in the spectral sequence. We get an ascending chain of subspaces

$$0\subset d(E_2^{r-3,\bullet-1})\subset d(E_3^{r-4,\bullet-2})\subset\cdots\subset d(E_{r-1}^{0,\bullet-r+2})\subset E_2^{r-1,\bullet}=\bigoplus_{w\in\mathcal{S}[\lambda]}\mathbb{Q}e_w. \tag{52}$$

The quotient embeds $E_2^{r-1,\bullet}/d(E_{r-1}^{0,\bullet-r+2})=E_\infty^{r-1,\bullet}\subset H^{\bullet+r-1}(\partial(\Gamma\backslash X),\tilde{\mathcal{M}})$. If $d=\dim(X)$ is then dimension of X then $d-1$ is the dimension of the boundary $\partial(\Gamma\backslash X)$ and Poincare duality provides a non degenerate pairing (we assume $w_0\cdot\lambda|_{T(\mathbb{Z})}=1$)

$$H^{\bullet+r-1}(\partial(\Gamma\backslash X),\tilde{\mathcal{M}})\times H^{d-r-\bullet}((\partial(\Gamma\backslash X),\tilde{\mathcal{M}}^\vee)\to\mathbb{Q} \tag{53}$$

We get a diagram

$$\begin{array}{ccccc} H^{\mathfrak{q}+r-1}(\partial(\Gamma\backslash X),\tilde{\mathcal{M}}) & \times & H^{d-r-q}((\partial(\Gamma\backslash X),\tilde{\mathcal{M}}^\vee) & \to\mathbb{Q} \\ \cup & & \downarrow & \\ E_\infty^{r-1,qt} & & E_\infty^{0,d-r-q} & \\ \uparrow u_q & & \cap & \\ \bigoplus_{w:l(w)=q}\mathbb{Q}e_w & \times & \bigoplus_{w:l(w')=d-r-q}\mathbb{Q}e_{w'} & \to\mathbb{Q} \end{array} \tag{54}$$

and the pairing in the bottom line is the obvious one. It seems to be desirable to have a computer program which computes all these modules, it is a Weyl group issue but we also have to compute the local operators $T(w,\theta,0)$ for this I refer to my Mumbai paper 2008 where I did this for Sl_3.

We consider the Eisenstein cohomology classes in 50 and restrict them to the boundary., we intersect this restriction with $E_\infty^{r-1,q}$in the left column 54 , call this intersection $X^{(q)}$, and project it to $E_\infty^{0,d-r-q}$ and call this projection $Y^{(q)}$..

Now we want to prove that $X^{(q)}$ *and* $Y^{(q)}$ *are mutual orthogonal complements of each other.*

Examples: If λ is regular then the only interesting value for q is $q=0$,in this case $X^{(0)}=0, Y^{(0)}.=\mathbb{Q}e_{w_0}$.

If $\lambda=0$ the situation is more interesting, In this case we the value $q=d-r$ is of interest and it is not difficult to show that $Y^{(q)}=\mathbb{Q}e_1$ and $X^{(q)}=0$. But we also have to take the values $w\in\mathcal{S}[0]$ into account. Already in this case it still has to be verified that (30) is the correct E_3 page but we ve not yet computed-only speculated- the modules $X^{(5)},,Y^{(3)}$. Exactly one of them should be zero, the other one $\mathbb{Q}$.

Final remark. It is certainly justified to call the cohomology groups $H^\bullet(\Gamma\backslash X,\tilde{\mathcal{M}}_\lambda)[\lambda]$ the "abelian" part of the Eisenstein cohomology..

We can also consider quasisplit groups over $\mathbb{Q}$. If $F/\mathbb{Q}$ ist an imaginary quadratic extension then $G = R_{F/\mathbb{Q}}(G_0/F)$ where G_0/F is split, is such a group.. But we may also start from a totally isotropic hermitian form f on a vector space V/F and consider the group $G/\mathbb{Q} = \mathrm{SU}(f)$.. Let $B/Q, T/Q$ be a Borel and a maximal torus such that $B \times_{\mathbb{Q}} F, T \times_{\mathbb{Q}} F$ are the standard Borel and its maximal torus. If $\pi_F = \{\alpha_1^F, \dots, \alpha_i^F, \dots, \alpha_{r_1}^F\} \subset X^*(T \times_Q F)$ is the set of simple roots over F then $\pi = \pi_F/\sigma$ is the system of simple roots in $X^*(T)$.. Hence a simple root is either

1) a simple root $\alpha_i = \alpha_i^F$ if $\sigma(\alpha_i^F) = \alpha_i^F$,

2) or a pair $\{\alpha_\nu^F, \sigma(\alpha_\nu^F)\}$

To any α_i we have the little semi-simple $H^{(1)}_{\alpha_i}$ which is a $\mathrm{Sl}_2/\mathbb{Q}$ in the first case, in the second case it depends on whether $\alpha_\nu^F + \sigma(\alpha_\nu^F)$ is a root or not. In the case

2a) H_{α_i} is the quasisplit $\mathrm{SU}(2,1)$
in the case

2b) $H_{\alpha_i} = R_{F/\mathbb{Q}}(\mathrm{Sl}_2)$.

These $H^{(1)}_{\alpha_i}$ are the semi simple parts of the reductive quotients of the next to minimal parabolic subgroups (called $P^{(i)}$). The symmetric space attached to $H^{(1)}_{\alpha_i}$ is either the upper half plane, the three dimensional hyperbolic space or the two dimensional complex ball. The resulting locally symmetric space is denoted by $\mathcal{S}^{H^{(1)}_{\alpha_i}}$. The maximal torus $T_{\alpha_i} \subset H^{(1)}_{\alpha_i}$ is isomorphic to $\mathbb{G}_m/\mathbb{Q}$ in the first case, to $R_{F/\mathbb{Q}}(\mathbb{G}_m)$ in the second and third case. The simple roots $\alpha, \bar{\alpha}$ are $t \to t^2, t \ \to \bar{t}^2$ in the first case and $t \to t^2/\bar{t}, t \to \bar{t}^2/t$ in the second case.

In my inventiones, Math. Annalen 1984-87 papers I give a description of the Eisenstein restriction map

$$H^{\mathrm{q}}(\mathcal{S}^{H^{(1)}_{\alpha_i}}, \tilde{\mathcal{M}}_\lambda \otimes \tilde{F}) \xrightarrow{r_\partial} H^q\partial(\mathcal{S}^{H^{(1)}_{\alpha_i}}), \tilde{\mathcal{M}}_\lambda \otimes \tilde{F})) = \bigoplus_{w \in W: l(w)=q} \bigoplus_{\phi: type(\phi) = w\cdot\lambda} \tilde{F} e_\phi \tag{55}$$

Here $\tilde{F}$ is finite extension of F which is generated by the values of the values of ϕ on $T(\mathbb{A}_f)$.

If $H^{(1)}_{\alpha_i}$ is a $\mathrm{Sl}_2/\mathbb{Q}$ we described the image in the first part of this note. In this case W is the group $\mathbb{Z}/2\mathbb{Z} = \{e, s_1\}$ and if $\lambda \neq 0$ the image is

$$r_\partial(H^\bullet(\mathcal{S}^{H^{(1)}_{\alpha_i}}, \tilde{\mathcal{M}}_\lambda \otimes \tilde{F})) = \bigoplus_{\phi: type(\phi) = s_1\cdot\lambda} \tilde{F} e_\phi, \tag{56}$$

the image is exactly the cohomology in degree 1.

If $\lambda = 0$ then the image is slightly different, then we have to special Hecke characters namely the trivial Hecke character $\mathbf{1}$ which is of type 0 and $|\alpha_1|$ which is of type $\alpha_1 = 2\rho$. Now

$$r_\partial(H^\bullet(\mathcal{S}^{H^{(1)}_{\alpha_i}}, \tilde{\mathcal{M}}_0 \otimes \tilde{F})) = \tilde{F} e_{\mathbf{1}} \oplus \bigoplus_{\phi: type(\phi) = s_1\cdot\lambda, \phi \neq |\alpha_1|} \tilde{F} e_\phi, \tag{57}$$

hence one summand is missing in degree one and it is replaced by a term in degree 0. (We have seen this already in the first section of this note)

If $H^{(1)}_{\alpha_i} = R_{F/\mathbb{Q}}(\mathrm{Sl}_2/F)$ then the situation is a little bit different. We restrict r_∂ to the cohomology in even degrees, then the answer is completely analogous to the previous case. For $\bullet \equiv 0 \mod 2$ and $w_0 = s_1 s_2$

$$r_\partial(H^\bullet(\mathcal{S}^{H^{(1)}_{\alpha_i}}, \tilde{\mathcal{M}}_0 \otimes \tilde{F})) = \tilde{F}e_{\mathbf{1}} \oplus \bigoplus_{\phi:type(\phi)=w_0\cdot\lambda,\phi\neq|\alpha_1||\alpha_2|} \tilde{F}e_\phi, \tag{58}$$

In odd degree, i.e. $\bullet = 1$ we have two elements of length one in the Weyl group, namely s_1 and s_2 and we have to look at

$$H^1(\mathcal{S}^{H^{(1)}_{\alpha_i}}, \tilde{\mathcal{M}}_0 \otimes \tilde{F})) \xrightarrow{r_\partial} \bigoplus_{\phi:type(\phi)=s_1\cdot\lambda} \tilde{F}e_\phi \oplus \bigoplus_{\phi':type(\phi')=s_2\cdot\lambda} \tilde{F}e_{\phi'} \tag{59}$$

Now the theory of Eisenstein series tells us that the ϕ, ϕ' come in pairs, i.e. to each ϕ we have a unique ϕ' such that

$$r_\partial(H^1(\mathcal{S}^{H^{(1)}_{\alpha_i}}, \tilde{\mathcal{M}}_0 \otimes \tilde{F})) = \bigoplus_{\phi:type(\phi)=s_1\cdot\lambda} \tilde{F}(e_\phi + \mathcal{L}(\phi)e_{\phi'}) \tag{60}$$

Here $\mathcal{L}(\phi) \in \tilde{F}$ is a ratio of L-values, it can be zero and also ∞.

It remains the case that $H^{(1)}_{\alpha_i}$ is the quasisplit $\mathrm{SU}(2,1)$. In this case the Weyl group $W = S_3$, it is generated by the two reflections s_1, s_2. We divide the W into three sets of two elements

$$V_0 = \{e, s_1s_2s_1\}, V_1 = \{s_1, s_2s_1\}, V_2 = \{s_2, s_1s_2\} \tag{61}$$

and then (55) becomes

$$H^\bullet(\mathcal{S}^{H^{(1)}_{\alpha_i}}, \tilde{\mathcal{M}}_\lambda \otimes \tilde{F}) \xrightarrow{r_\partial} \bigoplus_{i=1}^{3} \bigoplus_{w_i\in V_i} \bigoplus_{\phi:type(\phi)=w_i\cdot\lambda} \tilde{F}e_\phi \tag{62}$$

Now we rearrange the terms on the right hand side, we write $V_i = \{w_i^l, w_i^s\}$ where $l(w_i^l) > l(w_i^s)-$ Then it is again clear that the ϕ with $type(\phi) = w_i^l$ and $type(\phi') = w_i^s$ come in pairs and we can rewrite the previous diagram

$$H^\bullet(\mathcal{S}^{H^{(1)}_{\alpha_i}}, \tilde{\mathcal{M}}_\lambda \otimes \tilde{F}) \xrightarrow{r_\partial} \bigoplus_{i=1}^{3} \bigoplus_{\phi:type(\phi)=w_i^l\cdot\lambda} (\tilde{F}e_\phi \oplus \tilde{F}e_{\phi'}) \tag{63}$$

Now it follows from the computations in the Annalen paper that the restriction map r_∂ respects this directs sum and hence we have to understand the projection

$$H^\bullet(\mathcal{S}^{H^{(1)}_{\alpha_i}}, \tilde{\mathcal{M}}_\lambda \otimes \tilde{F}) \xrightarrow{p_\phi\circ r_\partial} \tilde{F}e_\phi \oplus \tilde{F}e_{\phi'} \tag{64}$$

Now we know

The image of $p_\phi \circ r_\partial$ is the summand $\tilde{F}e_\phi$ unless we are in one of the following cases

a) We have $\lambda = 0$ and $\phi = |\ |_F$

b) We have $w = s_1s_2($ resp. $w = s_2s_1)$ and $\lambda = n_1\gamma_1($ resp.$\lambda = n_2\gamma_2.)$

Moreover we have: The restriction of ϕ to $I_\mathbb{Q}$ is equal to $\chi_{F/\mathbb{Q}}|\ |_\mathbb{Q}^3$. (of course $\chi_{F/\mathbb{Q}}$ is the Dirichlet character attached to $F/\mathbb{Q}$) and $L(\phi,-1)\neq 0$.

If these conditions are fulfilled the image of $p_\phi \circ r_\partial$ is the second summand $\tilde{F}e_{\phi'}$.

Here a minor point has to be clarified. The evaluation point $s=-1$ is the central point for the L-function $L(\phi,s)$. But I think that the condition $\phi|_\mathbb{Q}=\chi_{F/\mathbb{Q}}|\ |_\mathbb{Q}^3$ also forces that this evaluation point is critical.

I think that again we should be able to compute the $E_2^{r-1,\bullet}$ term in the spectral sequence. This is the kokernel of the map

$$\bigoplus_{i=1}^{r}\bigoplus_{w\in W^{P^{(i)}}} H^{\bullet-l(w)}(\mathcal{S}^{H_{\alpha_i}},\tilde{\mathcal{M}}_{w\cdot\lambda})\to \bigoplus_{w\in W}\bigoplus_{\phi:type(\phi)=w\cdot\lambda} H^0(\mathcal{S}^T,\tilde{\mathcal{M}}_{w\cdot\lambda})=\bigoplus_{\phi:type(\phi)=w\cdot\lambda}\tilde{F}e_\phi \tag{65}$$

We are essentially in the same situation as in the first part. We have to find an analogue for (18), In principle this is a Weyl group issue, we have to write a algorithm which produces lists of the elements in W and produces lists of the Kostant representatives $\{\dots,w^{(i)},\dots\}=W^{P^{(i)}}$. Then our above computations will help us to describe the images

$$H^{\bullet-l(w^{(i)})}(\mathcal{S}^{H_{\alpha_i}},\tilde{\mathcal{M}}_{w^{(i)}\cdot\lambda})\xrightarrow{r_\partial^{w^{(i)}}}\bigoplus_{\phi:type(\phi)=w\cdot\lambda}\tilde{F}e_\phi. \tag{66}$$

Here we observe that for a given $w^{(i)}$ this image lies in a "very small" subspace, more precisely

$$\mathrm{Im}(r_\partial^{w^{(i)}})\subset\bigoplus_{v\in W_i}\bigoplus_{\phi:type(\phi)=vw^{(i)}\cdot\lambda}\tilde{F}e_\phi. \tag{67}$$

here W_i is the "small" Weyl group of H_{α_i}.

Now we have to understand the space generated by the $\mathrm{Im}(r_\partial^{w^{(i)}})$, I think that this will become much more complicated than in the split case. There is of course the combinatorial aspect, we have to understand the intersections of the sets of summation indices $\{vw^{(i)}\}_{v\in W_i}$ and $\{vw^{(j)}\}_{v\in W_j}$. But we also have understand the influence of the values of the L function, i.e. the value $\mathcal{L}(\phi)$ in (60) and the influence of the vanishing of $L(\phi,s)$ at $s=-1$.

Hence we see that we can compute the $E_2^{r-1,\bullet}$ and it is clear that this term depends on whether $\mathcal{L}(\phi)\neq 0,\infty$ (in (60) and on the vanishing of $L(\phi,s)$ at $s=-1$. Again the next step will be to compute (the image of) the map

$$d_2^{r-3,\bullet}:E_2^{r-3,\bullet}\to E_2^{r-1,\bullet-1} \tag{68}$$

Finally I want to raise a question which haunts me since many years and which I formulated in my Luminy paper.

The structure of the (abelian part) of the cohomology groups

$$H^\bullet(\Gamma\backslash X,\tilde{\mathcal{M}})\xrightarrow{r_\partial}H^\bullet(\partial(\Gamma\backslash X),\tilde{\mathcal{M}})$$

and the structure of the spectral sequence depend on the vanishing or non vanishing of certain L-function at their central point, hence we can read off from these cohomology groups whether we have vanishing or not.

Can we even read off the order of vanishing if we look deeper into the spectral sequence or can we see the order of vanishing if we look at $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) \xrightarrow{r_\partial} H^\bullet(\partial(\Gamma\backslash X), \tilde{\mathcal{M}})$ **for larger groups?**