Eisenstein Cohomology, Arithmetic Applications

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1 The basic object, some general and fundamental facts.

My basic object of interest is the following diagram

$$\longrightarrow \quad H^{i-1}(\overset{\bullet}{\mathcal{N}}\, \mathcal{S}^{G,\infty}_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Z}}) \quad \longrightarrow \quad H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Z}}) \quad \longrightarrow \quad H^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Z}}) \quad \xrightarrow{res} \quad H^i(\overset{\bullet}{\mathcal{N}}\, \mathcal{S}^{G,\infty}_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Z}})$$

$$p_c \searrow \qquad H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Z}}) \qquad \nearrow q_!$$

Here $G/\mathbb{Z}$ is a split reductive group scheme over $\mathbb{Z}$. We fix a split maximal torus $T/\mathbb{Z}$ and a Borel subgroup $B \supset T$, i.e. a set of simple roots. We choose an open compact subgroup $K_f \subset G(\mathbb{A}_f)$, and assume that $K_f = \prod_p K_p, K_p \subset G(\mathbb{Z}_p)$. Depending on the choice of K_f a prime is called unramified if $K_p = G(\mathbb{Z}_p)$, the set of ramified primes is finite. Let $X = G(\mathbb{R})/K_\infty$ be the symmetric space attached to the real group $G(\mathbb{R})$. Now we can define the associated locally symmetric space

$$\mathcal{S}^G_{K_f} = G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty \times K_f = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f.$$

We choose a compactification $\mathcal{S}^G_{K_f} \hookrightarrow \mathcal{S}^{G,\vee}_{K_f}$ and we put $\mathcal{S}^{G,\infty}_{K_f} = \mathcal{S}^{G,\vee}_{K_f} \setminus \mathcal{S}^G_{K_f}$, let $\mathcal{N}\mathcal{S}^{G,\infty}_{K_f}$ be a "tubular" neighborhood of $\mathcal{S}^{G,\infty}_{K_f}$ and $\overset{\bullet}{\mathcal{N}}\, \mathcal{S}^{G,\infty}_{K_f} = \mathcal{N}\mathcal{S}^{G,\infty}_{K_f} \cap \mathcal{S}^G_{K_f}$ the "punctured" tubular neighborhood. Let $G \to \mathrm{Gl}(\mathcal{M})$ be an absolutely irreducible representation of $G/\mathbb{Z}$ (given by a highest weight λ) let $\tilde{\mathcal{M}}_{\mathbb{Z}}$ the resulting local system of free $\mathbb{Z}$ modules of finite rank.

Now we have all the data necessary to define the cohomology groups above, here $H^\bullet_c$ is the cohomology with compact support and $H^\bullet_!$ the image of the cohomology with compact support in the cohomology. All these cohomology modules are finitely generated $\mathbb{Z}$ modules, if $\mathbb{Z} \to R$ is any ring with identity the same assertion holds for the corresponding cohomology groups with coefficients in $\tilde{\mathcal{M}}_R = \tilde{\mathcal{M}}_{\mathbb{Z}} \otimes R$.

A theorem due to Raghunathan asserts that all these cohomology groups $H^\bullet_?(?, \tilde{\mathcal{M}}_{\mathbb{Z}})$ are finitely generated $\mathbb{Z}-$ modules.

We still have more structure on these cohomology groups. Let $\mathcal{H}(G(\mathbb{A}_f)//K_f) = \otimes' \mathcal{H}(G(\mathbb{Q}_p)//K_p)$ be the Hecke algebra. We can define an action of the Hecke algebra

$$h_{\mathcal{M}} : \mathcal{H}(G(\mathbb{A}_f)//K_f) \to \mathrm{End}(H^\bullet_?(?, \tilde{\mathcal{M}}_{\mathbb{Z}})). \tag{1}$$

We know that for unramified primes p the local Hecke algebra $\mathcal{H}(G(\mathbb{Q}_p)//K_p)$ is commutative and generated by the characteristic functions of $K_p\chi_i(p)K_p$ where χ_i are the coroots defined by $< \chi_i, \alpha_j >= \delta_{i,j}$. Let $T_{p,\chi} : H^\bullet_?, (?, \tilde{\mathcal{M}}_{\mathbb{Z}}) \to H^\bullet_?, (?, \tilde{\mathcal{M}}_{\mathbb{Z}})$ be the resulting endomorphism. The sub algebra

$$\mathcal{H}^{(S)} = \bigotimes_{p \notin S} \mathcal{H}(G(\mathbb{Q}_p)//K_p)$$

is a central sub-algebra of the Hecke algebra.

Since our modules are finitely generated it is clear that we can find a finite extension $F/\mathbb{Q}$ such that any $H^\bullet_?(?, \tilde{\mathcal{M}}_F)$ has a Jordan-Hölder filtration

$$\subset \mathcal{JH}^{\nu+1}(H^\bullet_?(?, \tilde{\mathcal{M}}_F)) \subset \mathcal{JH}^{\nu}(H^\bullet_?(?, \tilde{\mathcal{M}}_F)) \subset \cdots \subset H^\bullet_?(?, \tilde{\mathcal{M}}_F)$$

with absolutely irreducible quotients $\mathcal{JH}^{\nu}(H^\bullet_?(?, \tilde{\mathcal{M}}_F))/\mathcal{JH}^{\nu+1}(H^\bullet_?(?, \tilde{\mathcal{M}}_F))$. Let us denote by $\mathrm{Coh}_?(?, \tilde{\mathcal{M}}_F)$ the set of isomorphism types π_f of Hecke-modules, which occur with multiplicity > 0. Each such isomorphism type is a product of local Hecke modules $\pi_f = \otimes \pi_p$.

Finally we define $H^\bullet_?(?, \tilde{\mathcal{M}}_{\mathcal{O}_F})_{\mathrm{int}}$ as the image of $H^\bullet_?(?, \tilde{\mathcal{M}}_{\mathcal{O}_F})$ in $H^\bullet_?(?, \tilde{\mathcal{M}}_F)$. Our Jordan-Hölder filtration induces a Jordan-Filtration

$$\mathcal{JH}^{\nu}(H^\bullet_?(?, \tilde{\mathcal{M}}_{\mathcal{O}_{\mathcal{F}}}))_{\mathrm{int}} = \mathcal{JH}^{\nu}(H^\bullet_?(?, \tilde{\mathcal{M}}_F)) \cap H^\bullet_?(?, \tilde{\mathcal{M}}_{\mathcal{O}_F})_{\mathrm{int}}$$

Hence we see that an isomorphism type π_f which occurs in the Jordan Hölder filtration of some $H^\bullet_?(?, \tilde{\mathcal{M}}_F))$ gets a natural structure of a $\mathcal{O}_F \times$Hecke module structure where the $\mathcal{O}_F$ module is locally free of finite rank. If we restrict the module to $\mathcal{H}^{(S)}$ then $\pi_f^{(S)}$ is simply a homomorphism

$$\pi_f^{(S)} : \mathcal{H}^{(S)} \to \mathcal{O}_F.$$

Of course $\pi_f^{(S)} = \otimes_{p \notin S} \pi_p$.

If $F/\mathbb{Q}$ is normal then the action of the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ on $\mathcal{M}_F$ induces an action of the Galois group on $H^\bullet_?(?, \tilde{\mathcal{M}}_F)$.

We have a first

Theorem 1.1. *The Hecke module $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Q}})$ is semi simple we have an isotypical decomposition*

$$H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F) = \bigoplus_{\pi_f \in Coh_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F)} H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F)(\pi_f)$$

The proof requires some sort of Hodge-theoretic arguments: We have to extend the coefficients to $\mathbb{C}$. We choose a model space H_{π_f} for our isomorphism type π_f. We have finitely many (up to isomorphism) irreducible unitary $(\mathfrak{g}, K_\infty)$ modules H_{π_∞} for which $H^\bullet(\mathfrak{g}, K_\infty, H_{\pi_\infty} \otimes \mathcal{M}_{\mathbb{C}}) \neq 0$. Let $L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{R}) \times$

$G(\mathbb{A}_f)/K_f)$ the discrete spectrum in the space of automorphic forms. We introduce the finite dimensional vector space

$$W^{(2)}_{\pi_\infty \times \pi_f} = \mathrm{Hom}_{\mathfrak{g},K_\infty}(H_{\pi_\infty} \otimes H_{\pi_f}, L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{R}) \times G(\mathbb{A}_f)/K_f)$$

and then we know

The natural homomorphism

$$\bigoplus_{\pi_\infty} W^{(2)}_{\pi_\infty \times \pi_f} \otimes H^\bullet(\mathfrak{g}, K_\infty, H_{\pi_\infty} \otimes \mathcal{M}_\mathbb{C}) \otimes H_{\pi_f} \to H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{C})(\pi_f) \qquad (2)$$

is surjective.

This is the representation theoretic version of the fact that on a complete Riemannian manifold with finite volume every square integrable cohomology class is represented by a harmonic square integrable form. (See [Ha-book], 4.1.4)

1.1 The cohomological L- functions

To any $\pi_f \in \mathrm{Coh}(\mathcal{S}^G_{K_f}\tilde{\mathcal{M}}_F)$ and any cocharacter $\chi : \mathbb{G}_m \to T$ we attach the cohomological L- function

$$L^{\mathrm{coh}}(\pi_f, r_\chi, s) = \prod_{p \in \S} L^{\mathrm{coh}}_p(\pi_p, r_\chi, s) \prod_{p \notin S} \frac{1}{1 - \pi_p(A_1(p,\lambda,\chi))p^{-s} + \pi_p(A_2(p,\lambda,\chi))p^{-2s} \dots}. \qquad (3)$$

Here the $A_i(p,\lambda,\chi)$ are certain elements in the Hecke - algebra $\mathcal{H}_p$, so by definition we have for the local factors at unramified places

$$L^{\mathrm{coh}}_p(\pi_f, r_\chi, s)^{-1} \in \mathcal{O}_F[p^{-s}].$$

Only after choosing an embedding $\iota : F \hookrightarrow \mathbb{C}$ we can view $L^{\mathrm{coh}}(\pi_f \circ \iota, r_\chi, s)$ as an honest holomorphic-or meromorphic function in the variable s at least if we assume $\Re(s) >> 0$. (See [Ha-book], Chap. III, 3.1.3.)

1.2 The role of automorphic forms

Here the theory of automorphic forms enters the stage. To our Hecke-module $\pi_f \in \mathrm{Coh}(\mathcal{S}^G_{K_f}\tilde{\mathcal{M}}_F)$ and an embedding $\iota : F \hookrightarrow \mathbb{C}$ we can find π_∞ as above such that $H_{\pi_\infty} \times H_{\pi_f}$ occurs somewhere in the space of automorphic forms, i. e. $\pi_\infty \times \pi_f$ becomes an automorphic representation of the group $G(\mathbb{A})$. Then Langlands has attached automorphic L -functions $L^{\mathrm{aut}}(\pi, r_\chi, s)$ to π. These automorphic L-functions differ from the cohomological L function by a factor obtained from π_∞ and a shift in the variable s.

In some cases the theory of automorphic forms provides tools to prove that these L -functions have a meromorphic or even holomorphic continuation into the entire complex plane.

The second important class of results of are multiplicity formulas. More precisely

The aim of this note is to show that an understanding of these cohomology groups and especially an understanding of the restriction map res is a source for number theoretic results.

a) We get rationality results for special values of L-functions attached to automorphic forms. Using Eisenstein cohomology we get algebraicity results for certain expressions

$$\frac{1}{\Omega(\sigma_f)}\frac{L(\sigma_f, r_\chi, \nu)}{L(\sigma_f, r_\chi, \nu+1)} \tag{4}$$

Here $\Omega(\sigma_f)$ is a period which is well defined modulo a group of (S)- units in a specific number field.

b) Furthermore we can formulate conjectures:

The divisibility of certain expressions of special L values as above by primes or prime powers implies that certain Eisenstein classes have a denominator divisible by that prime (power). These conjectures imply certain congruences modulo that same prime between eigenvalues of Hecke operators on eigenclasses on different groups and congruence conjectures have been tested in many cases. In principle it is also possible to check the stronger assertion concerning the denominator in a given case using a computer.

2 The spectral sequence for the cohomology of the boundary

We have to deal with the $G(\mathbb{Q})$ conjugacy classes of parabolic subgroups, we choose the parabolic subgroups containing $B/\mathbb{Z}$ as a set of representatives. We have a spectral sequence which computes the cohomology of the boundary in terms of the cohomology of reductive groups M which are the Levi-quotients of parabolic subgroups, i. e. $M = P/U$. We put $d(P) = \dim(C_M) - \dim(C_G)$ this is the rank of the center of the semi-simple part $M^{(1)}$ of M.

Then we put

$$E_1^{p,q} = \bigoplus_{P:d(P)=p+1} \bigoplus_{w \in W^P} I_P^G H^{q-l(w)}(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}_P, \mathcal{M}_\mathbb{Q})) \tag{5}$$

here W^P is the set of Kostant representatives, I_P^G denotes the induction of Hecke modules from P to G. Furthermore $l(w)$ is the length of w, the cohomology $H^n(\mathfrak{u}_P, \mathcal{M}_\mathbb{Q})) = 0$ in all degrees $n \neq l(w)$ and for $n = l(w)$ the M module $H^{l(w)}(\mathfrak{u}_P, \mathcal{M}_\mathbb{Q}) = \mathcal{M}(w \cdot \lambda)$ where $w \cdot \lambda = w(\lambda + \rho) - \rho$.

We get a complex

$$\to E_1^{p-1,q} \to E_1^{p,q} \to E_1^{p+1,q} \to \tag{6}$$

and the cohomology of this complex yields the next term $E_2^{p,q}$ in the spectral sequence which converges to the cohomology of the boundary.

On the E_2 level we have differentials $d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$ and we have the following fact (Schwermer)

If the highest weight λ is regular, then the $d_2^{p,q}$ and also the higher differentials are zero, the spectral sequence degenerates at level E_2

Our previous considerations also apply to the cohomology groups $H^{q-l(w)}(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}_P, \mathcal{M}_{\mathbb{Q}}))$ and we can define the inner cohomology

$$H_!^{q-l(w)}(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}_P, \mathcal{M}_{\mathbb{Q}})) \subset H^{q-l(w)}(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}_P, \mathcal{M}_{\mathbb{Q}})) \qquad (7)$$

Of course we have again

$$H_!^{\bullet-l(w)}(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}_P, \mathcal{M}_{\mathbb{Q}})) = \bigoplus_{\sigma_f \in \mathrm{Coh}(\mathcal{S}^M_{K_f^M}, \mathcal{M}(w\cdot\lambda))} H_!^{\bullet-l(w)}(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}_P, \mathcal{M}_{\mathbb{Q}}))(\sigma_f)$$

We define

$$E_{!,1}^{p,q} = \bigoplus_{P:d(P)=p+1} \bigoplus_{w\in W^P} I_P^G H_!^{q-l(w)}(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}_P, \mathcal{M}_{\mathbb{Q}}))$$

then it is almost clear from the definition that the differentials $d_1^{p,q}: E_{!,1}^{p,q} \to E_{!,1}^{p+1,q}$ vanish. We define $E_{!,2}^{p,q}$ as the image of $E_{!,1}^{p,q}$ in $E_2^{p,q}$ and again it is clear that the differentials $d_2^{p,q}: E_{!,2}^{p,q} \to E_2^{p+2,q-1}$ are zero. This goes on forever. We say that the elements in $E_{!,1}^{p,q}$ are universally closed.

This implies that we get an inclusion

$$\bigoplus_{P:d(P)=1} \bigoplus_{w\in W^P} I_P^G H_!^{q-l(w)}(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}_P, \mathcal{M}_{\mathbb{Q}})) \hookrightarrow H^q(\overset{\bullet}{\mathcal{N}}\, \mathcal{S}^{G,\infty}_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Q}}) \qquad (8)$$

3 The Eisenstein cohomology

The Eisenstein cohomology is designed to understand the image of the restriction map *res* to the cohomology of the boundary, which amounts to understand the difference between the $H_!$ and the cohomology without supports.

In this context we have a general theorem which is a consequence of Poincare duality. The boundary cohomology is the cohomology of a compact manifold of dimension $d-1 = \dim(\mathcal{S}^G_{K_f}) - 1$ and Poincare duality yields a non degenerate pairing

$$H^i(\overset{\bullet}{\mathcal{N}}\, \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{F}}) \times H^{d-i-1}(\overset{\bullet}{\mathcal{N}}\, \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^{\vee}_{\mathbb{F}}) \to F \qquad (9)$$

here F may be any field.

Theorem 3.1. *With respect to this pairing the images of res and res$^{\vee}$ are mutual orthogonal complements of each other.*

If we want a more precise information concerning the image, then we need a better understanding of the above spectral sequence. In this context I want to formulate a question, which seems to me of great importance

Is it always the case, that the above spectral sequence degenerates at E_2 level, or can it happen that we have non zero differentials $d_r^{p,q}$: $E_r^{p,q} \to E_r^{p+r,q-r+1}$ for $r \geq 2$? Can the number r become large?

It would be very interesting, if r could become large.

We discuss our issue for the submodule described in equation (8), we want to describe the image of the restriction map res in

$$\bigoplus_{P:d(P)=1}\bigoplus_{w\in W^P}\bigoplus_{\sigma_f\in \mathrm{Coh}(\mathcal{S}^M_{K_f^M},w\cdot\lambda)} I_P^G H_!^{q-l(w)}(\mathcal{S}^M_{K_f^M},\mathcal{M}_{\mathbf{w}\cdot\lambda,F}))(\sigma_f) \tag{10}$$

where I recall that $\mathcal{M}_{\mathbf{w}\cdot\lambda} = H^{l(w)}(\mathfrak{u}_P,\mathcal{M}_{\mathbb{Q}})$ is a cohomology group in degree $l(w)$.

3.1 The Eisenstein summation

We start from a maximal parabolic subgroup P. We extend the scalars to $\mathbb{C}$ and start from a summand

$$I_P^G H_!^{q-l(w)}(\mathcal{S}^M_{K_f^M},\mathcal{M}_{\mathbf{w}\cdot\lambda,\mathbb{C}}))(\sigma_f) \tag{11}$$

classes in this summand can be represented by differential forms

$$\omega_\infty\otimes\psi_f \in \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),\mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}H_{\sigma_\infty}\otimes\mathcal{M})\otimes I_P^G H_{\sigma_f}$$

which can be viewed as elements in $\Omega^\bullet(P(\mathbb{Q})U_P(\mathbb{A})\backslash G(\mathbb{A})/K_\infty\times K_f)$. We see that it is only invariant under $P(\mathbb{Q})$ whereas cohomology classes on $\mathcal{S}^G_{K_f}$ should be invariant under $G(\mathbb{Q})$. Therefore we make an attempt to lift this form by writing the infinite sum

$$\sum_{a\in P(\mathbb{Q})\backslash G(\mathbb{Q})}(\omega_\infty\otimes\psi_f)(a\underline{g}) \tag{12}$$

where $\underline{g}\in G(\mathbb{A})$.This sum may be divergent. This problem can be remedied if we twist our representation by a character, i.e. we consider $\mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}H_{\sigma_\infty}\otimes I_P^G H_{\sigma_f})\otimes|\gamma_P|^z$ and for $\Re(z)>>0$ the infinite sum

$$\mathrm{Eis}(\omega_\infty\otimes\psi_f,z)=\sum_{a\in P(\mathbb{Q})\backslash G(\mathbb{Q})}(\omega_\infty\otimes\psi_f)\otimes|\gamma_P|^z)a \tag{13}$$

is absolutely convergent and provides a holomorphic function in z. Then $\mathrm{Eis}(\omega_\infty\otimes\psi_f,z)\in\Omega^\bullet(G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty\times K_f)$ is a differential form. It follows from the work of Langlands that it extends to a meromorphic function in the entire $z-$plane.

Now it is clear

If $\mathrm{Eis}(\omega_\infty\otimes\psi_f,z)$ *is holomorphic at* $z=0$ *then* $\mathrm{Eis}(\omega_\infty\otimes\psi_f,0)$ *is a closed form and hence it represents a cohomology class* $[\mathrm{Eis}(\omega_\infty\otimes\psi_f,0)]$. *If* $\mathrm{Eis}(\omega_\infty\otimes\psi_f,z)$ *is not holomorphic at* $z=0$ *then we have to analyze what happens.*

This gives us a method to construct "global" cohomology classes starting from "local" cohomology classes at infinity. We have to understand the relation between the class $[\mathrm{Eis}(\omega_\infty\otimes\psi_f,0)]$ and our original class $[\omega_\infty\otimes\psi_f]$.

3.2 The constant term

If we want to understand what happens at $z = 0$ we have to consider the constant terms along the maximal parabolic subgroups. They are given by integrals

$$\mathcal{F}^Q(\mathrm{Eis}(\omega_\infty \otimes \psi_f, z))(\underline{g}) = \int_{U_Q(\mathbb{A})} \mathrm{Eis}(\omega_\infty \otimes \psi_f, z))(\underline{v}\underline{g})d\underline{v} \tag{14}$$

We assume that the representation $H_{\sigma_\infty} \otimes H_{\sigma_f}$ is embedded in the cuspidal spectrum. Furthermore we assume that $w \cdot \lambda = \mu^{(1)} + a(w, \lambda)\gamma_P + \delta$ is in the negative chamber, i.e. $a(w, \lambda) \leq -f_P$.

We have two cases

a) The parabolic P is conjugate to its opposite P_-. Then the constant term is "essentially" of the following form

$$\mathcal{F}^P : \omega_\infty \otimes \psi_f \otimes |\gamma_P|^z \mapsto \omega_\infty \otimes \psi_f \otimes |\gamma_P|^z + \frac{\mathcal{L}(\sigma, z)}{\mathcal{L}(\sigma, z+1)} T_\infty^{\mathrm{loc}}(\omega_\infty)(z) \bigotimes_p T_p^{\mathrm{loc}}(\psi_p)(z)|\gamma_P|^{2f_P - z} \tag{15}$$

b) The conjugacy class of the opposite parabolic subgroup is Q and $Q \neq P$. Then we have to compute two constant terms

$$\begin{gathered} \mathcal{F}^P(\mathrm{Eis}(\omega_\infty \otimes \psi_f, z))(\underline{g}) = (\omega_\infty \otimes \psi_f, z))(\underline{g}); \\ \mathcal{F}^Q(\mathrm{Eis}(\omega_\infty \otimes \psi_f, z))(\underline{g}) = \tfrac{\mathcal{L}(\sigma,z)}{\mathcal{L}(\sigma,z+1)} T_\infty^{\mathrm{loc}}(\omega_\infty)(z) \bigotimes_p T_p^{\mathrm{loc}}(\psi_p)(z)|\gamma_Q|^{2f_P - z}. \end{gathered} \tag{16}$$

Here $T_\infty^{\mathrm{loc}}(z)$ resp.$T_p^{\mathrm{loc}}(z)$ are "local" intertwining operators between Harish-Chandra modules (resp.) modules for the Hecke algebra which depend meromorphically on z and should be holomorphic for $\Re(z) \geq 0$. Then $\mathcal{L}(\sigma, z)$ is a certain product of L functions attached to the representation $\sigma = \sigma_\infty \times \sigma_f$.

We know

Under the assumption that $\sigma_\infty \times \sigma_f$ is cuspidal and $w \cdot \lambda$ in the negative chamber $\mathrm{Eis}(\omega_\infty \otimes \psi_f, z)$ *is holomorphic at $z = 0$ if and only if $\mathcal{L}(\sigma, z)$ is holomorphic at $z = 0$.*

4 The arithmetic applications

4.1 Rationality results for special values of L-functions

We pick a maximal parabolic subgroup P in the following the parabolic subgroup $Q = P$ if P is conjugate to its opposite, otherwise Q is the parabolic subgroup containing the Borel and which is conjugate to the opposite of P. We have a one-to-one correspondence between $W^P \leftrightarrow W^Q, w \mapsto w'$ which satisfies $l(w) + l(w') = \dim U_P$. We start from an isotypical summand

$$I_P^G H_!^{\bullet - l(w)}(\mathcal{S}^M_{K_f^M}, \mathcal{M}_{\mathbf{w} \cdot \lambda, F})(\sigma_f)$$

in (10), recall that this is a vector space over F. We have a corresponding isotypical summand

$$I_Q^G H_!^{\bullet - l(w')}(\mathcal{S}^M_{K_f^M}, \mathcal{M}_{\mathbf{w}'\cdot\lambda,F})(\sigma'_f).$$

Here the $\bullet$ may vary in a certain range where the cohomology is non zero.

We choose an embedding $\iota : F \to \mathbb{C}$. We assume that our Eisenstein form $\mathrm{Eis}(\omega_\infty \otimes \psi_f, z))$ is holomorphic at $z = 0$, hence we can evaluate it. The restriction of this class is of the form

$$[\omega_\infty] \otimes \psi_f + \frac{1}{\Omega(\sigma)} \frac{\mathcal{L}(\sigma,0)}{\mathcal{L}(\sigma,1)} [T_\infty^{\mathrm{loc}}(\omega_\infty)(0)] \bigotimes_p T_p^{\mathrm{loc}}(\psi_p)(0) \tag{17}$$

This needs some explanation.

i) The operator $\bigotimes_p T_p^{\mathrm{loc}} = T_f^{\mathrm{loc}} : H_{\sigma_f} \to H_{\sigma'_f}$ and where σ'_f is the corresponding isomorphism type, both Hecke modules are defined over F and T_f^{loc} is also defined over F.

ii) We can put a $\mathbb{Q}$-structure on these modules $H_{\sigma_\infty}, H_{\sigma'_\infty}$ and T_∞^{loc} is a $\mathbb{Q}-$linear map

$$T_\infty^{\mathrm{loc}} : \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma_\infty} \otimes \mathcal{M}) \to \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_{Q(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma'_\infty} \otimes \mathcal{M}) \tag{18}$$

Therefore these operators also induce endomorphisms in cohomology

$$T_\infty^{\mathrm{loc}} : H^\bullet(\mathfrak{g}, K_\infty, \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma_\infty} \otimes \mathcal{M}) \to H^\bullet(\mathfrak{g}, K_\infty, \mathrm{Ind}_{Q(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma'_\infty} \otimes \mathcal{M}) \tag{19}$$

iii) The period $\Omega(\sigma)$ is number which comes from the comparison of two isomorphisms $H_{\sigma_f} \xrightarrow{\psi_{\mathrm{alg}}} H_{\sigma'_f}$ and $H_{\sigma_f} \otimes \mathbb{C} \xrightarrow{\psi_{\mathrm{trans}}} H_{\sigma'_f} \otimes \mathbb{C}$.

We consider the commutative diagram

$$\begin{array}{ccc} H^\bullet(\mathfrak{g}, K_\infty, \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma_\infty} \otimes \mathcal{M}) \otimes H_{\sigma_f} \otimes_{F,\iota} \mathbb{C} & \xrightarrow{T_\infty^{\mathrm{loc}} \otimes T_f^{\mathrm{loc}}} & H^\bullet(\mathfrak{g}, K_\infty, \mathrm{Ind}_{Q(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma'_\infty} \otimes \mathcal{M}) \otimes H_{\sigma'_f} \otimes_{F,\iota} \mathbb{C} \\ \downarrow & & \downarrow \\ I_P^G H_!^{\bullet - l(w)}(\mathcal{S}^M_{K_f^M}, \mathcal{M}_{\mathbf{w}\cdot\lambda,\mathbb{C}})(\sigma_f) \otimes \mathbb{C} & \xrightarrow{\overline{T_\infty^{\mathrm{loc}} \otimes T_f^{\mathrm{loc}}}} & I_Q^G H_!^{\bullet - l(w')}(\mathcal{S}^M_{K_f^M}, \mathcal{M}_{\mathbf{w}\cdot\lambda,\mathbb{C}})(\sigma'_f) \otimes \mathbb{C} \end{array} \tag{20}$$

We assume that the kernels of the to vertical arrows are of the form $K^\bullet \otimes H_\sigma$... where $K^\bullet \subset H^\bullet(...)$ is a rational subspace.

We have to distinguish cases:

a) Of course it can happen that the operator $\overline{T_\infty^{\mathrm{loc}} \otimes T_f^{\mathrm{loc}}} = 0$ in this case we get that the restriction of $[\mathrm{Eis}(\omega_\infty \otimes \psi_f, 0)]$ to the cohomology of the boundary is the original class, or in other words restricted to the summand $I_P^G H_!^{q-l(w)}(\mathcal{S}^M_{K_f^M}, \mathcal{M}_{\mathbf{w}\cdot\lambda,\mathbb{C}}))(\sigma_f) \otimes \mathbb{C}$ the Eisenstein series provide a section to the map *res*.

b) If the operator $\overline{T_\infty^{\mathrm{loc}} \otimes T_f^{\mathrm{loc}}} \neq 0$ then we can define a period matrix $\Omega^\dagger(\sigma)$ such that $\Omega(\sigma)\overline{T_\infty^{\mathrm{loc}} \otimes T_f^{\mathrm{loc}}}$ is defined over F.

If we now can show that the image of res intersected with

$$I_P^G H_!^{\bullet - l(w)}(\mathcal{S}^M_{K_f^M}, \mathcal{M}_{\mathbf{w}\cdot\lambda,F})(\sigma_f) \otimes_{F,\iota} \mathbb{C} \oplus I_Q^G H_!^{\bullet - l(w')}(\mathcal{S}^M_{K_f^M}, \mathcal{M}_{\mathbf{w}\cdot\lambda,F})(\sigma'_f) \otimes_{F,\iota} \mathbb{C}$$

is given by the restrictions of the classes $[\mathrm{Eis}(\omega_\infty \otimes \psi_f, 0)]$ then we get a rationality result for special values of L functions

$$\frac{1}{\Omega(\sigma)} \frac{\mathcal{L}(\sigma,0)}{\mathcal{L}(\sigma,1)} \in F \tag{21}$$

where the period $\Omega(\sigma)$ is a suitable non zero matrix coefficient of $\Omega^\dagger(\sigma)$.

Now we have to sit down and try to realize situations where the condition b) is verified. This has been done in my paper with A. Raghuram (See [Ha-Rag]). We observe that there is a little obstacle. The inner cohomology $H_!^k(\mathcal{S}^M_{K_f^M}, \mathcal{M}_{\mathbf{w}\cdot\lambda,F})$ is non zero for k moving in a certain interval around the middle dimension. This implies: If the difference $|l(w) - l(w')|$ becomes to large then $\overline{T_\infty^{\mathrm{loc}} \otimes T_f^{\mathrm{loc}}} = 0$. Therefore our first attempt is to look for balanced pairs where $l(w) = l(w')$.

We start from the group $\mathrm{Gl}_n/\mathrm{Spec}(\mathbb{Z})$ (or over $\mathrm{Spec}(\mathcal{O}_F)$ for a totally real number field F) Our parabolic subgroups P resp. Q have as reductive quotients $M = \mathrm{Gl}_{n_1} \times \mathrm{Gl}_{n_2}$ resp. $\mathrm{Gl}_{n_2} \times \mathrm{Gl}_{n_1}$ where $n_1 + n_2 = n$ at least one of the summands is even.

We consider coefficient systems $\mathcal{M} = \mathcal{M}_\lambda$, we choose balanced Kostant representatives w, w' with $l(w) = l(w')$ We get coefficient systems

$$w(\lambda+\rho) - \rho = \mu^{(1)} + a(w,\lambda)\gamma_P + d\delta, w'(\lambda+\rho) - \rho = \mu^{(\prime 1)} + a(w',\lambda)\gamma_Q + d\delta \tag{22}$$

where we want to keep the semi simple components $\mu^{(1)}, \mu^{(,\prime 1)}$ fixed. (We could say that we start from a pair of self dual weights on $\mu^{(1)}, \mu^{(\prime,1)}$ and then vary λ or w) Then we have $a(w,\lambda) + a(w',\lambda) = -2f_P$.

The combinatorial lemma says:

If we fix a self dual $\mu^{(1)}$ and vary w, λ always requiring $l(w) = l(w')$ then the numbers $a(w,\lambda)$ run over an interval of integers $[q(mu^{(1)}), p(\mu^{(1)})]$ which is determined by the so called cuspidal parameters of $\mu^{(1)}$ (it may be empty).

This is a non trivial combinatorial fact, which we conjectured to be true and was proved by my former student U. Weselmann.

Now we choose any highest weight $\mu = \mu^{(1)} + a\gamma_P + d\delta$ for our reductive group M, our semi simple component $\mu^{(1)}$ is fixed, but we allow any $a \equiv f_P \mod \mathbb{Z}$. If we have an isotypical summand ($_{!!}$ means strongly inner and this is equivalent to cuspidal)

$$H_{!!}^{\bullet - l(w)}(\mathcal{S}^M_{K_f^M}, \mathcal{M}_{\mu,F})(\sigma_f)$$

then we get a string of such summands

$$H_{!!}^{\bullet - l(w)}(\mathcal{S}^M_{K_f^M}, \mathcal{M}_{\mu+b\gamma_P,F})(\sigma_f \otimes |\gamma_P|_f^b)$$

For a minimal degree $\bullet = b_{n_1} + b_{n_2}$ the multiplicity is one. We have two isomorphisms

$$\Psi_{alg} : H_{!!}^{\bullet - l(w)}(\mathcal{S}^M_{K^M_f}, \mathcal{M}_{\mu,F})(\sigma_f) \xrightarrow{\sim} H_{!!}^{\bullet - l(w)}(\mathcal{S}^M_{K^M_f}, \mathcal{M}_{\mu+\gamma_P,F})(\sigma_f \otimes |\gamma_P|) \tag{23}$$

$$\Psi_{trans} : H_{!!}^{\bullet - l(w)}(\mathcal{S}^M_{K^M_f}, \mathcal{M}_{\mu,F})(\sigma_f) \otimes \mathbb{C} \xrightarrow{\sim} H_{!!}^{\bullet - l(w)}(\mathcal{S}^M_{K^M_f}, \mathcal{M}_{\mu+\gamma_P,F})(\sigma_f \otimes |\gamma_P|) \otimes \mathbb{C} \tag{24}$$

The first isomorphism is unique up to an element in $F^\times$ (if we argue a little bit more carefully up to an element in $\mathcal{O}_F^\times$), the second one is explicitly given and hence unique. The comparison between these two gives a period $\Omega(\sigma_f)$ we have $\Omega(\sigma_f \otimes |\gamma_P|_f) = \Omega(\sigma_f \otimes |\gamma_P|_f)^{-1}$.

To our σ_f we attached a cohomological L function

$$L^{\mathrm{coh}}(\sigma, \mathrm{Ad}, s) = L^{\mathrm{coh}}(\sigma_f \otimes |\gamma_P|_f^b, \mathrm{Ad}, s) \tag{25}$$

in other words it is invariant under twisting. If we keep in mind that by the Künneth-theorem our $\sigma_f = \sigma_{1,f} \times \sigma_{2,f}$, this is a Rankin-Selberg L -function but with a shift in the variable s. Then our principle above yields:

For $m \in [q(mu^{(1)}), p(\mu^{(1)}) - 1]$

$$\Omega(\sigma_f)^{\epsilon(b)} \frac{L^{coh}(\sigma_f \otimes |\gamma_P|_f^b, \mathrm{Ad}, m)}{L^{coh}(\sigma_f \otimes |\gamma_P|_f^b, \mathrm{Ad}, m+1)} \in \iota(E) \tag{26}$$

This interval of integers covers exactly the set of integers m *for which* m *and* $m+1$ *are critical arguments in the sense of Deligne's conjecture, provided we believe that there is a motive* $M(\sigma_{1,f}) \times M(\sigma_{2,f})$. *We know the Hodge numbers of this motive - after Serre they can be computed from the* Γ *factors in the functional equation for* $L^{coh}(\sigma_f \otimes |\gamma_P|_f^b, \mathrm{Ad}, s)$*- these Hodge numbers provide an interval of critical arguments.*

In a certain sense our result is weaker than the Deligne conjecture, but not too much. The Deligne conjecture predicts rationality for

$$\Omega_\pm(\sigma_f)^{\epsilon(b)} L^{\mathrm{coh}}(\sigma_f \otimes |\gamma_P|_f^b, \mathrm{Ad}, m) \in \iota(E), \tag{27}$$

where $\Omega_\pm(\sigma_f)$ are two periods attached to the motive $M(\sigma_{1,f}) \times M(\sigma_{2,f})$, to define them we need algebraic geometry. But since we do not have this motive we may forget these periods. We look at the extremal critical argument m_0 and put $\Omega_+(\sigma_f) = L^{\mathrm{coh}}(\sigma_f \otimes |\gamma_P|_f^b, \mathrm{Ad}, m_0)$ and then $\Omega_-(\sigma_f) = \Omega(\sigma_f)$. With this choice of the period the above result implies Deligne's conjecture for the given L-functions.

Our result has another agreeable aspect. Our period comes from topology and not from algebraic geometry. Since we have integral structures on the cohomology we may choose our isomorphism Ψ_{alg} to be an isomorphism of the integral structure, at least locally in Spec($\mathcal{O}_F$). In other words if $\mathfrak{p}$ is a prime, then we find a Zariski open neighborhood $U_\mathfrak{p}$ such that we can choose

$$\Psi_{alg,U_\mathfrak{p}} : H_{!!}^{\bullet-l(w)}(\mathcal{S}^M_{K_f^M},\mathcal{M}_{\mu,\mathcal{O}_F(U_\mathfrak{p})})(\sigma_f) \xrightarrow{\sim} H_{!!}^{\bullet-l(w)}(\mathcal{S}^M_{K_f^M},\mathcal{M}_{\mu+\gamma_P,\mathcal{O}_F(U_\mathfrak{p})})(\sigma_f\otimes|\gamma_P|) \tag{28}$$

Then this pins down a period $\Omega_{U_\mathfrak{p}}(\sigma_f)$ up to a unit in $\mathcal{O}_F(U_\mathfrak{p})^\times$. This means that we get a Zariski sheaf of periods, the ratio

$$\frac{\Omega_{U_\mathfrak{p}}(\sigma_f)}{\Omega_{U_\mathfrak{q}}(\sigma_f)} \in \mathcal{O}_F(U_\mathfrak{p}\cap U_\mathfrak{q})^\times \tag{29}$$

This makes it possible to speak of the factorization of the numbers

$$\Omega(\sigma_f)^{\epsilon(b)}\frac{L^{coh}(\sigma_f\otimes|\gamma_P|_f^b,\mathrm{Ad},m)}{L^{coh}(\sigma_f\otimes|\gamma_P|_f^b,\mathrm{Ad},m+1)} \tag{30}$$

into prime ideals.

4.2 The denominators of Eisenstein classes

This plays a role in the formulation of the conjectures about denominators of Eisenstein classes and congruences between eigenvalues of Hecke operators. For certain other groups expressions of the form (30) occur as factors in $\frac{1}{\Omega(\sigma)}\frac{\mathcal{L}(\sigma,0)}{\mathcal{L}(\sigma,1)}$ in our formula (17) in other words the constant term is of the form

$$[\omega_\infty]\otimes\psi_f+\Omega(\sigma_f)^{\epsilon(b)}\frac{L^{\mathrm{coh}}(\sigma_f\otimes|\gamma_P|_f^b,\mathrm{Ad},m)}{L^{\mathrm{coh}}(\sigma_f\otimes|\gamma_P|_f^b,\mathrm{Ad},m+1)}\frac{L^{\mathrm{mot}}(2m)}{L^{\mathrm{mot}}(2m+1)}[T_\infty^{\mathrm{loc}}(\omega_\infty)(0)]\otimes_p T_f^{\mathrm{loc}}(\psi_f)(0) \tag{31}$$

where the first ratio is still a ratio of two critical values and the second ratio of $L-$ values can be interpreted as extension class in an Ext^1 of Betti-de-Rham structures. This happens for instance if our group is $\mathrm{GSp}_2,\mathrm{GSp}_3,..,U(n,m),....$

Then we will be in the case a) and we also should know (Manin-Drinfeld principle) that the section, which is provided by the Eisenstein classes, is defined over F. Then

$$[\mathrm{Eis}(\omega_\infty\otimes\psi_f,0)]\in H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F) \text{ and } res([\mathrm{Eis}(\omega_\infty\otimes\psi_f,0)])\in I_P^G H_{!!}^{\bullet-l(w)}(\mathcal{S}^M_{K_f^M},\mathcal{M}_{\mu,\mathcal{O}_F(U_\mathfrak{p})})(\sigma_f) \tag{32}$$

Let $\varpi_\mathfrak{p}$ be a uniformizer at $\mathfrak{p}$, we may ask for the denominator of the Eisenstein class, this is the smallest integer $\Delta_\mathfrak{p}(\sigma)\geq 0$ such that

$$\varpi_\mathfrak{p}^{\Delta_\mathfrak{p}(\sigma)}[\mathrm{Eis}(\omega_\infty\otimes\psi_f,0)]\in H^\bullet_{\mathrm{st}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_{F,\mathfrak{p}}}) \tag{33}$$

where $H^\bullet_{\mathrm{st}}$ is the image of $H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_F})$ in $H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)$, and $\mathcal{O}_{F,\mathfrak{p}}$ is the local ring at $\mathfrak{p}$ (not completed). Then we ask the question:

Is the denominator $\varpi_{\mathfrak{p}}^{\Delta_{\mathfrak{p}}(\sigma)}$ equal to the power of $\varpi_{\mathfrak{p}}$ in the denominator of

$$\Omega(\sigma_f)^{\epsilon(b)}\frac{L^{\mathbf{coh}}(\sigma_f\otimes|\gamma_P|_f^b,\mathrm{Ad},m)}{L^{\mathbf{coh}}(\sigma_f\otimes|\gamma_P|_f^b,\mathrm{Ad},m+1)}\ ?$$

In this generality we are asking to much, we should make some further assumption. In earlier expositions I required $\mathfrak{p}$ "large" which is not such a reasonable assumption, perhaps it is is better to assume that σ is $\mathfrak{p}$ ordinary, or does not divide some other L values...

Let us assume for simplicity that σ_f is unramified everywhere and $H_{!!}^{q-l(w)}(\mathcal{S}^M_{K_f^M},\mathcal{M}_{\mu,\mathcal{O}_F(U_{\mathfrak{p}})})(\sigma_f)$ is of rank one over $\mathcal{O}_F(U_{\mathfrak{p}})$ (multiplicity one). Let us also assume that

$$H^q(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_{F,\mathfrak{p}}}),H^{q+1}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_{F,\mathfrak{p}}})$$

are torsion free so that

$$H^q(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_{F,\mathfrak{p}}})\otimes\mathcal{O}_{F,\mathfrak{p}}/(\varpi_{\mathfrak{p}}^{\Delta_{\mathfrak{p}}(\sigma)})=H^q(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_{F,\mathfrak{p}}}\otimes(\mathcal{O}_{F,\mathfrak{p}}/(\varpi_{\mathfrak{p}}^{\Delta_{\mathfrak{p}}(\sigma)}))\tag{34}$$

If we choose an ordering of the π_f in our Theorem 1 and this induces a filtration

$$0\subset F^1(H^q_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_{F,\mathfrak{p}}})_{\mathrm{\ st}})\subset F^2(H^q_!(\mathcal{S}^G_{K_f}\tilde{\mathcal{M}}_{\mathcal{O}_{F,\mathfrak{p}}})_{\mathrm{\ st}})\subset\cdots\subset F^t(H^q_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_{F,\mathfrak{p}}})_{\mathrm{\ st}})\tag{35}$$

such that the successive quotient are free and

$$F^j(H^q_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_{F,\mathfrak{p}}})_{\mathrm{\ st}})/F^{j-1}(H^q_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_{F,\mathfrak{p}}})_{\mathrm{\ st}})\otimes F=H^\bullet_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)(\pi_{j,f})\tag{36}$$

Then we get an induced filtration if we reduce modulo $\varpi_{\mathfrak{p}}^{\Delta_{\mathfrak{p}}(\sigma)}$:

$$0\subset F^1(H^q_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_{F,\mathfrak{p}}/\varpi_{\mathfrak{p}}^{\Delta_{\mathfrak{p}}(\sigma)}}))\subset F^2(H^q_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_{F,\mathfrak{p}}/\varpi_{\mathfrak{p}}^{\Delta_{\mathfrak{p}}(\sigma)}}))\subset\cdots\subset F^t(H^q_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_{F,\mathfrak{p}}/\varpi_{\mathfrak{p}}^{\Delta_{\mathfrak{p}}(\sigma)}}))\tag{37}$$

If now $\varpi_{\mathfrak{p}}^{\Delta_{\mathfrak{p}}(\sigma)}$ is the denominator of the Eisenstein class then we get a Hecke equivariant injection

$$H_{!!}^{q-l(w)}(\mathcal{S}^M_{K_f^M},\mathcal{M}_{\mu,\mathcal{O}_F(U_{\mathfrak{p}})})(\sigma_f)\otimes(\mathcal{O}_{F,\mathfrak{p}}/(\varpi_{\mathfrak{p}}^{\Delta_{\mathfrak{p}}(\sigma)}))\hookrightarrow H^q_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_{F,\mathfrak{p}}/\varpi_{\mathfrak{p}}^{\Delta_{\mathfrak{p}}(\sigma)}})\tag{38}$$

Now we look how $H_{!!}^{q-l(w)}(\mathcal{S}^M_{K_f^M},\mathcal{M}_{\mu,\mathcal{O}_F(U_{\mathfrak{p}})})(\sigma_f)\otimes\mathcal{O}_{F,\mathfrak{p}}/(\varpi_{\mathfrak{p}}^{\Delta_{\mathfrak{p}}(\sigma)}))\xrightarrow{\sim}(\mathcal{O}_{F,\mathfrak{p}}/(\varpi_{\mathfrak{p}}^{\Delta_{\mathfrak{p}}(\sigma)})$ sits as submodule in the filtration. We take the intersections $(\mathcal{O}_{F,\mathfrak{p}}/(\varpi_{\mathfrak{p}}^{\Delta_{\mathfrak{p}}(\sigma)})\cap F^j(H^q_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_{F,\mathfrak{p}}}))=(\varpi_{\mathfrak{p}})^{e_j}\mathcal{O}_{F,\mathfrak{p}}/(\varpi_{\mathfrak{p}}^{\Delta_{\mathfrak{p}}(\sigma)}))$. Here $e_j\geq e_{j+1},\ e_t=0,e_0=\Delta_{\mathfrak{p}}(\sigma)$. Then we get congruences between the eigenvalues of Hecke operators

$$\pi_{j,f}(T_{\ell,\chi})\equiv\sigma_f(T_{\ell,\chi})\mod\varpi_{\mathfrak{p}}^{e_i-e_{i+1}}\tag{39}$$

If our locally symmetric space $\mathcal{S}^G_{K_f}$ is a Shimura variety defined over $\mathbb{Q}$ then we may consider the etale cohomology groups $H^\bullet_{et}(\mathcal{S}^G_{K_f}\times\bar{\mathbb{Q}},\tilde{\mathcal{M}}_{\hat{\mathcal{O}}_{F,\mathfrak{p}}}$, on these groups we have the action of the Galois group and the above filtration induces a Galois invariant filtration on these cohomology groups. Hence we get a Galois actions on

$$F^j(H^q_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\hat{\mathcal{O}}_{F,\mathfrak{p}}})/F^{j-1}(H^q_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\hat{\mathcal{O}}_{F,\mathfrak{p}}}))=\mathcal{H}(\pi_{j,f}). \tag{40}$$

We also have a Galois action on $H^{q-l(w)}_{!!}(\mathcal{S}^M_{K^M_f},\mathcal{M}_{\mu,\mathcal{O}_F(U_\mathfrak{p})})(\sigma_f)\otimes\hat{\mathcal{O}}_{F,\mathfrak{p}}$ and hence a Galois action on the quotient $(\mathcal{O}_{F,\mathfrak{p}}/(\varpi_\mathfrak{p}^{\Delta_\mathfrak{p}(\sigma)})$. This gives us injections

$$\varpi_\mathfrak{p}^{e_j-e_{j+1}}\mathcal{O}_{F,\mathfrak{p}}/\varpi_\mathfrak{p}^{\Delta_\mathfrak{p}(\sigma)}\hookrightarrow\mathcal{H}(\pi_{j,f})\otimes\mathcal{O}_F/\varpi_\mathfrak{p}^{\Delta_\mathfrak{p}(\sigma)} \tag{41}$$

Hence we see that our conjecture about denominators imply congruences and reducibility of certain Galois-modules $\mod \varpi_p$.

4.3 Experimental aspects

The congruences have been checked in many examples for a certain range of values ℓ, but this does not imply the conjectures about denominators. The denominator conjecture is stronger.

On the other hand the conjecture on denominators can be checked -with a little bit of luck- in a given case. We only need a program which computes the modules and the arrows in the diagram

$$0\to H^q_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\mathbb{Z})\to H^q(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\mathbb{Z})\to H^i(\overset{\bullet}{\mathcal{N}}\,\mathcal{S}^{G,\infty}_{K_f},\tilde{\mathcal{M}}_\mathbb{Z})\to$$

This is a purely combinatorial problem, such a program certainly exists in principle and in simple cases it should work in practice.

The next - and much more difficult problem - is to write a program that computes the action of Hecke operators $T_{\ell,\chi}$ for some small ℓ, possibly only for $\ell=2$.

Assume we find a submodule $\mathbb{Z}e_{\sigma_f}\subset H^i(\overset{\bullet}{\mathcal{N}}\,\mathcal{S}^{G,\infty}_{K_f},\tilde{\mathcal{M}}_\mathbb{Z})$ with the following properties

a) It is a direct summand, it is in the image of res.

b)It is an eigenspace for the $T_{\ell,\chi}$, i.e. we have $T_{\ell,\chi}e_{\sigma_f}=\sigma_f(T_{\ell,\chi})e_{\sigma_f}$ (Such an e_{σ_f} can be given by an elliptic modular form f)

c) There exists an ℓ such that the operator $\partial(T_{\ell,\chi},\sigma_f)=T_{\ell,\chi}-\sigma_f(T_{\ell,\chi})\mathrm{Id}$ is injective on $H^q_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\mathbb{Z})/Torsion$ (Manin-Drinfeld).

We introduce the notation $H^\bullet_{\mathrm{int}}$ for the image of $H^q_?(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\mathbb{Z})$ in $H^q_?(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\mathbb{Q})$. Furthermore let $H^q(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\mathbb{Z})(\sigma_f)$ be inverse image of $\mathbb{Z}e_{\sigma_f}$ under res. This means that we get an exact sequence

$$0\to H^q_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\mathbb{Z})\to H^q(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\mathbb{Z})(\sigma_f)\to\mathbb{Z}e_{\sigma_f}\to 0 \tag{42}$$

and this gives the exact sequence

$$0\to H^q_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\mathbb{Z})_{\mathrm{int}}\to H^q(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\mathbb{Z})(\sigma_f)_{\mathrm{int}}\to\mathbb{Z}e_{\sigma_f}\to 0 \tag{43}$$

Choosing a suitable ℓ we get from c) a splitting

$$H^q(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathbb{Q}})(\sigma_f)=H^q_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathbb{Q}})\oplus\mathbb{Q}\tilde{e}_{\sigma_f} \tag{44}$$

and hence we get a splitting up to isogeny

$$H^q(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathbb{Z}})_{\text{int}}(\sigma_f)\supset H^q_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathbb{Z}})_{\text{int}}\oplus\mathbb{Z}\tilde{e}'_{\sigma_f} \tag{45}$$

Then we have $res(\tilde{e}'_{\sigma_f})=\Delta(\sigma)e_{\sigma_f}$ and

$$H^q(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathbb{Z}})_{\text{int}}(\sigma_f)/(H^q_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathbb{Z}})_{\text{int}}\oplus\mathbb{Z}\tilde{e}'_{\sigma_f})\xrightarrow{\sim}\mathbb{Z}/\Delta(\sigma)\mathbb{Z} \tag{46}$$

This means:

A single Hecke operator $T_{\ell,\chi}$ satisfying c) detects the denominator of the Eisenstein class.

The denominator implies the congruences for all $T_{\ell,\chi}$, the programs which check the congruences cover only a finite number of ℓ. But even in the smallest case which is discussed in [Ha-Cong] where σ_f is the modular cusp form of weight 22 and $\Delta=41$ this computation of the action of the cohomology and the action of a Hecke operator has not yet been carried out. It seems to be incredibly difficult. (The congruence has been verified by different methods by Chenevier-Lannes.)

4.3.1 A still stronger "conjecture"

We stick to notations in the previous section. Let us pick an element $\tilde{e}_{\sigma_f}\in H^q(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathbb{Z}})_{\text{int}}$ which is a preimage of e_{σ_f}, i.e. $res(\tilde{e}_{\sigma_f})=e_{\sigma_f}$. Then for all ℓ,χ we have

$$(T_{\ell,\chi}-\sigma_f(T_{\ell,\chi})\mathrm{Id})(\tilde{e}_{\sigma_f})\in H^q_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathbb{Z}})_{\text{int}} \tag{47}$$

and clearly this induces a Hecke equivariant homomorphism

$$\partial(T_{\ell,\chi},\sigma_f):\mathbb{Z}e_{\sigma_f}/\Delta(\sigma_f)\to H^q_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathbb{Z}})_{\text{int}}/\partial(T_{\ell,\chi},\sigma_f)H^q_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathbb{Z}})_{\text{int}} \tag{48}$$

We can ask

Can we find a prime ℓ such that $\partial(T_{\ell,\chi},\sigma_f)$ is injective?

This is the kind of question where it is very likely that the answer is "YES." But finding a proof seems to be difficult or may be even impossible. It is connected via the congruence relations to Galois representations and the the Tschebotareff density suggests the the ℓ for which this is the case have positive density. But our question may be in the range of experimental verification.

In my book [Ha-book], Chapter2, I discuss the cohomology of the group $\Gamma=\mathrm{Sl}_2(\mathbb{Z})$ with coefficients in $\mathcal{M}_n$, this is of course the module of homogenous polynomials of degree n in two variables. For this case I give a complete description of the cohomology groups and the arrows in 1.2.7 (modulo the exercise at the end) (The notion of orbiconvex covering has to be revised, it works only if the symmetric space is of rank one). So the first part is easy in this special situation. What I do not do is to compute Hecke operators, but this should be possible in this case. (See the Final remark in chap2, 1.2.7) (See also the dissertation of X. Wang " Die Eisensteinklasse in $H^1(\mathrm{Sl}_2(\mathbb{Z}), \mathcal{M}_{n,\mathbb{Z}})$ und die Arithmetik spezieller Werte von L Funktionen., 1989 Bonner Mathematische Schriften 202)

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