Eisenstein Cohomology and the Construction of mixed Motives

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March 18, 2017

Contents

1 Motives and their cohomological realizations

In this manuscript I use the concept of motives without defining what I really mean by that. Basically a motive should be a piece in the cohomology of an algebraic variety, but the rules how I get such pieces are not fixed. In any case a motive must have various cohomological realizations, namely the Betti realization, the de-Rham realization and the ℓ-adic realizations for all primes ℓ.

1.1 Pure Motives

I consider smooth projective schemes $X/\mathrm{Spec}(\mathbb{Q})$, we know that we can find a nonempty open subset $V = \mathrm{Spec}(\mathbb{Z}_S) \subset \mathrm{Spec}(\mathbb{Q})$ such that X extends to a smooth projective scheme $\mathcal{X} \to V$. Let us choose such an extension. We consider the cohomology of this scheme, I denote it by $H^\bullet(X)$ and by this I mean the various realizations

The Betti-cohomology:

$$H^\bullet_B(X) = H^\bullet_B(X(\mathbb{C}), \mathbb{Z})$$

This a finitely generated $\mathbb{Z}$-module together with the involution F_∞ induced by the complex conjugation on $X(\mathbb{C})$

The de-Rham-cohomology is defined as the hypercohomology of the complex of coherent sheaves $0 \to \mathcal{O}_X \to \Omega^1_X \to \dots \Omega^d_X \to 0$. This cohomology is the cohomology of a double complex

$$\begin{array}{ccccccc} 0 & \to & \mathcal{O}_X & \to & \Omega^1_X & \to \dots & \Omega^d_X \to 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & \to & \mathcal{O}^{0,1}_X & \to & \Omega^{1,1}_X & \to \dots & \Omega^{d,1}_X \to 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & \to & \emptyset^{0,2}_X & \to & \Omega^{1,2}_X & \to \dots & \Omega^{d,2}_X \to 0 \\ & & \downarrow & & \downarrow & & \downarrow \end{array}$$

here the vertical complexes are resolutions of the coherent sheaves in the top line by coherent acyclic sheaves. Then

$$H^\bullet_{dRh}(X) = H^\bullet(\Omega^{\bullet,\bullet}(X))$$

These cohomology groups are finite dimensional $\mathbb{Q}$ vector space together with the descending filtration. In degree $\bullet = n$ it is of the form

$$H^n_{dRh}(X) = F^0 H^n_{dRh}(X) \supset F^1 H^n_{dRh}(X) \supset \dots \supset F^n H^n_{dRh}(X) = H^0(X, \Omega^n_X) \supset F^{n+1} H^n_{dRh}(X) = 0.$$

We have the comparison isomorphism

$$I_{B-dRh} : H^\bullet_B(X) \otimes \mathbb{C} \xrightarrow{\sim} H^\bullet_{dRh}(X) \otimes \mathbb{C}$$

The ℓ-adic cohomology: For any prime ℓ the etale cohomology groups

$$H^\bullet_{et,\ell}(X) = H^\bullet(X \times \bar{\mathbb{Q}}, \mathbb{Z}_\ell)$$

they are modules for the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

For any embedding $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$ we have the comparison isomorphism

$$I_\ell : H^\bullet_{et,\ell}(X) \otimes \mathbb{C} \xrightarrow{\sim} H^\bullet_B(X) \otimes \mathbb{C}$$

which is compatible with F_∞

Furthermore the complex conjugation acts on $H^\bullet_B(X(\mathbb{C}), \mathbb{C})$ and $H^\bullet(\Omega^\bullet(X)) \otimes \mathbb{C}$ via the complex conjugations c_B and c_{DR}. The comparison isomorphisms satisfy in addition

$$\begin{array}{c} I \circ c_B \otimes F_\infty = c_{DR} \circ I \\ I_\ell \circ c = F_\infty \circ I_\ell \end{array} \qquad (conj)$$

If I consider the cohomology in a fixed degree n then I want to call the object $H^n(X)$ a pure motive of weight n. This weight is visible as the length of the de-Rham cohomology filtration.

It is also visible in the etale cohomology: For $p \notin S$ and $p \neq \ell$ the modules $H^n(X \times \bar{\mathbb{Q}}, \mathbb{Q}_\ell)$ are unramified at p. The characteristic polynomial

$$\det(T - \Phi_p^{-1} | H^\bullet(X \times \bar{\mathbb{Q}}, \mathbb{Q}_\ell)) \in \mathbb{Z}[T]$$

is independent of ℓ and its roots (the eigenvalues of Frobenius Φ_p^{-1}) are of absolute value $p^{n/2}$.

1.2 Some simple pure motives

Now $\mathbb{Z}(-n) = H^{2n}(\mathbb{P}^n, \mathbb{Z})$ is the following object

$$\mathbb{Z}(-n) = \begin{cases} H_B^{2n}(\mathbb{P}^n) = \mathbb{Z}\cdot 1_B \ , F_\infty(1_B) = (-1)^n 1_B \\ H_{DR}^{2n}(\mathbb{P}^n) = \mathbb{Q}\cdot 1_{DR} + \text{ Filtration}, F^n\mathbb{Q}(-n) = \mathbb{Q}(-n), F^{n+1}\mathbb{Q}(-n) = 0 \\ I : H_B^{2n} \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} H_{DR}^{2n} \otimes_{\mathbb{Q}} \mathbb{C} \\ I : 1_B \longrightarrow (\frac{1}{2\pi i})^n \ 1_{DR} \\ I \circ F_\infty \circ c_B = c_{DR} \circ I \\ H_{ét}^{2n}(\mathbb{P}^n) = \mathbb{Z}_\ell(-n) \ \text{ Galoismodul} \\ I_\ell : H_B^{2n}(\mathbb{P}^n) \otimes \mathbb{Z}_\ell \xrightarrow{\sim} H_{\text{ét}}^{2n}(\mathbb{P}^n, \mathbb{Z}_\ell) \\ \text{ compatible with the action of } F_\infty. \end{cases}$$

It will important that the comparison isomorphism gives us a canonical generator in $\mathbb{Z}_\ell(-n-1)$. This generator can also be seen in the following way. For all m we have the privileged ℓ^m-s root of unity

$$\zeta_{\ell^m} = e^{\frac{2\pi i}{\ell^m}}$$

and the canonical generator in $\mathbb{Z}_\ell(-n)$ is given by $\zeta_{\ell^m}^{\otimes(-n)}$. These motives $\mathbb{Z}(-n)$ for $n \in \mathbb{Z}$ are called *pure Tate motives.*

If we have a finite extension $K/\mathbb{Q}$, then we can consider $\mathbb{P}^n/K$ and the Weil restriction $R_{K/\mathbb{Q}}\mathbb{P}^n$. Then we can consider the motive $H^n(R_{K/\mathbb{Q}}\mathbb{P}^n) = \mathbb{Z}(-n)^{K/\mathbb{Q}}$. Its Betti-cohomology and *é*tale cohomology

$$\mathbb{Z}(-n)_B^{K/\mathbb{Q}} = \sum_{\sigma:K\to\mathbb{C}} \mathbb{Z}1_B, \ \ \mathbb{Z}_\ell(-n)^{K/\mathbb{Q}} = \mathrm{Ind}_{\mathrm{Gal}(\bar{\mathbb{Q}}/K)}^{\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})} \mathbb{Z}_\ell(-n).$$

element $1_B^{K/\mathbb{Q}} = (\ldots, 1_B, \ldots)_{\sigma:K\to\mathbb{C}}$

These motives are Tate motives which are twisted by an Artin motive.

We also want to consider correspondences on $T \subset X \times_{\mathbb{Q}} X$, they induces endomorphisms on the cohomology $H^\bullet(X)$, of course by this we mean that they induce endomorphisms in any of the cohomological realizations. We consider the ring generated by these endomorphisms and we try to find correspondences which are projectors in all cohomological realizations. If we have such an endomorphism q then we also want that $(H^\bullet(X), q)$ is also a pure motive. In any case it has all the cohomological realizations and this is my basic criterion for something being a motive.

If we are lucky then we can find such projectors, which induce the identity on the cohomology $H^n(X)$ in degree n and which are zero in the other degrees. Then we may speak of $H^n(X)$ as a pure motive of weight n. It is also possible that we can find only "projectors with denominators", i.e. endomorphism p which satisfy $p^2 = mp$ with some non zero integer m. In such a case we get a motive with coefficients. (See 1.8.)

1.3 Mixed motives

Now I want to remove a closed subscheme $Y \subset X$. Let $U = X \setminus Y$, this is now quasi projective. I want to consider the cohomology $H^\bullet(U)$ of U and I want explain that we may consider this (under certain conditions) as a mixed motive.

We denote the inclusions $j : U \hookrightarrow X, i : Y \hookrightarrow X$.

Let me assume for simplicity that $Y \subset X$ is smooth, then $Y = \mathrm{Spec}(\mathcal{O}_X/\mathcal{J})$ and we consider the completion

$$\mathcal{N}Y = \mathrm{Spec}(\lim_{\leftarrow}(\mathcal{O}_X/\mathcal{J}^n))$$

which I consider as being a tubular neighborhood of Y. Locally on Y this is of the form $\mathrm{Spec}(\mathcal{O}_Y(W')[[f_1,\dots,f_r]])$, where the f_i are generators of the ideal $\mathcal{J}$ which form a system of local parameters.

If I remove the zero section Y from this scheme I get

$$\overset{\bullet}{\mathcal{N}}\, Y = \mathcal{N}Y \setminus Y$$

I want that the cohomology $H^\bullet(\overset{\bullet}{\mathcal{N}}\, Y)$ is a mixed motive and I will explain why this is not an entirely absurd idea.

Let us start from the case that Y is just a finite number of $\mathbb{Q}$-rational points. In this case and our completion is simply a disjoint union of $B_d = \mathrm{Spec}\mathbb{Q}[[x_1,\dots,x_d]]$ where $d = \dim(X)$. If we stick to one of these points P then we have to understand the cohomology of $B_d \setminus P = \overset{\bullet}{B}_d$.

It is clear that from the point of view of Betti-cohomology this is just a sphere of dimension $2d-1$ and we say

$$H^p_B(\overset{\bullet}{B}_d) = \begin{cases} \mathbb{Z} \text{ if } p = 0, 2d-1 \\ 0 \ \text{ else} \end{cases}$$

The involution F_∞ acts by the identity in degree zero and by $(-1)^d$ in degree $2d-1$.

If we want to understand the de-Rham and the etale realization I begin with the case $d = 1$. In this case we consider $\dot{B}_1$ as "homotopy equivalent" to the multiplicative group scheme $\mathbb{G}_m$. If we cover the projective line $\mathbb{P}^1$ by two affine planes U_0, U_1 then $\mathbb{G}_m = U_0 \cap U_1$ and we consider the resulting Mayer-Vietoris sequence in cohomology, it provides and isomorphism

$$H^1(\mathbb{G}_m) \xrightarrow{\sim} H^2(\mathbb{P}^1)$$

Now we remember how we compute the cohomology of a sphere by using the Mayer-Vietoris sequence. In B_d we can define the subschemes $B_d[x_i \neq 0]$ and we can cover B_d by these subschemes. Writing down certain Mayer-Vietoris sequences provides some convincing evidence that

$$H^{2d-1}(B_d) = \mathbb{Z}(-d)$$

Now we consider the general case, our subscheme Y is still smooth. We can view $\overset{\bullet}{\mathcal{N}}\, Y$ as a fibre bundle over Y where the fibres are $\overset{\bullet}{B}_d$ where d is the codimension of Y in X. If we consider the sheaf $\mathbb{Z}$ on $\overset{\bullet}{\mathcal{N}}\, Y$ and the inclusion $\overset{\bullet}{\mathcal{N}}\, Y \hookrightarrow \mathcal{N}Y$ then the direct image functor is not exact we have

$$R^q j_*(\mathbb{Z}) = 0 \text{ if } q \neq 2d-1, 0$$

and in degree zero

$$j_*(\mathbb{Z}) \text{ is the constant sheaf } \mathbb{Z}$$

and

$$R^{2d-1} j_*(\mathbb{Z})$$

is a local system of sheaves with stalk at a point isomorphic to $\mathbb{Z}(-d)$. This is just the local system of the cohomology groups of the fibres $\overset{\bullet}{B}_d$. I claim that this local system is trivial because if we consider the Betti-cohomology, then we have an orientation on the normal bundle and the stalk $R^{2d-1} j_*(\mathbb{Z})_x = \mathbb{Z}$. In the other realizations we get trivializations from the comparison isomorphisms.

We get a spectral sequence for the cohomology with E_2-term

$$H^p(Y, R^q j_*(\mathbb{Z})) \Rightarrow H^n(\overset{\bullet}{\mathcal{N}}\, Y, \mathbb{Z})$$

and since there are only two columns we get the Gysin sequence

$$\to H^n(Y, R^0 j_*(\mathbb{Z})) \to H^n(\overset{\bullet}{\mathcal{N}}\, Y, \mathbb{Z}) \to H^{n-2d+1}(Y, R^{2d-1} j_*(\mathbb{Z})) \to H^{n+1}(Y, R^0 j_*(\mathbb{Z}))$$

Now we have to assume that the kernel and the cokernel of

$$H^{n-2d+1}(Y, R^{2d-1} j_*(\mathbb{Z})) \to H^{n+1}(Y, R^0 j_*(\mathbb{Z}))$$

are pure motives. Since the local system $R^{2d-1} j_*(\mathbb{Z})$ is trivial this map

$$H^{n-2d+1}(Y, R^{2d-1} j_*(\mathbb{Z})) = H^{n-2d+1}(Y, \mathbb{Z}(-2d)) \to H^{n+1}(Y, R^0 j_*(\mathbb{Z}))$$

is given by the multiplication by d-th the Chern class of the normal bundle and we see that the map is induced by an algebraic cycle and this makes it clear that we can consider $H^n(\overset{\bullet}{\mathcal{N}}\, Y, \mathbb{Z})$ as a mixed motive. Of course the kernel and the cokernel are just the terms $E_3^{n,0}, E_3^{n-2d+1,2d-1}$.

We want to say a few words about the de-Rham realization. At first we consider the case that Y is of codimension one. We return to the global situation and consider $Y \hookrightarrow X$. In a suitable neighborhood of a point $y_0 \in Y$ the subscheme Y is given by an equation $x_1 = 0$, let $x_1, x_2, \ldots, x_{d_0}$ be a set of local coordinates in this neighborhood. Then we define two modified de-Rham complexes:

The first one is

$$j_{*,\log}(\Omega^\bullet(Y))_{y_0} = 0 \to \mathcal{O}_{X,y_0,\log}(Y) \to \Omega^1_{y_0,\log}(Y) \to \ldots \Omega^\nu_{y_0,\log}(Y) \to$$

where $\Omega^\nu_{y_0,\log}$ is $\mathcal{O}_{X,y_0}-$ module generated by the forms

$$\frac{dx_1}{x_1} \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_{\nu-1}},$$

it is slightly large than $\Omega^\bullet_X$ in all degrees > 0.

The second one is

$$j_{!,\text{zero}}(\Omega^\bullet)(Y)_{y_0} = 0 \to x_1\mathcal{O}_{X,y_0} \to \mathcal{O}_{X,y_0}dx_1 \oplus x_1\mathcal{O}_{X,y_0}dx_2 \oplus \cdots \to \ldots$$

where the differentials in degree ν are generated by the differentials $dx_1 \wedge dx_{i_2} \wedge \ldots dx_{i_\nu}$ and $x_1 dx_{i_1} \wedge dx_{i_2} \wedge \ldots dx_{i_\nu}$ if all the i_μ are different from the index 1. Now we can define

$$H^\bullet_{DR}(U) = H^\bullet(X, j_{*,\log}(\Omega^\bullet)(Y))$$

and

$$H^\bullet_{DR,c}(U) = H^\bullet(X, j_{*,\text{zero}}(\Omega^\bullet)(Y))$$

and

$$H^\bullet_{DR}(\overset{\bullet}{\mathcal{N}}\, Y) = H^\bullet(X, j_{*,\log}(\Omega^\bullet)(Y)/j_{*,\text{zero}}(\Omega^\bullet)(Y)).$$

We define the Hodge filtration in the standard way, then we can verify that

$$\begin{gathered}\operatorname{Im}(H^n_{DR}(U,\mathbb{Z}) \to H^n_{DR}(\overset{\bullet}{\mathcal{N}}\, Y, j_*(\mathbb{Z}))) \cap F^m(H^n_{DR}(\overset{\bullet}{\mathcal{N}}\, Y, j_*(\mathbb{Z}))) = \\ \operatorname{Im}(F^m(H^n_{DR}(U,\mathbb{Z})) \to F^m(H^n_{DR}(\overset{\bullet}{\mathcal{N}}\, Y, j_*(\mathbb{Z})))\end{gathered} \tag{1}$$

If Z is not of codimension one, then we blow up X along Y, we get a diagram

$$\begin{array}{ccccc} \hat{Y} & \hookrightarrow & \hat{X} & \leftarrow & U \\ \downarrow & & \downarrow & & \| \\ Y & \hookrightarrow & X & \leftarrow & U \end{array},$$

where now $\hat{Y}$ is of codimension one. The reasoning in SGA4$\frac{1}{2}$ IV.5 shows that we have $H^n(\overset{\bullet}{\mathcal{N}}\, Y,\mathbb{Z}) = H^n(\overset{\bullet}{\mathcal{N}}\, \hat{Y},\mathbb{Z})$ in the Betti and the ℓ adic realizations, hence we define

$$H^n_{DR}(\overset{\bullet}{\mathcal{N}}\, Y,\mathbb{Z}) = H^n_{DR}(\overset{\bullet}{\mathcal{N}}\, \hat{Y},\mathbb{Z})$$

and we have constructed all the realizations of our mixed motive $H^n(\overset{\bullet}{\mathcal{N}}\, Y,\mathbb{Z})$. It has a weight filtration coming from our spectral sequence, this weight filtration is visible on all realizations and compatible with the comparison isomorphisms. The weights are n and $n+1$.We get a long exact sequence

$$\to H^n_c(U,\mathbb{Z}) \to H^n(U,\mathbb{Z}) \to H^n(\overset{\bullet}{\mathcal{N}}\, Y,\mathbb{Z}) \to$$

Now we encounter a problem which we have seen in milder form before. We certainly should try to show that the image of

$$H^n(U,\mathbb{Z}) \to H^n(\overset{\bullet}{\mathcal{N}}\, Y,\mathbb{Z})$$

is the cohomology of a mixed motive and we also should show a similar assertion for the kernel the map

$$H^{n-1}(\overset{\bullet}{\mathcal{N}}\, Y,\mathbb{Z}) \to H^n_c(U,\mathbb{Z}).$$

As far as I understand this is one of the major obstacles if we want to construct an abelian category of mixed motives. If we can show that this is so under certain assumptions or in a given concrete situation, then we might be justified to say that

$$\text{image } (H^n_c(U,\mathbb{Z}) \to H^n(U,\mathbb{Z})) = H_!(U,\mathbb{Z})$$

is a pure motive and it sits in an exact sequence

$$0 \to H^n_!(U,\mathbb{Z}) \to H^n(U,\mathbb{Z}) \to \ker(H^n(\overset{\bullet}{\mathcal{N}}\, Y,\mathbb{Z}) \xrightarrow{\delta} H^{n+1}_c(U,\mathbb{Z})) \to 0.$$

The motive $H^n_!(U,\mathbb{Z})$ is pure of weight n the kernel $\ker(\delta) = \ker(H^n(\overset{\bullet}{\mathcal{N}}\, Y,\mathbb{Z}) \xrightarrow{\delta} H^{n+1}_c(U,\mathbb{Z}))$ is mixed of weights $n, n+1$. Hence $H^n(U,\mathbb{Z})$ has a weight filtration with weights $n, n+1$.

If d_0 is the dimension of U, then the dimension of Y is $d_0 - d$. If we assume that $n > 2(d_0 - d)$ then the weight n part in $H^n(\overset{\bullet}{\mathcal{N}}\, Y,\mathbb{Z})$ becomes zero and we have $H^n_!(U,\mathbb{Z}) = H^n(X,\mathbb{Z})$ independently of Y. Furthermore $\ker(H^n(\overset{\bullet}{\mathcal{N}}\, Y,\mathbb{Z}) \to H^{n+1}_c(U,\mathbb{Z}))$ is pure of weight $n+1$ and we get

$$[H^n(U,\mathbb{Z})] \in \mathrm{Ext}^1_{\mathcal{MM}}(\ker(\delta), H^n(X,\mathbb{Z})).$$

Now we assume that the codimension of Y is one and we look at the cohomology in degree $n = 2d_0 - 1$ In this case $H^{n+1}_c(U,\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}(-d_0)$ and $\ker(\delta) \xrightarrow{\sim} \mathbb{Z}(-d_0)^{r-1}$ where r is the number of connected components of Y. Therefore we end up with an element

$$[H^{2d+1}(U,\mathbb{Z})] \in \mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-d_0)^{r-1}, H^{2d_0-1}(X,\mathbb{Z})).$$

.

For any element $D \in \ker(\delta), D \neq 0$ we can consider the line $\mathbb{Z}D \in \ker(\delta)$, and the inverse image of this line provides a subextension

$$[H^n(U,\mathbb{Z})][D] \in \mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-d_0), H^n(X,\mathbb{Z})).$$

1.4 Bloch's Idea

This construction is due to S. Bloch. If $X/\mathbb{Q}$ is a smooth, projective curve, then the only choice we have is $d = 1$ and Y is simply a set of closed points $\{P_1, P_2, \dots, P_r\}$. Let $\mathbb{Q}(P_i)$ be the residue-field then we put $n_i = \deg(P_i) = [\mathbb{Q}(P_i) : \mathbb{Q}]$. If all the P_i are rational then $H^0(Y, R^1 j_*(\mathbb{Z})) = \mathbb{Z}(-1)^r$. In the general case we have to twist these Tate motives by a finite dimensional representation of the Galois group.

Now $Y(\mathbb{C})$ is a set of $n = \sum n_i$ points and

$$H^1_B(\overset{\bullet}{\mathcal{N}}\, Y) = H^1(\overset{\bullet}{\mathcal{N}}\, Y(\mathbb{C}),\mathbb{Z}) = \mathbb{Z}^n = \bigoplus_I \Big(\bigoplus_{\sigma:\mathbb{Q}(P_i)\to\mathbb{C}} \mathbb{Z}\Big)$$

and an element $D \in H^1(\overset{\bullet}{\mathcal{N}}\, Y(\mathbb{C}),\mathbb{Z})$ is simply a divisor, this divisor is rational over $\mathbb{Q}$ if its coefficients at the points lying over a given closed point P_i are constant and hence all equal to an integer d_i. Hence a divisor $D = \sum_i n_i P_i$ can be viewed as an element in $H^1_B(\overset{\bullet}{\mathcal{N}}\ Y)$ and this element is in the kernel of δ if and only if the degree $\deg(D) = \sum n_i d_i = 0$. Therefore we may remove the points P_i from X, we get an open subscheme $U = X \setminus \{P_1, \dots, P_r\}$ and

$$[H^n(U,\mathbb{Z})][D] \in \mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-1), H^1(X,\mathbb{Z})).$$

Now we can send the divisor D to its class $[D]$ in the Picard group $\mathrm{Pic}(X)(\mathbb{Q})$ and we get a diagram

$$\begin{array}{ccc} \ker(\delta)_{\mathbb{Q}} = \{D = \sum n_i P_i, \deg(D) = 0\} & \rightarrow & \mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-1), H^1(X,\mathbb{Z})) \\ & \searrow & \downarrow \\ & & \mathrm{Pic}(X)(\mathbb{Q}) \end{array}$$

S. Bloch formulated the idea, that for a good theory of an abelian category mixed motives the horizontal arrow in the top line should become surjective provided the set of points removed is large enough. The vertical arrow is well defined and should be an isomorphism.

We will also allow subvarieties $Y \subset X$ which are singular, we should have some control over the singularities. For instance the case that Y is a divisor with normal crossings should be accepted.

We modify our construction slightly. We define the derived sheaf $j_*(\mathbb{Z})$. We choose an injective resolution of our sheaf $\mathbb{Z}$ on U

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathbb{Z} & \rightarrow & 0 & \rightarrow & 0 & \rightarrow \\ & & \downarrow & & & & & \\ 0 & \rightarrow & J^0 & \rightarrow & J^1 & \rightarrow & J^2 & \rightarrow \end{array}$$

the complex of sheaves

$$j_*(\mathbb{Z}) := 0 \rightarrow j_*(J^0) \rightarrow j_*(J^1) \rightarrow j_*(J^2) \rightarrow$$

we restrict this sheaf to Y and hope that we can show that

$$H^\bullet(Y, j_*(\mathbb{Z}))$$

is a mixed motive. I think that this has been proved by Deligne in his papers Weil II and Hodge I-III.. Furthermore we hope that can can identify the kernel and the image of

$$\delta : H^\bullet(Y, j_*(\mathbb{Z})) \rightarrow H_c^{\bullet+1}(U,\mathbb{Z})$$

as mixed motives, i.e. we can find certain projectors obtained from correspondences which cut out these kernels and cokernels. I do not think that there is a general theorem which asserts this, so it has to be decided in the given concrete case. If we can do this we again get exact sequences

$$0 \rightarrow H^n_!(U,\mathbb{Z}) \rightarrow H^n(U,\mathbb{Z}) \rightarrow \ker(\delta : H^n(Y, j_*(\mathbb{Z})) \rightarrow H_c^{n+1}(U,\mathbb{Z})) \rightarrow 0.$$

Now the mixed motives will have longer weight filtrations, because $H^\bullet(Y, j_*(\mathbb{Z}))$ has a weight filtration with many different weights $\geq n$.

We get a second mixed motive, if we consider the cohomology with compact supports, namely

$$0 \to \mathrm{koker}(H^{n-1}(X,\mathbb{Z}) \to H^{n-1}(Y, j_*\mathbb{Z})) \to H^n_c(U,\mathbb{Z}) \to H^n_!(U,\mathbb{Z}) \to 0.$$

At this point it is not clear what it means that we have exact sequences of mixed motives. But in any case we can look at the different realizations of these motives and then we get exact sequences in the category $\mathcal{M}_{\mathcal{B}-\lceil\mathcal{R}\langle}$ and $\mathcal{M}_{\mathrm{Gal}}$ and this are abelian categories.

We briefly discuss an example. We may for instance remove three lines in general position from $X = \mathbb{P}^2$, i.e. we consider

$$U = \mathbb{P}^2 \setminus (l_0 \cup l_1 \cup l_2) = \mathbb{P}^2 \setminus \Delta \xrightarrow{\sim} \mathbb{G}_m \times \mathbb{G}_m$$

Then $U = \mathbb{G}_m \times \mathbb{G}_m$, and the Künneth-formula yields

$$H^\bullet_c(U,\mathbb{Z}) = H^2_c(U,\mathbb{Z}) \oplus H^3_c(U,\mathbb{Z}) \oplus H^4_c(U,\mathbb{Z}) = \mathbb{Z}(0) \oplus \mathbb{Z}(-1)^2 \oplus \mathbb{Z}(-2).$$

For the cohomology without supports we get

$$H^\bullet(U,\mathbb{Z}) = H^0(U,\mathbb{Z}) \oplus H^1(U,\mathbb{Z}) \oplus H^2(U,\mathbb{Z}) = \mathbb{Z}(0) \oplus \mathbb{Z}(-1)^2 \oplus \mathbb{Z}(-2).$$

Hence the map $H^\bullet_c(U,\mathbb{Z}) \to H^\bullet(U,\mathbb{Z})$ is zero and this yields short exact sequences

$$0 \to H^\bullet(U,\mathbb{Z}) \to H^\bullet(\Delta, i^*(j_*(\mathbb{Z})) \to H^{\bullet+1}_c(U,\mathbb{Z}) \to 0$$

The computation of the cohomology sheaves $R^\bullet(i^*(j_*(\mathbb{Z}))$ becomes a little bit more complicated, but we can easily compute the E^2 terms $H^p(\Delta, R^q(i^*(j_*(\mathbb{Z}))$ and get

$$H^n(\Delta, i^*(j_*(\mathbb{Z})) = \begin{cases} \mathbb{Z}(0) & \text{for n=0} \\ \mathbb{Z}(-1)^2 \oplus \mathbb{Z}(0) & \text{for n=1} \\ \mathbb{Z}(-2) \oplus \mathbb{Z}(-1)^2 & \text{for n=2} \\ \mathbb{Z}(-2) & \text{for n=3} \end{cases},$$

1.5 Construction principles

Now we give a vague outline how we may extend our construction principles to construct certain objects, which may be called mixed motives. These principles will be applied in concrete situations.

We may consider subvarieties Y which are singular, an interesting case is when Y is a divisor with normal crossings. We may also replace the system of coefficients $\mathbb{Z}$ by something more complicated namely a motivic sheaf $\mathcal{F}$ on U. These motivic sheaves are obtained as follows. We may for instance have a smooth, projective morphism $\pi : Z \to U$. Then the cohomology $R^\bullet\pi_*(\mathbb{Z})$ provides such a motivic sheaf. It may happen that certain correspondences of this morphism define idempotents on $R^\bullet\pi_*(\mathbb{Z})$. In this case the cohomology of $R^\bullet\pi_*(\mathbb{Z})$ decomposes into a direct sum, the summands again define motivic sheaves. Finally we may extend a motivic sheaf $\mathcal{F}$ from U to X. This may be done by requiring support conditions for the extensions. If for instance Y is the disjoint union of to subschemes $Y = Y_1 \cup Y_2$, then we may extend to the points in Y_1 by taking the direct image without support conditions and to Y_2 by taking compact supports. (See the construction of Anderson motives in [Ha-Eis] and also later in this paper.) Then we get certain sheaves $\mathcal{F}^\#$ on X and we consider their cohomology $H^\bullet(X,\mathcal{F}^\#)$. These objects will be "mixed motives." We have to take care that these mixed motives still have cohomological realizations, they must have Betti-de-Rham realizations which are mixed Hodge-structures and the ℓ-adic realisations must be modules for the Galois group. Of course we must be aware that we encounter incredibly complicated objects. These mixed motives have very long weight filtration with with many different weights.

But if we are lucky then we can find correspondences, i.e. finite to finite correspondences $T \subset X \times X$ which respect the subset U and then also the subscheme Y. We have the two projections $p_1, p_2 : T \to X$. If we now have a motivic sheaf $\mathcal{F}^\#$ on X and the resulting mixed motive $H^\bullet(X,\mathcal{F}^\#)$ (or a piece cut out by an idempotent) then we get a morphism $[T] : H^\bullet(X,\mathcal{F}^\#) \to H^\bullet(T, p_1^*(\mathcal{F}^\#))$ Now we have $H^\bullet(T, p_1^*(\mathcal{F}^\#)) = H^\bullet(X, R^\bullet p_{2*}(p_1^*(\mathcal{F}^\#))$ and now we hope or assume that we have a natural morphism $\phi : R^\bullet p_{2*}(p_1^*(\mathcal{F}^\#)) \to \mathcal{F}^\#$. Then it is clear that the pair (T,ϕ) induces an endomorphism

$$[T,\phi] : H^\bullet(X,\mathcal{F}^\#) \to H^\bullet(X,\mathcal{F}^\#).$$

These endomorphisms induce endomorphisms in the realizations we can consider the ring of endomorphism generated by these correspondences. An element q in this ring is called an idempotent if it induces an idempotent in any of the realizations.

Then it is tempting to decompose

$$H^\bullet(X,\mathcal{F}^\#) = H^\bullet(X,\mathcal{F}^\#)[q=1] \oplus H^\bullet(X,\mathcal{F}^\#)[q=0],$$

we do not know what the individual summands are. But we consider them as mixed motives and we know that they have cohomological realizations.

If we look at examples of these $H^\bullet(X,\mathcal{F}^\#)$ then we see that they become very large, they have weight filtrations with many steps and the dimensions of the cohomology groups become very large.

We hope to find projectors which cut out summands in our mixed motives $H^\bullet(X,\mathcal{F}^\#)$. For instance we can try to construct mixed motives which have only two steps in the weight filtration and where the filtration steps are Tate motives, i.e. we want construct extension of Tate motives of the form

$$\mathcal{X} = \{0 \longrightarrow \mathbb{Z}(0) \longrightarrow X \longrightarrow \mathbb{Z}(-n-1) \longrightarrow 0\}, \qquad (MT)$$

we write $\mathcal{X} \in \mathrm{Ext}^1_{\mathcal{M},\mathcal{M}}(\mathbb{Z}(-n-1), \mathbb{Z}(0))$. Such objects have been constructed in [Ha-Eis], these are the Anderson motives. In the second volume of [Ha-Eis] we will extend this construction of Anderson-motives to other groups.

A remark :*These mixed motives are Grothendieck motives and as far as I understand we do not know whether there is a abelian category of mixed Grothendieck motives. There are constructions the derived category of mixed motives and an abelian subcategory $\mathcal{MT}$ of mixed Tate motives (over number fields), but it is by no means clear whether we can consider the objets above as elements $\mathcal{X} \in \mathrm{Ext}^1_{\mathcal{M},\mathcal{T}}(\mathbb{Z}(-n-1), \mathbb{Z}(0))$. The point is that the above condition*

$q = q^2$ for projectors has to be true in the cohomological realizations. In the architecture of a category $\mathcal{MM}$ the equivalence relation one puts on cycles is much finer.

But anyway our objects above $\mathcal{X}$ have a Betti-de-Rham realization $\mathcal{X}_{B-dRh}$ and etale realizations $\mathcal{X}_{et,\ell}$ and these realizations are objets in abelian categories. Therefore it makes sense to attach the extension classes

$$\mathcal{X}_{B-dRh} \in \mathrm{Ext}^1_{B-dRh}(\mathbb{Z}(-n-1), \mathbb{Z}(0)), \mathcal{X}_{et,\ell} \in \mathrm{Ext}^1_{et,\ell}(\mathbb{Z}_\ell(-n-1), \mathbb{Z}_\ell(0)) \quad (2)$$

to these objects. We discuss these Ext^1 *groups in section (1.7).*

1.6 Some constructions of mixed Tate motives

We may also do the following. Let k be an arbitrary field of characteristic zero. As above we remove the triangle Δ from $\mathbb{P}^2$. Now we pick points $Q_i \in l_i$, these point should be different from the intersection points of the lines $P_c = l_a \cap l_b$. We get a second triangle Δ_2 whose sides are the lines passing through the pairs of points Q_i, Q_j. We blow up the three points Q_i, we get a surfaces X. The triangle Δ_1 can be viewed as a subscheme of X, the inverse image of Δ_2 is a hexagon $\tilde{\Delta}_2$ inside of X. Each line of the triangle Δ_1 meets intersects the hexagon in two points.

We put $V = X \setminus \tilde{\Delta}_2 \cap \Delta_1$ and we introduce the notation

$$j_2 : V \hookrightarrow X \setminus \Delta_1 \xrightarrow{j_1} X.$$

On X we define the sheaf $\mathbb{Z}^\# = j_{1,*}j_{2,!}(\mathbb{Z})$. Now I hope that the cohomology $H^2(X, \mathbb{Z}^\#)$ is a very interesting Tate motive which has a three step filtration

$$0 \subset \mathbb{Z}(0) \subset M \subset H^2(X, \mathbb{Z}^\#),$$

where $M/\mathbb{Z}(0) \xrightarrow{\sim} \mathbb{Z}(-1), H^2(X, \mathbb{Z}^\#)/M = \mathbb{Z}(-2)$. (This hope is supported by some tentative computations). Furthermore I hope that

$$0 \to \mathbb{Z}(0) \to M \to \mathbb{Z}(-1) \to 0$$

is a Kummer motive, hence it corresponds to a number $t \in k^\times$. The other quotient $H^2(X, \mathbb{Z}^\#)/\mathbb{Z}(0)$ is also a Kummer-motive $\otimes\mathbb{Z}(-1)$. This Kummer-motive should be given by the number $1 - t \in k^\times$. The number t should correspond to the position of the third point $Q_2 \in l_2$. We denote this motive by $\mathcal{M}_{x,1-x}$.

Such a motive is of course not in $\mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-n-1), \mathbb{Z}(0))$, but we may form "framed" direct sums

$$framed(\bigoplus_{i=1}^{r} \mathcal{M}_{x_i,1-x_i}).$$

If now $\sum x_i \wedge (1-x_i) = 0$ in $\Lambda^2 k^\times$ then we may hope that we can change the basis in $M/\mathbb{Z}(0) = \oplus\mathbb{Z}(-1)e_i$ in such a way that $0 \to \mathbb{Z}(0) \to M_i \to \mathbb{Z}(-1)e_i \to 0$ splits for $i = 1, \ldots, [r/2]$ and $0 \to M_i/\mathbb{Z}(0) \to H^2(X, \mathbb{Z}^\#)_i \to \mathbb{Z}(-2) \to 0$ splits for $i = [r/2] + 1, \ldots, r$.

This seems to indicate that in some sense (??)

$$framed(\bigoplus_{i=1}^{r} \mathcal{M}_{x_i,1-x_i}) = \{0 \to \mathbb{Z}(0) \to X \to \mathbb{Z}(-2) \to 0\} \oplus \mathbb{Z}(-2)^r.$$

1.7 Extensions

Let us assume, that we produced extensions $\mathcal{X}$, which are sequences

$$0 \longrightarrow \mathbb{Z}(0) \longrightarrow X \longrightarrow \mathbb{Z}(-n-1) \longrightarrow 0.$$

We consider their realizations:

The Betti realization X_B is a free $\mathbb{Z}$-module which sits in an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow X_B \longrightarrow \mathbb{Z} \longrightarrow 0.$$

We have an involution F_∞ on X_B which acts by 1 on the left copy of $\mathbb{Z}$ and by $(-1)^{n+1}$ on the right. The extremal modules have canonical generators, in other words as modules they are equal to $\mathbb{Z}$.

The *de-Rham realization* yields is an exact sequence of $\mathbb{Q}$ vector spaces

$$0 \longrightarrow \mathbb{Q}(0) \longrightarrow X_{DR} \longrightarrow \mathbb{Q}(-n-1) \longrightarrow 0$$

together with a descending filtration

$$\begin{array}{ccccccc} F^0X_{DR} & \supset & F^1X_{DR} & = \cdots = & F^{n+1}X_{DR} & \supset & F^{n+2}X_{DR} = 0 \\ & & & & \downarrow & & \\ & & & & F^{n+1}\mathbb{Q}(-n-1) = \mathbb{Q}. & & \end{array}$$

where the downwards arrow is an isomorphism.

We have a comparison isomorphism between the two exact sequences

$$I : X_B \otimes \mathbb{C} \longrightarrow X_{DR} \otimes \mathbb{C}.$$

an this comparison isomorphism satisfies

$$I \circ c_B \circ F_\infty = c_{dRh} \circ I$$

where the $c_{??}$ is always the action of the complex conjugation on the coefficients.
We want to consider these objects $(X_B, F_\infty, X_{DR}, F, I)$ as objects of an abelian category $B-dRh$ it is related to the category of mixed Hodge-structures.

Finally we have the *p-adic realizations.* For each prime p we have an action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $X_B \otimes \mathbb{Z}_p$ and we get an exact sequence

$$0 \longrightarrow \mathbb{Z}_p(0) \longrightarrow X_B \otimes \mathbb{Z}_p \longrightarrow \mathbb{Z}_p(-n-1) \longrightarrow 0$$

and this action is unramified outside of $S \cup \{p\}$.
Again we notice that the comparison isomorphism gives us canonical generators in $\mathbb{Z}_p(0)$ and $\mathbb{Z}_p(-n-1)$

1.7.1 The Betti- de-Rham extension class

We can associate an extension class

$$[X]_{B-dRh} \in \mathrm{Ext}^1_{B-dRh}(\mathbb{Z}(-n-1), \mathbb{Z}(0))$$

to our objects X. To do this we have to understand $\mathrm{Ext}^1_{B-dRh}(\mathbb{Z}(-n-1), \mathbb{Z}(0))$. We distinguish two cases.

In the first we assume that n even.

We know that $\mathbb{Z}(-n-1)$ has a canonical generator $1_B^{(-n-1)}$. We have a unique lift of this generator to an element $e_B^{(-n-1)} \in X_B \otimes \mathbb{Q}$ which lies in the -1 eigenspace for F_∞. We also find a unique $e_{DR}^{(-n-1)} \in F^{n+1}X_{DR} \otimes \mathbb{C}$ which maps to the image of $I(1_B^{(-n-1)})$. Then $e_B^{(-n-1)} - I^{-1}(e_{DR}^{(-n-1)})$ maps to zero in $\mathbb{C}(-n-1)$ and therefore $e_B^{(-n-1)} - I^{-1}(e_{DR}^{(-n-1)}) \in \mathbb{Z}_B(0) \otimes \mathbb{C}$.

Finally we look at the action of c_B on this class. Since F_∞ acts trivially on $\mathbb{Z}(0)$ we get from the compatibility condition

$$\begin{aligned} c_B(e_B^{(-n-1)} - I^{-1}(e_{DR}^{(-n-1)})) &= F_\infty \circ c_B(e_B^{(-n-1)} - I^{-1}(e_{DR}^{(-n-1)})) \\ &= F_\infty(e_B^{(-n-1)} - I^{-1}(c_{DR}(e_{DR}^{(-n-1)}))) = \\ &-(e_B^{(-n-1)} - I^{-1}(e_{DR}^{(-n-1)})) \end{aligned}$$

and therefore we conclude that the extension class lies in $i\mathbb{R}$. If we choose $\frac{1}{2\pi i}$ as a basis element for $i\mathbb{R}$ then we get an identification

$$\mathrm{Ext}^1_{B-dRh}(\mathbb{Z}(-n-1), \mathbb{Z}(0)) = \mathbb{R}.$$

Now we consider the case that n is odd.

Again we know that $\mathbb{Z}(-n)$ has a canonical generator $1_B^{(-n)}$. We have a non unique lift of this generator to an element $e_B^{(-n)} \in X_B \otimes \mathbb{Q}$. We find a unique $e_{DR}^{(-n)} \in F^n X_{DR} \otimes \mathbb{C}$ which maps to the image of $I(1_B^{(-n)})$. Then $1_B^{(-n)} - I^{-1}(e_{DR}^{(-n)})$ maps to zero in $\mathbb{C}(-n)$. Hence we see that $1_B^{(-n)} - I^{-1}(e_{DR}^{(-n)}) \in \mathbb{C}(0) \mod \mathbb{Z} = \mathbb{C} \mod \mathbb{Z}$.

We compute the action of c_B on this class and this time we get

$$\begin{aligned} c_B(1_B^{(-n)} - I^{-1}(e_{DR}^{(-n)})) &= F_\infty \circ c_B(1_B^{(-n)} - I^{-1}(e_{DR}^{(-n)})) \\ &= (1_B^{(-n)} - I^{-1}(e_{DR}^{(-n)})) \end{aligned}$$

and hence

$$\mathrm{Ext}^1_{B-dRh}(\mathbb{Z}(-n), \mathbb{Z}(0)) = \mathbb{R}/\mathbb{Z}.$$

Now we encounter the fundamental question: What are the classes which come from a mixed motive over $\mathbb{Q}$, in other words what is the image

$$\mathrm{Ext}^1_{\mathcal{M}}(\mathbb{Z}(-n-1), \mathbb{Z}(0)) \to \mathrm{Ext}^1_{B-dRh}(\mathbb{Z}(-n-1), \mathbb{Z}(0))?$$

Since the group on the left hand side is not really defined we may ask: How many objects of the form

$$\mathcal{X} = \{0 \longrightarrow \mathbb{Z}(0) \longrightarrow X \longrightarrow \mathbb{Z}(-n-1) \longrightarrow 0\}$$

can we find somewhere in the cohomology of an algebraic variety over $\mathbb{Q}$ and what are the possible values for their extension class in the category $B-dRh$.

The general conjectures about the connection between K-theory and the hypothetical category of mixed motives seems to suggest the following question:

The case $n > 0$ even:
Is it true that for any such object $\mathcal{X}$ the extension class

$$[\mathcal{X}_{B-dRH}] = \zeta'(-n)a(\mathcal{X}) \qquad (Ext_{B-dRh})$$

with some rational number $a(\mathcal{X})$? What are the possible denominators of $a(\mathcal{X})$, are they bounded?

The case n odd:
Is it true that for any such object $\mathcal{X}$ the extension class

$$[\mathcal{X}_{B-dRH}] \in \mathbb{Q}/\mathbb{Z}$$

in other words we get only torsion elements?

I think that we must be aware, that it is by no means clear, that our construction principles do not go beyond the construction of mixed motives constructed by K-theory, or or other known approaches to the category of mixed Tate-motives.

1.7.2 The Galois-module extension class

Now we consider the attached sequences of Galois modules

$$0 \longrightarrow \mathbb{Z}_p(0) \longrightarrow X \otimes \mathbb{Z}_p \longrightarrow \mathbb{Z}_p(-n-1) \longrightarrow 0$$

We consider exact sequences of Galois-modules $\mathbb{Z}_p \times \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-Moduln

$$0 \to \mathbb{Z}_p(0) \to X \to \mathbb{Z}_p(-n-1) \to 0.$$

We assume that n is even and $p > 2$. Then we know especially $p-1 \nmid n+1$.

Such a module provides an element X

$$\begin{aligned}[X] \in \mathrm{Ext}^1_{\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})}(\mathbb{Z}_p(-n-1), \mathbb{Z}_p(0)) = H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbb{Z}_p(n+1)) = \\ \lim_{\underset{m}{\leftarrow}} H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbb{Z}/p^m\mathbb{Z}(n+1)).\end{aligned} \qquad (3)$$

It may be helpful if we introduce the notation

$$\mathbb{Z}_p/p^m\mathbb{Z}_p(n+1) = \mu_{p^m}^{\otimes(n+1)}$$

To understand this cohomology we pass to the cyclotomic extensions $\mathbb{Q}(\zeta_{p^m})/\mathbb{Q}$ and we denote their Galoisgroups over $\mathbb{Q}$ by $\bar{\Gamma}_m$. We have the canonical isomorphism

$$\alpha : \bar{\Gamma}_m = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^m})/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/p^m\mathbb{Z})^*.$$

Our assumptions on p, n imply that we can find an $x \in (\mathbb{Z}/p^m\mathbb{Z})^*$ such that $x^{n+1} \not\equiv 1 \mod p$ and this implies that

$$H^1(\bar{\Gamma}_m, \mu_{p^m}^{\otimes(n+1)}) = H^2(\bar{\Gamma}_m, \mu_{p^m}^{\otimes(n+1)}) = 0$$

and the Hochschild-Serre spectral sequence yields an isomorphism

$$H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mu_{p^m}^{\otimes(n+1)}) \simeq H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_{p^m})), \mu_{p^m}^{\otimes(n+1)})^{\bar{\Gamma}_m}.$$

Since the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_{p^m}))$-module $\mu_{p^m}^{\otimes(n+1)}$ is trivial we have the Kummer isomorphism

$$H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_{p^m})), \mu_{p^m}^{\otimes(n+1)})^{\bar{\Gamma}_m} \simeq ((\mathbb{Q}(\zeta_{p^m})^* \otimes \mu_{p^m}^{\otimes n})^{\bar{\Gamma}_m} =$$

$$\big(\mathbb{Q}(\zeta_{p^m})^* \otimes \mathbb{Z}/p^m\mathbb{Z}\big)(-n) = \{x | x \in \mathbb{Q}(\zeta_{p^m})^* \otimes \mathbb{Z}/p^m\mathbb{Z}, x^\sigma = x^{\alpha(\sigma)^{-n}}\}.$$

An element $\xi \in H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbb{Z}_p(n+1))$ is a sequence of elements

$$\xi = (\ldots, \xi_m, \ldots)$$

which satisfy

$$\xi_m \in (\mathbb{Q}(\zeta_{p^m})^* \otimes \mu_{p^m}^{\otimes n}))^{\Gamma_m} = H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mu_{p^m}^{\otimes(n+1)})$$

and are mapped to eachother by the transition map: The homomorphism $\mathbb{Z}/p^{m+1}\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$ yields the projective system and consequently we get a homomorphism

$$(\mathbb{Q}(\zeta_{p^{m+1}})^* \otimes \mu_{p^{m+1}}^{\otimes n})^{\Gamma_{m+1}} \to (\mathbb{Q}(\zeta_{p^m})^* \otimes \mu_{p^m}^{\otimes n})^{\Gamma_m}$$

and we have to identify this homomorphism. An easy computation yields

$$N_{\mathbb{Q}(\zeta_{p^{m+1}})/\mathbb{Q}(\zeta_{p^m})}(\xi_{m+1}) = \xi_m^p,$$

and we conclude that our homomorphism is given by

$$N_{m+1,m}^{1/p} : (\mathbb{Q}(\zeta_{p^{m+1}})^* \otimes \mu_{p^{m+1}}^{\otimes n})^{\Gamma_{m+1}} \to (\mathbb{Q}(\zeta_{p^m})^* \otimes \mu_{p^m}^{\otimes n})^{\Gamma_m}.$$

With respect to these homomorphisms we have

$$H^1(\mathbb{Q}, \mathbb{Z}_p(n+1)) = \lim_{\leftarrow}(\mathbb{Q}(\zeta_{p^m})^* \otimes \mu_{p^m}^{\otimes n}))^{\Gamma_m}$$

We consider the restriction

$$H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbb{Z}_p(n+1)) \to H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p), \mathbb{Z}_p(n+1)).$$

Since $\bar{\Gamma}_m = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^m}/\mathbb{Q}) = \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^m}/\mathbb{Q}_p)$ our considerations above using the Hochschild -Serre sequence also apply to this situation: we may replace $\mathbb{Q}$ by $\mathbb{Q}_p$. Let

$$U_m^{(1)} \subset \mathcal{O}^* = (\mathbb{Z}_p[\zeta_{p^m}])^*$$

be the group of units congruent $1 \mod \mathfrak{p}$, then

$$(U_m^{(1)} \otimes \mu_{p^m}^{\otimes n}))^{\Gamma_m} \subset (\mathbb{Q}_p(\zeta_{p^m})^* \otimes \mathbb{Z}/p^m\mathbb{Z})(-n).$$

The projective limit

$$\lim_{\substack{\leftarrow \\ n}} (U_m^{(1)} \otimes \mathbb{Z}/p^m)(-n) = V_p(-n).$$

and I claim that $V(-n)$ is a free $\mathbb{Z}_p$ -module of rank one .

(The Hilbert symbol yields a pairing

$$(U_m^{(1)} \otimes \mathbb{Z}/p^m\mathbb{Z}) \times (U_m^{(1)} \otimes \mathbb{Z}/p^m\mathbb{Z}) \to \frac{1}{p^m}\mathbb{Z}/\mathbb{Z}(1)$$

and a generator in $V_m(-n)$ yields a homomorphism

$$\delta_{n+1} : U_m^{(1)} \otimes \mathbb{Z}/p^m\mathbb{Z} \to \frac{1}{p^m}\mathbb{Z}/\mathbb{Z}$$

which satisfies

$$\delta_{n+1}(u^\sigma) = \alpha(\sigma)^{n+1}\delta_{n+1}(u)$$

and this must be the Coates-Wiles homomorphism. (Washington, Chap. 13). This has to be clarified.)

Now we assume that n is even. We introduce the subring $\mathcal{O}_m = \mathbb{Z}[\frac{1}{p}, \zeta_{p^m}]$. We define elements $\zeta_{p^m}^{\otimes n} := \zeta_{p^m} \otimes \zeta_{p^m} \cdots \otimes \zeta_{p^m} \in \mu_p^{\otimes n}$ and we construct the Soulé elements in $(\mathcal{O}_m^* \otimes \mu_{p^m}^{\otimes n})^{\Gamma_m}$:

$$c_{n,m}(p) = \prod_{\substack{(a,p)=1 \\ a \mod p^m}} (1 - \zeta_{p^m}^a)^{a^n} \otimes \zeta_{p^m}^{\otimes n}$$

and $N_{m+1,m}^{1/p}(c_{n,m+1}) = c_{n,m}$. We get an element in the projective limit

$$c_n(p) = (\ldots, c_{n,m}(p), \ldots) \in H^1(\mathbb{Q}, \mathbb{Z}_p(n+1)).$$

These elements $c_{n,m}(p)$ and $c_n(p)$ do not depend on the choice of the primitive p^m-th root of unity, they are canonical elements in $H^1(\mathbb{Q}, \mathbb{Z}_p(n+1))$.

If we send the elements $c_p(n)$ into the local Galois cohomology then they become a multiple of a generator e_n

$$c_n(p) = \ell_p(n) \cdot e_n$$

with $\ell_p(n) \in \mathbb{Z}_p$. I think, that the results on p-adic L-functions and Iwasawas results imply (Washington 13.56) that

$$\ell_p(n) = \zeta_p(n+1) \mod \mathbb{Z}_p^*$$

where

$$\zeta_p(n+1) = \lim_{\alpha \to \infty} \zeta(n+1-(p-1)p^\alpha).$$

I also assume at this point that $\zeta_p(n+1) \neq 0$. In any case it is not clear whether $\lim\limits_{\leftarrow}(\mathcal{O}_m^* \otimes \mu_{p^m}^{\otimes n})^{\Gamma_m}$ has a non zero image in $H^1(\mathbb{Q}, \mathbb{Z}_p(n+1))$ without such an assumption.

1.8 The p-adic extension classes

We assume that $n > 0$, even and that we have constructed a mixed motive over $\mathbb{Q}$

$$\mathcal{X} = \{0 \to \mathbb{Z}(0) \to X \to \mathbb{Z}(-n-1) \to 0\}$$

it provides an extension class

$$[\mathcal{X}]_p \in H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbb{Z}_p(n+1))$$

for all primes p. We can say that these Galois-modules form a "compatible system" of representations for the Galois group, because all the Galois-modules come from the same global object. The Soulé elements allow us to formulate an assertion which makes the above statement precise.

We ask:

Let $\mathcal{X}$ be a mixed motive as above. Is it true that for all primes p

$$[\mathcal{X}]_p = c_p(n)^{a(\mathcal{X})} \qquad (Ext_{p-etale})$$

where $a(\mathcal{X})$ is the same number which occurred in our formula for the Hodge-de Rham extension class? Perhaps it is more reasonable to ask the weaker question whether this relation holds for the image of $[\mathcal{X}]_p$ in $H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p), \mathbb{Z}_p(n+1))$

It is the aim of this work to present constructions of mixed Tate motives, which "live" inside the cohomology of Shimura varieties, for which the Betti-de-Rham extension class satisfies the above relation (Ext_{B-dRh}). But we are not able to show, that the Galois-module extension class also satisfies $Ext_{p-etale}$.

This raises the next question:

1.9 Do exotic mixed Tate motives exist?

We call a mixed Tate motive $\mathcal{X}$ exotic if one of the above assertions fails. Clearly there are various qualities of being exotic. In the course of these notes we will construct mixed motives for which we know that (Ext_{B-dRh}) is true, but where we do not know how to prove ($Ext_{p-etale}$). In my lecture notes volume "Eisenstein Kohomologie und die Konstruktion gemischter Motive" I gave the construction of the Anderson motives. For these motives I computed the Hodge-de-Rham extension class and showed that in fact they are of the predicted form $a(\mathcal{X})\zeta'(-n)$.

I also hope that I can compute the p-adic extension classes, so both questions can be answered positively for these motives. In their paper "Dirichlet motives via modular curves. " Ann. Sci. cole Norm. Sup. (4) 32 (1999) A. Huber and G. Kings prove that the p-adic classes have the right form. But they use K-theory and I do not understand completely how the object in K-theory can be compared to the object which I construct.

I also will construct mixed Anderson motives $\mathcal{X}(f)$ for the symplectic group GSp_2, they will be labeled by classical elliptic modular forms f. Again I will compute the Hodge-de-Rham extension class, the computation of the p-adic extension class seems to be even much more difficult.

On the other hand in the volume "The 1-2-3 of modular forms" I pointed out that the non existence of exotic mixed Tate-motives gives us a hint to prove the conjectural congruences in my article " A congruence between a Siegel and an elliptic modular form."

Of course the non existence of exotic Tate motives would be an interesting theorem in arithmetic algebraic geometry. But it seems to me also interesting that it has such concrete consequences which can be checked in examples.

I also will construct mixed Anderson motives for the symplectic group GSp_2, they will be labeled by classical elliptic modular forms. Again I will compute the Hodge-de-Rham extension class, the computation of the p-adic extension class seems to be much more difficult.

1.9.1 Final remarks ?

At this point I am always a little bit confused. Experts in K-theory keep telling me that the answer to both questions is clearly yes, i.e. there are no exotic mixed Tate motives. They say that this follows if we work in the category of mixed Tate-motives over number fields, which has been constructed by Voevodsky. In this category the computation of the extension groups $\mathrm{Ext}_{\mathcal{MM}/\mathbb{Q}}$ is reduced to the computation of K-groups of number fields, which has been done by Borel.

But it is not clear to me whether the mixed Tate motives which I constructed above can be viewed as objects in Voevodsky's category.

We add some further speculations: Assume that we have always the above relation between the Hodge-de-Rham extension class to the p-adic extension class. If this would be the case then we would have a tool to attack the question concerning the denominators of $a(\mathcal{X})$. If for instance $\zeta_p(n+1-(p-1)) \not\equiv 0 \mod p$ we have seen that the image of $c_p(n)$ in $H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p), \mathbb{Z}_p(n+1))$ is a generator and therefore we can not have a p in the denominator of $a(\mathcal{X})$. But if the $\zeta-$value is zero $\mod p$ we get that $c_p(n)$ is locally at p a $p-$ th power. In such a case the Vandiver conjecture would still imply that $c_p(n)$ itself is not a p-th power.

But independently of the validity of the Vandiver conjecture we can consider we can pick an n as above and a prime p. If $\zeta_p(n+1) \neq 0$ then we would know that the p-denominator in $a(\mathcal{X})$ is at most $p^{\delta_p(n)}$.

But of course at this point we cannot say anything for a single value $n > 0$. We have the remarkable result by Christophe Soulé that for $p > v(n)$ the Vandiver conjecture is true for the n-component which means that we know that $c_p(n,1) \neq 0$ and this is equivalent to $c_p(n)$ is not a p-th power. Hence we have to check a finite number of primes. hence we see that we can bound the denominators and we have to check a finite set of primes. But this finite set is so enormously large that we can not check them all.

We can also speculate what happens if n is odd. Then we have seen that the p adic extension classes are all zero. This applies only to the projective limit, the cohomology on the finite level may be non zero. This supports the idea that the Hodge-de Rham classes should be torsion classes in this case.

1.9.2 Anderson motives and Euler systems ??

These Anderson motives depend on the choice of an auxiliary prime p_0 and they have very little ramification (See my Lecture Notes volume and section 2. in this paper) Hence we see that for almost all primes p the image of $[\mathcal{X}(p_0)]_p$ in $H^1(\mathbb{Q},\mu_{p^m}^{\otimes(n+1)})$ will be in the kernel of r'. The number $a([\mathcal{X}(p_0)])$ is even very close to one in this case (See LN) and $c_p(n)$ is related to the value of the p-adic L-function. Now the method of Euler systems (Thaine, Kolyvagin and Rubin) yields that the index of the subgroup $\mathbb{Z}_p[\mathcal{X}_n(p_0)_p] \subset H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p),\mathbb{Z}_p(n+1))$ provides also an estimate for the cokernel of the map

$$r : H^1(\mathbb{Q},\mu_{p^m}^{\otimes(n+1)}) \to H^1(\mathbb{Q}_p,\mu_{p^m}^{\otimes(n+1)}) \oplus \bigoplus_{\ell\neq p} H^1_{ram}(\mathbb{Q}_\ell,\mu_{p^m}^{\otimes(n+1)}).$$

The Euler systems give us a tool, starting from $[\mathcal{X}(p_0)]_p$ which is rather unramified, to construct new classes which are unramified everywhere except at a specifically selected prime ℓ and which are definitely ramified at this selected prime ℓ. This supplies a tool to bound the cokernel of r.

What I propose is to use the construction of Anderson motives at this point. We can construct Anderson motives $\mathcal{X}(p_0\ell_1)$, $\mathcal{X}(p_0\ell_1\ell_2)\dots$ which then would pick up ramification at ℓ_1,ℓ_2 and would help us to make the image large and this would also allow to get the same or similar results.

2 Kummer-Anderson motives

2.1 Curves over $\mathbb{Q}$ and the construction of mixed Kummer-Tate-motives

We start from a smooth absolutely irreducible curve $S/\mathrm{Spec}(\mathbb{Q})$. Let $\mathfrak{d} = \sum n_i x_i$ be a divisor which is of degree zero- i.e. $\sum n_i = 0$ and which is rational over $\mathbb{Q}$. Let us denote the support of the divisor by Σ_∞, let Σ_0 a second set of (at least 2) rational points. We assume $\Sigma_0 \cap \Sigma_1 = \emptyset$. We have inclusions

$$S \setminus \Sigma_0 \setminus \Sigma_\infty \xrightarrow{i_0} S \setminus \Sigma_\infty \xrightarrow{i_\infty} S. \tag{4}$$

On $S \setminus \Sigma_0 \setminus \Sigma_\infty$ we have the constant (motivic) sheaf $\mathbb{Z}$ and we extend it by support conditions to a sheaf $\mathbb{Z}^\#$ on S. In a first step we extend it to a sheaf $i_{0,!}(\mathbb{Z})$ on $S\setminus\Sigma_\infty$, i.e. we extend it by zero in the points in Σ_0. We extend $i_{0,!}(\mathbb{Z})$ to a sheaf on S by taking the derived direct image $i_{\infty,*}(i_{0,!})(\mathbb{Z})$.

We study the mixed motive $H^1(S,\mathbb{Z}^\#)$. Of course this is equal to the relative cohomology $H^1(S\setminus\Sigma_\infty,\Sigma_0,\mathbb{Z})$, perhaps a less scaring object. Let us put $U = S\setminus\Sigma_\infty$, we have the sheaf $\mathbb{Z}_U$ on U, we get an exact sequence of sheaves on U:

$$0 \to i_{0,!}(\mathbb{Z})|_U \to \mathbb{Z}_U \to \oplus_{y_i}\mathbb{Z}_{y_i} \to 0 \tag{5}$$

and hence

$$0 \to \mathbb{Z}^\# \to i_{\infty,*}(\mathbb{Z}_U) \to \oplus_{y_i}\mathbb{Z}_{y_i} \to 0 \tag{6}$$

We are now in the situation described in section 1.4 but we have removed two sets Σ_0,Σ_∞. Let us assume for simplicity that the points x_i, y_i are rational.

From our two exact sequences we get maps

$$\begin{array}{c} H^0(S,\mathbb{Z}) \to \oplus_{y_i} H^0(y_i,\mathbb{Z}) \xrightarrow{\delta_0} H^1(S,\mathbb{Z}^{\#}) \xrightarrow{\Theta} H^1(S,i_{\infty,*}(\mathbb{Z}_U)) \to 0 \\ \to H^1(S,i_{\infty,*}(\mathbb{Z}_U)) \xrightarrow{r} \oplus_{x_i} H^1(\overset{\bullet}{\mathcal{N}}\, x_i,\mathbb{Z}) \to H^2_c(S,\mathbb{Z}) \to 0 \end{array} \tag{7}$$

Both sequences are exact, the $H^0(y_i,\mathbb{Z}) = \mathbb{Z}(0)$, the $\overset{\bullet}{\mathcal{N}}\, x_i$ are punctured discs and $H^1(\overset{\bullet}{\mathcal{N}}\, x_i,\mathbb{Z}) = \mathbb{Z}(-1)$.

(If the points y_i or x_i are not rational we have to include a twist by the Galois group) Let us now return to our divisor $\mathfrak{d} = \sum n_i x_i$, we assume that it is primitive, that means it is not divisible or the n_i are coprime. We can view $\mathfrak{d}$ as a rational point on the Jacobian, in section 1.4 we explained how we can interpret $\mathfrak{d}$ as an extension class in $\mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-1),H^1(S,\mathbb{Z}))$. The cohomology groups of the punctured discs $H^1(\overset{\bullet}{\mathcal{N}}\, x_i,\mathbb{Z}) = \mathbb{Z}e_i$ and hence our divisor is also a cohomology class $e_{\mathfrak{d}} \in H^1(\overset{\bullet}{\mathcal{N}}\, \Sigma_\infty,\mathbb{Z})$.

Now we make the assumption that this extension class is zero, at least after we tensor by $\mathbb{Q}$. This means that $\mathfrak{d}$ is a torsion point. We invoke Abels theorem. First we look at the comparison isomorphism

$$H^1_{B-dRh}(S\setminus\Sigma_\infty)\otimes\mathbb{C} = H^1(S(\mathbb{C})\setminus\Sigma_\infty,\mathbb{Z})\otimes\mathbb{C} \tag{8}$$

The splitting of the extension class means that we can find a differential form $\omega_{\mathfrak{d}} \in H^0(S,\Omega^1_{\log}(\Sigma_\infty))$ which has residue n_i at x_i and for any homology class $[\mathfrak{z}] \in H_1(S(\mathbb{C})\setminus\Sigma_\infty,\mathbb{Z})$ which is represented by a cycle $\mathfrak{z}$ we have

$$\int_{\mathfrak{z}} \omega_{\mathfrak{d}} \in 2\pi i\mathbb{Q} \tag{9}$$

Hence the cohomology class $\frac{1}{2\pi i}[\omega_{\mathfrak{d}}] \in H^1(U(\mathbb{C}),\mathbb{Q})$ its restriction to the boundary is $e_{\mathfrak{d}} \in \bigoplus_{x_i} H^1(\overset{\bullet}{\mathcal{N}}\, x_i,\mathbb{Z})$. It is clear that the differential $\omega_{\mathfrak{d}}$ is unique, because $\omega_{\mathfrak{d}} - \omega'_{\mathfrak{d}}$ would be holomorphic and hence it cannot have periods in $2\pi i\mathbb{Q}$.

Hence we see that $\mathbb{Q}\omega_{\mathfrak{d}} \in H^1_{dRh}(U)$ and $\frac{1}{2\pi i}[\omega_{\mathfrak{d}}] \in H^1(U(\mathbb{C}),\mathbb{Q})$ provide a copy of $\mathbb{Q}[e_{\mathfrak{d}}] = \mathbb{Q}(-1) \in H^1(U,\mathbb{Q})$. We have our map Θ in (7) and

$$\mathcal{K}[\omega_{\mathfrak{d}}] = \Theta^{-1}(\mathbb{Q}[e_{\mathfrak{d}}])$$

will the Betti-de-Rham realization of a mixed Kummer-Tate motive

$$0 \to (\bigoplus_{y_i} H^0(y_i,\mathbb{Q})) \to \mathcal{K}[\omega_{\mathfrak{d}}] \to \mathbb{Q}[e_{\mathfrak{d}}] \to 0 \tag{10}$$

The term on the left is a direct sum of copies of $\mathbb{Q}(0)$, if we chose a non zero linear form

$$\kappa : (\bigoplus_{y_i} H^0(y_i,\mathbb{Q}))/H^0(S,\mathbb{Q}) \to \mathbb{Q}(0)$$

and divide in (10) by the kernel of κ then we get a mixed motive

$$< \kappa, e_{\mathfrak{d}} > \in \mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Q}(-1),\mathbb{Q}(0)). \tag{11}$$

which is given by the exact sequence

$$0 \to\to (\bigoplus_{y_i} H^0(y_i,\mathbb{Q}))/\ker(\kappa) \to \mathcal{K}[\kappa,\omega_{\mathfrak{d}}] \to \mathbb{Q}[e_{\mathfrak{d}}] \to 0 \tag{12}$$

We believe that

$$\mathrm{Ext}^1_{\mathcal{M}\mathcal{M}}(\mathbb{Q}(-1),\mathbb{Q}(0)) = \mathbb{Q}^\times \otimes \mathbb{Q} = \bigoplus_{p:primes} \mathbb{Q} \tag{13}$$

and hence the question arises to compute $< \kappa, e_{\mathfrak{d}} > \in \mathbb{Q}^\times \otimes \mathbb{Q}$.

To do this we go back to integral cohomology. We look at the denominator $\Delta(\mathfrak{d})$ of $\frac{1}{2\pi i}[\omega_{\mathfrak{d}}]$, this is the smallest positive integer for which

$$\Delta(\mathfrak{d})\frac{1}{2\pi i}[\omega_{\mathfrak{d}}] \in H^1(U(\mathbb{C}),\mathbb{Z}), \tag{14}$$

we will see in a moment that this is also the order of the torsion point $\mathfrak{d}$. Let us consider the differential $\tilde{\omega}_{\mathfrak{d}} = \Delta(\mathfrak{d})\omega_{\mathfrak{d}}$, we fix a base point and write down a meromorphic function on $S(\mathbb{C})$

$$F_{\mathfrak{d}}(z) = e^{\int_{z_0}^{z} \tilde{\omega}_{\mathfrak{d}}} \tag{15}$$

where we integrate along any path avoiding Σ_∞ and joining z_0 and z. This is what Abel did to construct a meromorphic function with divisor $\mathfrak{d}$.

If we fix two points $y_j, y_k \in \Sigma_0$ then this defines a linear form $\kappa_{j,k}$: $(\bigoplus_{y_i} H^0(y_i,\mathbb{Z}))/H^0(S,\mathbb{Z}) \to \mathbb{Z}(0)$ namely $\kappa_{j,k} : (n_1,n_2,\dots,n_t) \mapsto n_j - n_k$. Then it seems to be clear that

$$< \kappa_{j,k}, e_{\mathfrak{d}} > = \frac{F_{\mathfrak{d}}(x_j)}{F_{\mathfrak{d}}(x_k)} \in \mathbb{Q}^\times \tag{16}$$

Of course this equality is disputable since the isomorphism (13) is disputable. We refer to the remark at the end of section (1.5) and discuss the extension classes

$$\begin{gathered} \frac{F_{\mathfrak{d}}(x_j)}{F_{\mathfrak{d}}(x_k)} \in H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}),\mathbb{Z}_\ell(1)) = \mathbb{Q}_\ell^\times \\ < \kappa_{j,k}, e_{\mathfrak{d}} >_{B-dRh} \in \mathrm{Ext}^1_{B-dRh}(\mathbb{Z}(-1),\mathbb{Z}(0)) \end{gathered} \tag{17}$$

It seems to me that here we are in the only situation where the computation of the extension class in Galois cohomology is easy, we get the tautological answer

$$< \kappa_{j,k}, e_{\mathfrak{d}} >_{et;\ell} = \frac{F_{\mathfrak{d}}(x_j)}{F_{\mathfrak{d}}(x_k)}$$

We compute the class $< \kappa_{j,k}, e_{\mathfrak{d}} >_{B-dRh}$. To do this we apply the recipes from section (1.7.1). Our generators $1_B^{(-1)}$, resp. $1_{DR}^{(-1)}$ are the classes $e_{\mathfrak{d}}$ resp. $2\pi i e_{\mathfrak{d}}$. Hence we see that the class $e_{DR}^{(-1)}$ in section (1.7.1) is given by $\omega_{\mathfrak{d}}$. The class $e_B^{(-1)}$ has the same restriction to the boundary, but it has to satisfy $F_\infty(e_B^{(-1)}) = -e_B^{(-1)}$. Therefore it becomes clear that

$$e_B^{(-1)} = \frac{1}{4\pi i}(\omega_{\mathfrak{d}} - F_\infty(\omega_{\mathfrak{d}}))$$

here we view the comparison isomorphism as being the identity. Hence

$$e_B^{(-1)} - e_{DR}^{(-1)} = -\frac{1}{4\pi i}(\omega_{\mathfrak{d}} + F_\infty(\omega_{\mathfrak{d}})) \in (\bigoplus_{y_i} H^0(y_i, \mathbb{Q}))/H^0(S, \mathbb{Q}) \tag{18}$$

Now we remember that $\tilde{\omega}_{\mathfrak{d}} = \Delta(\mathfrak{d})\omega_{\mathfrak{d}} = \frac{dF_{\mathfrak{d}}}{F_{\mathfrak{d}}}$. To identify the class

$$-\frac{1}{\Delta(\mathfrak{d})4\pi i}(\tilde{\omega}_{\mathfrak{d}} + F_\infty(\tilde{\omega}_{\mathfrak{d}})) = -\frac{1}{\Delta(\mathfrak{d})4\pi i}\frac{dF_{\mathfrak{d}}}{F_{\mathfrak{d}}}$$

we have to evaluate it on relative 1-cycles from y_i to y_k, therefore we get

$$< \kappa_{j,k}, e_{\mathfrak{d}} >_{B-dRh} = \frac{1}{\Delta(\mathfrak{d})}(\log(F_{\mathfrak{d}}(x_j) - \log(F_{\mathfrak{d}}(x_k)) \in \mathbb{R} \tag{19}$$

remember that we have chosen $\frac{1}{2\pi i}$ as basis element for $i\mathbb{R}$. The difference of logarithms is of the form

$$\log(F_{\mathfrak{d}}(x_j) - \log(F_{\mathfrak{d}}(x_k) = \sum n(p, \mathfrak{d}) \log(p) \tag{20}$$

where of course $n(p, \mathfrak{d})$ is the power of p in $\frac{F_{\mathfrak{d}}(x_j)}{F_{\mathfrak{d}}(x_k)}$. Hence we get

$$< \kappa_{j,k}, e_{\mathfrak{d}} >_{B-dRh} = \frac{1}{\Delta(\mathfrak{d})}(\sum n(p, \mathfrak{d}) \log(p)) \tag{21}$$

2.2 The cohomology of $\overset{\bullet}{\mathcal{N}}\,\Sigma$

Our group G is $G = \mathrm{Gl}_2/\mathrm{Spec}(\mathbb{Z})$, let B, T be the standard Borel subgroup and the standard diagonal torus. Let $K_\infty = \mathrm{SO}(2)Z^0(\mathbb{R}) \subset \mathrm{Gl}_2(\mathbb{R})$. We choose an open compact subgroup $K_f \subset \prod \mathrm{Gl}_2(\mathbb{Z}_p)$ and consider the space

$$\mathcal{S}^G_{K_f} = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f \tag{22}$$

where $X = \mathbb{H}_+ \cup \mathbb{H}_- = \mathrm{Gl}_2(\mathbb{R})/\mathrm{SO}(2)Z^0(\mathbb{R}) = \mathrm{Gl}_2(\mathbb{R})/K_\infty$ is the union of an upper and a lower half plane. We assume that the determinant homomorphism $\det K_f \to \hat{\mathbb{Z}}^\times$ is surjective. Therefore we have only one connected component and our space is of the form

$$\mathcal{S}^G_{K_f} = \Gamma\backslash\mathbb{H}_+ \tag{23}$$

where $\Gamma = \mathrm{Gl}_2(\mathbb{Q}) \cap \mathbb{K}_f$. The curve has a canonical model $Y_{K_f}/\mathbb{Q}$ and if we add the cusps, we get a projective curve $X_{K_f} = Y_{K_f} \cup \Sigma$ where $\Sigma/\mathbb{Q}$ is finite. We want to apply the considerations in the previous section to these curves.

The set

$$\Sigma(\mathbb{C}) = U(\mathbb{A}_f)T(\mathbb{Q})\backslash\pi_0(\mathrm{Gl}_2(\mathbb{R})) \times G(\mathbb{A}_f)/K_f = U(\hat{\mathbb{Z}})T(\mathbb{Z})\backslash\{\pm 1\} \times \mathrm{Gl}_2(\hat{\mathbb{Z}})/K_f.$$

The factor $\{\pm 1\}$ corresponds to the two connected components $\mathbb{H}_\pm$ since $T(\mathbb{Q})$ acts transitively on $\{\pm 1\}$ we also have

$$\Sigma(\mathbb{C}) = U(\mathbb{A}_f)T^+(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f$$

On this set we have an action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which we describe next.

We define the *coarse* set of cusps

$$\bar{\Sigma}(\mathbb{C}) = B(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K_f = \mathbb{P}^1(\hat{\mathbb{Z}})/K_f,$$

we have the obvious projection

$$p_B : \Sigma(\mathbb{C}) \to \bar{\Sigma}(\mathbb{C}). \tag{24}$$

On $\Sigma(\mathbb{C})$ we have an action of $\pi_0(\mathbb{R}) \times T(\mathbb{A}_f)/T(\mathbb{Q})$ by multiplication from the left and the orbits of this action are exactly the fibers of p_B. Our Shimura datum which gives us the $\mathbb{Q}$ structure on Y_{K_f} yields a Shimura datum for the boundary scheme $\Sigma/\mathbb{Q}$ and this is the homomorphism $t \mapsto \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$. Hence we get an action of $\{\pm 1\} \times \mathbb{G}_m(\mathbb{A}_f)/\mathbb{Q}^\times$ on $\Sigma(\mathbb{C})$. For any $y \in \bar{\Sigma}$ we there is an open subgroup $\mathfrak{U}_y$ which acts trivially on $b_P(y)$, and hence we get an action of $\{\pm 1\} \times \mathbb{G}_m(\mathbb{A}_f)/\mathbb{Q}^\times \mathfrak{U}_y$. The reciprocity homomorphism $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \{\pm 1\} \times \mathbb{G}_m(\mathbb{A}_f)/\mathbb{Q}^\times \mathfrak{U}_y$ gives us the Galois action on $p_B^{-1}(y)$, this gives us the explicit description of our scheme $\Sigma/\mathbb{Q}$,

In the next step give an explicit description of the cohomology $H^1(\overset{\bullet}{\mathcal{N}}\,\Sigma, \mathbb{Z})$. We describe the tubular neighborhood of a point $x \in \Sigma(\mathbb{C})$. The point $y = p_B(x) \in \mathbb{P}^1(\hat{\mathbb{Z}})/K_f$ can be represented by a rational point $\tilde{y} \in \mathbb{P}^1(\mathbb{Q})$ which defines a Borel subgroup $B_{\tilde{y}}/\mathrm{Spec}(\mathbb{Z})$. This Borel subgroup yields the Farey disks $D(c, \tilde{y}) \subset \mathbb{H}_+$ (See chap.2, 1.7) and the punctured tubular neighborhood $\overset{\bullet}{\mathcal{N}}\,x$ of x is

$$\overset{\bullet}{\mathcal{N}}\,x = \Gamma_{U,y}\backslash D(c, \tilde{y}) \tag{25}$$

where $\Gamma_{U,\tilde{y}} = U_{\tilde{y}}(\mathbb{Q}) \cap K_f \subset U_{\tilde{y}}(\mathbb{Z})$. (The stabilizer of $\tilde{y}$ is of course $B_y(\mathbb{Z})$ but since we selected the $+$ component at the infinite place it reduces to the subgroup $B^+(\mathbb{Z}) = \Gamma_{U,\tilde{y}}(\mathbb{Z}) \times \{-\mathrm{Id}\}$ of elements with determinant $+1$ and the central element $-\mathrm{Id}$ acts trivially) The subgroup $\Gamma_{U,y}$ is of finite index $d(\tilde{y})$ in $U_y(\mathbb{Z})$. (Spitzenbreite.)

Now it is clear that $\overset{\bullet}{\mathcal{N}}\,x$ is a punctured disk and since this disk has a complex structure we get

$$H^1(\overset{\bullet}{\mathcal{N}}\,x, \mathbb{Z}) = \mathbb{Z}e_x \tag{26}$$

where e_x is the positive generator of the cohomology. We need to represent the cohomology classes by a closed 1-forms.

Let $\alpha \in X^*(T)$ be the simple positive root, let $|\alpha|$ the induced character $T(\mathbb{A})/T(\mathbb{Q}) \to \mathbb{R}^\times$. It has the infinite component $|\alpha|_\infty : T(\mathbb{R}) \to \mathbb{R}^\times_{>0}$ and the finite component $|\alpha|_f : T(\mathbb{A}_f) \to \mathbb{Q}^\times$. We consider the induced representation $\mathrm{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})} |\alpha|_\infty$. We also introduce the space of functions

$$\mathcal{C}_\alpha = \{f : \{\pm 1\} \times U(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K_f \to \mathbb{Q} | f(t\underline{g}) = \alpha(t) f(\underline{g}) \text{ for all } t \in T(\mathbb{Q})\} \tag{27}$$

this is of course a sophisticated description of the space of functions on $\Sigma(\mathbb{C})$.

Then we get an inclusion

$$\mathrm{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})}|\alpha|_\infty \otimes \mathcal{C}_\alpha \hookrightarrow \mathcal{A}(U(\mathbb{A})T(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \tag{28}$$

and from this we get a homomorphism

$$\mathrm{Hom}_{K_\infty}(\Lambda^1(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})}|\alpha|_\infty) \otimes \mathcal{C}_\alpha \to \Omega^1(\overset{\bullet}{\mathcal{N}}\ \Sigma) \tag{29}$$

We have the standard decomposition of the induced representation into K_∞ types

$$\mathrm{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})}|\alpha|_\infty = \bigoplus_{n:n \text{ even}} \mathbb{C}\psi_n \tag{30}$$

where

$$\psi_n(\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix}\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}) = |\alpha|_\infty(t)(\cos(\phi) + i\sin(\phi))^n \tag{31}$$

The Lie algebra $\mathfrak{g}$ is a direct sum $\mathfrak{g}_\mathbb{R} = \mathbb{R}H \oplus \mathbb{R}E_+ \oplus \mathbb{R}E_- \oplus \mathbb{R}Z_0$, where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, Z_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{32}$$

Let $V = E_+ + E_-$, we define the elements

$$P_+ = H + i \otimes V,\ P_- = H - i \otimes V \in \mathfrak{g} \otimes \mathbb{C} \tag{33}$$

which form a basis of $\mathfrak{g}/\mathfrak{k} \otimes \mathbb{C}$. Under the adjoint action of K_∞ they are eigenvectors

$$\mathrm{Ad}(\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix})P_\pm = (\cos(\phi) + i\sin(\phi))^{\pm 2}P_\pm \tag{34}$$

Let us denote by $P_\pm^\vee$ the dual elements in $(\mathfrak{g}/\mathfrak{k})^\vee \otimes \mathbb{C}$ then

$$\omega_{\mathrm{hol}} = 4i\ P_+^\vee \otimes \psi_2,\ \bar{\omega}_{\mathrm{hol}} = -4i\ P_-^\vee \otimes \psi_{-2} \tag{35}$$

form a basis of $\mathrm{Hom}_{K_\infty}(\Lambda^1(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})}|\alpha|_\infty)$. As in [Eis] 4.3.2 we define the element

$$\omega_{\mathrm{top}} = \frac{1}{2}(\omega_{\mathrm{hol}} + \bar{\omega}_{\mathrm{hol}})$$

and we know that ω_{hol} and ω_{top} define the same class in $H^1(\mathfrak{g}, K_\infty, \mathrm{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})}|\alpha|_\infty)$ and this class is a basis of this one dimensional cohomology group.

If now $\omega_\infty = \omega_{\mathrm{hol}}$ or ω_{top}, then following is clear (See also section 2.4.3 below) :

Proposition 2.1. *If we choose an element in $h \in \mathcal{C}_\alpha$ then the image of $\omega_\infty \otimes h$ under the map (29) represents the cohomology class*

$$[\omega_\infty \otimes h] = \sum_{x \in \Sigma(\mathbb{C})} \frac{h(x)}{d(y)} e_x \tag{36}$$

where of course $y = p_B(x)$.

2.3 The Eisenstein lift

Let us now assume that $\sum_x \frac{h(x)}{d(y)} = 0$, then we know that $[\omega_\infty \otimes h]$ is in the image of the restriction map

$$H^1(\mathcal{S}^G_{K_f}, \mathbb{Z}) \xrightarrow{\text{res}} H^1(\overset{\bullet}{\mathcal{N}}\,\Sigma(\mathbb{C}), \mathbb{Z}). \tag{37}$$

The theory of Eisenstein series gives us a lifting of this class in the cohomology of the boundary to a class $\text{Eis}(\omega_\infty \otimes h, 0) \in H^1(\mathcal{S}^G_{K_f}, \mathbb{C})$. We recall the construction of this Eisenstein class. Let $\gamma = \alpha/2$ be the dominant weight. We twist the induced representation by $|\gamma|^s$ and get an embedding

$$(\text{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})} |\alpha|_\infty \otimes \mathcal{C}_\alpha) \otimes |\gamma|^s \hookrightarrow \mathcal{A}(U(\mathbb{A})T(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \tag{38}$$

and by summation we get an embedding

$$\begin{array}{c} \text{Eis} : (\text{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})} |\alpha|_\infty \otimes \mathcal{C}_\alpha) \otimes |\gamma|^s \hookrightarrow \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \\ \text{Eis}(f, s)(\underline{g}) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} f(\gamma \underline{g}) \end{array}. \tag{39}$$

This series converges for $\Re(s) > 0$ and extends to a meromorphic function in the entire complex plane. From this we get a homomorphism

$$\begin{array}{c} \text{Hom}_{\text{SO}(2)}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \text{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})} |\alpha|_\infty) \otimes \mathcal{C}_\alpha \otimes |\gamma|^s \to \text{Hom}_{\text{SO}(2)}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \\ \text{Eis} : \omega \otimes h \otimes |\gamma|^s \mapsto \text{Eis}(\omega \otimes h, s) \end{array} \tag{40}$$

It follows from the theory of Eisenstein series, that this operator has a simple pole at $s = 0$. But we have

Proposition 2.2. *For all $\omega \in \text{Hom}_{\text{SO}(2)}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \text{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})} |\alpha|_\infty)$ and all h for which $\sum_x \frac{h(x)}{d(y)} = 0$, the function $\text{Eis}(\omega \otimes h, s)$ is holomorphic at $s = 0$. If $\omega_\infty \otimes h$ is as in (37) and $\sum_x \frac{h(x)}{d(y)} = 0$ then we get for the restriction of the cohomology class res$([\text{Eis}(\omega_\infty \otimes h, 0)]) = [\omega_\infty \otimes h]$.*

It follows from the Manin-Drinfeld principle that the class

$$[\text{Eis}(\omega_\infty \otimes h, 0)] \in H^1(\mathcal{S}^G_{K_f}, \mathbb{Q}).$$

Now we are ready to apply our consideration in the previous section to construct the mixed Kummer-Anderson motives. We divide the set $\Sigma = \Sigma_\infty \cup \Sigma_0$ (both subsets should contain at least two elements). We choose a divisor $\mathfrak{d} = \sum n_i x_i$ of degree zero which is supported on Σ_∞. We have the element $h_\mathfrak{d} \in \mathcal{C}_\alpha$ which is given by $h(x_i) = n_i b(y_i)$. Then we can choose for our element $\omega_\mathfrak{d}$ the Eisenstein differential

$$\omega_\mathfrak{d} = \text{Eis}(\omega_{\text{hol}} \otimes h_\mathfrak{d}) \in H^0(\mathcal{S}^G_{K_f}, \Omega_{\log}(\Sigma_\infty)) \tag{41}$$

Then we multiply by the denominator such that

$$\Delta(\mathfrak{d})\text{Eis}(\omega_{\text{hol}} \otimes h_\mathfrak{d}) \in H^1(\mathcal{S}^G_{K_f}, \mathbb{Z}) \tag{42}$$

and we find a meromorphic function $F_{\mathfrak{d}}$ on $X_{K_f}(\mathbb{C})$ such that

$$\Delta(\mathfrak{d})\mathrm{Eis}(\omega_{\mathrm{hol}} \otimes h_{\mathfrak{d}}) = \frac{dF_{\mathfrak{d}}}{F_{\mathfrak{d}}} \tag{43}$$

We have to discuss the field of definition of this function and to evaluate it at the points of Σ_0 to compute the resulting mixed Kummer motives.

To attain this goal we have to compute some intertwining operators.

2.4 Local intertwining operators

2.4.1 The local operator at a finite place

I want to resume a computation which appears already in my Lecture Notes volume [Eis] on p. 128. Unfortunately it contains some errors which are not relevant for the following, but which have to be corrected.

Our underlying group is still $G = \mathrm{Gl}_2$. We consider a specific representations of $G(\mathbb{Q}_p)$. The root α provides a character

$$|\alpha|_p = |\gamma|_p^2 \quad : \quad B(\mathbb{Q}_p) \longrightarrow \mathbb{Q}^*$$

$$|\alpha|_p \qquad : \quad \begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \longmapsto |\frac{t_1}{t_2}|.$$

The induced representation $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} |\alpha|_p$ admits a $G(\mathbb{Q}_p)$ invariant linear map to $\mathbb{C}$, the kernel is the Steinberg module. Here we take the algebraic induction and not the unitary induction.

Let $\eta : T(\mathbb{F}_p) \to \mathbb{C}^\times$ be a character, we extend η to a character on $T(\mathbb{Q}_p)$ by putting $\eta(\begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix}) = 1$. Let $\varphi = |\alpha|_p \eta$. By $\varphi^{(1)}$ we denote the restriction of φ to the subgroup $\left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right\}$. We identify this subgroup to $\mathbb{Q}_p^\times$ by $t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$. Then we get

$$\varphi_p^{(1)} = |t|_p^2 \frac{\eta_1(t_1)}{\eta_2(t_2)}$$

We consider the induced module

$$I_{\varphi_p \otimes |\gamma|^s} = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \varphi_p \otimes |\gamma|^s$$

Let $K_{1,1}(p) \subset GL_2(\mathbb{Z}_p)$ the subgroup of matrices

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \;\middle|\; \gamma \equiv 0 \mod p, \alpha \text{ and } \delta \equiv 1 \mod p \right\} \subset GL_2(\mathbb{Z}_p).$$

We have a basis $f_{0,\eta}$, $f_{\infty,\eta}$ for the vector space

$$I_{\phi_s}^{K_{1,1}(p)},$$

these are the same elements as in [Eis] p. 107, but we interchange the subscripts $_0$ and $_\infty$ so it may be better to write the definition again: The restriction to $GL_2(\mathbb{Z}_p)$ is given by

$$f_{\eta,0}: \begin{cases} \begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix} \longrightarrow \eta_1(t_1)\eta_2(t_2) \\ \begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \longrightarrow 0 \end{cases}$$

$$f_{\eta,\infty}: \begin{cases} \begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix} \longrightarrow 0 \\ \begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \longrightarrow \eta_1(t_1)\eta_2(t_2) \end{cases}$$

Let η^w the conjugate under the non trivial Weyl group element, then we define the standard intertwining operator

$$T_p^{\,\mathrm{st}} = I_{\varphi_p|\gamma|^s} \longrightarrow I_{\eta^w|\gamma|^{-s}}$$

which is given by the integral

$$T_p^{\,\mathrm{st}}(f) = \int_{U(\mathbb{Q}_p)} f(wug)du \tag{44}$$

where $\mathrm{vol}(U(\mathbb{Z}_p)) = 1$.

We want to see what happens to the elements $f_{\eta,0}$, $f_{\eta,\infty}$. We have an obvious identification

$$\left(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \varphi \otimes s\right)^{K_{1,1}(p)} = \left(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} w \cdot \varphi_p \otimes (-s)\right)^{K_{1,1}(p)}$$

where we identify $f_{\eta,0} = f_{\eta^w,0} = f_0$, $f_{\eta,\infty} = f_{\eta^w,\infty} = f_\infty$.

We compute the integrals and during this process we correct the mistakes in [Eis]. We have

$$\begin{aligned} T_p^{\mathrm{int}}(f_0) &= a(s)f_0 + b(s)f_\infty \\ T_p^{\mathrm{int}}(f_\infty) &= c(s)f_0 + d(s)f_\infty \end{aligned} \tag{45}$$

where

$$a(s) = T_p^{\mathrm{int}}(f_0)(e)\ ,\ b(s) = T_p^{\mathrm{int}}(f_0)(w)\ ,\ c(s) = T_p^{\mathrm{int}}(f_\infty)(e)\ ,\ d(s) = T_p^{\mathrm{int}}(f_\infty)(w)$$

Now

$$T_p^{\mathrm{int}}(f_0)(e) = \int_{\mathbb{Z}_p} f_0(wu)du + \sum_{\nu=1}^{\infty}(p^\nu - p^{\nu-1}) \int_{\mathbb{Z}_p^\times} f_0\left(w \begin{pmatrix} 1 & \epsilon p^{-\nu} \\ 0 & 1 \end{pmatrix}\right) d\epsilon.$$

(The volume factor $p^\nu - p^{\nu-1}$ is missing in [Eis], p. 128, $\mathrm{vol}_{d\epsilon}(\mathbb{Z}_p^\times) = 1$.) The first summand is zero. To compute the second term we decompose

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \epsilon p^{-\nu} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\epsilon p^{-\nu} \end{pmatrix} = \begin{pmatrix} \epsilon p^{\nu} & -1 \\ 0 & \epsilon^{-1} p^{-\nu} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -\epsilon p^{\nu} & -1 \end{pmatrix} \tag{46}$$

and hence

$$f_0\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \epsilon p^{-\nu} \\ 0 & 1 \end{pmatrix} \right) = f_0\left(\begin{pmatrix} \epsilon p^{\nu} & -1 \\ 0 & \epsilon^{-1} p^{-\nu} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -\epsilon p^{\nu} & -1 \end{pmatrix} \right) = \eta^{(1)}(\epsilon) p^{-\nu(s+2)}.$$

Taking the factor $p^\nu - p^{\nu-1}$ into account we get

$$a(s) = \left(1 - \frac{1}{p}\right) \frac{p^{-1-s}}{1 - p^{-1-s}}$$

if $\eta^{(1)} = 1$ and zero otherwise.

Now we show that $b(s) = \frac{1}{p}$, to do this we compute

$$T_p^{\mathrm{int}}(f_0)(w) = \int_{\mathbb{Z}_p} f_0(wuw)du + \sum_{\nu=1}^{\infty} (p^\nu - p^{\nu-1}) \int_{\mathbb{Z}_p^\times} f_0\left(w \begin{pmatrix} 1 & \epsilon p^{-\nu} \\ 0 & 1 \end{pmatrix} w \right) d\epsilon.$$

In the first integral the integrand is zero for $u \notin p\mathbb{Z}_p$ and equal to 1 on $p\mathbb{Z}_p$ hence the integral gives $\frac{1}{p}$. To compute the second term we decompose

$$w \begin{pmatrix} 1 & \epsilon p^{-\nu} \\ 0 & 1 \end{pmatrix} w = \begin{pmatrix} -1 & 0 \\ \epsilon p^{-\nu} & -1 \end{pmatrix} = \begin{pmatrix} p^{\nu} & * \\ 0 & p^{-\nu} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & -p^{\nu} \end{pmatrix}, \tag{47}$$

where $\gamma \not\equiv 0 \mod p$. Therefore the value of f_0 is zero on this matrix and the infinite sum is zero. We summarize and put $\eta^{(1)}(p) = 1$ if $\eta^{(1)} = 1$ and 0 else, then

$$T_p^{\mathrm{st}}(f_0) = \left(1 - \frac{1}{p}\right) \frac{\eta^{(1)}(p) p^{-1-s}}{1 - \eta^{(1)}(p) p^{-1-s}} \; f_0 + \frac{1}{p} f_\infty \tag{48}$$

The same computation for f_∞ yields

$$T_p^{\mathrm{st}}(f_\infty)(e) = \int_{\mathbb{Z}_p} f_\infty(wu)du + \sum_{\nu=1}^{\infty} (p^\nu - p^{\nu-1}) \int_{\mathbb{Z}_p^\times} f_\infty\left(w \begin{pmatrix} 1 & \epsilon p^{-\nu} \\ 0 & 1 \end{pmatrix} \right) d\epsilon. \tag{49}$$

The integrand in the first integral is equal to one hence the value of the integral is one. The value of the second integral is zero. Hence

$$T_p^{\mathrm{st}}(f_\infty)(e) = 1 \tag{50}$$

The second summation is a little bit more delicate.

$$T_p^{\,\mathrm{st}}(f_\infty)(w) = \int_{\mathbb{Z}_p} f_\infty(wuw)du + \sum_{\nu=1}^{\infty}(p^\nu - p^{\nu-1})\int_{\mathbb{Z}_p^\times} f_\infty\left(w\begin{pmatrix}1 & \epsilon p^{-\nu}\\ 0 & 1\end{pmatrix}w\right)d\epsilon. \tag{51}$$

For $u \in \mathbb{Z}_p$ we have

$$f_\infty(wuw) = \begin{cases} 0 & \text{if } u \in p\mathbb{Z}_p \\ \eta^{(1)}(-u^{-1}) & \text{if } u \in \mathbb{Z}_p^\times \end{cases} \tag{52}$$

Therefore

$$\int_{\mathbb{Z}_p} f_\infty(wuw)du = \eta^{(1)}(p)(1 - \frac{1}{p}) \tag{53}$$

For the second term we use again (47) and get

$$f_\infty\left(\begin{pmatrix}p^\nu & * \\ 0 & p^{-\nu}\end{pmatrix}\begin{pmatrix}\alpha & \beta \\ \gamma & -p^\nu\end{pmatrix}\right) = \eta^{(1)}(\epsilon)p^{-(s+2)\nu} \tag{54}$$

Summing up yields

$$(1-\frac{1}{p})\sum_{\nu=1}^{\infty} p^\nu \int_{\mathbb{Z}_p^\times} f_\infty\left(w\begin{pmatrix}1 & \epsilon p^{-\nu}\\ 0 & 1\end{pmatrix}w\right)d\epsilon = (1-\frac{1}{p})\frac{\eta^{(1)}(p)p^{-1-s}}{1-\eta^{(1)}(p)p^{-1-s}} \tag{55}$$

Adding up the two contributions we get

$$T_p^{\,\mathrm{st}}(f_\infty)(w) = (1-\frac{1}{p})\frac{1}{1-\eta^{(1)}(p)p^{-1-s}} \tag{56}$$

and we get

$$T^{\,\mathrm{st}}(f_\infty) = f_0 + (1-\frac{1}{p})\frac{1}{1-\eta^{(1)}(p)p^{-1-s}}f_\infty \tag{57}$$

Let us summarize:

$$T_p^{\,\mathrm{st}}(f_0) = \begin{cases} \left(1-\frac{1}{p}\right)\frac{p^{-1-s}}{1-p^{-1-s}}\; f_0 + \frac{1}{p}f_\infty & \text{if } \eta^{(1)} = 1 \\ \frac{1}{p}f_\infty & \text{else} \end{cases}$$
$$T_p^{\,\mathrm{st}}(f_\infty) = \begin{cases} f_0 + (1-\frac{1}{p}\Big)\frac{1}{1-p^{-1-s}}f_\infty & \text{if } \eta^{(1)} = 1 \\ f_0 & \text{else} \end{cases} \tag{58}$$

In the meanwhile we may have forgotten our character φ and its restriction $\varphi^{(1)}$ to $\mathbb{Q}_p^\times$. We have the local Euler factor attached to $\mathfrak{e}$ it is defined by

$$L_p(\mathfrak{e}, s) = \frac{1}{1-\eta^{(1)}(p)\mathfrak{e}(p)^{-s}} \tag{59}$$

so it is equal to 1 if $\mathfrak{e}$ is non trivial.

In view of the computation of the constant term of the Eisenstein series we define the local intertwining operator by the relation

$$T_p^{\mathrm{st}}(f) = \frac{L_p(\mathfrak{e}, s+1)}{L_p(\mathfrak{e}, s+2)} T_p^{\mathrm{loc}}(f) \tag{60}$$

so the numerator of the $L-$ ratios in front swallows the pole at $s = -1$ in the operator T^{st}, the operator T^{st} is holomorphic for $\Re(s) \geq -2$. We get the following expression for the operator T^{loc}

$$\begin{aligned} T_p^{\mathrm{loc}}(f_0) &= \begin{cases} \left(1-\frac{1}{p}\right)\frac{p^{-1-s}}{1-p^{-2-s}}\ f_0 + \frac{1}{p}\frac{1-p^{-1-s}}{1-p^{-2-s}} f_\infty & \text{if } \eta^{(1)} = 1 \\ \frac{1}{p} f_\infty & \text{else} \end{cases} \\ T_p^{\mathrm{loc}}(f_\infty) &= \begin{cases} \frac{1-p^{-1-s}}{1-p^{-2-s}}\ f_0 + (1-\frac{1}{p}\Big)\frac{1}{1-p^{-2-s}} f_\infty & \text{if } \eta^{(1)} = 1 \\ f_0 & \text{else} \end{cases} \end{aligned} \tag{61}$$

2.4.2 The unramified case

Now we assume for the moment that η is trivial. In this case the function $h_0 = f_0 + f_\infty$ is the spherical function, it is invariant under the maximal compact subgroup $G(\mathbb{Z}_p)$. In this the local operator is so designed that independently of the value of s

$$T_p^{\mathrm{loc}}(h_0) = T_p^{\mathrm{loc}}(f_0 + f_\infty) = f_0 + f_\infty = h_0. \tag{62}$$

If we choose $s = 0$, then T_p^{loc} annihilates the element $f_0 - \frac{1}{p} f_\infty$ and hence we get in accordance with what we know that T_p^{loc} is not injective if $s = 0$. Its kernel is the Steinberg module. We see that the final result in [Eis] on p. 128 is correct up to a factor $\frac{1}{p_0}$ on the right hand side.

If we keep the parameter s and consider the image of $f_0 - \frac{1}{p} f_\infty$ under T_p^{loc} then we get

$$\begin{gathered} T_p^{\mathrm{loc}}(f_0 - \tfrac{1}{p} f_\infty) = \\ \left(1-\tfrac{1}{p}\right)\tfrac{1}{1-p^{-2-s}}\ f_0 + \tfrac{1}{p}\tfrac{1-p^{-1-s}}{1-p^{-2-s}} f_\infty\ - \tfrac{1}{p}\left(\tfrac{1-p^{-1-s}}{1-p^{-2-s}}\ f_0 + \left(1-\tfrac{1}{p}\right)\cdot \tfrac{1}{1-p^{-2-s}}\ f_\infty)\right) = \\ \tfrac{p^{-1-s}-p^{-1}}{1-p^{-2-s}} f_0 + \tfrac{p^{-2}-p^{-2-s}}{1-p^{-2-s}} f_\infty. \end{gathered} \tag{63}$$

This expression vanishes at $s = 0$ and we obtain as expansion in the variable s

$$\frac{p^{-1-s}-p^{-1}}{1-p^{-2-s}} f_0 + \frac{p^{-2}-p^{-2-s}}{1-p^{-2-s}} f_\infty = s\left(\frac{-p}{p^2-1}\log p\ f_0 + \frac{1}{p^2-1}\log p\ f_\infty\right) + \dots \tag{64}$$

For a later reference we note that for $s = 0$ we have

$$T_p^{\mathrm{loc}}(f_0 + \frac{1}{p} f_\infty) = \frac{2}{p+1} h_0 \tag{65}$$

2.4.3 The local operator at the infinite place.

The same kind of integral also defines an intertwining operator

$$\begin{aligned} T_\infty^{\text{st}} : \text{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})} |\alpha|_\infty \otimes |\gamma|_\infty^s &\to \text{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})} |\gamma|_\infty^{-s} \\ f(g) &\mapsto \int_{U(\mathbb{R})} f(wug) du \end{aligned} \tag{66}$$

where the measure is the standard Lebesgue measure. We have seen that

$$\text{Hom}_{K_\infty}(\Lambda^1(\mathfrak{g}/\mathfrak{k}), \text{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})} |\alpha|_\infty \otimes |\gamma|_\infty^s) = \mathbb{C}\omega_{\text{hol}} \oplus \mathbb{C}\bar{\omega}_{\text{hol}}, \tag{67}$$

We evaluate at $s = 0$. We know that for $s = 0$ the intertwining operator maps all K_∞ types ψ_n to zero except ψ_0 and

$$T^{\text{ st}}(\psi_0) = \pi \psi_0 \tag{68}$$

(See slzweineu.pdf 4.1.5.) The kernel is the discrete series representation $\mathcal{D}_2$, and $\text{Hom}_{K_\infty}(\Lambda^1(\mathfrak{g}/\mathfrak{k}), \mathcal{D}_2) = H^1(\mathfrak{g}, K_\infty, \mathcal{D}_2) = \mathbb{C}[\omega_{\text{hol}}] \oplus \mathbb{C}[\bar{\omega}_{\text{hol}}]$. We consider the complexes

$$\begin{array}{ccc} \text{Hom}_{K_\infty}(\Lambda^0(\mathfrak{g}/\mathfrak{k}), \text{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})} |\alpha|_\infty) \to & \text{Hom}_{K_\infty}(\Lambda^1(\mathfrak{g}/\mathfrak{k}), \text{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})} |\alpha|_\infty)) & \to \\ & \uparrow & \\ & \text{Hom}_{K_\infty}(\Lambda^1(\mathfrak{g}/\mathfrak{k}), \mathcal{D}_2) & \end{array} \tag{69}$$

We have $\mathfrak{g}/\mathfrak{k} = \mathbb{C}H \oplus \mathbb{C}E_+ = \mathfrak{t} \oplus \mathfrak{u}$ and the Delorme isomorphism gives us

$$\text{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \text{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})} |\alpha|_\infty) = \text{Hom}_{K_\infty^T}(\Lambda^\bullet(\mathfrak{t} \oplus \mathfrak{u}), \mathbb{C}|\alpha|_\infty) \tag{70}$$

A straightforward computation shows

$$(\omega_{\text{hol}} - \bar{\omega}_{\text{hol}})(H) = 4i \text{ and } (\omega_{\text{hol}} - \bar{\omega}_{\text{hol}})(E_+) = 0 \tag{71}$$

and this implies that this implies that for the element $\psi_0 \in \text{Hom}_{K_\infty}(\Lambda^0(\mathfrak{g}/\mathfrak{k}), \text{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})} |\alpha|_\infty)$ we get

$$-i \, d\psi_0 = \frac{1}{2}(\omega_{\text{hol}} - \bar{\omega}_{\text{hol}}) \tag{72}$$

and hence ω_{hol} and $\bar{\omega}_{\text{hol}}$ represent the same class in $H^1(\mathfrak{g}, K_\infty, \text{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})} |\alpha|_\infty)$.

We see easily that

$$(\omega_{\text{hol}} + \bar{\omega}_{\text{hol}})(H) = 0 \text{ and } (\omega_{\text{hol}} + \bar{\omega}_{\text{hol}})(E_+) = 2 \tag{73}$$

and hence we get

$$\omega_{\text{hol}}(E_+) = \omega_{\text{top}}(E_+) = 1 \tag{74}$$

and this justifies the assertion in proposition 2.1.

The author admits that keeping track of the powers of 2 as factors was painful for him. But they are important.

2.5 The simplest modular Anderson-Kummer motives

Now I want to return to [Eis] and I will take up the discussion on p. 139. I consider the case $n = 0$, but now I consider two primes p_0 and ℓ. In principle I keep the notations from Kap. IV. We apply our considerations to the curve

$$Y(p_0\ell) \hookrightarrow X_0(p_0\ell).$$

We have four cusps. These cusps are obtained from the diagram

$$\begin{array}{ccc} & X_0(p_0\ell) & \\ \swarrow & & \searrow \\ X_0(p_0) & & X_0(\ell). \end{array}$$

The curves $X_0(p_0)$ (resp. $X_0(\ell)$) have the cusps $\{\infty_{p_0}, 0_{p_0}\}$ (resp. $\{\infty_\ell, 0_\ell\}$) and hence

$$X_0(p_0\ell) \setminus Y_0(p_0\ell) = \Sigma = \{\infty_{p_0}, 0_{p_0}\} \times \{\infty_\ell, 0_\ell\}.$$

We construct sheaves $\mathbb{Z}^\#$ on $X_0(p_0\ell)$ which will be obtained as extensions with support conditions of the sheaf $\mathbb{Z}$ to $X_0(p_0\ell)$. To do this we divide the set of cusps into two subsets

$$\begin{aligned} \Sigma_\infty &= \{(\infty_{p_0}, 0_\ell), (0_{p_0}, \infty_\ell)\} \\ \Sigma_0 &= \{(\infty_{p_0}, \infty_\ell), (0_{p_0}, 0_\ell)\} \end{aligned}$$

and we have

$$Y_0(p_0\ell) \xrightarrow{i_0} Y_0(p_0\ell) \cup \Sigma_0 \xrightarrow{i_1} X_0(p_0\ell).$$

We put

$$\mathbb{Z}^\# = i_{1,!} \circ i_{0,*}(\mathbb{Z})$$

and as usual we consider the cohomology $H^1(X_0(p_0\ell), \mathbb{Z}^\#)$. Our above scheme $U = Y_0(p_0\ell) \cup \Sigma_0$.

Now we choose $\mathfrak{d} = (\infty_{p_0}, 0_\ell) - (0_{p_0}, \infty_\ell)$ and $\kappa(x, y) = x - y$ and define

$$H^1_{\mathrm{Eis}}(X_0(p_0\ell), \mathbb{Q}^\#) = \mathcal{K}(\kappa, \omega_{\mathfrak{d}}) \tag{75}$$

We compute the extension class of this motive in $\mathbb{Q}^\times \otimes \mathbb{Q}$.

Now we choose

$$\tilde{\psi}_f = \bigotimes_{p \neq p_0, \ell} \psi_p \otimes \psi_{\{p_0,\ell\}},$$

where of course again the factors different from p_0, ℓ are given by the normalized spherical function and

$$\begin{aligned} \psi_{\{p_0,\ell\}} &= \tfrac{1}{2}\left(f_0^{(p_0)} - \tfrac{1}{p_0} f_\infty^{(p_0)}\right) \otimes \left(f_0^{(\ell)} + \tfrac{1}{\ell} f_\infty^{(\ell)}\right) - \\ & \tfrac{1}{2}\left(f_0^{(p_0)} + \tfrac{1}{p_0} f_\infty^{(p_0)}\right) \otimes \left(f_0^{(\ell)} - \tfrac{1}{\ell} f_\infty^{(\ell)}\right) = \\ &= \tfrac{1}{\ell} \cdot f_0^{(p_0)} \otimes f_\infty^{(\ell)} - \tfrac{1}{p_0} f_\infty^{(p_0)} \otimes f_0^{(\ell)} \end{aligned} \tag{76}$$

Then the class $[\omega_\infty \otimes \tilde{\psi}_f]$ has a non zero restriction to the cusps $(\infty_{p_0}, 0_\ell)$ and $(0_{p_0}, \infty_\ell)$ and in fact the are generators of opposite sign.

Now we know that

$$\mathrm{Eis}[\omega_\infty \otimes \tilde{\psi}_f] \in H^1(\mathcal{S}^G_{K_0(p_0\ell)}, \mathbb{Q})$$

and we want to determine the number δ which multiplies it into the integral cohomology.

Here ω_∞ is as above, we may choose for ω_∞ the holomorphic form ω_{hol} or the topological form ω_{top}. The form $\mathrm{Eis}(\omega_{\mathrm{hol}} \otimes \tilde{\psi}_f)$ will then be a holomorphic 1-form on $\mathcal{S}^G_{K_0(p_0\ell)}$ which has a first order pole with residue 1resp. -1 in the two cusps $(\infty_{p_0}, 0_\ell)$ resp. $(0_{p_0}, \infty_\ell)$ and which is holomorphic in the two other cusps. On the other hand $\mathrm{Eis}(\omega_{\mathrm{hol}} \otimes \tilde{\psi}_f)$ is the lift for the Betti-de-Rham realization. If we multiply this form by δ the residues become $\pm\delta$ and it represents an integral class. But then this form will be the logarithmic derivative of the function

$$F(z) = e^{\delta 2\pi i \int_{z_0}^{z} Eis(\omega_{\mathrm{hol}} \otimes \tilde{\psi}_f})$$

and this function has the divisor $\delta((\infty_{p_0}, 0_\ell) - (0_{p_0}, \infty_\ell))$.

Now we recall that the extension class of our mixed motive $H^1_{\mathrm{Eis}}(X_0(p_0\ell), \mathbb{Z}^{\#})$ is given by the ratio of values

$$\frac{F((\infty_{p_0}, \infty_\ell))}{F((0_{p_0}, 0_\ell))}$$

We have outlined in [Eis] how such a value can be computed.

We choose a 1-cycle $\mathfrak{z}$ joining the two points $z_0 = (\infty_{p_0}, \infty_\ell)$ and $z_1 = (0_{p_0}, 0_\ell)$ and we compute the integral

$$\int_{\mathfrak{z}} \mathrm{Eis}(\omega_{\mathrm{hol}} \otimes \tilde{\psi}_f).$$

As in [Eis] we also consider the integral $\int_{\mathfrak{z}} \mathrm{Eis}(\omega_{\mathrm{top}} \otimes \psi_f)$. If we apply complex conjugation c to this cycle and if we observe that complex conjugation map ω_{top} to $-\omega_{\mathrm{top}}$, then we see that

$$\int_{\mathfrak{z}} \mathrm{Eis}(\omega_{\mathrm{top}} \otimes \tilde{\psi}_f) = -\int_{c\mathfrak{z}} \mathrm{Eis}(\omega_{\mathrm{top}} \otimes \tilde{\psi}_f)),$$

and hence

$$2\int_{\mathfrak{z}} \mathrm{Eis}(\omega_{\mathrm{top}} \otimes \tilde{\psi}_f)) = \int_{-c\mathfrak{z}} \mathrm{Eis}(\omega_{\mathrm{top}} \otimes \tilde{\psi}_f)).$$

But now we have a closed cycle and hence we know that

$$2\delta \int_{\mathfrak{z}} \mathrm{Eis}(\omega_{\mathrm{top}} \otimes \tilde{\psi}_f) \in \mathbb{Z}.$$

We conclude that

$$\int_{\mathfrak{z}} \mathrm{Eis}(\omega_{\mathrm{top}} \otimes \tilde{\psi}_f) - \mathrm{Eis}(\omega_{\mathrm{top}} \otimes \tilde{\psi}_f)) \equiv \int_{\mathfrak{z}} \mathrm{Eis}(\omega_{\mathrm{top}} \otimes \psi_f) \bmod \frac{1}{2\delta}\,\mathbb{Z},$$

and hence we compute

$$\int_{\mathfrak{z}} \mathrm{Eis}(\omega_{\mathrm{hol}} - \omega_{\mathrm{top}}) \otimes \tilde{\psi}_f).$$

Now we have seen that in $\mathrm{Hom}_K(\Lambda^1(\mathfrak{g}/\mathfrak{k}), I_{\varphi_\infty})$ (See (72))

$$-i\ d\psi_0 = \frac{1}{2}(\omega_{\mathrm{hol}} - \bar{\omega}_{\mathrm{hol}})$$

and hence we get

$$\int_{\mathfrak{z}} \mathrm{Eis}((\omega_{\mathrm{hol}} - \omega_{\mathrm{top}}) = -\mathrm{Eis}(\psi_0 \otimes \tilde{\psi}_f))((\infty_{p_0}, \infty_\ell)) + \mathrm{Eis}(\psi_0 \otimes \tilde{\psi}_f))((\infty_{p_0}, \infty_\ell)).$$

To evaluate these integrals we have to twist the Eisenstein series by a complex parameter s and evaluate at $s = 0$. We have to compute the constant term at $s = 0$ and get

$$-\psi_0 \otimes \psi_f((\infty_{p_0}, \infty_\ell)) + (\psi_0 \otimes \tilde{\psi}_f))((0_{p_0}, 0_\ell))$$

$$-\ \frac{\Gamma(\frac{s+1}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{s+2}{2})}\ \cdot\ \frac{\zeta(s+1)}{\zeta(s+2)}\ T^{\mathrm{loc}}(\tilde{\psi}_f))((\infty_{p_0}, \infty_\ell))+$$

$$-\ \frac{\Gamma(\frac{s+1}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{s+2}{2})}\ \cdot\ \frac{\zeta(s+1)}{\zeta(s+2)}\ T^{\mathrm{loc}}(\tilde{\psi}_f))((0_{p_0}, 0_\ell))+$$

The first two summands give zero by the construction of $\tilde{\psi}_f$ as a product over all primes. To compute the other two contributions we have to observe that $\zeta(s+1)$ has a pole at $s = 0$ of residue 1 and this pole cancels because $T^{\mathrm{loc}}(\tilde{\psi}_f)$ has a zero at $s = 0$. We have to evaluate the result of this cancellation. We have to compute

$$-\ \frac{1}{s} \cdot T^{\mathrm{loc}}(\tilde{\psi}_f))((\infty_{p_0}, \infty_\ell))\ |_{s=0} + \frac{1}{s} \cdot T^{\mathrm{loc}}(\tilde{\psi}_f))((0_{p_0}, 0_\ell))\ |_{s=0}, \tag{77}$$

and multiply the result by

$$\frac{\Gamma(\frac{1}{2}) \cdot \Gamma(\frac{1}{2})}{\Gamma(1)} \cdot \frac{1}{\zeta(2)} = \frac{6}{\pi}.$$

To compute $T^{\mathrm{loc}}(\tilde{\psi}_f)$ we only have to consider the two factors ℓ, p_0 and then we have to apply the formulas from section to the original definition of $\psi_{\{p_0,\ell\}}$. We get

$$T^{\mathrm{loc}}(\psi_{\{p_0,\ell\}}) =$$

$$s\Big(\log p_0(\tfrac{-p_0}{p_0^2-1} f_\infty^{(p_0)} + \tfrac{1}{p_0^2-1} f_0^{(p_0)}) \otimes \tfrac{1}{\ell+1} h_0^{(\ell)} - \log \ell(\tfrac{1}{p_0+1} h_0^{(p_0)} \otimes (\tfrac{-\ell}{\ell^2-1} f_\infty^{(\ell)} + \tfrac{1}{\ell^2-1} f_0^{(\ell)})\Big) \tag{78}$$

We have to evaluate at $((\infty_{p_0}, \infty_\ell))$ and $((0_{p_0}, 0_\ell))$ and to take the difference. The first term in the expression above yields

$$\begin{aligned} -\log p_0(\tfrac{-p_0}{p_0^2-1} f_\infty^{(p_0)} + \tfrac{1}{p_0^2-1} f_0^{(p_0)}) \otimes \tfrac{1}{\ell+1} h_0^{(\ell)}((\infty_{p_0}, \infty_\ell)) &= \tfrac{p_0}{p_0^2-1} \tfrac{1}{\ell+1} \\ -\log p_0(\tfrac{-p_0}{p_0^2-1} f_\infty^{(p_0)} + \tfrac{1}{p_0^2-1} f_0^{(p_0)}) \otimes \tfrac{1}{\ell+1} h_0^{(\ell)}((0_{p_0}, 0_\ell) &= -\tfrac{1}{p_0^2-1} \tfrac{1}{\ell+1} \end{aligned} \tag{79}$$

and the difference is

$$\frac{p_0}{p_0^2-1}\frac{1}{\ell+1}+\frac{1}{p_0^2-1}\frac{1}{\ell+1}=\frac{1}{p_0-1}\frac{1}{\ell+1}$$

For the second term in our formula above we get the same expression with p_0, ℓ interchanged and therefore we get the final result

$$\begin{gathered}-\tfrac{1}{s}\cdot T^{\mathrm{loc}}(\tilde{\psi}_f))((\infty_{p_0},\infty_\ell))\mid_{s=0}+\tfrac{1}{s}\cdot T^{\mathrm{loc}}(\tilde{\psi}_f))((0_{p_0},0_\ell))\mid_{s=0}= \\ \tfrac{1}{p_0-1}\tfrac{1}{\ell+1}\log p_0-\tfrac{1}{\ell-1}\tfrac{1}{p_0+1}\log \ell\end{gathered} \tag{80}$$

Hence our ratio of values which gives the extension class of our mixed motive is given by

$$\left(e^{2\pi i\frac{6}{\pi i}(\frac{1}{p_0-1}\frac{1}{\ell+1}\log p_0-\frac{1}{\ell-1}\frac{1}{p_0+1}\log\ell)}\right)^{\delta}=\left(p_0^{\frac{1}{p_0-1}\frac{1}{\ell+1}}\ell^{-\frac{1}{\ell-1}\frac{1}{p_0+1}}\right)^{12\delta}$$

We reached our goal: Since we know that the result must be in $\mathbb{Q}^*$ we can say something about δ:

The exponents in the powers of p_0, ℓ are

$$\frac{(p_0+1)(\ell-1)}{(p_0^2-1)(\ell^2-1)} \text{ and } \frac{(p_0-1)(\ell+1)}{(p_0^2-1)(\ell^2-1)}$$

If we multiply them by 12δ they must become integers. If we define $D(p_0,\ell)$ as the greatest common divisor of the two numbers $(p_0+1)(\ell-1), (p_0-1)(\ell+1)$ then we get the divisibility relation

$$\frac{(p_0^2-1)(\ell^2-1)}{D(p_0,\ell)} \mid 12\delta \tag{81}$$

We want to comment briefly on the factor $12 = 2\cdot 2\cdot 3$: The first factor 2 comes from the factor 2 in $2\pi i$ and the comparison with π in equation (68) and the factor $2\cdot 3$ comes from the number 6 in the value of $\zeta(2)$.

2.5.1 Some examples

We want to discuss a few examples. We have the action of $\pi_0(\mathrm{Gl}_2(\mathbb{R}))$ on $H^1_!(Y_0(p_0\ell)(\mathbb{C}),\mathbb{Z})$ which commutes with the Hecke operators, hence the inner cohomology has a $+$ and a $-$ eigenspace. We also know that the characteristic polynomial $\mathrm{Ch}(T_p)[x]$ of the Hecke operator acting $H^1_!(Y_0(p_0\ell(\mathbb{C}),\mathbb{Z})_-$ is equal to the characteristic polynomial of T_p acting on the space of cusp forms $S_2(\Gamma_0(p_0\ell))$. For these characteristic polynomials we have the tables of W. Stein. Let $\mathbb{Z}e_0$ the Hecke module on which the Hecke operator T_p acts by the eigenvalue $T_pe_0=(p+1)e_0$. At various places we explained that the denominator δ produces congruences, more precisely we get an embedding

$$\mathbb{Z}e_0\otimes\mathbb{Z}/\delta\mathbb{Z}\hookrightarrow H^1_!(Y_0(p_0\ell)(\mathbb{C}),\mathbb{Z})_-\otimes\mathbb{Z}/\delta\mathbb{Z} \tag{82}$$

This implies that we must have

$$\mathrm{Ch}(T_p)(p+1)\equiv 0 \mod \delta \tag{83}$$

If we take $p_0 = 2, \ell = 11$ the the number on the left in the above divisibility relation is 60 and since we have factor $12 = 2^2 \cdot 3$ on the right hand side we get $5|\delta$. In fact looking at the tables we see that $\mathrm{Ch}(T_3)[\lambda] = (\lambda+1)^2$ and hence $\mathrm{Ch}(T_3)[3+1] = 25$. Here we notice that this congruence already occurs in $H^1_!(Y_0(11)\mathbb{C}), \mathbb{Z})_-$ and hence it occurs twice in $H^1_!(Y_0(22)(\mathbb{C}), \mathbb{Z})_-$ which explains the square.

We want to stick to the case $p_0 = 2$. We want to discuss examples where the congruences are modulo divisors of $\ell+1$ because the others are well known.

We take $\ell = 149$ then we get $5^2 \cdot 37|\delta$. We only consider the characteristic polynomials on the space of new forms

$$\mathrm{Ch}(T_3^{\mathrm{new}})(x) = x(x+2)(x^2-2x-2)(x^3+5x^2+4x-5)(x^5-x^4-10x^3+11x^2+12x-2)$$
$$\mathrm{Ch}(T_5^{\mathrm{new}})(x) = (x+2)(x+4)(x^2-2x-2)(x^3-x^2-4x-1)(x^5-5x^4+2x^3+9x^2-2)$$
$$\vdots$$
$$\mathrm{Ch}(T_{13}^{\mathrm{new}})(x) = (x^2+4x+1)(x^5-6x^4-37x^3+236x^2+32x-704)(x+5)^2(x)^3$$

We see that in all cases $5^2|\mathrm{Ch}(T_p^{\mathrm{new}}(p+1)$ and for $p = 13$ we have $5^2||\mathrm{Ch}(T_{13}^{\mathrm{new}}(14)$. We may be even be a little bit more precise. We see that the characteristic polynomial is always a product of two linear factors and then one factor of degree 2, 3 and 5. Then it is the factor of degree 5 which provides the divisibility relation.

At this point a more scrutinized examination of the relationship between the structure of $H^1_!(Y_0(p_0\ell)(\mathbb{C}), \mathbb{Z})_-$ and the spaces of new and old forms should be in order. This applies also to the next example.

We consider the case $p_0 = 2, \ell = 499$ then $3 \cdot 5^3 \cdot 83|\delta$ we are interested in the power of 5. Again we look up the tables, the characteristic polynomial of the Hecke operator T_{53}^{new} factors

$$\mathrm{Ch}(T_{53}^{\mathrm{new}})(x) = (x^2+12x+16)(x^3-6x^2-28x-16)$$
$$(x^9+10x^8-74x^7-836x^6+1107x^5+18650x^4+2332x^3-95517x^2+41365x-751) \times R[x]$$

We know that $5^3|\mathrm{Ch}(T_{54}^{\mathrm{new}})$ and we find that $5 \not| R[54]$ and $5||P[54]$ for any of the three factors in front. We conclude that $5^3||\delta$ and we say that the congruence is spread out over three factors

Our last example is

$$p_0 = 2, \ell = 127.$$

In this case the denominator δ is divisible by $2^5 \cdot 3^2 \cdot 7$. where the Hecke operator T_p acts on $\mathbb{Z}/\delta\mathbb{Z}[p+1]$ by the eigenvalue $p+1$ ($p \neq p_0, \ell$). This implies for the characteristic polynomial $T_p[X]$ of the Hecke operator that we have

$$T_p[p+1] \equiv 0 \mod \delta$$

We looked up William Stein's tables and found

$$\mathrm{Ch}(T_3)(x) = x^2(x+2)^2(x-2)^2(x^5+2x^4-10x^3-16x^2+10x+16)\times$$
$$(x^3+3x^2-3)^2(x^7-3x^6-12x^5+39x^4+26x^3-128x^2+64x+16)^2$$

$$\mathrm{Ch}(T_5)(x) = x(x-2)(x+3)(x+1)(x^2+x-4)(x^5+x^4-20x^3-18x^2+54x+54)\times$$
$$(x^3+6x^2+9x+1)^2(x^7-8x^6+11x^5+53x^4-146x^3+32x^2+128x-48)^2$$

$$\mathrm{Ch}(T_7)(x) = x(x+1)(x-4)(x+3)(x^2-x-4)(x^5-3x^4-20x^3+40x^2+96x-32)\times$$
$$(x^3+3x^2-3)^2(x^7+3x^6-20x^5-41x^4+114x^3+64x^2-112x-16)^2$$

$$\mathrm{Ch}(T_{11})(x) = x(x+3)(x-4)(x-1)(x^2+7x+8)(x^5-x^4-44x^3+72x^2+480x-1056)\times$$
$$(x^3-21x-37)^2(x^7-28x^5-17x^4+88x^3-37x^2-5x+3)^2$$

where the square factors in the second line are "old". We should expect that $2^5|\mathrm{Ch}(T_p)(p+1)$ and we find that this is always the case, in general we find a divisibility by a much higher power of 2.

We also find that $\mathrm{Ch}(T_{11}^{\mathrm{old}})(12)$ is an odd number, hence we should expect that we get an inclusion

$$\mathbb{Z}/2^5\mathbb{Z}e_0 \hookrightarrow H_!^{1,\mathrm{new}}(Y_0(254)(\mathbb{C}),\mathbb{Z})_- \otimes \mathbb{Z}/2^5\mathbb{Z}$$

and this in turn implies that

$$2^5|\mathrm{Ch}(T_p^{\mathrm{new}})(p+1)$$

Of course this true, actually we almost always find a divisibility by a very high power of two. Only in the two cases $p=5$ and $p=53$ we find the exact divisibility by 2^5.

We can say that the Jacobian $J^{\mathrm{new}}(X_0(254))$ is up to isogeny a product of four elliptic curves, an abelian surface and an abelian 5-fold. Two of the elliptic curves admit a congruence mod 2 (resp.) mod 4 and both abelian varieties admit a congruence mod 2. So the congruence mod 2^5 is spread out over several cusp forms.

2.5.2 Euler systems ?

We want to indicate how we can use these mixed motives $H^1_{\mathrm{Eis}}(X_0(p_0\ell),\mathbb{Q}^{\#})$ to bound ideal class groups. I

We see that we have to multiply this motive by δ, if we want it to become an integral motive. From this it follows that we have a (Hecke-invariant) inclusion

$$\mathbb{Z}/\delta\mathbb{Z}(-1) \hookrightarrow H_!^1(Y_0(p_0\ell),\mathbb{Z}/\delta\mathbb{Z}) \tag{84}$$

We apply the arguments from [book] 3.3.8. For a suitable finite extension $F/\mathbb{Q}$ we have a decomposition

$$H_!^1(Y_0(p_0\ell),F) = \bigoplus_{\pi_f} H_!^1(Y_0(p_0\ell),F)(\pi_f) \tag{85}$$

where π_f runs over a set of isomorphism classes of absolutely irreducible modules for the Hecke algebra. These Hecke modules are unramified at all places different from p_0,ℓ. This decomposition induces a Jordan-Hölder filtration on the integral cohomology (See [book], 3.3.8)

$$(0) \subset \mathcal{JH}^{(1)}H^1_{\mathrm{int},!}(Y_0(p_0\ell),\mathcal{O}_F) \subset \mathcal{JH}^{(2)}H^1_{\mathrm{int},!}(Y_0(p_0\ell),\mathcal{O}_F) \subset \cdots \subset \mathcal{JH}^{(r)}H^1_{\mathrm{int},!}(Y_0(p_0\ell),\mathcal{O}_F) \tag{86}$$

if we tensorize the subquotients by F we get the $H_!^1(Y_0(p_0\ell),F)(\pi_f)$. The filtration depends on an ordering of the π_f.

We can tensorize this Jordan-Hölder filtration by $\mathbb{Z}/\delta\mathbb{Z}$ and we get

$$\begin{array}{ccc} & & \mathcal{O}_F \otimes \mathbb{Z}/\delta\mathbb{Z}(-1) \\ & & \downarrow \\ (0) \subset \quad \mathcal{JH}^{(1)} H^1_{\mathrm{int},!}(Y_0(p_0\ell), \mathcal{O}_F) \otimes \mathbb{Z}/\delta\mathbb{Z} \subset \cdots \subset & & \mathcal{JH}^{(r)} H^1_{\mathrm{int},!}(Y_0(p_0\ell), \mathcal{O}_F) \otimes \mathbb{Z}/\delta\mathbb{Z} \end{array} \tag{87}$$

where the vertical arrow is an inclusion.

We choose a prime p which divides δ and we choose a prime $\mathfrak{p} \subset \mathcal{O}_F$ above p. Let $\mathfrak{p}^d || \delta$ then we can localize at $\mathfrak{p}$ and our diagram becomes

$$\begin{array}{ccc} & & \mathcal{O}_F/\mathfrak{p}^d(-1) \\ & & \downarrow \\ (0) \subset \quad \mathcal{JH}^{(1)} H^1_{\mathrm{int},!}(Y_0(p_0\ell), \mathcal{O}_F/\mathfrak{p}^d) \subset \cdots \subset & & \mathcal{JH}^{(r)} H^1_{\mathrm{int},!}(Y_0(p_0\ell), \mathcal{O}_F/\mathfrak{p}^d) \end{array} \tag{88}$$

Now we refer to the considerations in [book] 3.3.8. For simplicity we assume that there is exactly one π_f^{Eis} among the above π_f which is congruent to the Eisenstein class $\mod \mathfrak{p}$. Then we can order the π_f in such a way that the vertical arrow factors through the first step in the filtration, i.e. we get an inclusion

$$\mathcal{O}_F/\mathfrak{p}^d(-1) \hookrightarrow \mathcal{JH}^{(1)} H^1_{\mathrm{int},!}(Y_0(p_0\ell), \mathcal{O}_F/\mathfrak{p}^d)(\pi_f^{\mathrm{Eis}}) =: H^1_{\mathrm{Eis!}}(Y_0(p_0\ell), \mathcal{O}_F/\mathfrak{p}^d)(\pi_f^{\mathrm{Eis}}) \tag{89}$$

The module on the right hand side is now a free $\mathcal{O}_F/\mathfrak{p}^d$ module of rank 2, it is a module under the action of the Galois group. If we divide it by $\mathcal{O}_F/\mathfrak{p}^d(-1)$ then the Weil pairing implies that the quotient is $\mathcal{O}_F/\mathfrak{p}^d(0)$in other words it sits in an exact sequence

$$0 \to \mathcal{O}_F/\mathfrak{p}^d(-1) \to H^1_{\mathrm{Eis!}}(Y_0(p_0\ell), \mathcal{O}_F/\mathfrak{p}^d)(\pi_f^{\mathrm{Eis}}) \to \mathcal{O}_F/\mathfrak{p}^d(0) \to 0 \tag{90}$$

hence we have constructed an element

$$[H^1_{\mathrm{Eis!}}(Y_0(p_0\ell), \mathcal{O}_F/\mathfrak{p}^d)(\pi_f^{\mathrm{Eis}})) \in H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathcal{O}_F/\mathfrak{p}^d(-1)) \tag{91}$$

Now we encounter several questions:

A) Is this class non trivial? Is this class even a class of order $\mathfrak{p}^d$, i.e. does it generate a cyclic submodule $\mathcal{O}_F/\mathfrak{p}^d \in H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathcal{O}_F/\mathfrak{p}^d(-1))$?

B) We can restrict this class to $H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell), \mathcal{O}_F/\mathfrak{p}^d(-1))$. Can we compute this restriction? Is perhaps this restriction an element of order $\mathfrak{p}^d$?.

At this point I actually get stuck for the moment. Recall that we have $\ell \equiv 1 \mod p^\delta$ and hence $\zeta_{p^\delta} \in \mathbb{Q}_\ell$. Then the twist disappears and we get the Kummer isomorphism

$$H^1(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell, \mathbb{Z}/p^\delta\mathbb{Z}) = \mathbb{Q}_\ell^\times \otimes \mathbb{Z}/p^\delta\mathbb{Z} \tag{92}$$

and the hope, which is expressed in B), is that the cohomology class which we constructed is as ramified as possible, i.e.

$$[H^1_{\mathrm{Eis!}}(Y_0(p_0\ell), \mathcal{O}_F/\mathfrak{p}^d)(\pi_f^{\mathrm{Eis}})) = \ell \times \text{ a unit} \tag{93}$$

I think it suffices to prove this for $\delta = 1$, I am somewhat optimistic.

2.5.3 Denominators and modular symbols

We notice that need the two primes ℓ, p_0 to produce denominators of Eisenstein classes and therefore inclusions like (84).

In the [book] section 4.5 we will discuss another method to construct Eisenstein classes with denominator by using the theory of modular symbols.

We start from a prime ℓ and consider the modular curve $Y_0(\ell)/\mathbb{Q}$. It has two cusps $\infty_\ell, 0_\ell = \{x_1, x_2\}$ and we have the exact sequence (see (7))

$$0 \to H^1_!(Y_0(\ell)(\mathbb{C}),\mathbb{Z}) \to H^1_!(Y_0(\ell)(\mathbb{C}),\mathbb{Z}) \xrightarrow{r} \oplus_i H^1(\overset{\bullet}{\mathcal{N}}_{x_i},\mathbb{Z}) \xrightarrow{\delta_1} H^2_c(Y_0(\ell)/(\mathbb{C}),\mathbb{Z}) \to 0, \tag{94}$$

here $H^1(\overset{\bullet}{\mathcal{N}}_{x_i}) = \mathbb{Z}(-1), H^2_c(Y_0(\ell)/(\mathbb{C}),\mathbb{Z}) = \mathbb{Z}(-1)$ and $\delta_1 : (x,y) \mapsto x-y$. Hence, if $h(x_1) = 1, h(x_2) = 1$ the Eisenstein lift (See (2.3) $\mathrm{Eis}(\omega_\infty \otimes h) \in H^1_!(Y_0(\ell)(\mathbb{C}),\mathbb{Q})$ has a denominator $\Delta(\ell)$. This denominator is of course equal to the order of the torsion point $(x_1) - (x_2)$ in the Jacobian of $X_0(\ell)$. Now we can use the method of modular symbols to get estimates: We can evaluate $\mathrm{Eis}(\omega_\infty \otimes h)$ on certain cycles (the modular symbols), the result will be a rational number, and the denominator of this rational number gives us an estimate for $\Delta(\ell)$.

Using this method we should get

For a prime $p > 2$ which satisfies $p^\delta | \ell - 1$ we have $p^\delta | \Delta(\ell)$

This occurs in principle already in Mazur [Ma]- IHES .

Hence we see that this approach using modular symbols actually gives us elements in

$$[H^1_{\mathrm{Eis}!}(Y_0(\ell),\mathcal{O}_F/\mathfrak{p}^d)(\pi_f^{\mathrm{Eis}})] \in H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}),\mathcal{O}_F/\mathfrak{p}^d(-1)) \tag{95}$$

In a sense these considerations tell us that we may chose $p_0 = 1$.

Of course we can also get this from a slight modification of our previous consideration if we work with an auxiliary prime p_0 and consider the projection $Y_0(p_0\ell) \to Y_0(\ell)$. We pull back the divisor $(x_1) - (x_2)$ to a divisor on $Y_0(p_0\ell)$ and our previous arguments also give the denominator estimate.

2.6 Poitou-Tate duality and bounding cohomology groups.

We want to indicate how the existence of these cohomology classes can be used to bound some cohomology groups. The method is basically the methods of Euler systems. We stick to our example. Let us also assume that the prime p is completely split in F and hence we may assume that $\mathcal{O}_F/\mathfrak{p}^d = \mathbb{Z}/p^d$.

We pick a prime $p > 2$. We want to study the Galois cohomology

$$H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}),\mathbb{Z}/p^\delta\mathbb{Z}(-1)). \tag{96}$$

more precisely we want to consider only classes which satisfy a certain local condition at p we consider

$$H^1_{\{p\}}(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}),\mathbb{Z}/p^\delta\mathbb{Z}(-1)) = \{\xi|\xi_p \text{ restricted to } H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p),(\zeta_{p^\delta}),\mathbb{Z}/p^\delta\mathbb{Z}(-1)) = 0\} \tag{97}$$

If S is a finite set of primes different from p then we denote by $H^1_{S,\{p\}}(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbb{Z}/p^\delta\mathbb{Z}(-1))$ those classes which are unramified outside S.

We also consider $H^1_{S,\{p\}}(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbb{Z}/p^\delta(2))\delta)$ at the same time and get the diagram

$$\begin{array}{ccc} H^1_{S,\{p\}}(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}, \mathbb{Z}/p^\delta(2)) & \xrightarrow{r_2} & \bigoplus_{v\in S,\{p\}} H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_v/\mathbb{Q}_v, \mathbb{Z}/p^\delta(2)) \\ & & \times \\ H^1_{S,\{p\}}(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}, \mathbb{Z}/p^\delta(-1)) & \xrightarrow{r_{-1}} & \bigoplus_{v\in S,\{p\}} H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_v/\mathbb{Q}_v, \mathbb{Z}/p^\delta(-1)) \\ & & \downarrow \\ & & \mathbb{Q}/\mathbb{Z}. \end{array} \tag{98}$$

The vertical arrow $\times$ is the direct sum of local pairings, i. e. $\times = \oplus_v \cup_v$ where

$$\cup_v : H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_v/\mathbb{Q}_v, \mathbb{Z}/p^\delta(2)) \times H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_v/\mathbb{Q}_v, \mathbb{Z}/p^\delta(-1)) \to \mathbb{Q}/\mathbb{Z}. \tag{99}$$

We know

Theorem 2.1. *(Poitou-Tate) The vertical arrow is a non degenerate pairing and the images of r_2 and r_{-1} are mutual orthogonal complements of each other.*

We analyze the local pairing $\cup_v$. We have two cases

i) $v = \ell \neq p$.

Let $\mathbb{Q}_\ell^{\mathrm{nr}}$ be a maximal unramified extension, then $\zeta_{p^\delta} \in \mathbb{Q}_\ell^{\mathrm{nr}}$. Let $\Phi_\ell \in \mathrm{Gal}(\mathbb{Q}_\ell^{\mathrm{nr}}/\mathbb{Q}_\ell)$ be the Frobenius element. We have the Hochschild-Serre spectral sequence

$$0 \to H^1(\mathrm{Gal}(\mathbb{Q}_\ell^{\mathrm{nr}}/\mathbb{Q}_\ell), \mathbb{Z}/p^\delta(k))) \to H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell, \mathbb{Z}/p^\delta(k))) \to (H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell^{\mathrm{nr}}), \mathbb{Z}/p^\delta(k)))^{\phi_\ell} \to 0. \tag{100}$$

Here k can be any integer. We denote the first term in the filtration by $H^1_{\mathrm{nr}}(\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell, \mathbb{Z}/p^\delta(k)))$ and the second term (the quotient) by $H^1_{\mathrm{ram}}(\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell, \mathbb{Z}/p^\delta(k)))$. As before we have the Kummer isomorphism

$$H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell^{\mathrm{nr}}), \mathbb{Z}/p^\delta(k)) = (\mathbb{Q}_\ell^{\mathrm{nr}})^\times \otimes \mathbb{Z}/p^\delta(k-1), \tag{101}$$

the group $(\mathbb{Q}_\ell^{\mathrm{nr}})^\times = \mathfrak{U}_\ell \times < \ell >$, clearly $\mathfrak{U}_\ell \otimes \mathbb{Z}/p^\delta = 0$ and hence our exact sequence becomes

$$0 \to \mathbb{Z}/p^\delta(k))/(\mathrm{Id} - \Phi_\ell)\mathbb{Z}/p^\delta(k)) \to H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell, \mathbb{Z}/p^\delta(k))) \to (\mathbb{Z}/p^\delta(k-1))^{\Phi_\ell} \to 0 \tag{102}$$

Since Φ_ℓ acts on $\mathbb{Z}/p^\delta(k)$ by $\Phi_\ell x = \ell^k x$ we can rewrite the sequence

$$0 \to \mathbb{Z}/p^\delta(k)/(1-\ell^k)\mathbb{Z}/p^\delta(k) \to H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell, \mathbb{Z}/p^\delta(k))) \to \{x \in, \mathbb{Z}/p^\delta(k-1)))|(1-\ell^{k-1})x = 0\} \to 0 \tag{103}$$

and hence finally

$$H^1_{\mathrm{nr}}(\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell), \mathbb{Z}/p^\delta(k))) = \mathbb{Z}/p^\delta(k)/(1-\ell^k)\mathbb{Z}/p^\delta(k);\ H^1_{\mathrm{ram}}(\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell), \mathbb{Z}/p^\delta(k))) = \ker(1-\ell^{k-1}). \tag{104}$$

Now it is clear how the pairing in (99) looks like. We consider the diagram

$$\begin{array}{ccccc} \to & \mathbb{Z}/p^\delta(k)/(1-\ell^k)\mathbb{Z}/p^\delta(k) & \to H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell),\mathbb{Z}/p^\delta(k))) & \to & \ker(1-\ell^{k-1}) \to \\ & & \times & & \\ \to & \mathbb{Z}/p^\delta(1-k)/(1-\ell^{1-k})\mathbb{Z}/p^\delta(1-k) & \to H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell),\mathbb{Z}/p^\delta(1-k))) & \to & \ker(1-\ell^{-k}) \to \\ & & \downarrow \cup_\ell & & \\ & & \mathbb{Q}/\mathbb{Z} & & \end{array} \tag{105}$$

and for the pairing $\cup_\ell$ the modules

$$\mathbb{Z}/p^\delta(k)/(1-\ell^k)\mathbb{Z}/p^\delta(k) \text{ and } \mathbb{Z}/p^\delta(1-k)/(1-\ell^{1-k})\mathbb{Z}/p^\delta(1-k)$$

are mutual orthogonal complements of each other. Recall that these modules consist of the unramified classes H^1_{nr}. The pairing $\cup_\ell$ now induces a pairings between the ramified quotient of one sequence with the unramified submodule of the other. This is given by the multiplication

$$\begin{gathered}\ker(1-\ell^{k-1}) \times \mathbb{Z}/p^\delta(1-k)/(1-\ell^{1-k})\mathbb{Z}/p^\delta(1-k) \to \mathbb{Q}/\mathbb{Z} \\ (x,y) \mapsto \tfrac{xy}{p^\delta}\end{gathered} \tag{106}$$

ii) $v = p$.

Of course we start again from the Hochschild-Serre spectral sequence and apply the Kummer isomorphism

$$0 \to H^1(\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\delta})\mathbb{Q}_p),\mathbb{Z}/p^\delta\mathbb{Z}(k)) \to H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p,\mathbb{Z}/p^\delta\mathbb{Z}(k)) \to (\mathbb{Q}(\zeta_{p^\delta})^\times \otimes \mathbb{Z}/p^\delta\mathbb{Z}(k-1))^{\Gamma_\delta} \to H^2(\dots \tag{107}$$

Let us assume that $p-1 \not| k,\ p-1 \not| k-1$ then $H^i(\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\delta})/\mathbb{Q}_p),\mathbb{Z}/p^\delta\mathbb{Z}(k)) = 0, H^i(\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\delta})\mathbb{Q}_p),\mathbb{Z}/p^\delta\mathbb{Z}(1-k)) = 0$ and

$$(\mathbb{Q}(\zeta_{p^\delta})^\times \otimes \mathbb{Z}/p^\delta\mathbb{Z}(k-1)) = \mathbb{Z}[\zeta_{p^\delta}]^\times(1) \otimes \mathbb{Z}/p^\delta\mathbb{Z}(k-1) \tag{108}$$

and these cohomology classes are totally ramified. This says that

$$H^1_{\{p\}}(\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p),\mathbb{Z}/p^\delta\mathbb{Z}(k)) = H^1_{\{p\}}(\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p),\mathbb{Z}/p^\delta\mathbb{Z}(k)) = 0 \tag{109}$$

Now we describe the potential strategy to bound Galois-cohomology groups, it is essentially the the same strategy as in the theory of Euler systems. We start from a class

$$\xi \in H^1_{\emptyset,\{p\}}(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q},\mathbb{Z}/p^\delta(2)),$$

let p^a be its order. We choose a $\delta \geq a$. We show that Tschebotareff density implies that there exists a prime $\ell \equiv 1 \mod p^\delta$ such that the restriction $\xi_\ell \in H^1_{\mathrm{nr}}(\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell,\mathbb{Z}/p^\delta(2))$ still has order p^a. Now we look at the elements $[H^1_{\mathrm{Eis!}}(Y_0(p_0\ell),\mathcal{O}_F/\mathfrak{p}^d)(\pi_f^{\mathrm{Eis}}))$ in (93). Let us assume that we really can show that (93) holds. Then this class is only ramified at ℓ, the sum defining $\times$ has only one term and Poitou Tate implies

$$\xi \times [H^1_{\mathrm{Eis!}}(Y_0(p_0\ell),\mathcal{O}_F/\mathfrak{p}^d)(\pi_f^{\mathrm{Eis}}))] = p^a\frac{1}{p^\delta} = 0 \tag{110}$$

and it follows that a must be zero, hence $\xi = 0$.

Hence we see that the success of strategy hinges on the assumption that the classes $[H^1_{\mathrm{Eis!}}(Y_0(p_0\ell),\mathcal{O}_F/\mathfrak{p}^d)(\pi_f^{\mathrm{Eis}}))]$ are very ramified at ℓ and unramified at all other places.

2.7 More ramification

We choose a slightly smaller open compact subgroup. Let us pick two different primes p_0, ℓ and consider the open compact subgroup $K_f = K_f^{p_0,\ell} = \prod K_p$ where $K_p = \mathrm{Gl}_2(\mathbb{Z}_p)$ for $p \neq p_0, \ell$ and where

$$K_{p_0} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | c \equiv 0 \mod p_0\}, K_\ell = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \equiv 1 \mod \ell,\ c \equiv 0 \mod \ell\} \tag{111}$$

We denote the the resulting Shimura varieties by $Y_{0,1}(p_0, \ell) \subset X_{0,1}(p_0, \ell)$. We still have

$$\bar{\Sigma}(\mathbb{C}) = \{(\infty_{p_0}, \infty_\ell), (\infty_{p_0}, 0_\ell), (0_{p_0}, \infty_\ell), (0_{p_0}, 0_\ell)\}$$

The group $T(\mathbb{F}_\ell) = \mathbb{F}_\ell^\times \times \mathbb{F}_\ell^\times$ acts transitively on the fibers of $p_B : \Sigma(\mathbb{C}) \to \bar{\Sigma}(\mathbb{C})$. We can be more precise: We have two embeddings $j_1 : \mathbb{F}_\ell^\times \hookrightarrow T(\mathbb{F}_\ell)($ resp. $, j_2 : \mathbb{F}_\ell^\times \hookrightarrow T(\mathbb{F}_\ell)$) by the first resp. the second entry on the diagonal. We get two quotients

$$\Xi^{(0)} = T(\mathbb{F}_\ell)/j_2(\mathbb{F}_\ell^\times) \times \{\pm 1\}\ , \Xi^{(\infty)} = T(\mathbb{F}_\ell)/j_1(\mathbb{F}_\ell^\times) \times \{\pm 1\} \tag{112}$$

Let $\Xi^{(0)\vee}, \Xi^{(\infty\vee)}$ be the character modules. The element $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ interchanges these two character modules. We can summarize:

The fibers over $(y, 0_\ell) \in \bar{\Sigma}(\mathbb{C})$ (resp. $(y, \infty_\ell) \in \bar{\Sigma}(\mathbb{C})$) are torsors for $\Xi^{(0)}$ (resp. $\Xi^{(\infty)}$) They have natural base points: The fiber over $(y, 0_\ell)$ is $\Xi^{(0)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the fiber over (y, ∞_ℓ) is $\Xi^{(\infty)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

From this we get give an even more explicit description of $H^1(\overset{\bullet}{\mathcal{N}}\,\Sigma, \mathbb{Z})$.

We have to be pedantic at this point. We define the ring

$$\mathbb{Z}[\zeta_{\ell-1}] = \mathbb{Z}[X]/(X^{\ell-2} + \cdots + 1) \tag{113}$$

it comes with a distinguished $(\ell-1)$-th root of unity. We also choose a distinguished embedding $j : \mathbb{Z}[\zeta_{\ell-1}] \hookrightarrow \mathbb{C}$, we send $\zeta_{\ell-1} \mapsto e^{\frac{2\pi i}{\ell-1}}$. We consider characters $\chi_\ell : T(\mathbb{F}_\ell)/\{\pm 1\} \to \mu_{\ell-1} \subset \mathbb{C}^\times$ It is clear that each such character is the ℓ-component of a unique Dirichlet character $\chi_f : T(\mathbb{A}_f)/T(\mathbb{Q}) \to \mu_{\ell-1}$ which is unramified outside ℓ. To χ_f we attach the algebraic Hecke character

$$\varphi_f = |\alpha|_f \chi_f : T(\mathbb{A}_f)/T(\mathbb{Q}) \to \mathbb{Q}[\zeta_{\ell-1}]^\times$$

and the induced representation $I_{\varphi_f} = \mathrm{Ind}_{B(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \varphi_f$, where we require that the functions $f \in I_{\varphi_f}$ are $\mathbb{Q}(\zeta_{\ell-1})$ valued.

Then our space of functions (See (27)) - now with values in $\mathbb{Q}(\zeta_{\ell-1})$ - is

$$\mathcal{C}_\alpha \otimes \mathbb{Q}(\zeta_{\ell-1}) = \bigoplus_{\chi_\ell \in \Xi^{(0)\vee} \cup \Xi^{(\infty\vee)}} (I_{\varphi_f})^{K_f} \tag{114}$$

and we get the same equality if we replace α by the trivial character

$$\mathcal{C}_{\alpha^0}\otimes\mathbb{Q}(\zeta_{\ell-1})=\bigoplus_{\chi_\ell\in\Xi^{(0)\vee}\cup\Xi^{(\infty\vee)}}(I_{\chi_f^{-1}})^{K_f}. \tag{115}$$

The module of invariants under K_f is

$$(I_{\varphi_f})^{K_f}=(\mathbb{Q}[\zeta_{\ell-1}]f_0^{(p_0)}\oplus\mathbb{Q}[\zeta_{\ell-1}]f_\infty^{(p_0)})\otimes(\mathbb{Q}[\zeta_{\ell-1}]f_{0,\chi_\ell}^{(\ell)}\oplus\mathbb{Q}[\zeta_{\ell-1}]f_{\infty,\chi_\ell}^{(\ell)}) \tag{116}$$

and the two factors have been described in section 2.4.1

We defined the intertwining operator (remember $\varphi_f=|\alpha|_f\chi_f$)

$$T^{\rm loc}:(I_{\varphi_f})^{K_f}\to(I_{w\chi_f})^{K_f} \tag{117}$$

it is essentially the tensor product

$$T_{p_0}^{\rm loc}\otimes T_\ell^{\rm loc}:(I_{\varphi_{p_0}})^{K_{p_0}})\otimes(I_{\varphi_\ell})^{K_\ell}\to(I_{w\chi_{p_0}})^{K_{p_0}}\otimes(I_{w\chi_\ell})^{K_\ell} \tag{118}$$

The formulas for these local operators are in section 2.4.1

Now we apply the principles from section 2.2 and 2.3. We extend the coefficients from $\mathbb{Z}$ to $\mathbb{Z}[\zeta_{\ell-1}]$ and consider the cohomology $H^1(\overset{\bullet}{\mathcal{N}}\ \Sigma,\mathbb{Z}[\zeta_{\ell-1}])$ We start from an function $h\in\mathcal{C}_\alpha\otimes\mathbb{Q}(\zeta_{\ell-1})$ then we can write it as the sum $h=\sum_{\varphi_f}\hat{h}_{\chi_f}$ with $\hat{h}(\chi_f)\in(I_{\varphi_f})^{K_f}$. If $\frac{h(x)}{d(y)}\in\mathbb{Z}$ we get a cohomology class $[\omega_{\rm hol}]\otimes h\in H^1(\overset{\bullet}{\mathcal{N}}\ \Sigma,\mathbb{Z})$. If the sum of the residues $\sum\frac{h(x)}{d(y)}=0$ then we have seen that $\mathrm{Eis}(\omega_{\rm hol}\otimes h)$ is a meromorphic 1 -form with simple poles at the cusps in the support of h and residue $\frac{h(x)}{d(y)}$. We have to write a suitable multiple of this form as the logarithmic derivative of a function H and to evaluate H at the cusps outside the support of h.

2.8 The computation of the extension class

We have to consider the constant term

$$\mathrm{Eis}\,(\psi_0\otimes h,0)\simeq\alpha_\infty\otimes h+\sum_\chi\frac{\Lambda(\chi,,1)}{\Lambda(\chi,2)}T_\infty^{\rm loc}(\psi_0)\otimes T_f^{\rm loc}(\hat{h}(\chi_f)) \tag{119}$$

(Here we notice that the character χ in the argument of the completed L function Λ should be $\chi^{(1)}$, the restriction of χ to the torus $T^{(1)}$ in Sl_2. But this torus is identified to $\mathbb{G}_m$ and under this identification we have $\chi=\chi^{(1)}$). We consider the ratio of the two values of the L-function. For a non trivial χ the value $L(\chi,1)$ is computed by the usual process. We write the character χ_ℓ as a linear combination

$$\chi_\ell(n)=\sum_{a\in\mathbb{F}_\ell}g(a,\chi_\ell)e^{\frac{2\pi i}{\ell}an}$$

where

$$g(a,\chi)=\frac{1}{\ell}\sum_{n\in\mathbb{Z}/\ell\mathbb{Z}}\chi_\ell(n)e^{-\frac{2\pi i}{\ell}an}=\frac{1}{\ell}G(a,\chi_\ell),$$

and where $G(a,\chi_\ell)$ is a Gaussian sum. We put $\chi_\ell(0) = 0$ and since χ_ℓ is non-trivial, we get $G(0,\chi_\ell) = 0$.

Then we obtain the classical formula

$$L(1.\chi) = \frac{1}{\ell} \sum_{a\in\mathbb{F}_\ell^\times} G(a,\chi_\ell)\cdot \log(1 - e^{\frac{2\pi i}{\ell}a}).$$

To compute $L(\chi_\ell, 2)$, we apply the functional equation. We have

$$\ell^{s/2}\frac{\Gamma(s/2)}{\pi^{s/2}} L(\chi,s) = \frac{G(1,\chi_\ell)}{\sqrt{\ell}} \cdot L(\chi^{-1}, 1-s)\cdot \frac{\Gamma(\frac{1-s}{2})}{\pi^{\frac{1-s}{2}}}\ell^{(1-s)/2}$$

and evaluate at $s = 2$

$$L(\chi,2) = -\frac{1}{\ell^2}\cdot G(1,\chi_\ell))\cdot L(\chi^{-1},-1)\cdot \frac{\pi^2}{2}.$$

and hence we get for the ratio of the values of Λ at 1 and 2

$$-\frac{i\ell}{\pi}\sum_{a\in\mathbb{F}_\ell^\times} \frac{G(a,\chi_\ell)}{G(1,\chi_\ell)}\cdot \frac{1}{L(\chi^{-1},-1)}\log(1-e^{\frac{2\pi i}{p}a}).$$

and since $G(a,\chi_\ell) = \chi_\ell(a)^{-1}G(1,\chi_\ell))$ we get

$$-\frac{i\ell}{\pi L(\chi^{-1},-1)}\sum_{a\in\mathbb{F}_\ell^\times}\chi_\ell(a)^{-1}\log(1-e^{\frac{2\pi i}{p}a}).$$

We plug this into equation (119) and get

$$\begin{gathered}\mathrm{Eis}\,(\psi_0\otimes h,0) \simeq \alpha_\infty\otimes h + \tfrac{\Lambda(\chi_0,,1)}{\Lambda(\chi_0,2)}T_\infty^{\mathrm{loc}}(\psi_0)\otimes T_f^{\mathrm{loc}}(\hat h(\chi_{0,f})) = \\ \alpha_\infty\otimes h - \tfrac{i\ell}{\pi}T_\infty^{\mathrm{loc}}(\psi_0)\otimes\left(\textstyle\sum_{a\in\mathbb{F}_\ell^\times}\log(1-e^{\frac{2\pi i}{\ell}a})\sum_{\chi_\ell\neq\chi_0}\frac{\chi_\ell(a)^{-1}T_f^{\mathrm{loc}}(\hat h(\chi_f))}{L(\chi^{-1},-1)}\right)\end{gathered} \tag{120}$$

Here a few comments are in order.

i) The character χ_0 is the trivial character, then $\Lambda(\chi_0, 1+s)$ is the Riemann ζ function completed by the Γ factor. It has a pole at $s = 0$ which cancels if the sum of residues is zero. To evaluate the second term in the first line we proceed as in the previous section.

ii) If $h \in \mathcal{C}_\alpha$ then the inner sum $\sum_{\chi_\ell\neq\chi_0}\frac{\chi_\ell(a)^{-1}T_f^{\mathrm{loc}}(\hat h(\chi_f))}{L(\chi^{-1},-1)} \in \mathcal{C}_\alpha$ i.e. its values are rational.

Now we play with different choices of $h \in \mathcal{C}_\alpha$. We choose a non trivial character $\chi_\ell \in \Xi^{(0)\vee}$. In this situation we do not need the place p_0, hence we choose $h = 1^{(p_0)} \otimes f^{(\ell)}_{0,\chi_\ell}$, where $1^{(p_0)}$ is the spherical function. Then we know $T^{\mathrm{loc}}(f^{(\ell)}_{0,\chi_\ell}) = \frac{1}{\ell}f^{(\ell)}_{\infty,w\chi_\ell}$. The function h has support in the fibers over $(y, 0_\ell)$ and $f^{(\ell)}_{\infty,w\chi_\ell}$ has support in the fibers over (y,∞_ℓ). Our expression for the constant term of the Eisenstein class becomes

$$\mathrm{Eis}(\psi_0 \otimes h, 0) \simeq \psi_0 \otimes 1^{(p_0)} \otimes f^{(\ell)}_{0,\chi_\ell} - \tfrac{i}{\pi}\psi_0 \otimes 1^{(p_0)} \otimes \Big(\textstyle\sum_{a \in \mathbb{F}_\ell^\times} \log(1 - e^{\frac{2\pi i}{\ell}a})\frac{\chi_\ell(a)^{-1} f^{(\ell)}_{\infty, w\chi_\ell}}{L(\chi^{-1},-1)}\Big) \tag{121}$$

We apply our previous considerations. Our function h can be interpreted as a divisor with coefficients in $\mathbb{Z}[\zeta_{\ell-1}]$, its sum of the residues is zero. Hence we know that it becomes a principal divisor if we multiply it by a suitable integer $\Delta(h) \in \mathbb{Z}[\zeta_{\ell-1}]$ i.e it becomes a divisor of a function $H \in \mathbb{Q}(\zeta_\ell)(X_{K_f}) \otimes \mathbb{Z}[\zeta_{\ell-1}]$. To get an estimate for the denominator $\Delta(h)$ we have to evaluate the function H at two points say x_1 and 1 in the fiber over (y, ∞_ℓ) and look at the ratio, this has to be a number in $\mathbb{Q}(\zeta_\ell)^\times \otimes \mathbb{Z}[\zeta_{\ell-1}]$. We know how two compute this ratio is is equal to

$$\frac{H(x_1)}{H(1)} = \Big(\prod_{a \in \mathbb{F}_\ell^\times} (1 - e^{\frac{2\pi i}{\ell}a})^{\chi_\ell(a)^{-1}}\Big)^{2\frac{\chi_\ell(x_1)-1}{L(\chi^{-1},-1)}\Delta(h)} \tag{122}$$

Now we encounter a typical problem in the theory of cyclotomic fields. The number

$$c(\chi_\ell) = \prod_{a \in \mathbb{F}_\ell^\times} (1 - e^{\frac{2\pi i}{\ell}a})^{\chi_\ell(a)^{-1}} \tag{123}$$

is a cyclotomic unit in $\mathbb{Z}[\zeta_\ell]^\times \otimes \mathbb{Z}[\zeta_{\ell-1}]$. It is not a root of unity but we do not know whether or not it is a non trivial power of another unit. This is certainly not the case if the class number $h^+(\ell)$ of the totally real field $\mathbb{Q}_+(\zeta_\ell)$ is one, this is probably very often a case. On the other hand it is known that the class number $h^+(\ell)$ is the index of the cyclotomic units in the group of all units. Therefore we get in any case

Theorem 2.2. *The exponent*

$$2h^+(\ell)\frac{\chi_\ell(x_1) - 1}{L(\chi^{-1}, -1)}\Delta(h) \in \mathbb{Z}[\zeta_{\ell-1}]$$

If our prime $\ell \equiv 1 \mod 4$ then we may take for $\chi_\ell \in \Xi^{(0)\vee}$ the quadratic character $\left(\frac{}{\ell}\right)$. Then our cyclotomic unit is

$$c(\left(\frac{}{\ell}\right)) = \prod_a (1 - e^{\frac{2\pi i}{\ell}a})^{(\frac{a}{\ell})} \in \mathbb{Z}[\sqrt{\ell}]^\times \tag{124}$$

We take for x_1 a non residue and assume $h(\mathbb{Z}[\sqrt{\ell}]) = 1$ then we get

$$\frac{4}{L(\chi^{-1}, -1)}\Delta(h) \in \mathbb{Z} \tag{125}$$

We may of course choose the function $h = f_0^{(p_0)} \otimes f^{(\ell)}_{0,\chi_\ell}$, then we have to replace $1^{(p_0)}$ the second summand on the right hand side in (121) by

$$T^{\mathrm{loc}}(f_0^{(p_0)}) = \Big(1 - \frac{1}{p_0}\Big)\frac{p_0^{-1}\chi_{p_0}(p_0)}{1 - p_0^{-2}\chi_{p_0}(p_0)} f_0^{(p_0)} + \frac{1}{p_0}\frac{1 - p_0^{-1}\chi_{p_0}(p_0)}{1 - p_0^{-2}\chi_{p_0}(p_0)} f_\infty^{(p_0)} \tag{126}$$

in the notation of (45) we have

$$a(\chi_{p_0}) = \frac{(p_0-1)\chi_{p_0}(p_0)}{p_0^2 - \chi_{p_0}(p_0)}, b(p_0) = \frac{p_0 - \chi_{p_0}(p_0)}{p_0^2 - \chi_{p_0}(p_0)} \tag{127}$$

(Comment: This is of course our formula (61), the functions $f_0^{(p_0)}, f_\infty^{(p_0)}$ do not depend on the unramified character χ_{p_0} for our parameter s we have to choose $p^{-s} = \chi_{p_0}(p_0)$.)

As before $\mathfrak{d}_h$ denotes the divisor attached to h, it becomes principal if we multiply it by the denominator $\Delta(h)$ and we find a function H with $\mathrm{Div}(H) = \Delta(h)$. We evaluate H at two different points in Σ which are not in the support of h and compute the ratio of the values. The support of h is the fiber over $(0_{p_0}, 0_\ell)$ so we may evaluate in $(0_{p_0}, x)$ and (∞_{p_0}, x) where $x \in \mathbb{F}_\ell^\times/\{\pm 1\}$. We get for the values

$$H((0_{p_0}, x)) = c(\chi_\ell)^{2\frac{a(\chi_{p_0})\chi_\ell^{-1}(x_1)}{L(\chi^{-1},-1)}\Delta(h)}, H((\infty_{p_0}, x)) = c(\chi_\ell)^{2\frac{b(\chi_{p_0})\chi_\ell^{-1}(x_1)}{L(\chi^{-1},-1)}\Delta(h)} \tag{128}$$

Taking ratios of two such values we get expressions of the form

$$c(\chi_\ell)^{\frac{c(\kappa)}{(p_0^2 - \chi_{p_0}(p_0))L(\chi^{-1},-1)}\Delta(h)} \tag{129}$$

where κ is our pair of points and $c(\kappa)$ the resulting numerator. We denote by $\mathfrak{n}(p_0, \chi_\ell)$ the integral ideal generated by these $c(\kappa)$ then we find

Theorem 2.3.

$$h^+(\ell)\frac{\mathfrak{n}(p_0,\chi_\ell)}{(p_0^2 - \chi_{p_0}(p_0))L(\chi^{-1},-1)}\Delta(h) \subset \mathbb{Z}[\zeta_{\ell-1}] \tag{130}$$

2.9 Different interpretation using sheaves with support conditions

Starting from characters $\chi_\ell \in \Xi^{(0)\vee} = \mathrm{Hom}(\Xi^{(0)}, \mathbb{C}^\times)$ we consider the induced representation of $\mathrm{Gl}_2(\mathbb{F}_\ell)$

$$I_{\chi_\ell} = \mathrm{Ind}_{\mathcal{B}(\mathbb{F}_\ell)}^{G(\mathbb{F}_\ell)} \chi_\ell$$

and since this is also a representation of $\mathrm{Gl}_2(\mathbb{Z})$ we can construct a local system $\tilde{I}_{\chi_\ell}$ on $X_0(p_0)(\mathbb{C})$. As usual extend it to a sheaf $\tilde{I}^\#_{\chi_\ell}$ by taking the direct image at 0_{p_0} and extending by zero to ∞_{p_0}. Discuss $H^1_{\mathrm{Eis}}(X_0(p_0), \tilde{I}^\#_{\chi_\ell})$ and relate it to the above mixed motives.(Needs some improvement)

3 Higher Tate- Anderson motives

3.0.1 The coefficient systems

We recall the construction of mixed Anderson motives in [Eis] Kap. VI. Our basic data are as in section 2.2. In addition we consider the representation of Gl_2 on the space

$$\mathcal{M}_{n,\mathbb{Z}} = \{P(X,Y) = a_n X^n + a_{n-1}X^{n-1}Y + \cdots + a_0 Y^n | a_\nu \in \mathbb{Z}\} \tag{131}$$

of homogenous polynomials in two variables X, Y of degree n and with coefficients $a_i \in \mathbb{Z}$. We assume $n > 0$ and even. The center acts by the character $t \mapsto t^n$. We may twist the representation by a power of the determinant then $\mathcal{M}_{n,\mathbb{Z}}[\nu] = \mathcal{M}_{n,\mathbb{Z}} \otimes \det^\nu$, the twisted representation has the central character $t \mapsto t^{n+2\nu}$. We have an Gl_2 invariant pairing

$$< \, , \, >_{\mathbb{Q}}: \mathcal{M}_{n,\mathbb{Q}}[-n] \times \mathcal{M}_{n,\mathbb{Q}} \to \mathbb{Q} \tag{132}$$

is given by

$$< X^\nu Y^{n-\nu}, X^{n-\mu} Y^\mu >_{\mathbb{Q}} = \binom{n}{\mu}^{-1} \delta_{\nu,\mu}. \tag{133}$$

This gives an isomorphism of Gl_2 modules

$$\Phi_{\mathcal{M}} : \mathcal{M}_{n,\mathbb{Q}}[-n] \to \mathcal{M}_{n,\mathbb{Q}}^\vee \tag{134}$$

Hence we get for the dual module

$$\mathcal{M}_{n,\mathbb{Z}}[-n]^\vee = \{\sum \binom{n}{\mu} a_\mu X^\mu Y^{n-\mu} | a_\mu \in \mathbb{Z}\} \tag{135}$$

In the following we will omit the stupid twist and only require that our pairings and isomorphism are Sl_2 invariant, or for the Harish-Chandra modules that the are $(\mathfrak{g}^{(1)}, K_\infty)$ invariant.

3.1 The construction of mixed Anderson motives

We choose an auxiliary prime $p_0 > n$ and consider the curves $Y_0(p_0) \subset X(p_0)$. We have the two cusps $\{0_{p_0}, \infty_{p_0}\}$ and we define the sheaf $\tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}}$: We extend the sheaf on $Y_0(p_0)$ by the direct image to 0_{p_0} and by zero to ∞_{p_0}. We consider the cohomology (the mixed motive) (See [Eis], 4.2.2)

$$H^1(X_0(p_0), \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}})$$

We call it a mixed motive because it has different realizations:

a) We have the Betti realization

$$H^1_B(X_0(p_0), \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}}) = H^1(X_0(p_0)(\mathbb{C}), \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}})$$

which is a finitely generated $\mathbb{Z}$ module with an involution F_∞.

b) We have the de-Rham realization

$$H^1_{d-Rh}(X_0(p_0), \tilde{\mathcal{M}}^{\#}_{n,d-Rh})$$

which are finite dimensional $\mathbb{Q}$ vector spaces together with a filtration of weights $0, n+1, 2n+2$.

c) For all primes ℓ we have the $\ell-$ adic realizations

$$H^1_{et}(X_0(p_0) \times \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}_\ell})$$

these are finitely generated $\mathbb{Z}_\ell$ modules with an action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. We have the usual comparison isomorphisms. We make the remark that in contrast to [Eis] we do not make any attempt to discuss an integral structure on the de-Rham realization.

We know the boundary cohomology $H^1(\overset{\bullet}{\mathcal{N}}\ \Sigma_0, \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}}) = H^1(\overset{\bullet}{\mathcal{N}}\ \Sigma_0, \tilde{\mathcal{M}}_{\mathbb{Z}})$. The tubular neighborhood $\overset{\bullet}{\mathcal{N}}\ \Sigma_0 = \Gamma_U \backslash\ \mathbb{H}_+(c)$ where $\mathbb{H}_+(c) = \{z | \Im(z) > c >> 0\}$ and $\Gamma_U = \{\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\}$. The group Γ_U is generated by the element $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and

$$H^1(\overset{\bullet}{\mathcal{N}}\ \Sigma_0, \tilde{\mathcal{M}}_{\mathbb{Z}}) = \mathcal{M}_{\mathbb{Z}}/(1-T)\mathcal{M}_{\mathbb{Z}} = \mathbb{Z}Y^n \oplus \mathrm{Tors} = \mathbb{Z}(-n-1) \oplus \mathrm{Tors}$$

Then the following holds

We can find a polynomial $P_n(X,Y) = a_0X^n + a_1X^{n-1}Y + \cdots + Y^n$ such that its image $\Omega_n \in \mathcal{M}_{\mathbb{Z}}/(1-T)\mathcal{M}_{\mathbb{Z}}$ satisfies

$$T_p(\Omega_n) = (p^{n+1}+1)\Omega_n \textit{ for all Hecke operators } T_p \textit{ with } p \neq p_0 \tag{136}$$

and the Hecke operator T_p is nilpotent on the p-torsion of the group Tors *and on the cohomology $H^2_c(X_0(p_0), \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}})$. For any $p \neq p_0$ the Hecke operator T_p acts nilpotently on all p torsion subgroups in our cohomology groups. The only non zero p-torsion occurs for $p < n$. From this we get especially that Ω_n is in the image of the restriction map $H^1_B(X_0(p_0)(\mathbb{C}), \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}}) \to H^1(\overset{\bullet}{\mathcal{N}}\ \Sigma_0, \tilde{\mathcal{M}}_{\mathbb{Z}})$.*

This tells us that inside of our mixed motive $H^1(X_0(p_0), \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}})$ we have a sub motive

$$H^1_{\mathrm{Eis}}(X_0(p_0), \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}}) \tag{137}$$

which is defined by the condition that for all $p \neq p_0$ the Hecke operator T_p acts on all realizations by the (generalized) eigenvalue $p^{n+1}+1$. This motive is of rank 2. Hence we get a diagram

$$\begin{array}{ccc} H^1(X_0(p_0), \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}}) & \xrightarrow{\mathrm{res}} & H^1(\overset{\bullet}{\mathcal{N}}\ \Sigma_0, \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}})/\mathrm{Tors} = \mathbb{Z}(-1-n) \\ \uparrow & & \uparrow \\ H^1_{\mathrm{Eis}}(X_0(p_0), \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}}) & \xrightarrow{\mathrm{res}_{\mathrm{Eis}}} & H^1_{\mathrm{Eis}}(\overset{\bullet}{\mathcal{N}}\ \Sigma_0, \tilde{\mathcal{M}}_{\mathbb{Z}})/\mathrm{Tors} = \mathbb{Z}(-1-n) \end{array} \tag{138}$$

where the two horizontal arrows are surjective an the upwards arrow on the right is the multiplication by $\Delta(n)$, the *denominator of the Eisenstein class.*

This denominator can be computed by using the theory of modular symbols ([Ha], [Hab], [Kai]), in the following we discuss another strategy to get information about this denominator.

By construction the motive $H^1_{\mathrm{Eis}}(X_0(p_0), \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}})$ sits in an exact sequence

$$0 \to \mathbb{Z}(0) \to H^1_{\mathrm{Eis}}(X_0(p_0), \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}}) \to \mathbb{Z}(-n-1) \to 0 \tag{139}$$

and hence we can view it as an element in

$$[H^1_{\mathrm{Eis}}(X_0(p_0), \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}})] \in \mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-n-1), \mathbb{Z}(0)) \tag{140}$$

Of course this last equation does not make sense at this very moment, but we have the extension classes for the Betti-de-Rham cohomology and for the p-adic Galois-modules.

3.2 The Betti-de-Rham extension class

3.2.1 The intertwining operator

To compute the Betti-de-Rham extension class we have to apply the rules in section 1.7.1. There we show that the extension class is simply a real number.

In principle this is also what we do in [Eis], 4.3.3. We essentially repeat the calculation because the computation in [Eis], gives some wrong powers of 2 in the final formula and it is also much to complicated because of the totally superfluous integrality considerations in the de-Rham cohomology.

We introduce the characters

$$\varphi, \varphi' = T(\mathbb{A}) \to \mathbb{R}^\times;\ \varphi : \begin{pmatrix} \underline{t}_1 & 0 \\ 0 & \underline{t}_2 \end{pmatrix} \mapsto |\underline{t_1}||\underline{t_2}|^{-n-1}, \varphi' : \begin{pmatrix} \underline{t}_1 & 0 \\ 0 & \underline{t}_2 \end{pmatrix} \mapsto |\underline{t_1}|^{-n},$$

then the induced representations $I_{\varphi_\infty}, I_{\varphi'_\infty}$ have non trivial $(\mathfrak{g}, K_\infty)-$ cohomology with coefficients in $\mathcal{M}_{n,\mathbb{Q}}$. We have the standard intertwining operator given by an integral

$$T^{\mathrm{st}} : I_\varphi \to I_{\varphi'} \tag{141}$$

between these two representations. We also have the local operators $T_v^{\mathrm{loc}} : I_{\varphi_v} \to I_{\varphi'_v}$ and these operators are related by

$$T^{\mathrm{st}} = \frac{\xi(n+1)}{\xi(n+2)} \bigotimes T_v^{\mathrm{loc}} \tag{142}$$

where $\xi(s) = \frac{\Gamma(s/2)}{\pi^{s/2}}\zeta(s)$ is the completed Riemann $\zeta-$ function. It satisfies the functional equation $\xi(s) = \xi(1-s)$ and then the functional equation for the Gamma-functionl yields

$$\frac{\xi(n+1)}{\xi(n+2)} = \frac{\frac{\Gamma(-\frac{n}{2}-s)}{\pi^{-\frac{n}{2}-s}}\zeta(-n-s)|_{s=0}}{\frac{\Gamma(\frac{-1-n}{2})}{\pi^{-\frac{-1-n}{2}}}\zeta(-1-n)} = \frac{1}{\pi}\frac{(\frac{-1-n}{2})((\frac{+1-n}{2})\ldots\frac{1}{2}}{(-\frac{n}{2})(-\frac{n}{2}+1)\ldots 1}\frac{\zeta'(-n)}{\zeta(-1-n)} = -2^{-n+1}(n+1)\binom{n}{\frac{n}{2}}\frac{\zeta'(-n)}{\zeta(-1-n)} \tag{143}$$

3.2.2 The $(\mathfrak{g}, K_\infty)-$ cohomology

We have the local intertwining operator $T_\infty^{\mathrm{loc}} : I_{\varphi_\infty} \to I_{\varphi'_\infty}$, its kernel is the discrete series representation $\mathcal{D}_{n+2}$. The operator induces a homomorphism between complexes

$$\begin{array}{ccccc} & & & \mathrm{Hom}_{K_\infty}(\Lambda^1(\mathfrak{g}/\mathfrak{k}), \mathcal{D}_{n+2} \otimes \mathcal{M}_n) & \\ & & & \downarrow & \\ 0 \to & \mathrm{Hom}_{K_\infty}(\Lambda^0(\mathfrak{g}/\mathfrak{k}), I_{\varphi_\infty} \otimes \mathcal{M}_n) & \to & \mathrm{Hom}_{K_\infty}(\Lambda^1(\mathfrak{g}/\mathfrak{k}), I_{\varphi_\infty} \otimes \mathcal{M}_n) & \to \\ & \downarrow T_\infty^{0,\mathrm{loc}} & & \downarrow T_\infty^{1,\mathrm{loc}} & \\ 0 \to & \mathrm{Hom}_{K_\infty}(\Lambda^0(\mathfrak{g}/\mathfrak{k}), I_{\varphi'_\infty} \otimes \mathcal{M}_n) & \to & \mathrm{Hom}_{K_\infty}(\Lambda^1(\mathfrak{g}/\mathfrak{k}), I_{\varphi'_\infty} \otimes \mathcal{M}_n) & \to \end{array} \tag{144}$$

Following our procedure in [Eis] 4.3.3 we define differential forms $\omega_{\text{hol}}, \bar{\omega}_{\text{hol}} \in \text{Hom}_{K_\infty}(\Lambda^1(\mathfrak{g}/\mathfrak{k}), I_\varphi \otimes \mathcal{M}_{n,\mathbb{Z}[i]})$:

$$\begin{aligned}\omega_{\text{hol}}(P_+) = 4i^{n+1}\psi_{n+2} \otimes (X - iY)^n, \ \bar{\omega}_{\text{hol}}(P_-) = 0 \\ \bar{\omega}_{\text{hol}}(P_+) = 0, \ \bar{\omega}_{\text{hol}}(P_-) = 4i^{-n-1}\psi_{-n-2} \otimes (X - iY)^n\end{aligned} \tag{145}$$

(They differ by a factor 4 from the differential forms defined in [Eis]). As before we define

$$\omega_{\text{top}} = \frac{1}{2}(\omega_{\text{hol}} + \bar{\omega}_{\text{hol}}), \ \omega_{\text{null}} = \frac{1}{2}(\omega_{\text{hol}} - \bar{\omega}_{\text{hol}}). \tag{146}$$

Here we carry out the straightforward calculation for equation (71), we have

$$P_+ + P_- = 2H \ ; \ P_+ - P_- = 4iE_+ + 2iV \text{ where } V \in \mathfrak{k} \tag{147}$$

Hence $\omega_{\text{top}}(4iE_+) = \omega_{\text{top}}(P_+ - P_-)$ and

$$\begin{gathered}\omega_{\text{top}}(P_+ - P_-) = 2i^{n+1}\psi_{n+2} \otimes (X - iY)^n - 2i^{-n-1}\psi_{-n-2} \otimes (X + iY)^n = \\ 2i^{n+1}\psi_{n+2} \otimes (\cdots + i^n Y^n) - 2i^{-n-1}\psi_{-n-2} \otimes (\cdots + (-i)^n Y^n) = \\ 2i(\psi_{n+2} + \psi_{-n-2}) \otimes Y^n + \text{ terms without } Y^n\end{gathered} \tag{148}$$

The term $\omega_{\text{top}}(H)$ looks similar but the monomial Y^n does not occur.

The differential forms $\omega_{\text{hol}}, \bar{\omega}_{\text{hol}}$ are closed and hence they define cohomology classes. We have the Delorme isomorphism: Let $\mathfrak{t}, \mathfrak{u}$ be the Lie algebras of the torus and the unipotent radical of the Borel subgroup, then

$$H^\bullet(\mathfrak{g}, K_\infty, I_\varphi \otimes \mathcal{M}_{n,\mathbb{Z}[i]}) = H^\bullet(\Lambda^q(\mathfrak{t}), H^{\bullet - q}(\mathfrak{u}, \mathcal{M}_{n,\mathbb{Z}[i]})) \tag{149}$$

Then our formula above implies that ω_{top} represents the class in $H^1(\mathfrak{u}, \mathcal{M}_{n,\mathbb{Z}})$ which is represented by the form $E_+ \mapsto Y^n, H \mapsto 0$.

The form ω_{null} gives the trivial class in cohomology. Hence it is in the image of the boundary map

$$\text{Hom}_{K_\infty}(\Lambda^0(\mathfrak{g}/\mathfrak{k}), I_\varphi \otimes \mathcal{M}_{n,\mathbb{Q}[i]}) \xrightarrow{d_0} \text{Hom}_{K_\infty}(\Lambda^1(\mathfrak{g}/\mathfrak{k}), I_\phi \otimes \mathcal{M}_{n,\mathbb{Q}[i]}) \tag{150}$$

we have to write down an explicit bounding element. We know that I_φ contains the discrete series representation $\mathcal{D}_{n+2}$ as a submodule and $I_\varphi/\mathcal{D}_{n+2} \xrightarrow{\sim} \mathcal{M}_{n,\mathbb{Q}[i]}$ hence we can write an element

$$\alpha = a\psi_n \otimes (X - iY)^n + \cdots + a\psi_{-n} \otimes (X + iY)^n$$

in $I_\varphi \otimes \mathcal{M}_{n,\mathbb{Q}[i]}$ whose image in $I_\varphi/\mathcal{D} \otimes \mathcal{M}_{n,\mathbb{Q}[i]}$ is invariant under Gl_2. Then it is clear that $d\alpha$ must be a multiple of ω_{null}. We know how P_+, P_- act on the ψ_ν and get

$$P_+(\psi_n) = 2(n+1)\psi_{n+2}, P_-(\psi_{-n}) = 2(n+1)\psi_{-n-2}. \tag{151}$$

and hence

$$d\alpha(H) = \frac{1}{2}d\alpha(P_+ + P_-) = (n+1)(a\psi_{n+2} \otimes (X - iY)^n + \cdots + a\psi_{-n-2} \otimes (X + iY)^n) \tag{152}$$

On the other hand we have

$$\omega_{\text{null}}(H) = \frac{1}{4}(\omega_{\text{hol}} - \bar{\omega}_{\text{hol}})(P_+ + P_-) = i^{n+1}\psi_{n+2} \otimes (X - iY)^n - i^{-n-1}\psi_{-n-2} \otimes (X + iY)^n \tag{153}$$

So we get the answer that our a in the definition of α satisfies $a(n+1) = i^{n+1}$ therefore we define

$$\alpha_0 = \frac{i(-1)^{n/2}}{n+1}(\psi_n \otimes (X - iY)^n + \cdots + \psi_{-n} \otimes (X + iY)^n)$$

and then

$$d\alpha_0 = \omega_{\text{null}}$$

3.2.3 The secondary class

Since ω_{null} is annihilated by $T_\infty^{1,\text{loc}}$ it follows that the differential form α_0 maps to a closed form $T^{\text{loc}}(\alpha_0) \in \text{Hom}_{K_\infty}(\Lambda^0(\mathfrak{g}/\mathfrak{k}), I_{\varphi'_\infty} \otimes \mathcal{M}_{n,\mathbb{Q}[i]})$, and hence it defines a cohomology class in

$$[T^{\text{loc}}(\alpha_0)] \in H^0(\mathfrak{g}, K_\infty, I_{\varphi'_\infty} \otimes \mathcal{M}_{n,\mathbb{Q}[i]}),$$

we call it the secondary class of α_0. We need to compute this class: To do this we observe

i) We have a $(\mathfrak{g}, K_\infty)-$ invariant pairing

$$< \, , \, >_\infty : I_{\varphi_\infty} \times I_{\varphi'_\infty} \to \mathbb{C}; \; < \psi_\nu, \psi_{-\mu} >_\infty = \delta_{\nu,\mu} \tag{154}$$

ii) We have the canonical inclusion $\iota_{\mathcal{M}} : \mathcal{M}_{n,\mathbb{Q}(i)} \hookrightarrow I_{\varphi'_\infty}$ which is defined by

$$P(X,Y) \mapsto f_P = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto P(c,d)\} \tag{155}$$

The pairing in i) induces a non degenerate pairing

$$< \, , \, >_{\bar{\infty}} : I_{\varphi_\infty}/\mathcal{D}_{n+2} \times \mathcal{M}_{n,\mathbb{Q}(i)} \to \mathbb{C} \tag{156}$$

iii)The local intertwining operator yields an isomorphism $\bar{T}_\infty^{\text{loc}} : I_{\varphi_\infty}/\mathcal{D}_{n+2} \xrightarrow{\sim} \mathcal{M}_{n,\mathbb{C}}$.

The pairing i) and the inclusion ii) together yield an isomorphism

$$\Psi_{\mathcal{M}} : \mathcal{M}_{n,\mathbb{C}} \xrightarrow{(\bar{T}_\infty^{\text{loc}})^{-1}} I_{\varphi_\infty}/\mathcal{D}_{n+2} \xrightarrow{\sim} \mathcal{M}_{n,\mathbb{C}}^\vee = \text{Hom}(\mathcal{M}_{n,\mathbb{C}}, \mathbb{C}) \tag{157}$$

which is defined by

$$\text{For } m_1 \in \mathcal{M}_{n,\mathbb{C}}, m \in \mathcal{M}_{n,\mathbb{Q}(i)} \text{ we have } \Psi_{\mathcal{M}}(m_1)(m) =< (\bar{T}_\infty^{\text{loc}})^{-1}(m_1), m >_{\bar{\infty}}. \tag{158}$$

We have a second $\mathfrak{g}^{(1)}$ invariant isomorphism (134)

$$\Phi_{\mathcal{M}} : \mathcal{M}_{n,\mathbb{Q}} \xrightarrow{\sim} \mathcal{M}_{n,\mathbb{Q}}^\vee \tag{159}$$

and these two isomorphisms differ by a scalar factor, $\Phi_{\mathcal{M}} = c_n \Psi_{\mathcal{M}}$. The polynomials $g_\mu = (X+iY)^\mu (X-iY)^{n-\mu} \in \mathcal{M}_{n,\mathbb{Q}(i)}$ are sent to $\psi_{2\nu-n}$ under $\iota_{\mathcal{M}}$. Since our local operator is normalized such that $T^{\text{loc}}_\infty(\psi_0) = \psi_0$ it follows that

$$\Psi_{\mathcal{M}}(g_{\frac{n}{2}})(g_{\frac{n}{2}}) = 1$$

On the other hand we can easily check that

$$\Phi_{\mathcal{M}}(g_{\frac{n}{2}})(g_{\frac{n}{2}}) =< g_{\frac{n}{2}}, g_{\frac{n}{2}} >_{\mathbb{Q}}= \binom{n}{\frac{n}{2}}^{-1} 2^n (-1)^{\frac{n}{2}}$$

Our differential form $T^{\text{loc}}_\infty(\alpha_0) \in \mathcal{M}_{n,\mathbb{Q}} \otimes \mathcal{M}_{n,\mathbb{Q}} \subset I_{\varphi'_\infty} \otimes \mathcal{M}_{n,\mathbb{Q}}$, the non degenerate form $< \, , \, >_{\mathbb{Q}} \in \mathcal{M}_{n,\mathbb{Q}} \otimes \mathcal{M}_{n,\mathbb{Q}}$ and then our computation yields

$$T^{\text{loc}}_\infty(\alpha_0) = \frac{i}{n+1} \binom{n}{\frac{n}{2}}^{-1} 2^n < \, , \, >_{\mathbb{Q}} \tag{160}$$

Remark: Here we observe that something (seemingly) miraculous happens. The scalar factor in front is -up to a sign and a factor 2 -the inverse of the factor in front of the ratio of ζ values in formula (143).

3.2.4 The extension class

We choose the open compact subgroup $K_f = \prod K_p$ where now $K_p = \mathrm{Gl}_2(\mathbb{Z}_p)$ for all $p \neq p_0$ and K_{p_0} is as in (188). We apply the considerations in section 2.2, and see that we can represent cohomology classes in $H^1(\overset{\bullet}{\mathcal{N}}\,\Sigma, \tilde{\mathcal{M}}_{n,\mathbb{Q}})$ by elements

$$\omega_\infty \otimes h \in \mathrm{Hom}_{K_\infty}(\Lambda^1(\mathfrak{g}/\mathfrak{k}), I_{\varphi_\infty} \otimes \tilde{\mathcal{M}}_{\mathbb{Q}}) \otimes I^{K_f}_{\varphi_f} \tag{161}$$

We have $I^{K_f}_{\varphi_f} = I^{K_{p_0}}_{\varphi_{p_0}} = \mathbb{Q} f_0^{(p_0)} \oplus \mathbb{Q} f_\infty^{(p_0)}$. We see that the two forms $\omega_{\text{hol}}, \omega_{\text{top}}$ define the same class in $H^1(\mathfrak{g}, K_\infty, I_{\varphi_\infty} \otimes \mathcal{M}_{n,\mathbb{Q}})$ and it clear that the forms $\omega_{\text{top}} \otimes f_0^{(p_0)}, \omega_{\text{hol}} \otimes f_0^{(p_0)}$ represent the class Ω_n. The Eisenstein differential forms $\mathrm{Eis}(\omega_{\text{top}} \otimes f_0^{(p_0)}, 0)$ resp. $\mathrm{Eis}(\omega_{\text{hol}} \otimes f_0^{(p_0)}, 0)$ represent classes in

$$\begin{gathered}[\mathrm{Eis}(\omega_{\text{top}} \otimes f_0^{(p_0)}, 0)] \in H^1_{\mathrm{Eis},B}(X_0(p_0), \tilde{\mathcal{M}}^{\#}_{n,\mathbb{C}}), \\ [\mathrm{Eis}(\omega_{\text{hol}} \otimes f_0^{(p_0)}, 0)] \in H^1_{\mathrm{Eis},d-Rh}(X_0(p_0), \tilde{\mathcal{M}}^{\#}_{n,d-Rh} \otimes \mathbb{C})\end{gathered} \tag{162}$$

The class Ω_n is the generator $1_B^{(-n-1)}$ in section 1.7.1 and $[\mathrm{Eis}(\omega_{\text{top}} \otimes f_0^{(p_0)}, 0]$?(resp.) $[\mathrm{Eis}(\omega_{\text{hol}} \otimes f_0^{(p_0)}, 0]$ are the Betti lift $e_B^{(-n-1)}$ (resp.) the de-Rham lift e_{DR}^{-n-1} and therefore the difference

$$\mathrm{Eis}(\omega_{\text{top}} \otimes f_0^{(p_0)}, 0) - \mathrm{Eis}(\omega_{\text{hol}} \otimes f_0^{(p_0)}, 0) \in \mathbb{C}(0) \subset H^1_{\mathrm{Eis}}(X_0(p_0), \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}}) \otimes \mathbb{C} \tag{163}$$

represents the Betti-de-Rham extension class. To compute this class we have to look at the homology groups with coefficients in $\underline{\mathcal{M}}^\vee_{n,\mathbb{Q}}$ and to consider the exact sequence for the relative of the pair $(X_0(p_0)(\mathbb{C}), \overset{\bullet}{\mathcal{N}}\,\Sigma_\infty)$

$$H_1(X_0(p_0)(\mathbb{C}), \underline{\mathcal{M}}^\vee_{n,\mathbb{Q}}) \to H_1((X_0(p_0)(\mathbb{C}), \overset{\bullet}{\mathcal{N}}\Sigma_\infty), \underline{\mathcal{M}}^\vee_{n,\mathbb{Q}})) \xrightarrow{\partial_1} H_0(\overset{\bullet}{\mathcal{N}}\Sigma_\infty, \underline{\mathcal{M}}^\vee_{n,\mathbb{Q}})) \to \tag{164}$$

The relative homology $H_0(\overset{\bullet}{\mathcal{N}}\Sigma_\infty, \underline{\mathcal{M}}^\vee_{n,\mathbb{Q}}) = (\underline{\mathcal{M}}^\vee_{n,\mathbb{Q}})_{\Gamma_{U_\infty}}$ where U_∞ is the unipotent radical of the opposite (with respect to our standard torus) Borel subgroup B_∞, i.e. $\Gamma_{U_\infty} = \{\begin{pmatrix} 1 & 0 \\ p_0 m & 1 \end{pmatrix} \mid m \in \mathbb{Z}\}$. The module of coinvariants is generated by the element X^n. We realize this homology class as the boundary of an explicit 1 cycle $\mathfrak{z} \in C_1((X_0(p_0)(\mathbb{C}), \overset{\bullet}{\mathcal{N}}\Sigma_\infty), \underline{\mathcal{M}}_\mathbb{Z})$. We refer to chap2.pdf section 1.7. The standard maximal torus T is contained in our two Borel subgroups B_0, B_∞ and the group $T_1(\mathbb{R})^{(0)} = \{\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} |t \in \mathbb{R}^\times_{>0}\}$ acts simply transitively on the unique geodesic joining these two points in $\mathbb{P}^1(\mathbb{Q})$. Let y_0 be any point on this geodesic, then we can identify it to $T_1(\mathbb{R})^{(0)}$ If $t \to 0$ then ty_0 goes to B_0, if $t \to \infty$ then ty_0 moves to B_∞. We choose a small number $1 >> c > 0$ and consider the interval $I_c = \{ty_0 | \; c \le t \le c^{-1}\}$ and the chain $I_c \otimes X^n$. The point $cy_0 \in \overset{\bullet}{\mathcal{N}}\Sigma_\infty$ and $c^{-1}y_0 \in \overset{\bullet}{\mathcal{N}}\Sigma_0$. The zero cycle $c^{-1}y_0 \otimes X^n \in C_0(\overset{\bullet}{\mathcal{N}}\Sigma_0, \mathcal{M}_{n,\mathbb{Z}})$ is a boundary of a cycle $\mathfrak{z}_0 \in C_1(\overset{\bullet}{\mathcal{N}}\Sigma_0, \mathcal{M}_{n,\mathbb{Z}})$ because we have

$$X^n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} X^{n-1}Y - X^{n-1}Y \tag{165}$$

Therefore the chain $I_c \otimes X^n - \mathfrak{z}_0 = \mathfrak{z}$ bounds the zero chain $cy_0 \otimes X^n$. Then it follows that the real number which gives our extension class is given by the integral

$$[H^1_{\mathrm{Eis}}(X_0(p_0), \tilde{\mathcal{M}}^\#_{n,\mathbb{Z}})]_{B-dRh} = \int_\mathfrak{z} \mathrm{Eis}(\omega_{\mathrm{null}} \otimes f^{(p_0)}, 0) \tag{166}$$

Then the theorem of Stokes yields

$$[H^1_{\mathrm{Eis}}(X_0(p_0), \tilde{\mathcal{M}}^\#_{n,\mathbb{Z}})]_{B-dRh} = \mathrm{Eis}(\alpha_0 \otimes f^{(p_0)}, 0)(cy_0 \otimes X^n) \tag{167}$$

We look at the constant term of the Eisenstein series

$$\alpha_0 \otimes f^{(p_0)} + T^{\mathrm{st}}(\alpha_0 \otimes f^{(p_0)}) \in I_\varphi \oplus I_{\varphi'} \tag{168}$$

we know from the theory of Eisenstein series that the

$$(\alpha_0 \otimes f^{(p_0)})(cy_0 \otimes X^n) + T^{\mathrm{st}}(\alpha_0 \otimes f^{(p_0)})(cy_0 \otimes X^n) - \mathrm{Eis}(\alpha_0 \otimes f^{(p_0)}, 0)(cy_0 \otimes X^n) \to 0 \tag{169}$$

converges very rapidly to zero if $c \to 0$. The first of the three terms tends to zero because $f^{(p_0)}$ has support in 0_{p_0}. Hence we have to compute $\lim_{c\to 0} T^{\mathrm{st}}(\alpha_0 \otimes f^{(p_0)})(cy_0 \otimes X^n)$. Now we have to invoke our local formulae for the intertwining operators (61),(160) and (143) and we get for the the extension class

$$[H^1_{\mathrm{Eis}}(X_0(p_0), \tilde{\mathcal{M}}^\#_{n,\mathbb{Z}})]_{B-dRh} = \frac{p_0^{n+1} - 1}{p_0^{n+2} - 1} \frac{1}{\zeta(-1-n)} (\frac{-2i}{\pi} \zeta'(-n)) \tag{170}$$

3.2.5 The p-adic extension class

We consider the realization in etale cohomology: For any prime $\ell \neq p_0$ we have the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})-$ modules $H^1_{\mathrm{Eis,\acute{e}t}}(X_0(p_0)\times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}} \otimes \mathbb{Z}_\ell)$ which sit in an exact sequence

$$0 \to \mathbb{Z}_\ell(0) \to H^1_{\mathrm{Eis,\acute{e}t}}(X_0(p_0) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}} \otimes \mathbb{Z}_\ell) \to \mathbb{Z}_\ell(-n-1) \to 0 \qquad (171)$$

and hence we get extension classes

$$[H^1_{\mathrm{Eis,\acute{e}t}}(X_0(p_0) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}} \otimes \mathbb{Z}_\ell] \in H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}, \mathbb{Z}_\ell(n+1)). \qquad (172)$$

At this very moment we can only formulate a conjecture which in some sense expresses the hope that these motives are not exotic. In other words we believe that the Betti-de-Rham extension class should determine the ℓ-adic extension class.

We constructed canonical elements $c_\ell(n) \in H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbb{Z}_\ell(n+1))$ and in a certain sense we should have

$$\frac{-2i}{\pi}\zeta'(-n) \text{ " = " } \log(c_\ell(n)) \qquad (173)$$

Then this leads to the conjecture

$$[H^1_{\mathrm{Eis,\acute{e}t}}(X_0(p_0) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}} \otimes \mathbb{Z}_\ell)] = c_\ell(n)^{\frac{p_0^{n+1}-1}{p_0^{n+2}-1}\frac{\Delta(n)}{\zeta(-1-n)}} \qquad (174)$$

3.2.6 The conjecture mod p

We can check this conjecture modulo p for almost all primes p. We choose a prime p and assume that there is no p torsion. Furthermore we assume that $\Delta(n)$ and $\zeta(-1-n)$ are units at p.

$$[H^1_{\mathrm{Eis,\acute{e}t}}(X_0(p_0) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}} \otimes \mathbb{Z}/p\mathbb{Z})] \in H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbb{Z}/p\mathbb{Z}(n+1)). \qquad (175)$$

and we want to show that this class is equal to (See section 1.7.2)

$$[H^1_{\mathrm{Eis,\acute{e}t}}(X_0(p_0) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}} \otimes \mathbb{Z}/p\mathbb{Z})] = c_{n,1}(p)^{\frac{p_0^{n+1}-1}{p_0^{n+2}-1}\frac{1}{\zeta(-1-n)}} \qquad (176)$$

The module $\mathcal{M}_n \otimes \mathbb{F}_p$ is a module for $\mathrm{Gl}_2(\mathbb{F}_p)$. We consider the character

$$\chi_n : B(\mathbb{F}_p) \to \mathbb{F}_p^\times \text{ given by } \begin{pmatrix} x & u \\ 0 & y \end{pmatrix} \mapsto x^n \qquad (177)$$

and consider the induced representation

$$I_{\chi_n} = \mathrm{Ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \chi_n \qquad (178)$$

We have an obvious inclusion of $\mathrm{Gl}_2(\mathbb{F}_p)$-modules $\mathcal{M}_n \otimes \mathbb{F}_p \to I_{\chi_n}$. These modules induce coefficient systems on $Y_0(p_0)(\mathbb{C})$ and we can perform the usual construction of extending them to sheaves on $X_0(p_0)(\mathbb{C})$. We get a homomorphism of cohomology groups

$$H^\bullet(X_0(p_0)(\mathbb{C}), \tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}} \otimes \mathbb{Z}/p\mathbb{Z}) \to H^\bullet(X_0(p_0)(\mathbb{C}), I^{\#}_{\chi_n}) \qquad (179)$$

In our paper on p-adic interpolation we show that these modules are modules for the Hecke algebra and we show that the Hecke operator T_p acts nilpotently on the cohomology of the quotient sheaf $(I^{\#}_{\chi_n}/\tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}})\otimes\mathbb{Z}/p\mathbb{Z}$. (See [Ip], section 3, 3.1) This allows us to define the ordinary cohomology and we get an isomorphism

$$H^{\bullet}_{\text{ord}}(X_0(p_0)(\mathbb{C}),\tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}}\otimes\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} H^{\bullet}_{\text{ord}}(X_0(p_0)(\mathbb{C}),I^{\#}_{\chi_n}) \tag{180}$$

Since the Eisenstein sub motive is defined by the condition that $T_p-(p+1)$ acts nilpotently on it we get an isomorphism

$$H^{\bullet}_{\text{Eis}}(X_0(p_0)(\mathbb{C}),\tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}}\otimes\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} H^{\bullet}_{\text{Eis}}(X_0(p_0)(\mathbb{C}),I^{\#}_{\chi_n}) \tag{181}$$

and this is an isomorphism of Galois modules if we pass to the p-adic realization. Hence we have to compute the class

$$[H^{\bullet}_{\text{Eis}}(X_0(p_0)\times_{\mathbb{Q}}\bar{\mathbb{Q}},I^{\#}_{\chi_n})]\in H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}),\mathbb{Z}/p\mathbb{Z}(n+1)), \tag{182}$$

this extension is the reduction mod p of a Kummer-Anderson motive. We apply the considerations from section 2.7 the ℓ will be replaced by p. The group of $p-1$-th roots of unity $\mu_{p-1}(\mathbb{Z}[\zeta_{p-1}])$ is the cyclic group of order $p-1$ generated by ζ_{p-1}. We identify it to $\mu_{p-1}(\mathbb{C})$ by sending $\zeta_{p-1}\mapsto e^{\frac{2\pi i}{p-1}}$. The Teichmüller character provides an inclusion $\omega:\mathbb{F}_p^{\times}\to\mathbb{Z}_p^{\times}$. The prime p splits completely in $\mathbb{Z}[\zeta_{p-1}]$. We choose a prime $\mathfrak{p}$ above p in $\mathbb{Z}[\zeta_{p-1}]$, this yields an inclusion $i_{\mathfrak{p}}:\mathbb{Z}[\zeta_{p-1}]\hookrightarrow\mathbb{Z}_p$ and hence an identification $i_{\mathfrak{p}}=\mu_{p-1}(\mathbb{Z}[\zeta_{p-1}])\xrightarrow{\sim}\mu_{p-1}(\mathbb{Z}_p)$. Then we define the character $\chi_{\mathfrak{p},p}=i_{\mathfrak{p}}^{-1}\circ\omega:\mathbb{F}_p^{\times}\to\mu_{p-1}(\mathbb{Z}[\zeta_{p-1}])=\mu_{p-1}(\mathbb{C})$. The notation indicates that $\chi_{\mathfrak{p},p}$ is the local component at p of a character $\chi_{\mathfrak{p}}$ which is unramified outside p. (See section 2.7).

From this we get the induced $\text{Gl}_2(\mathbb{F}_p)$ module $I_{\omega^n}=\text{Ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)}\omega^n$ which is a free $\mathbb{Z}_p$ module of rank $p+1$. We can define the Eisenstein motive $H^{\bullet}_{\text{Eis}}(X_0(p_0)\times_{\mathbb{Q}}\bar{\mathbb{Q}},I^{\#}_{\omega^n})$ which is a motive with coefficients in $\mathbb{Z}_p$.

This motive is actually the base extension of $H^1_{\text{Eis}}(X_0(p_0),\tilde{I}^{\#}_{\chi^n_{\mathfrak{p}}})$ via $i_{\mathfrak{p}}$ we computed its extension class in section 2.9. It is given by

$$[H^1_{\text{Eis}}(X_0(p_0),\tilde{I}^{\#}_{\chi^n_{\mathfrak{p}}})]=c(\chi_{\mathfrak{p},p})^{\frac{p_0^{-1}-p_0^{-2}\chi^n_{\mathfrak{p},p_0}(p_0)}{1-p_0^{-2}\chi^n_{\mathfrak{p},p_0}(p_0)}\frac{1}{L(\chi^n_{\mathfrak{p}},-1)}} \tag{183}$$

The Galois-module $H^{\bullet}_{\text{Eis}}(X_0(p_0)(\mathbb{C}),I^{\#}_{\chi_n})$ is the reduction of this class $\mod p$. Therefore we have evaluate the exponent $\mod \mathfrak{p}$ By definition (product formula) we have $\chi^n_{\mathfrak{p},p_0}(p_0)=\chi^{-n}_{\mathfrak{p},p}(p_0)$ and $\chi^{-n}_{\mathfrak{p},p}(p_0)\equiv p_0^{-n}\mod\mathfrak{p}$. Furthermore we know that $L_(\chi^n_{\mathfrak{p}},-1)\equiv\zeta(-1-n)\mod\mathfrak{p}$. Then a simple computation yields

$$[H^{\bullet}_{\text{Eis}}(X_0(p_0)\times_{\mathbb{Q}}\bar{\mathbb{Q}},I^{\#}_{\chi_n})]=c(\chi_{\mathfrak{p},p})^{\frac{p_0^{n+1}-1}{p_0^{n+2}-1}\frac{1}{\zeta(-1-n)}} \tag{184}$$

and this is our conjecture $\mod p$.

3.3 The $p-$ adic approximation of higher Anderson-Tate motives by Kummer motives

We describe a strategy to prove our conjecture by reducing it to the proof of a congruence relation between special values of L-functions which looks as follows

$$\sum_{\chi\in((\mathbb{Z}/\ell^r)^\times(1))^\vee}\frac{1}{\Lambda(\omega^n\chi,-1)}\sum_{a\in(\mathbb{Z}/p^r)^\times}\frac{p_0-\chi_{p_0}(p_0)}{p_0^2-\chi_{p_0}(p_0)}a^nT_\ell^{\mathrm{loc}}(\hat{\omega_n}(\chi_p))(a)\equiv\frac{p_0^{n+1}-1}{p_0^{n+2}-1}\frac{1}{\zeta(-1-n)}\mod \ell^r \tag{185}$$

Here the point is that that the inner sum is a local term obtained by a local contribution at the two primes p_0 (the first factor) and p the second factor (this is the reason why the local components of our character χ_{p_0},χ_p appear). The shape of this second factor is in principle correct, the computation of it explicit form is postponed.

We could also say that the second factor is a product of two factors

$$\frac{p_0-\chi_{p_0}(p_0)}{p_0^2-\chi_{p_0}(p_0)}(\sum_{a\in(\mathbb{Z}/\ell^r\mathbb{Z})^\times}a^nT_\ell^{\mathrm{loc}}(\hat{\omega_n}(\chi_\ell))(a))=\frac{p_0-\chi_{p_0}(p_0)}{p_0^2-\chi_{p_0}(p_0)}\tau(\chi_\ell,n,r) \tag{186}$$

We explain the term $\tau(\chi_\ell,n,r)$, to do this we extend the computation in section (2.7) to higher ramification.

3.3.1 Wildly ramified Kummer-Anderson motives

I want to get rid of the problems with the center. We pass to the adjoint group $G=\mathrm{PGl}_2/\mathbb{Z}$. We assume that n is even, then can define the Gl_2- module $\mathcal{M}_n[-\frac{n}{2}]/\mathbb{Z}$, we simply twist the Gl_2 action by $\det^{-n/2}$ then the representation becomes trivial on the center and hence $\mathcal{M}_n[-\frac{n}{2}]$ is a G-module.

$$K_f=\prod_{p\neq p_0,\ell}\mathrm{Gl}_2(\mathbb{Z}_p)\times K_{p_0}\times K_\ell \tag{187}$$

where we allow wild ramification at ℓ, i.e. we choose a number $r>1$ and then

$$K_{p_0}=K_0(p_0)=\{\begin{pmatrix}a&b\\c&d\end{pmatrix}|c\equiv 0\mod p_0\},K_\ell=K_1(\ell^r)=\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\mid a/d\equiv 1\mod \ell^r,\ c\equiv 0\mod \ell^r\} \tag{188}$$

The group K_ℓ is the inverse image of the unipotent group $U(\mathbb{Z}/\ell^r\mathbb{Z})$. For the group of real points we restrict to $G^+(\mathbb{R})$, this is the subgroup of elements with determinant >0, i.e the topological connected component of $G(\mathbb{R})$. We then have the subgroup $G^+(\mathbb{A})$ in the group of adeles and we define $G^+(\mathbb{Q}),T^+(\mathbb{A}),T^+(\mathbb{Q})$... accordingly. From now on we will suppress the superscript $^+$. We denote the the resulting Shimura varieties by $Y_{0,1}(p_0,\ell^r)\subset X_{0,1}(p_0,\ell^r)$. Again we consider the set of cusps

$$\Sigma_{p_0,\ell^r}=\{0_{p_0},\infty_{p_0}\}\times\bar{\Sigma}_{\ell^r}=U(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K_f=U(\mathbb{F}_{p_0})\backslash\mathbb{P}^1(\mathbb{F}_{p_0})\times(U(\mathbb{Z}/\ell^r\mathbb{Z})\backslash(G(\mathbb{Z}/\ell^r\mathbb{Z})/U(\mathbb{Z}/\ell^r\mathbb{Z}).$$

The quotient

$$G(\mathbb{Z}/\ell^r\mathbb{Z})/U(\mathbb{Z}/\ell^r\mathbb{Z})\xrightarrow{\sim}\mathbb{A}^2\setminus\{(0)\}$$

where of course $\mathbb{A}^2$ is the affine 2 space. We identify

$$(\mathbb{A}^2\setminus\{0\})(\mathbb{Z}/\ell^r\mathbb{Z})=\{(x,z,y)|xy+z^2=0,x,y\text{ coprime }\}.$$

Then we get a covering $(\mathbb{A}^2 \setminus \{0\}) = W_{x\neq 0} \cup W_{y\neq 0}$.

We get a section

$$\mathbf{c} : (\mathbb{A}^2 \setminus \{0\})(\mathbb{Z}/\ell^r\mathbb{Z}) \to G(\mathbb{Z}/\ell^r\mathbb{Z})$$

which is given by

$$\mathbf{c} : (x, y, z) \mapsto \begin{pmatrix} 1+z & x \\ y & 1-z \end{pmatrix}$$

We return to the situation discussed in (2.8) where in some sense we discussed tamely ramified Kummer-Anderson motives, now we allow higher ramification.

Again we have the projection map p_B from the set of cusps to the coarse set of cusps

$$\begin{array}{c} U(\mathbb{F}_{p_0})\backslash\mathbb{P}^1(\mathbb{F}_{p_0}) \times U(\mathbb{Z}/\ell^r\mathbb{Z})\backslash(\mathbb{A}^2 \setminus \{0\})(\mathbb{Z}/\ell^r\mathbb{Z}) \\ \downarrow p_B \\ B(\mathbb{F}_{p_0})\backslash\mathbb{P}^1(\mathbb{F}_{p_0}) \times (B(\mathbb{Z}/\ell^r\mathbb{Z})\backslash(\mathbb{A}^2 \setminus \{0\}))(\mathbb{Z}/\ell^r\mathbb{Z}). \end{array} \tag{189}$$

where the projection in the first factor is a bijection. The torus $T(\mathbb{Z}/\ell^r\mathbb{Z})$ acts transitively on the fibers.

We study the action of the diagonal torus $T(\mathbb{Z}/\ell^r\mathbb{Z})$ on $\mathbb{A}^2 \setminus \{0\})(\mathbb{Z}/\ell^r\mathbb{Z})$, it is of course the adjoint action on the section. We see

$$\mathrm{Ad}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix})(\begin{pmatrix} 1+z & x \\ y & 1-z \end{pmatrix}) = \begin{pmatrix} 1+z & tx \\ yt^{-1} & 1-z \end{pmatrix}$$

and this means under this action we can achieve that $x = 1$ on $W_{x\neq 0}(\mathbb{Z}/\ell^r\mathbb{Z})$.

If $x \equiv 0 \mod \ell$ then we can normalize $y = 1$. To get the complete description of the set $B(\mathbb{Z}/\ell^r\mathbb{Z})\backslash(\mathbb{A}^2 \setminus \{0\}))(\mathbb{Z}/\ell^r\mathbb{Z})$ will still have to divide by the action of $U(\mathbb{Z}/\ell^r\mathbb{Z})$ from the left. Then we see that the image of $W_{y\neq 0}(\mathbb{Z}/\ell^r\mathbb{Z})$ under p_B is simply a point which is represented by

$$u_\infty = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ the fiber } p_B^{-1}(u_\infty) \text{ is a torsor under } T(\mathbb{Z}/\ell^r\mathbb{Z}).$$

The other points where $y \equiv 0 \mod \ell$ are labelled according to the order $s = \mathrm{ord}_\ell(y) = \mathrm{ord}_\ell(z)$. Under the action of $B(\mathbb{Z}/\ell^r\mathbb{Z})$ every point can be brought into the form

$$u_{s,\epsilon} = \begin{pmatrix} 1+\ell^s\epsilon & 1 \\ -\epsilon^2\ell^{2s} & 1-\ell^s\epsilon \end{pmatrix} \text{ where } \epsilon \in (\mathbb{Z}/\ell^r\mathbb{Z})^\times/(\mathbb{Z}/\ell^r\mathbb{Z})^\times(r-2s)$$

where $(\mathbb{Z}/\ell^r\mathbb{Z})^\times(m) = \{a \in (\mathbb{Z}/\ell^r\mathbb{Z})^\times | a \equiv 1 \mod \ell^m\}$. Of course

$$(\mathbb{Z}/\ell^r\mathbb{Z})^\times/(\mathbb{Z}/\ell^r\mathbb{Z})^\times(m) = (\mathbb{Z}_\ell/\ell^m)^\times = T(\mathbb{Z}_\ell/\ell^m).$$

On the fibers of $p_B^{-1}(u_{s,\epsilon})$ we have the transitive action of $T(\mathbb{Z}/\ell^r\mathbb{Z})$. The stabilizer of $u_{s,\epsilon}$ is the subgroup $T(\mathbb{Z}/\ell^r\mathbb{Z})(m_{s,r})$ where $m_{s,r} = \min(s, r-s)$ so we find

$$p_B^{-1}(u_{s,\epsilon}) = T(\mathbb{Z}/\ell^r\mathbb{Z})/T(\mathbb{Z}/\ell^r\mathbb{Z})(m_{s,r})u_{s,\epsilon} = T(\mathbb{Z}_\ell/\ell^{m_{s,r}})u_{s,\epsilon}.$$

We pick an $u_{s,\epsilon}$ and a character $\chi \in T^\vee(\mathbb{Z}_\ell/\ell^{m_{s,r}})$ and define a function which is supported on $B(\mathbb{Z}/\ell^r\mathbb{Z})u_{s,\epsilon}$:

$$f_{s,\epsilon,\chi} : \mathrm{Gl}_2(\mathbb{Z}/\ell^r\mathbb{Z})/K_\ell \to \mathbb{Z}_\ell[\zeta_{\ell^{r-1}}], f_{s,\epsilon,\chi}(u_{s,\epsilon}) = 1, f_{s,\epsilon,\chi}(\begin{pmatrix} t & u \\ 0 & 1 \end{pmatrix} g) = \chi(t) f_{s,\epsilon,\chi} \tag{190}$$

Our previous considerations imply that

$$(\mathrm{Ind}_{B(\mathbb{Z}/\ell^r\mathbb{Z})}^{\mathrm{Gl}_2(\mathbb{Z}/\ell^r\mathbb{Z})} \chi)^{K_\ell} = \bigoplus_s \mathbb{Z}_\ell[\zeta_{\ell^{r-1}}] f_{s,\chi} \tag{191}$$

We consider the intertwining operator

$$T^{st} : \mathrm{Ind}_{B(\mathbb{Q}_\ell)}^{\mathrm{Gl}_2(\mathbb{Q}_\ell)} |\alpha| \chi \otimes \mathbb{Q} \to \mathrm{Ind}_{B(\mathbb{Q}_\ell)}^{\mathrm{Gl}_2(\mathbb{Q}_\ell)} \chi^{-1} \otimes \mathbb{Q} \tag{192}$$

which is defined by

$$T^{st}(f)(g) = \int_{U(\mathbb{Q}_\ell)} f(wug) du \tag{193}$$

here the measure is normalized such that the volume $U(\mathbb{Z}_\ell)$ becomes one. This of course induces an intertwining operator between the subspaces of K_ℓ invariants

$$T^{st} : \mathrm{Ind}_{B(\mathbb{Z}/\ell^r\mathbb{Z})}^{\mathrm{Gl}_2(\mathbb{Z}/\ell^r\mathbb{Z})} \chi \otimes \mathbb{Q} \to \mathrm{Ind}_{B(\mathbb{Z}/\ell^r\mathbb{Z})}^{\mathrm{Gl}_2(\mathbb{Z}/\ell^r\mathbb{Z})} w\chi \otimes \mathbb{Q} \tag{194}$$

We apply this to our basis and then we get

$$T^{st}(f_{s,\epsilon,\chi}) = \sum_{t,\eta} a_{s,\epsilon,t,\eta}(\chi) f_{t,\eta,\chi^{-1}} \tag{195}$$

of course we have to compute the matrix $a_{s,\epsilon,t,\eta}(\chi)$. In the case $r = 1$ this is done in (2.4.1), we postpone this computation.

We pass to the global situation. We assume that our character $\chi = \chi_\ell$, i.e. it is the local component at ℓ of a global character which is unramified outside ℓ.

We consider the function $f_0^{(p_0)} \times f_{s,\epsilon,\chi}$ and this provides a divisor D_{s,ϵ,χ,p_0} with coefficients $\mathbb{Z}_\ell[\zeta_{\ell^{r-1}}]$ which is supported in the fiber $p_P^{-1}(0_{p_0} \times \bar{\Sigma}_{\ell^r})$. Let us assume that χ is non trivial then degree of this divisor is zero, and hence we find a denominator $\Delta_{s,\epsilon,\chi,p_0} \in \mathbb{Z}_\ell[\zeta_{\ell^{r-1}}]$ such that $\Delta_{s,\epsilon,\chi,p_0} D_{s,\epsilon,\chi,p_0}$ becomes principal, we can write

$$\Delta_{s,\epsilon,\chi,p_0} D_{s,\epsilon,\chi,p_0} = \mathrm{Div}(H_{s,\epsilon,\chi,p_0}) \tag{196}$$

where

$$\mathrm{Div}(H_{s,\epsilon,\chi,p_0}) \in \mathbb{Q}(\zeta_{\ell^{r-1}})(X_{0,1}(p_0,\ell^r))^\times \otimes \mathbb{Z}_\ell[\zeta_{\ell^{r-1}}] \tag{197}$$

We play the same game as before: We pick two points $x, y \in p_P^{-1}(\infty_{p_0} \times \bar{\Sigma}_{\ell^r})$ then

$$\frac{H_{s,\epsilon,\chi,p_0})(x)}{H_{s,\epsilon,\chi,p_0})(y)} \in \mathbb{Q}(\zeta_{\ell^{r-1}})^\times \otimes \mathbb{Z}_\ell[\zeta_{\ell^{r-1}}] \tag{198}$$

and this numbers measures an extension class of a mixed Kummer-Anderson motive

$$[\Delta_{s,\chi,p_0} D_{s,\chi,p_0}[x-y]] \in \mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}_\ell[\zeta_{\ell^{r-1}}]\otimes\chi\otimes\mathbb{Z}(-1),\mathbb{Z}_\ell[\zeta_{\ell^{r-1}}]) \tag{199}$$

and our previous computation give us a formula for this extension class, it is given by

$$\prod_{a\in(\mathbb{Z}/\ell^{s_0}\mathbb{Z})^\times}(1-\zeta^a_{\ell^{s_0}})^{\chi_\ell(a)h(x,y,s,\epsilon,\chi)} \tag{200}$$

where

$$h(x,y,s,\chi) = (T^{st}(f_{s,\epsilon,\chi})(x) - T^{st}(f_{s,\epsilon,\chi})(y))\frac{1}{L(\chi,-1)} \tag{201}$$

3.3.2 The ℓ-adic approximation of higher Anderson-Tate motives by Kummer-motives.

We apply our considerations in (3.2.6) to higher ramified Kummer-Anderson motives. For any integer m we consider the homomorphisms

$$[e(m)]_r : T(\mathbb{Z}/\ell^r\mathbb{Z}) \to T(\mathbb{Z}/\ell^r\mathbb{Z}) \text{ which are defined by } x\mapsto x^m. \tag{202}$$

Our module $\mathcal{M}_n[-\frac{n}{2}]$ is as above, we introduce the induced module

$$I_{[e(\frac{n}{2})_r]} := \mathrm{Ind}^{G(\mathbb{Z}/\ell^r\mathbb{Z})}_{B(\mathbb{Z}/\ell^r\mathbb{Z})} e(\frac{n}{2})_r : \{f : G(\mathbb{Z}/\ell^r\mathbb{Z}) \to \mathbb{Z}/\ell^r\mathbb{Z} \mid f(\begin{pmatrix} t & u \\ 0 & 1\end{pmatrix} g)) = e(\frac{n}{2})_r(t)f(g)\} \tag{203}$$

As in our paper [Ha2] we can define the map

$$\mathcal{M}_n[-\frac{n}{2}]\otimes\mathbb{Z}/\ell^r\mathbb{Z} \to I_{e(\frac{n}{2})_r} : \; P\mapsto \{f_P : \begin{pmatrix} a & b \\ c & d\end{pmatrix} \mapsto P(c,d)\det\begin{pmatrix} a & b \\ c & d\end{pmatrix}^{-\frac{n}{2}}) \tag{204}$$

this induces a homomorphism

$$H^1(X_0(p_0)(\mathbb{C}),\mathcal{M}_n[-\frac{n}{2}]\otimes\mathbb{Z}/\ell^r\mathbb{Z}) \to H^1(X_0(p_0)(\mathbb{C}), I_{e(\frac{n}{2})_r}) \tag{205}$$

and we know that this induces an isomorphism if we restrict this map to the ordinary part (with respect to ℓ) H^1_{ord} on both sides. Since our module is induced we get

$$H^1(X_0(p_0)(\mathbb{C}), I_{e(\frac{n}{2})_r}) \xrightarrow{\sim} H^1(X_0(p_0\ell^r)(\mathbb{C}),\mathbb{Z}/\ell^r\mathbb{Z}\otimes e(\frac{n}{2})_r) \subset H^1(X_{0,1}(p_0,\ell^r)(\mathbb{C}), I_{[e(\frac{n}{2})_r]}) \tag{206}$$

The same applies to the cohomology of the boundary. Let $\overset{\bullet}{\mathcal{N}}\Sigma_{0_{p_0},\ell^r}$ be the tubular neighborhood of $p_P^{-1}(0_{p_0}\times\bar\Sigma_{\ell^r})$ then we get

$$H^1(\overset{\bullet}{\mathcal{N}}\Sigma_{0_{p_0}},\mathcal{M}_n[-\frac{n}{2}]\otimes\mathbb{Z}/\ell^r\mathbb{Z}) \to H^1(\overset{\bullet}{\mathcal{N}}\Sigma_{0_{p_0}}, I_{e(\frac{n}{2})_r}) \hookrightarrow H^1(\overset{\bullet}{\mathcal{N}}\Sigma_{0_{p_0},\ell^r},\mathbb{Z}/\ell^r\mathbb{Z}) \tag{207}$$

Our element Ω_n (See 136) gives us an element $\tilde\Omega_n \in H^1(\overset{\bullet}{\mathcal{N}}\Sigma_{0_{p_0},\ell^r},\mathbb{Z}/\ell^r\mathbb{Z})$ and this can be interpreted as a divisor module ℓ^r.

4 Anderson motives for the symplectic group

4.1 The basic situation

I may consider the group $G = GSp_g$ and consider the double quotient

$$G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f$$

where K_f is a suitable open compact subgroup in the group of finite adeles and X is the hermitian symmetric domain attached to this group. This quotient is the set of complex valued point of a quasiprojective scheme

$$\mathcal{S}^G_{K_f} = \mathcal{S} \longrightarrow \mathrm{Spec}(\mathbb{Z}[\frac{1}{N}])$$

where N is the product of primes occurring in the congruences defining K_f. Hence the topological space will now be denoted by

$$\mathcal{S}^G_{K_f}(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f.$$

Remark: In the following exposition we have a slight notational inconsistency. For any reductive group $M/\mathbb{Q}$ we can define the spaces

$$\mathcal{S}^M_{K^M_f} = M(\mathbb{Q})\backslash M(\mathbb{A})/K^M_\infty \times K^M_f \tag{208}$$

or more generally leads G leads to a Shimura variety, then we gave a different meaning to $\mathcal{S}^G_{K_f}$ in this case it is a scheme and the resulting locally symmetric space is $\mathcal{S}^G_{K_f}(\mathbb{C})$. Hence $\mathcal{S}^G_{K_f}$ has two different meanings. In the following text $\mathcal{S}^G_{K_f}$ will be most of the time the topological space, and only under certain circumstances we remember that the set of complex points of a scheme, which is denoted by the same letter.

If we consider an irreducible rational representation

$$\rho : G/\mathbb{Q} \longrightarrow GL(\mathcal{M}_\mathbb{Q})$$

of the algebraic group $G/\mathbb{Q}$. This representation is given by its highest weight $\lambda = \sum n_i\gamma_i + m\mu$, where the γ_i are the fundamental weights and where μ is the weight character. Then this representation provides a sheaf $\tilde{\mathcal{M}}_\mathbb{Q}$ of $\mathbb{Q}$-vector spaces on our complex variety $\mathcal{S}^G_{K_f}(\mathbb{C})$. If we are a little bit careful und if we write

$$\mathcal{S}^G_{K_f}(\mathbb{C}) = \bigcup \Gamma_i\backslash X$$

with some congruence subgroups $\Gamma_i \subset G(\mathbb{Q})$ (or maybe even better $\Gamma_i \subset G(\mathbb{Z})$), then we can choose Γ_i-invariant lattices in $\mathcal{M}_\mathbb{Z}$ in $\mathcal{M}$ and this provides sheaves $\tilde{\mathcal{M}}_\mathbb{Z}$ on $\mathcal{S}^G_{K_f}(\mathbb{C})$.

During the progress of this notes we have to enlarge the ring $\mathbb{Z}$ to a larger ring R at several occasions. This larger ring R will be obtained from $\mathbb{Z}$ by inverting some primes and and then we take the integral closure of this new ring in an algebraic extension $K/\mathbb{Q}$. We tensorize the sheaves $\tilde{\mathcal{M}}_\mathbb{Z}$ by R and the resulting sheaves will be denoted by $\tilde{\mathcal{M}}$.

At the beginning of our discussion we do not know how big we have to choose R, whenever I enlarge it I will say which new primes have to be inverted and which further algebraic extensions have to be taken. These primes will be called *small* primes.

These sheaves are obtained from the universal family of principally polarized abelian varieties and products, symmetric parts and so on – over $\mathcal{S}^G_{K_f}$. Let us denote this motivic sheaf also by $\tilde{\mathcal{M}}$ and rebaptize the old $\tilde{\mathcal{M}}$ to $\tilde{\mathcal{M}}_B$, i.e. $\tilde{\mathcal{M}}_B$ is the sheaf of Betti- cohomology groups of this motive. At this step we may have invert some primes, I think tat the primes which are smaller than the coefficients $n_i$of our highest weight are enough.

We may also consider the sheaf of de-Rham cohomology groups

$$\tilde{\mathcal{M}}_{\mathrm{dRh}}/\mathcal{S}^G_{K_f},$$

it comes with a filtration and the Gauss-Manin connection

$$\nabla : \tilde{\mathcal{M}}_{\mathrm{dRh}} \longrightarrow \tilde{\mathcal{M}}_{\mathrm{dRh}} \otimes \Omega^1_S$$

which is flat and satisfies a Griffith transversality condition.

Finally we can consider consider a prime $\mathfrak{l}$, which lies over a prime ℓ and $\tilde{\mathcal{M}}_{\mathfrak{l}} = \tilde{\mathcal{M}}_B \otimes R_{\mathfrak{l}}$, and this is a system of $\mathfrak{l}$-adic sheaves on $\mathcal{S}^G_{K_f}$. If ℓ is not invertible in R then $R_{\mathfrak{l}}$ is a field.

Let $\mathcal{S} \overset{i}{\hookrightarrow} \mathcal{S}^{\wedge\wedge}$ where $\mathcal{S}^{\wedge\wedge}$ is a smooth compactification obtained by the method of toroidal embeddings (Faltings-Chai). We have that

$$\mathcal{S}^{\wedge\wedge}\backslash\mathcal{S} = \bigcup_{[P]} \mathcal{S}^{\wedge\wedge}_{[P]} = \mathcal{S}^{\wedge\wedge}_{\infty}$$

where $[P]$ runs over the conjugacy classes of maximal parabolic subgroups. The Levi-quotients of these maximal parabolic subgroups are essentially products $Gl_{g-a} \times GSp_a \times G_m$ where a runs from 0 to $g-1$. We call the parabolic for $a = 0$ the Siegel parabolic and for $a = g-1$ the Klingen parabolic. The boundary stratum corresponding to the Klingen parabolic subgroup is a union of Shimura varieties attached to GSp_{g-1} together with their universal abelian variety over it. So it is of codimension 1 and a smooth divisor provided K_f is sufficiently small.

The $\mathcal{S}^{\wedge\wedge}_{[P]}$ attached to the Siegel parabolic is a configuration smooth toroidal varieties of dimension $\frac{g(g+1)}{2} - 1$ with transversal intersections. The combinatorics of this configuration is governed by taking certain cone decompositions for the action of congruence subgroups $\Gamma' \subset Gl_g(\mathbb{Z})$ on the positive definite symmetric matrices in $M_g(\mathbb{R})$. I will come back to this point later. For the other strata we get something in between.

We can construct "motivic sheaves" on $\mathcal{S}^{\wedge\wedge}$ by extending $\tilde{\mathcal{M}}$ from $\mathcal{S}$ to $\mathcal{S}_\infty$ where we require support conditions for these extensions. We are mainly interested in the Siegel parabolic and hence we extend somehow to the strata $S^{\wedge\wedge}_Q$ which are different from the Siegel stratum. Then we take an auxiliary prime p_0 and choose a congruence subgroup $K_f(p_0) \subset K_f$. (This is similar to the construction in my book.) We get a decomposition of $\mathcal{S}^{\wedge\wedge}_{[P]}$ into different connected components. And then according to certain rules we extend $\tilde{\mathcal{M}}$ to $\mathcal{S}^{\wedge\wedge}_{[P]}$.

Of course we may still take the full direct image $i_*(\tilde{\mathcal{M}})$ (here we take the derived functor) and we consider the cohomology

$$H^\bullet(S_\infty, i_*(\tilde{\mathcal{M}}))$$

as a mixed motive over $\mathbb{Z}[\frac{1}{p_0 N}]$ with coefficients in R.

If we consider the Betti cohomology of this motive then we can compute it using the Borel-Serre compactification and we apply our considerations from [MixMot 3.1].

We write the compactification

$$\mathcal{S}^G_{K_f}(\mathbb{C}) \longrightarrow \overline{\mathcal{S}^G_{K_f}}$$

and $\overline{\mathcal{S}^G_{K_f}}$ is a manifold with corners. We have

$$\overline{\mathcal{S}^G_{K_f}} \setminus \mathcal{S}^G_{K_f}(\mathbb{C}) = \bigcup_P \partial_P \mathcal{S} = \partial \mathcal{S}$$

where now P runs over all parabolic subgroups containing a fixed Borel subgroup.

We choose a Borel subgroup B and let us choose P to be the representative of P which contains B.

Then we have a finite coset decomposition

$$G(\mathbb{A}_f) = \bigcup_{\xi_f} P(\mathbb{A}_f)\xi_f K_f$$

and we recall from [MixMot 3.1] that we have

$$H^\bullet(\partial_P \mathcal{S}, \tilde{\mathcal{M}}_R) = \bigcup_{\xi_f} H^\bullet(\mathcal{S}^M_{K_f^M(\xi_f)}, \widetilde{H^\bullet(\mathfrak{u}, \mathcal{M})}_R),$$

$$H^\bullet(\mathfrak{u}, \mathcal{M}) = \bigoplus_{w \in W^P} H^{l(w)}(\mathfrak{u}, \mathcal{M})(w \cdot \lambda),$$

where W^P is the set of Kostant representatives of W/W^M and where $w \cdot \lambda = (\lambda + \rho)^w - \rho$ and ρ is the half sum of positive roots.

At this point I am rather imprecise about which primes should be inverted, a safe choice would be to invert all primes which ae less or equal to the numbers n_i which enter in our highest weight. But I am not so sure whether this choice is too cautious, I will discuss this problem later.

Remark. Let us assume for the moment that $g = 2$, Let P be the Siegel parabolic and Q be the Klingen. I have explained that the strata $S_P^{\wedge\wedge}$ and $S_Q^{\wedge\wedge}$ are different in nature. This is also reflected in the Borel-Serre compactification or better in the cohomology of the two strata. Let M (resp. M_1) be the reductive quotient for the Siegel (resp. Klingen) parabolic. In the following discussion I suppress the ξ_f because what I am saying does not depend on this variable. We form the symmetric spaces $\mathcal{S}^M_{K^M_f}$ and $\mathcal{S}^{M_1}_{K^{M_1}_f}$, then they are of the form

$$\begin{array}{ccc} M(\mathbb{Q})\setminus M(\mathbb{R})/K^M_\infty & \times & M(\mathbb{A}_f)/K^M_f \\ M_1(\mathbb{Q})\setminus M_1(\mathbb{R})/K^{M_1}_\infty & \times & M_1(\mathbb{A}_f)/K^{M_1}_f \end{array}$$

and the groups $K^M_\infty, K^{M_1}_\infty$ are the images of $P(\mathbb{R})\cap K_\infty$, $Q(\mathbb{R})\cap K_\infty$ respectively.

Both groups M, M_1 are naturally product of the form $GL_2 \times G_m = M^{(1)} \times G_m, M_1^{(1)} \times G_m$. Now $K_\infty \cap M^{(1)}(\mathbb{R})$ is not connected but $K_\infty \cap M_1^{(1)}(\mathbb{R})$ is. This has some influence on the structure of the cohomology. We consider the cohomology

$$H^\bullet(M(\mathbb{Q})\setminus M(\mathbb{R})/K^M_\infty \times M(\mathbb{A}_f)/K_f, H^\bullet(\mathfrak{u}, \tilde{\mathcal{M}}))$$

as a module under the Hecke algebra

$$\mathcal{H}^M = \mathcal{C}_c(K^M_f \setminus M(\mathbb{A}_f)/K^M_f).$$

If we replace K^M_∞ by its connected component $\overset{\circ}{K}{}^M_\infty$, then the cohomology becomes a $\mathcal{H}^M \times \pi_0(M(\mathbb{R}))$-module where $\pi_0(M(\mathbb{R}))$ is as usual the group of connected components. If we restrict to the action of the Hecke algebra, then

$$H^\bullet(M(\mathbb{Q})\setminus M(\mathbb{R})/\overset{\circ}{K}{}^M_\infty \times M(\mathbb{A}_f)/K_f, \widetilde{H^\bullet(\mathfrak{u}, \mathcal{M})})$$

decomposes under $\mathcal{H}^{\mathcal{M}}$ into

$$\bigoplus_{\sigma_f} H^\bullet(M(\mathbb{Q})\setminus M(\mathbb{R})/\overset{\circ}{K}{}^M_\infty \times M(\mathbb{A}_f)/K_f, \widetilde{H^\bullet(\mathfrak{u}, \mathcal{M})})(\sigma_f) \oplus$$
$$\bigoplus_{\tau_f} H^\bullet(M(\mathbb{Q})\setminus M(\mathbb{R})/\overset{\circ}{K}{}^M_\infty \times M(\mathbb{A}_f)/K_f, \widetilde{H^\bullet(\mathfrak{u}, \mathcal{M})})(\tau_f)$$

where σ_f, τ_f are irreducible R-modules under the Hecke algebra. Here we must enlarge our ring R. We have to be sure that the eigenvalues of the Hecke-operators (of course we only take $\mathbb{Z}$-valued functions in the Hecke-algebra) lie in R and we need that there are no congruence amoung the moular forms. The isotypical components σ_f have multiplicity two.

Then we know:

The σ_f are modules given by Hecke modules on the space of certain cusp forms where the weight and level are determined by $\mathcal{M}$ and K_f. The τ_f correspond to Hecke modules attached to Eisenstein series of the same weight and level.

Now we know that the σ_f components come with multiplicity two and the τ_f come with multiplicity one. These considerations are valid for M and for M_1.

But now we observe that we have replaced K^M_∞ by $\overset{\circ}{K}{}^M_\infty$. It is easy to understand the effect of this manipulation. We recall that we have an action of

$\pi_0(G(\mathbb{R}))$ and the image $\pi_0(K)$ of K_∞^M in $\pi_0(G(\mathbb{R}))$ is non trivial (The connected component $\overset{\circ}{K}{}_\infty^M$ goes to zero.). This means

$$H^\bullet(M(\mathbb{Q})\setminus(M(\mathbb{R})/K_\infty^M)\times M(\mathbb{A}_f)/K_f^M,\widetilde{H^\bullet(\mathfrak{u},\mathcal{M})}) =$$
$$H^\bullet(M(\mathbb{Q})\setminus(M(\mathbb{R})/\overset{\circ}{K}{}_\infty^M)\times M(\mathbb{A}_f)/K_f^M,\widetilde{H^\bullet(\mathfrak{u},\mathcal{M})})^{\pi_0(K)}. \tag{209}$$

In the case of the Klingen parabolic subgroup $\pi_0(K)=1$ but in the case of the Siegel parabolic subgroup the group $\pi_0(K)$ has both eigenvalues ±1 on the isotypic components

$$H^\bullet(\ ,H^\bullet(\ ))(\sigma_f)=H^\bullet_+(\ ,H^\bullet(\ ))(\sigma_f)\oplus H^\bullet_-(\ ,H^\bullet(\ ))(\sigma_f).$$

We return to the case of a general genus g, we will be mostly interested in the Siegel parabolic in the following we reserve the name P for it, let M be its reductive quotient it is a $\mathrm{Gl}_g\times\mathbb{G}_m$. If we consider the cohomology then we have the surjective map from the cohomology with compact support to the inner cohomology.

$$H_c^\bullet(\mathcal{S}^M_{K_f^M},\widetilde{H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)})\longrightarrow H_!^\bullet(\mathcal{S}^M_{K_f^M},\widetilde{H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)}).$$

These modules for the Hecke-algebra $\mathcal{H}^M(M(\mathbb{A})//K_f^M)$, according to a theorem of Franke and Schwermer the surjective map has a canonical rational splitting. If some congruence primes for the cohomology are invertible in R and the quotient field of R is large enough then we get an isotypical decompositions over R

$$H_c^\bullet(\mathcal{S}^M_{K_f^M},\widetilde{H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)})\xrightarrow{\sim}H^\bullet_{\mathrm{Eis}}\oplus H_!^\bullet(\mathcal{S}^M_{K_f^M},\widetilde{H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)}).$$

$$H_!^\bullet(\mathcal{S}^M_{K_f^M},\widetilde{H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)})=\bigoplus_{\sigma_f}H_!^\bullet(\mathcal{S}^M_{K_f^M},\widetilde{H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)})(\sigma_f),$$

where the σ_f are irreducible modules for the Hecke-algebra $\mathcal{H}^M(M(\mathbb{A})//K_f^M)$. We have also the Hecke -algebra $\mathcal{H}^G(G(\mathbb{A})//K_f)$ and I abbreviate the notation by calling them $\mathcal{H}^M,\mathcal{H}^G$.

$$\mathrm{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G}H_!^\bullet(\mathcal{S}^M_{K_f^M},\widetilde{H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)})(\sigma_f)\to H^\bullet(\partial\mathcal{S},\tilde{\mathcal{M}}),$$

and again we may have to invert a few more primes.

The modules σ_f have a central character $\omega(\sigma_f)$ which is an algebraic Hecke-character and the type of this character can be read off from the data λ,w.

From this algebraic Hecke character we get another algebraic Hecke character

$$\tilde{\omega}(\sigma_f):I_{\mathbb{Q},f}\to R^*$$

whose weight is equal something computed from $w\cdot\lambda$ and perhaps we call it simply $\mathbf{w}(w\cdot\lambda)$.

Now we invoke a theorem of R. Pink which tells us that

The isotypical component $H^\bullet_!(\mathcal{S}^M_{K^M_f}, \widetilde{H^{l(w)}(\mathfrak{u},\mathcal{M})}(w\cdot\lambda))(\sigma_f)$ *is pure Tate motive*

$$H^\bullet_!(\mathcal{S}^M_{K^M_f}, \widetilde{H^{l(w)}(\mathfrak{u},\mathcal{M})}(w\cdot\lambda))(\sigma_f)\otimes R(\tilde{\omega}(\sigma_f)).$$

We have to inquire whether this inner cohomology can be non zero. A necessary condition is that the representations of M on the cohomology $\widetilde{H^{l(w)}(\mathfrak{u},\mathcal{M})}(w\cdot\lambda)$ is self dual. This is of course not a problem if $g=2$ because in this case the semi simple type of M is A_1. We will discuss later what happens if $g\geq 3$

We want to discuss the construction of sheaves with support conditions on $S^{\wedge\wedge}$. I assume that my subgroup is of the form $K_f=\prod_p K_p$ where K_p is open in $G(\mathbb{Z}_p)$ and equal to it for almost all primes p. I choose an auxiliary prime p_0 for which $K_{p_0}=G(\mathbb{Z}_{p_0})$. I consider a group $K_0(p_0)\subset G(\mathbb{Z}_{p_0})$ whose reduction mod p_0 is a Borel subgroup $\bar{B}(\mathbb{F}_{p_0})\subset G(\mathbb{F}_{p_0})$. We define the new $K_f(p_0)$ to be the subgroup of K_f where I replace the component at p_0 by $K_f(p_0)$. With respect to this choice of the open compact subgroup I define my space $\mathcal{S}^G_{K_f}$.

The boundary of $\mathcal{S}^G_{K_f}$ will now have a certain combinatorial structure obtained from the prime p_0. We have an action of the group $B(\mathbb{F}_{p_0})$ on the sets of different types of parabolic subgroups. We form the simplicial set $\mathcal{T}$ whose vertices are the maximal parabolic subgroups in $G(\mathbb{F}_{p_0})$ modulo this action and the simplices of maximal dimension are the Borel subgroups modulo this action. If we consider the character module $X^*(T)$ of a maximal torus then the maximal simplices are just the chambers and so on.

We from reduction theory we get a projection map

$$\pi: S_\infty = S^{\wedge\wedge}\setminus S\to\mathcal{T}.$$

If we take a closed subset $\Xi\subset\mathcal{T}$ then the inverse image of this closed subset will be an open subset S_∞ and its union with the interior will provide an open subset $S_\Xi\subset S^{\wedge\wedge}$. We have the chain of inclusions

$$i^\Xi: S\hookrightarrow S_\Xi \text{ and } i_\Xi: S_\Xi\hookrightarrow\mathcal{T}$$

We extend our sheaf $\tilde{\mathcal{M}}$ from $\mathcal{S}^G_{K_f}$ to S_Ξ by zero, i.e. we take the sheaf $i^\Xi_!(\tilde{\mathcal{M}})$ and then we take the full direct image $i_{\Xi,*}(i^\Xi_!(\tilde{\mathcal{M}}))$ This gives us a sheaf $\tilde{\mathcal{M}}_\Xi$ on $S^{\wedge\wedge}$ and we can consider its cohomology

$$H^\bullet(S^{\wedge\wedge},\tilde{\mathcal{M}}_\Xi).$$

Now we will investigate this sheaf and we want to analyze to what extend we can find mixed Tate motives inside this cohomology.

To understand this we look at the middle dimension first. Let $d=\frac{g(g+1)}{2}$ and we consider the maps in the Betti cohomology

$$H^d(\mathcal{S}^G_{K_f}(\mathbb{C}),\tilde{\mathcal{M}}_B)\to H^d(\partial\mathcal{S},\tilde{\mathcal{M}}_B)$$
$$H^{d-1}(\partial\mathcal{S},\tilde{\mathcal{M}}_B)\to H^d_c(\mathcal{S}^G_{K_f}(\mathbb{C}),\tilde{\mathcal{M}}_B)$$

We have the Dynkin Diagram as above but now α_g will denote the long root at the right end. To this root corresponds an injective cocharacter $\chi_g: G_m\to T\subset G$ which is defined by $<\chi_g,\alpha_j>=2\delta_{jg}$ and by the requirement that it factors through the semisimple part $G^{(1)}$ of G. Hence it is clear that $\chi_g(G_m) = A$ is the central torus of M intersected with $G^{(1)}$.

Let $\gamma_0 : A \to G_m$ be the character for which $\gamma_0 \circ \chi_g(x) = x$.

If as usual $\gamma_1, \gamma_2, \ldots, \gamma_g$ are the dominant weights, the γ_g extends to a character on M and for the restriction to A we have $\gamma_g|A = \gamma_0^g$.

We select a σ_f which occurs in $H^{l(w)}\widetilde{(\mathfrak{u}, \mathcal{M})}(w\cdot\lambda)$ and we assume that

$$< \chi_g, w\cdot\lambda > < \ - < \chi_g, \rho_P >$$

so we are on the left hand side of the central point for cohomology. Then the general results on Eisenstein cohomology tell us that the subspace

$$\mathrm{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G} H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}\widetilde{(\mathfrak{u}, \mathcal{M})}(w\cdot\lambda)(\sigma_f) \subset H^{\bullet + l(w)}(\partial\mathcal{S}, \tilde{\mathcal{M}})_{\mathbb{C}}$$

is in fact in the image of the cohomology provided the associated Eisenstein series does not have a pole.

Actually what we have to do is to extend σ_f to a representation $\sigma = \sigma_\infty \times \sigma_f$ which now occurs in the cuspidal spectrum $\mathcal{A}_{cusp}(M(\mathbb{Q})\backslash M(\mathbb{A}))$. Let H_σ this isotypical submodule so that we have

$$H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}\widetilde{(\mathfrak{u}, \mathcal{M})}(w\cdot\lambda)(\sigma_f) = H^\bullet(\mathfrak{m}, K^M, H_\sigma \otimes H^{l(w)}(\mathfrak{u}, \tilde{\mathcal{M}}))$$

We twist this representation by a character

$$\mu_s : \underline{m} \mapsto |\gamma_g(\underline{m})|^s$$

and we consider the induced representation

$$I_{\sigma\otimes s} = \{f : G(\mathbb{A}) \to H_\sigma | f(\underline{pg}) = \sigma(\underline{m})\mu_s)(\underline{p})\}$$

where $\underline{m}$ is the image of $\underline{p}$ in $M(\mathbb{A})$. The functions should satisfy some finiteness conditions.

We can form the Eisenstein series

$$\mathrm{Eis} : I_{\sigma\otimes s} \to \mathcal{A}(\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})$$

given by

$$\mathrm{Eis}(f)(\underline{g}) = \sum_{P\mathbb{Q})\backslash G(\mathbb{Q})} f(g)(\underline{e})$$

which is convergent for $\Re(s) >> 0$.

Let us assume that we are in the holomorphic case, i.e. the Eisenstein operator is holomorphic at $s = 0$ Then we know that the Eisenstein series is actually an intertwining operator

$$\mathrm{Eis} : I_\sigma \to \mathcal{A}(\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A}))$$

we get a homomorphism

$$\mathrm{Eis}^\bullet : H^\bullet(\mathfrak{g}, K_\infty, I_{\sigma_\infty} \otimes \mathcal{M}_{\mathbb{C}}) \otimes \sigma_f \to H^\bullet(S, \tilde{\mathcal{M}}_{\mathbb{C}})$$

and if we compose this with the restriction to the boundary then I claim that this composition gives us a surjective map

$$r \circ \text{Eis} : H^\bullet(\mathfrak{g}, K_\infty, I_{\sigma_\infty} \otimes \mathcal{M}_\mathbb{C}) \otimes I_{\sigma_f} \to \text{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G} H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}, \mathcal{M})(w \cdot \lambda)(\sigma_f)$$

This means that as long as $w \cdot \lambda$ is far enough to the right we know that after tensorization by $\mathbb{C}$ the subspace

$$\oplus_{\sigma_f} \text{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G} H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}, \mathcal{M})(w \cdot \lambda)(\sigma_{f\mathbb{C}})$$

is in the image of the restriction map. If now $\Theta_P \in W^P$ is the longest element then we can consider $\Theta_P \cdot \sigma_f = \sigma_f^\vee$ and this module occurs in the cohomology $H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w')}(\mathfrak{u}, \mathcal{M})(w' \cdot \lambda))$ where w' is the dual partner to w. Here we have the opposite inequality

$$< \chi_g, w' \cdot \lambda >> \; - < \chi_g, \rho_P >$$

and for these weights we find that the map

$$\text{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G} H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}, \mathcal{M})(w \cdot \lambda)(\sigma_f^\vee) \to H_c^{\bullet+1+l(w)}(S^{\wedge\wedge}, \tilde{\mathcal{M}}_\mathbb{C})$$

is injective if we do not have a pole.

If we pick such a σ_f then the module $\text{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G}(\sigma_f)$ provides a module under the Hecke-algebra $\mathcal{H}^G$. This induced module is of course a restricted tensor product over all primes p of local modules. If we consider the local induced modules at p_0 then we can use our support condition Ξ define submodules

$$\text{Ind}_{\mathcal{H}^M}^{\Xi, \mathcal{H}^G} \subset \text{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G}(\sigma_f)$$

and quotients

$$u(\sigma_f) : \text{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G}(\sigma_f^\vee) \to \text{Ind}_{\mathcal{H}^M}^{\Xi', \mathcal{H}^G},$$

where Ξ' is the complementary support condition. These submodules are not modules for the full Hecke algebra, we have to take the identity element at the prime p_0. We define

$$H^\bullet(S^{\wedge\wedge}, \tilde{\mathcal{M}}_\Xi)(\sigma_f)$$

to be the inverse image of $\text{Ind}_{\mathcal{H}^M}^{\Xi, \mathcal{H}^G} H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}, \mathcal{M})(w \cdot \lambda)(\sigma_f)$ in $H^\bullet(S^{\wedge\wedge}, \tilde{\mathcal{M}}_\Xi)$ divided by the kernel of $u(\sigma_f)$. Then by construction we have a map

$$r(\sigma_f) : H^\bullet(S^{\wedge\wedge}, \tilde{\mathcal{M}}_\Xi)(\sigma_f) \to H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}, \mathcal{M})(w \cdot \lambda)(\sigma_f)$$

which is surjective up to torsion. We also get a map

$$\delta(\sigma_f) : \text{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G}(\sigma_f^\vee) \to H^\bullet(S^{\wedge\wedge}, \tilde{\mathcal{M}}_\Xi)(\sigma_f).$$

If we divide the kernel of $r(\sigma_f)$ by the image of $\delta(\sigma_f)$ then we get the inner cohomology.

Now I want to assume for a moment that Ξ is everything and $\Xi' = \emptyset$. I also assume that

$$r(\sigma_f) : H^\bullet(S^{\wedge\wedge}, \tilde{\mathcal{M}})(\sigma_f) \to H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}, \mathcal{M})(w \cdot \lambda)(\sigma_f)$$

is in fact surjective on the integral level. Then we have rationally the Manin-Drinfeld principle, this gives us a canonical section and a decomposition up to isogeny

$$H^\bullet(S^{\wedge\wedge},\tilde{\mathcal{M}})(\sigma_f)\supset H^\bullet_!(S^{\wedge\wedge},\tilde{\mathcal{M}})(\sigma_f)\oplus H^\bullet_{\mathrm{Eis}}(\mathcal{S}^M_{K^M_f},H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)(\sigma_f).$$

We can not expect that the restriction

$$H^\bullet_{\mathrm{Eis}}(\mathcal{S}^M_{K^M_f},H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)(\sigma_f)\to H^\bullet(\mathcal{S}^M_{K^M_f},H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)(\sigma_f)$$

is surjective. I explained in [Ha-book], chap3, 6.3 that I believe that the order of the cokernel should be related to a special values of the L function attached to σ_f. More precisely the "arithmetic" of the second constant term should tell us something about this kokernel.

4.2 The Anderson motive

I want to explain that the discussion of the mixed Anderson motives gives some further evidence that a result of this kind should be true. We take a suitable Ξ. If we divide $H^\bullet(S^{\wedge\wedge},\tilde{\mathcal{M}})(\sigma_f)$ by the image of $\delta(\sigma_f)$ then we get as a quotient a submodule of the cohomology namely the inverse image of $H^\bullet(\mathcal{S}^M_{K^M_f},H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)(\sigma_f,\Xi)$ in the cohomology. We get an almost decomposition

$$H^\bullet(S^{\wedge\wedge},\tilde{\mathcal{M}})(\sigma_f)/\ \mathrm{Im}(\delta(\sigma_f^\vee))\supset H^\bullet_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})\oplus H^\bullet_{\mathrm{Eis}}(\mathcal{S}^M_{K^M_f},H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)(\sigma_f,\Xi)$$

and this gives us the subobject $H^\bullet_{\mathrm{Eis}}(S^{\wedge\wedge},\tilde{\mathcal{M}})(\sigma_f)$ which sits in an exact sequence

$$0\to \mathrm{Ind}_{\mathcal{H}^M}^{\Xi'\mathcal{H}^G}H^\bullet(\mathcal{S}^M_{K^M_f},H^{l(w')}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)(\sigma_f^\vee)\qquad\xrightarrow{\delta}H^\bullet_{\mathrm{Eis}}(S^{\wedge\wedge},\tilde{\mathcal{M}}_\Xi)(\sigma_f)$$
$$\xrightarrow{r}\mathrm{Ind}_{\mathcal{H}^M}^{\Xi,\mathcal{H}^G}H^\bullet_{\mathrm{Eis}}(\mathcal{S}^M_{K^M_f},H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)(\sigma_f)\to 0$$

and the term in the middle is a mixed Tate motive $\mathcal{X}[\sigma_f]$ Here we have to observe that δ raises the degree by one and r respects the degree. The map

$$H^\bullet_{\mathrm{Eis}}(\mathcal{S}^M_{K^M_f},H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)(\sigma_f)\to H^\bullet(\mathcal{S}^M_{K^M_f},H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)(\sigma_f)$$

has a finite cokernel, this cokernel will be given by a number $\Delta(\sigma_f)$.

So we assume $g=2$ and we also assume that we do not have a pole of the Eisenstein series.

I want to give some indication how the Hodge-de Rham extension classes can be computed. We apply the same argument as in my SLN. Actually I think I made the computations unnecessarily complicated there. To simplify the considerations I also assume that σ_f is defined over $\mathbb{Q}$ otherwise I have to make a lot of noise about fields of definition and conjugation under Galois.

We follow the advice given by our general discussion of the computation of the Hodge de-Rham Ext-group. We can twist by a Tate motive so that the bottom becomes $\mathbb{Z}(0)$ and then the top will be $\mathbb{Z}(-n-1)$ with n. Let us also assume for simplicity that our choice of K_f is so that $\sigma_f^{K_f^M}$ is of rank one so that $H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)(\sigma_f) = \mathbb{Z}(-n-1)$ is of rank one. Now we assume that

$$H^\bullet(S^{\wedge\wedge},\tilde{\mathcal{M}}_\Xi)(\sigma_f) \xrightarrow{r} \mathrm{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G} H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)(\sigma_f,\Xi)$$

is surjective. Then $\frac{1}{\Delta(\sigma_f)}H^\bullet_{\mathrm{Eis}}(S^{\wedge\wedge},\tilde{\mathcal{M}}_\Xi)(\sigma_f) \to H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)(\sigma_f,\Xi)$ will be surjective. We take a suitable differential form

$$\omega_{\mathrm{top}} \in \mathrm{Hom}_{K_\infty}(\Lambda^3(\mathfrak{g}/\mathfrak{k}), I_{\sigma_\infty}\otimes\mathbb{C})$$

such that the Eisenstein intertwining operator maps $\omega_{\mathrm{top}}\otimes I_{\sigma_f}$ to

$$\mathrm{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G} H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)(\sigma_f,\Xi)$$

and such that complex conjugation acts by -1 since n is even. This is the canonical Betti lift which we described earlier. If we multiply be a denominator $d(n)$ then we will land in the integral cohomology of the boundary. Now we can find (at this point some details have to be fixed) a class ω_{hol} such that ω_{hol} and ω_{top} define the same class in $\mathrm{Hom}_{K_\infty}(\Lambda^3(\mathfrak{g}/\mathfrak{k}), I_{\sigma_\infty}\otimes\mathbb{C})$ and such that $\mathrm{Eis}(\omega_{hol})$ lies in the F^2 filtration step of the de Rham filtration. So this is the de-Rham-lift. According to our rules we have to look at the difference

$$(\mathrm{Eis}(\omega_{\mathrm{hol}}) - \mathrm{Eis}(\omega_{\mathrm{top}}))\times\psi_f \in \mathrm{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G} H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)(\sigma_f)$$

Again this can be computed as an integral against a relative cycle.

First of all we notice that we can write the difference $\omega_{hol}-\omega_{\mathrm{top}}$ as a $d\psi_\infty$ where

$$\psi_\infty \in \mathrm{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}), I_{\sigma_\infty}\otimes\mathbb{C})$$

This differential form can be interpreted as a form on

$$P(\mathbb{Q})\backslash X\times G(\mathbb{A}_f)/K_f$$

more or less by construction. We have the level function

$$P(\mathbb{Q})\backslash X\times G(\mathbb{A}_f)/K_f \xrightarrow{|\gamma_g|} \mathbb{R}_{>0}$$

and any level surface is homotopy equivalent to $\partial_P\mathcal{S}$. If we restrict this class to such a level hypersurface it becomes closed and $\psi\times\psi_f$ will be a non zero class in

$H^2(\partial_P\mathcal{S},\tilde{\mathcal{M}}) = \mathrm{Ind}_M^G H1(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)_\Xi)$. Now we can find a 2-cycle $\mathfrak{z}$ which represents a non zero class in

$$[\mathfrak{z}] \in H_2(\partial_P\mathcal{S},\tilde{\mathcal{M}}) = H_1(\mathcal{S}^M_{K_f^M}, H_1(\mathfrak{u},\tilde{\mathcal{M}}))$$

and this cycle can be bounded by a chain $\mathfrak{c}$ inside $\mathcal{S}^G_{K_f}(\mathbb{C})$. Then it is the definition that our extension class is given by the integral

$$\int_{\mathfrak{c}} \mathrm{Eis}((\omega_{hol} - \omega_{\mathrm{top}}) \times \psi_f) = \int_{\mathfrak{z}} \mathrm{Eis}(\psi_\infty \times \psi_f)$$

and as in [Ha-book] we find that integral can be computed from the second term in the constant term of the Eisenstein class. We copy the result from SecOPs.pdf

$$\mathcal{X}(f)_{H-dRh} = -C(\sigma_{p_0}, \lambda)\Big(\frac{1}{\Omega(\sigma_f)^{\epsilon(k,m)}} \frac{\Lambda^{\mathrm{coh}}(f, n_1 + n_2 + 2)}{\Lambda^{\mathrm{coh}}(f, n_1 + n_2 + 3)}\Big) \frac{1}{\zeta(-1 - n_1)} \frac{\zeta'(-n_1)}{i\pi} \tag{210}$$

The factor $C(\sigma_{p_0}, \lambda)$ is a local contribution which stems from the auxiliary prime p_0. I have not yet done the computation but I think that up to a power of p_0 it is equal to the inverse of the local Euler factor at p_0 in the ratio of L-values. If a_{p_0} is the p_0-th Fourier coefficient, i.e. the eigenvalue of T_{p_0} then $a_{p_0} = \alpha_{p_0} + \beta_{p_0}, \alpha_{p_0}\beta_{p_0} = p^{k-1}$ and we should have

$$\begin{aligned} C(\sigma_{p_0}, \lambda) = \frac{(1 - \alpha_{p_0} p_0^{-n_1-n_2-2})(1 - \beta_{p_0} p_0^{-n_1-n_2-2})}{(1 - \alpha_{p_0} p_0^{-n_1-n_2-3})(1 - \beta_{p_0} p_0^{-n_1-n_2-3})} \frac{1}{p_0} \frac{1 - p_0^{-n_1-1}}{1 - p_0^{-n_1-2}} = \\ \frac{1 - a_{p_0} p_0^{-n_1-n_2-2} + p_0^{-n_1-1}}{1 - a_{p_0} p_0^{-n_1-n_2-3} + p_0^{-n_1-3}} \frac{1}{p_0} \frac{1 - p_0^{-n_1-1}}{1 - p_0^{-n_1-2}} \end{aligned} \tag{211}$$

We should interpret the formula (212) as follows: The last factor factor $\frac{\zeta'(-n_1)}{i\pi}$ is an extension class in $\mathrm{Ext}^1_{B-dRh}(\mathbb{Z}(-2 - n_1 - n_2), \mathbb{Z}(-1 - n_2))$ and the rest of this expression is an algebraic number. Since the period is defined up to a unit, it makes sense to speak of the prime decomposition of this number. Under certain conditions we expect congruences modulo primes which occur in the denominator of this number.(See SecOps.pdf)

4.3 Non regular coefficients

So far we discussed only the regular case, this means the case where the Eisenstein series is holomorphic at $s = 0$. Our our special case this means that $n_1 > 0$. If we have $n_1 = 0$ then we have to study the behavior of the function

$$-C(\sigma_{p_0}, \lambda)\Big(\frac{1}{\Omega(\sigma_f)^{\epsilon(k,m)}} \frac{\Lambda^{\mathrm{coh}}(f, n_2 + 2 + s)}{\Lambda^{\mathrm{coh}}(f, n_2 + 3 + s)}\Big) \frac{\zeta(1 + s)}{\zeta(2 + s)} \tag{212}$$

at $s = 0$.

Let us recall that f can be viewed as a modular form of weight $k = 4 + n_1 + 2n_2 = 4 + 2n_2$. Hence we see in the numerator the expression $\Lambda(f, \frac{k}{2} + s)$ If this does not vanish at $s = 0$ then the Eisenstein series has a pole at $s = 0$. Taking the residue we get some non zero residual classes in $H^2(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda)$, they are square integrable.

At this moment we are more interested in the case where $\Lambda(f, \frac{k}{2}) = 0$. Then the Eisenstein class will be holomorphic at $s = 0$. Let us assume that we are in the unramified case.

It is already discussed in [Ha1] that in this case the induced module $\mathrm{Ind}(\sigma_f)$ has a unitary quotient $J(\sigma_f)$, and this may have the consequence that

$$\mathrm{Hom}_{\mathcal{H}^G}(\mathrm{Ind}(\sigma_f), H^3_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda) \neq 0$$

and hence the Manin-Drinfeld principle is not valid under these circumstances. This issue is discussed in [Ha1]. In the appendix (letter to Goresky and MacPherson) we carry out a lacunary computation which shows that

$$\begin{array}{c} J(\sigma_f) \textit{ occurs non trivially in } H^3_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda) \\ \textit{if and only if the sign in the functional equation is } -1 \end{array} \tag{213}$$

We also discuss the relation between this assertion and the Saito Kurokawa lift.

Remark: At this point we should remark that we tacitly assume that we are in the unramified case. This implies that actually $I(\sigma_f) = J(\sigma_f)$ The assertion that that $I(\sigma_f)$ has a unitary quotient, means that after tensorization with $\mathbb{C}$

a) we have an admissible representation $\tilde{I}(\sigma_f)$ of $G(\mathbb{A}_f)$ whose Hecke-module of K_f invariant vectors is $I(\sigma_f)$,

b) The $G(\mathbb{A}_f)-$ module has a non trivial quotient $\tilde{J}(\sigma_f)$ on which we have a positive definite $G(\mathbb{A}_f)$ invariant hermitian scalar product and $I(\sigma_f)$ injects.

In [Ha1] 3.1.4 we also discuss the construction of a mixed motive attached to σ_f. It follows from Piatetkii-Shapiro that $H^3_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda \otimes F)$ contains a submodule $\mathcal{SK}(\sigma_f)$ which consists of two copies of $J(\sigma_f)$ and we get an exact sequence

$$0 \to \mathcal{SK}(\sigma_f) \to H^3_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda \otimes F)(\sigma_f) \to \mathrm{Ind}(H^3(\mathcal{S}^M_{K^M_f}, \mathcal{M}_\lambda(w \cdot \lambda))(\sigma_f)) \to 0 \tag{214}$$

The motive $\mathcal{SK}(\sigma_f) = \mathbb{M}(\sigma_f, r_1)$ and it provides an extension class

$$\mathcal{Y}(\sigma_f) \in \mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-k), \mathbb{M}(\sigma_f, r_1)) = \mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-k), \mathcal{SK}(\sigma_f)) \tag{215}$$

In [Ha1] we do not discuss the question of computing this extension class, in a sense we did not know what that meant. But following T. Scholl we can give some kind of an answer to this question. We choose an auxiliary prime p_0 and modify K_f at p_0 to the Iwahori and the level will be $K_f(p_0)$. We modify our sheaf and construct a mixed motive $H^3(\mathcal{S}^G_{K_f}, \mathcal{M}^\#_\lambda)(\sigma_f)$. We have

$$\begin{array}{c} H^1(\mathcal{S}^M_{K^M_f}, \mathcal{M})(w' \cdot \lambda)(\sigma_f)^\# \subset H^3(\mathcal{S}^G_{K_f}, \mathcal{M}^\#_\lambda)(\sigma_f) \\ H^3(\mathcal{S}^G_{K_f}, \mathcal{M}^\#_\lambda)(\sigma_f) \xrightarrow{r} H^1(\mathcal{S}^M_{K^M_f}, \mathcal{M})(w \cdot \lambda)(\sigma_f)^\# \end{array} \tag{216}$$

The submodule in the top row is a Tate motive $\mathbb{Z}(-k+1)^a$ the quotient in the bottom row is $\mathbb{Z}(-k)^b$ where $a = 1, 2,\ b = 2, 1$ depending on the support conditions defining $\mathcal{M}^\#$. We can write two exact sequences

$$\begin{array}{c} 0 \to H^1(\mathcal{S}^M_{K^M_f}, \mathcal{M})(w' \cdot \lambda)(\sigma_f)^\# \to \ker(r) \to \mathcal{SK}(\sigma_f) \to 0 \\ 0 \to \mathcal{SK}(\sigma_f) \to H^3(\mathcal{S}^G_{K_f}, \mathcal{M}^\#_\lambda)(\sigma_f) \xrightarrow{r} H^1(\mathcal{S}^M_{K^M_f}, \mathcal{M})(w \cdot \lambda)(\sigma_f)^\# \to 0 \end{array} \tag{217}$$

these two sequences are obtained from the diagram (216). They provide extension classes

$$\mathcal{Y}(\sigma_f) \in \mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-k), \mathcal{SK}(\sigma_f), \mathcal{Y}'(\sigma'_f) \in \mathrm{Ext}^1_{\mathcal{MM}}(\mathcal{SK}(\sigma_f), \mathbb{Z}(-k+1)) \tag{218}$$

Such a pair is a biextension and to such a biextenion T. Scholl attaches an "intersection number " or "height pairing"

$$i[(\mathcal{Y}(\sigma_f), \mathcal{Y}'(\sigma'_f))]$$

, which is well defined modulo an "element in $\mathrm{Ext}^1(\mathbb{Z}(-1), \mathbb{Z}(0))$." To define this pairing Scholl "intergrates" the pair of extension classes $((\mathcal{Y}(\sigma_f), \mathcal{Y}'(\sigma'_f))$ into a diagram of type (216) let us call it

$$((\widetilde{\mathcal{Y}(\sigma_f), \mathcal{Y}'}(\sigma'_f)) \tag{219}$$

and to such an object Scholl attaches an honest number

$$i[((\widetilde{\mathcal{Y}(\sigma_f), \mathcal{Y}'}(\sigma'_f)]$$

The "integral" (219) is only defined modulo an element in $\mathrm{Ext}^1(\mathbb{Z}(-1), \mathbb{Z}(0))$ and this explains the ambiguity in the definition of $i[(\mathcal{Y}(\sigma_f), \mathcal{Y}'(\sigma'_f))]$. Moreover the existence of this integral is conjectural.

But in our case we have an integral $i[((\widetilde{\mathcal{Y}(\sigma_f), \mathcal{Y}'}(\sigma'_f)]$, this is simply the diagram (216). (It is like finding a primitive to a function f which is defined as the derivative of a function F.)

Now an extension of the computations in [Ha1] Kap. IV 4.3.3 and SecOPs.pdf to this case should yield

$$i[((\widetilde{\mathcal{Y}(\sigma_f), \mathcal{Y}'}(\sigma'_f)] \sim L^{\mathrm{coh},\prime}(\sigma_f, r_1, \frac{k}{2}) \tag{220}$$

where $\sim$ means up to some uninteresting non zero factors (JW). This is in a certain sense a formula of Gross-Zagier type.

4.4 $g \geq 3$

We consider the case $g = 3$. We start from a highest weight $\lambda = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3$ for simplicity we assume that this yields a representation of the group $\mathrm{GSp}_3/\mathbb{G}_m$, then we have $n_1+n_3 \equiv 0 \mod 2$. The group $M = \mathrm{Gl}_3 \cdot \mathbb{G}_m$ the locally symmetric space $\mathcal{S}^M_{K^M_f}$ is of dimension 5, we look for cohomology in degree 2 and 3. We have the two interesting Kostant representatives $w' = s_3s_2, w = s_3s_2s_3s_1$. For these two elements we consider the coefficient systems $\mathcal{M}_\lambda(w' \cdot \lambda), \mathcal{M}_\lambda(w \cdot \lambda)$ on $\mathcal{S}^M_{K^M_f}$. Since we want to have non trivial inner cohomology we need to assume that the coefficient systems are self dual and hence we need $n_1 = 1 + 2n_3$. Then we get for our coefficient systems

$$w' \cdot \lambda = (2+n_2+2n_3)(\gamma^M_1+\gamma^M_2)+(-1+n_3)\gamma_3,\ w \cdot \lambda = (2+n_2+2n_3)(\gamma^M_1+\gamma^M_2)+(-3-n_3)\gamma_3.$$

and we can look for isotypical summands

$$H^3(\mathcal{S}^M_{K^M_f}, \mathcal{M}_\lambda(w' \cdot \lambda))(\sigma'_f),\ H^2(\mathcal{S}^M_{K^M_f}, \mathcal{M}_\lambda(w \cdot \lambda))(\sigma_f) \tag{221}$$

We know that they provide motives, if we assume that K_f is unramified then they are Tate motives of weight $\mathbf{w}(w' \cdot \lambda)$ respectively. Now a simple computation shows that the difference $\mathbf{w}(w\cdot\lambda)-\mathbf{w}(w'\cdot\lambda)$ is even and therefore the extension classes should be torsion and our source for congruences dries out. But this is only good because at the time we can not expect a rationality result for the ratios

$$\frac{\Lambda^{\mathrm{coh}}(\sigma_f,\nu-1)}{\lambda^{\mathrm{coh}}(\sigma_f,\nu)}$$

because the motive $M(\sigma_f)$ should have a non zero middle Hodge number and this puts a parity condition on the critical values, (or kills them all).

The situation changes if we the take the parabolic subgroup given by

$$\alpha_1 \quad - \quad \times \quad <= \quad \alpha_3,$$

the semi simple part is $M=\mathrm{PSl}_2\times \mathrm{Sp}_1$. The first factor has to be viewed as the linear factor and corresponds to α_1, the other factor is the hermitian factor. Hence we see that our locally symmetric space is essentially a product

$$\mathcal{S}^M_{K^M_f}=\mathcal{S}_1\times\mathcal{S}_2. \tag{222}$$

We pick a Kostant representative $w\in W^P$ and write as usual

$$w(\lambda+\rho)-\rho=d_1\gamma_1^M+d_3\gamma_3^M+a(w,\lambda)\gamma_2 \tag{223}$$

The resulting coefficient system is a tensor product of coefficient systems on the two factors and hence we see that in the isotypical decomposition (after a suitable finite extension $F/\mathbb{Q}$)

$$H_!^{\bullet-l(w)}(\mathcal{S}^M_{K^M_f},\mathcal{M}_\lambda(w\cdot\lambda)\otimes F)=\bigoplus_{\sigma_f}H_!^{\bullet-l(w)}(\mathcal{S}^M_{K^M_f},\mathcal{M}_\lambda(w\cdot\lambda))(\sigma_f). \tag{224}$$

The $\sigma_f=\tau_f\times\sigma'_f$ where τ_f resp. σ'_f are simply modular forms f of weight $k_1=d_1+2$ and g of weight $k_3=d_3+2$. We simply write

$$\sigma_f=\tau_f\times\sigma'_f=(f,g).$$

If we now apply the Eisenstein intertwining operator then we have to look at the second term in the constant term. We find the formula for it in chap3.pdf section 6.3. The Dynkin diagram of the semi-simple part of the dual group $M^\vee=\mathrm{Gl}_2\times\mathrm{PSl}_2$ is

$$\alpha_1^\vee \quad - \quad \times \quad >= \quad \alpha_3^\vee,$$

the first factor corresponds to Gl_2 the second to PSl_2. We have to compute the action of ${}^LM$ on the Lie-algebra $\mathfrak{u}_P^\vee$. The roots in $\Delta^+_{U_P^\vee}$ are those $\beta^\vee=a_1\alpha_1^\vee+a\alpha_1^\vee+a_3\alpha_3^\vee$ for which $a>0$. By inspection we get 6 such roots with $a=1$ and one such root with $a=2$. We can easily check that $r_1^{\mathfrak{u}_P^\vee}=r_1\otimes\mathrm{Ad}$ and $r_2^{\mathfrak{u}_P^\vee}=\det$, where det is of course the determinant on the first factor. The highest weight for the representation Ad is $\chi_1=\alpha_1^\vee+\alpha_2^\vee+2\alpha_3^\vee$ and $\chi_2=\alpha_1^\vee+2\alpha_2^\vee+2\alpha_3^\vee$

We compute the second constant term. We are interested in cases where we can construct Anderson mixed motives and this means that we should deal with a pair of Kostant representatives w', w where $l(w) = 4$ and $l(w') = 3$. We have two such pairs

$$\begin{array}{ll} w_1 = s_2 s_1 s_3 s_2 & w_1' = s_2 s_1 s_3 \\ w_2 = s_2 s_3 s_2 s_1 & w_2' = s_2 s_3 s_2 \end{array}.$$

and then

$$w_1(\lambda+\rho) = (3+n_2+2n_3)\gamma_1^M + (3+n_1+n_2+n_3)\gamma_3^M + 1/2(-1-n_2)\gamma_2$$

$$w_1'(\lambda+\rho) = (3+n_2+2n_3)\gamma_1^M + (3+n_1+n_2+n_3)\gamma_3^M + 1/2(+1+n_2)\gamma_2$$

$$w_2(\lambda+\rho) = (5+n_1+2n_2+2n_3)\gamma_1^M + (1+n_3)\gamma_3^M + 1/2(-1-n_1)\gamma_2$$

$$w_2'(\lambda+\rho) = (5+n_1+2n_2+2n_3)\gamma_1^M + (1+n_3)\gamma_3^M + 1/2(1+n_1)\gamma_2 \tag{225}$$

The coefficients of γ_1^M resp.γ_3^M are the numbers d_1+1 resp. d_3+1 in equation (223). Then we find easily that

$$\begin{cases} 2d_3 - d_1 = 2 + 2n_1 + n_2 (\geq 2) & \text{if } w = w_1 \\ d_1 - 2d_3 = 4 + n_1 + 2n_2 (\geq 4) & \text{if } w = w_2 \end{cases} \tag{226}$$

In other words: We give ourselves d_1, d_3 and we look for $w = w_1$ resp. $w = w_2$ and for a solution of the equations in (225) with λ dominant. Then d_1, d_3 determine the choice of w.

In the case $w = w_1$ we have the further constraint $d_1 - d_3 = n_3 - n_1$ which in the case $d_1 \leq d_3$ implies $n_1 \geq -d_1 + d_3$.

Then it becomes clear that the possible solutions for n_2 resp. n_1 are even and $\frac{n_2}{2}$ resp. $\frac{n_1}{2}$ run through an interval

$$[0, c_\lambda] = \begin{cases} [0, \min(\frac{2d_3-d_1-2}{2}, \frac{d_1-2}{2})] & \text{if } w = w_1 \\ [0, \frac{d_1-2d_3-4}{2}] & \text{if } w = w_2 \end{cases} \tag{227}$$

We want to understand the expression in chap3.pdf (100). We get in the two cases

$$\begin{array}{rl} <\chi_1, \tilde{\mu_1}^{(1)}>= \frac{1}{2}(9+2n_1+3n_2+4n_3) & b(w_1,\lambda) = -\frac{1}{2}(1+n_2) \\ <\chi_2, \tilde{\mu_1}^{(1)}>= 0 & 2b(w_1,\lambda) = -(1+n_2) \\ <\chi_1, \tilde{\mu_2}^{(1)}>= \frac{1}{2}(7+n_1+2n_2+4n_3) & b(w_2,\lambda) = -\frac{1}{2}(1+n_1) \\ <\chi_2, \tilde{\mu_2}^{(1)}>= 0 & 2b(w_1,\lambda) = -(1+n_1) \end{array} \tag{228}$$

For w_1 this yields for the the following expression for the second constant term (chap3.pdf (100)and SecOps.pdf).

$$\frac{\pi}{\Omega(\sigma_f)^\epsilon} \frac{\Lambda^{\text{coh}}(\tau \times \sigma'_f, r_1 \times \text{Ad}, 5+n_1+2n_2+2n_3)}{\Lambda^{\text{coh}}(\tau \times \sigma'_f, r_1 \times \text{Ad}, 6+n_1+2n_2+2n_3)} \frac{\zeta(1+n_2)}{\zeta(2+n_2)} C^*(\sigma_\infty,\lambda) T_\infty^{\text{loc}}(\omega) \otimes T_f^{\text{loc}}(\psi_f)) \tag{229}$$

and for w_2

$$\frac{\pi}{\Omega(\sigma_f)^\epsilon}\frac{\Lambda^{\mathrm{coh}}(\tau\times\sigma'_f, r_1\times \mathrm{Ad}, 4+n_1+n_2+2n_3)}{\Lambda^{\mathrm{coh}}(\tau\times\sigma'_f, r_1\times \mathrm{Ad}, 5+n_1+n_2+2n_3)}\frac{\zeta(1+n_1)}{\zeta(2+n_1)}C^*(\sigma_\infty,\lambda)T^{\mathrm{loc}}_\infty(\omega)\otimes T^{\mathrm{loc}}_f(\psi_f)) \tag{230}$$

We give the "arithmetic interpretation" of these two second terms. For the beginning we forget the factors at the right we come from the infinite place.

Again we expect that these second terms give us the Betti-de-Rham extension class of a mixed Tate motive $\mathcal{X}(\sigma_f)$ and if we look at the formulae (225) then we see that we get

$$\mathcal{X}(\sigma_f)\in \mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-1-n),\mathbb{Z}(0)) \tag{231}$$

where $n=n_1$ or n_2 depending on the case in which we are. Since we need non torsion classes we have to assume that n is even. Then we apply the functional equation for the Riemann ζ- function to the ratio of ζ values and get

$$\frac{\zeta(n+1)}{\zeta(n+2)}=-\frac{n+1}{\pi^2}\frac{\zeta'(-n)}{\zeta(-1-n)}$$

and we get a factorization

$$\Big(\frac{1}{\Omega(\sigma_f)^\epsilon}\frac{\Lambda^{\mathrm{coh}}(\cdots)}{\Lambda^{\mathrm{coh}}(\cdots+1)}\Big)\Big(\frac{-n-1}{\zeta(-1-n)}\Big)\Big(\frac{\zeta'(-n)}{\pi}\Big) \tag{232}$$

We assume that n is even. The last factor on the right is interpreted as extension class in $\mathrm{Ext}^1_{B-deRham}(\mathbb{Z}(-1-n),\mathbb{Z}(0))$, the factor in the middle is a rational number. The first factor needs some more explanation. It depends on a pair (f,g) of cusp forms on $\mathrm{Gl}_2(\mathbb{Z})$ of weight k_1 resp. k_3. These weights are the coefficients of γ_1^M resp. γ_3^M in (225) augmented by 1. We have the symmetric square lift of the automorphic form σ' to an automorphic form Π_f on $\mathrm{Gl}_3/\mathbb{Z}$. Let $H=\mathrm{Gl}_2\times\mathrm{Gl}_3$ then this lift provides an isotypical subspace

$$H^\bullet(\mathcal{S}^H_{K^H_f},)(\tau_f\times\Pi_f)\subset H^\bullet(\mathcal{S}^H_{K^H_f},) \tag{233}$$

and then we have more or less by definition

$$\Lambda^{\mathrm{coh}}(\tau\times\sigma'_f, r_1\times\mathrm{Ad}, s)=\Lambda^{\mathrm{coh}}(\tau\times\Pi_f, r_1\times r_2, s) \tag{234}$$

where r_1, r_2 are the two tautological representations(**In chap3.pdf erklären).**

Now we have the results in [Ha-Rag] and we know that for integers ν in a certain interval $[c(w,\lambda),d(w,\lambda)]$ the ratios

$$\frac{1}{\Omega(\sigma_f)^\epsilon}\frac{\Lambda^{\mathrm{coh}}(\nu)}{\Lambda^{\mathrm{coh}}(\nu+1)} \tag{235}$$

are algebraic numbers in F. Here $\Omega(\sigma_f)$ is a period which is well defined up to a unit in $\mathcal{O}_F^\times$ (See [Ha-Rag]). The above intervall $[c(w,\lambda),d(w,\lambda)]$ can be determined from the data w,λ. It is called the interval of critical arguments.

4.5 Delignes conjectures

In [Ha-book] , chap3.pdf, 3.1 and 3.1.3. we discussed the hypothetical construction of motives attached to isotypical subspaces in the cohomology of arithmetic groups. In our situation here this is actually not so difficult, we have

$$\mathbb{M}(\sigma_f, r_1 \times \mathrm{Ad}) = \mathbb{M}(\tau_f, r_1) \times \mathbb{M}(\Pi_f, r_2) = \mathbb{M}(\tau_f, r_1) \times \mathrm{Sym}^2((\mathbb{M}(\sigma'_f, r_2)). \tag{236}$$

where the factors $\mathbb{M}(\tau_f, r_1), \mathbb{M}(\sigma'_f, r_1)$ are the Deligne-Scholl motives attached to the modular forms (f, g). Note that the motive attached to $(\sigma_f, r_1 \times \mathrm{Ad})$ does not change if we twist σ_f by a power of $|\delta_{P,f}|$.

For any pure motive $\mathbb{M}$ of weight $\mathbf{w} = \mathbf{w}(\mathbb{M})$ Deligne defines a set of critical arguments. To define this set we look a the Hodge-decomposition

$$\mathbb{M}_B \otimes \mathbb{C} = \bigoplus_{p,q:p+q=\mathbf{w}} \mathbb{M}_B^{p,q} \tag{237}$$

and we say that $\mathbb{M}$ has Hodge numbers (p, q) if $\mathbb{M}_B^{p,q} \neq 0$. We define this set only under the assumption that our motive does not have a middle Hodge number, i.e. $h^{\frac{\mathbf{w}}{2},\frac{\mathbf{w}}{2}} = 0$. We look for the Hodge number (p_c, q_c) with $p_c > q_c$ and p_c minimal, then the set of critical arguments is the interval $[q_c + 1, p_c]$.

Under these conditions Deligne formulates the following conjecture (here we assume that the motive is a motive with coefficients in $\mathbb{Q}$)

There exist two periods $\Omega_\pm \in \mathbb{C}^\times$ which are defined in terms of the comparison of Betti- and de-Rham cohomology and which are unique up to an element in $\mathbb{Q}^\times$ such that for all integers $\nu \in [q_c + 1, p_c]$

$$\frac{\Lambda(\mathbb{M}, \nu)}{\Omega_{\epsilon(\nu)}} \in \mathbb{Q} \tag{238}$$

In our situation the Hodge numbers are

$$(d_1 + 1, 0), (0, d_1 + 1) \text{ for } \mathbb{M}(\tau_f, r_1)$$

and

$$(2d_3 + 2, 0), (d_3 + 1, d_3 + 1), (0, 2d_3 + 2) \text{ for } \mathrm{Sym}^2((\mathbb{M}(\sigma'_f, r_2)).$$

The Hodge numbers for $\mathbb{M}(\sigma_f, r_1 \times \mathrm{Ad})$ are the sums of these Hodge numbers. The motive is pure of weight $\mathbf{w} = d_1 + 2d_3 + 3$ this number is odd and hence we know that for all Hodge numbers we have $p \neq q$. Therefore we get

$$p_c - \frac{\mathbf{w}+1}{2} = \begin{cases} \begin{cases} \frac{d_1}{2} & \text{if } d_1 \leq d_3 \\ d_3 - \frac{d_1}{2} & \text{if } d_1 > d_3 \end{cases} & \text{if } w = w_1 \\ \frac{d_1}{2} - d_3 - 1 & \text{if } w = w_2 \end{cases}. \tag{239}$$

Miraculously (?) this number is the number $c_\lambda + 1$ in (227). Our second term in the constant term becomes

$$\frac{\pi}{\Omega(\sigma_f)^\epsilon} \frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_1 \times \mathrm{Ad}, \frac{\mathbf{w}+1}{2} + \frac{n}{2})}{\Lambda^{\mathrm{coh}}(\sigma_f, r_1 \times \mathrm{Ad}, \frac{\mathbf{w}+1}{2} + \frac{n}{2} + 1)} \frac{\zeta(1+n)}{\zeta(2+n)} C^*(\sigma_\infty, \lambda) T_\infty^{\mathrm{loc}}(\omega) \otimes T_f^{\mathrm{loc}}(\psi_f)) \tag{240}$$

where $n = n_1$ or $n = n_2$ depending on the case. The argument $\frac{\mathbf{w}+1}{2} + \frac{n}{2} + 1$ runs exactly over the right half of the critical arguments.

If we believe in the existence of the motive $\mathbb{M}(\sigma_f, r_1 \times \mathrm{Ad})$ and the equality of the motivic and the cohomological L-function then the conjecture of Deligne predicts that in formula (240) the ratio

$$\frac{1}{\Omega(\sigma_f)^\epsilon} \frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_1 \times \mathrm{Ad}, \frac{\mathbf{w}+1}{2} + \frac{n}{2})}{\Lambda^{\mathrm{coh}}(\sigma_f, r_1 \times \mathrm{Ad}, \frac{\mathbf{w}+1}{2} + \frac{n}{2} + 1)} \tag{241}$$

is an algebraic number in F, provided ν and $\nu + 1$ are critical and we choose as period

$$\Omega(\sigma_f)^\epsilon = \frac{\Omega(\mathbb{M}(\sigma_f, r_1 \times \mathrm{Ad}))_{\epsilon(\nu+1)}}{\Omega(\mathbb{M}(\sigma_f, r_1 \times \mathrm{Ad}))_{\epsilon(\nu)}}$$

In this form Delignes conjectures are not available, already the existence of the motive is not clear. But there is still another drawback: The periods $\Omega(\mathbb{M}(\sigma_f, r_1 \times \mathrm{Ad}))_{\epsilon(\nu)}$ are only defined modulo an element in $F^\times$. The definition of the periods uses the comparison between the Betti and de-Rham cohomology.

In our paper with Raghuram [Ha-Rag] we prove a rationality result about special values of Rankin-Selberg L-functions which is weaker than Delignes conjecture but also in some sense stronger. Applied to our situation here it says that we can define a period $\Omega(\sigma_f)$ which is well defined up to an element in $\mathcal{O}_F^\times$. With this definition of the period the numbers

$$\frac{1}{\Omega(\sigma_f)^\epsilon} \frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_1 \times \mathrm{Ad}, \frac{\mathbf{w}+1}{2} + \frac{n}{2})}{\Lambda^{\mathrm{coh}}(\sigma_f, r_1 \times \mathrm{Ad}, \frac{\mathbf{w}+1}{2} + \frac{n}{2} + 1)} \frac{1}{\zeta(-1-n)} C^*(\sigma_\infty, \lambda) \tag{242}$$

are in $F^\times$ are their prime decomposition is well defined. In [Ha-Rag] we also show that the factor $C^*(\sigma_\infty, \lambda)$ is a non zero rational number. It is an important question to compute this number exactly. In the case $g = 2$ this number in SecOps.pdf and it turns out to be very simple. A similar question arises in [Ha-Mum] and has been solved by Don Zagier in the appendix to that paper.

We are again at the point where we can ask the question whether primes $\mathfrak{l}$ dividing the denominator of the algebraic number in (242) create denominators of the Eisenstein classes and therefore also congruences between eigenvalues of modular forms on different groups.

We return to the ratios of L-values on p.3. The L-functions which occur in these expressions are actually the "automorphic" or "unitary" L functions. But I think that I have strong reasons that we should express them in terms of the "cohomological" L-function. In the case discussed in "Eis-coh..." the arguments of evaluation are exactly the critical points of the Scholl-motive $M(f)$ attached to the automorphic form and this is equal to the cohomological L-function.

In the special case which we consider we started from two modular forms f, g of weights k_1, k_3 respectively. For both of them we have the Scholl-motive $M(f), M(g)$ and the two dimensional ℓ-adic Galois-representations

$$\rho(\tau) : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}(M(f))_\ell), \rho(\sigma) : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}(M(f))_\ell),$$

and we have for the Frobenii:

$$\rho(\tau)(\Phi_p^{-1}) \simeq \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix}, \ \alpha_p+\beta_p = a_p, \alpha_p\beta_p = p^{k_1-1} = p^{d_1+1}$$

$$\rho(\sigma)(\Phi_p^{-1}) \simeq \begin{pmatrix} \gamma_p & 0 \\ 0 & \delta_p \end{pmatrix}, \ \gamma_p+\delta_p = c_p, \gamma_p\delta_p = p^{k_3-1} = p^{d_3+1}$$

where a_p resp. c_p is the $p-th$ Fourier coefficient of f resp. g.

We take the symmetric square of $\rho(\sigma)$ and get

$$\rho(\mathrm{Sym}^2(\sigma)) : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to Gl_3(\mathbb{Z}_\ell)$$

(here we assume that f, g have coefficients in $\mathbb{Z}$.) Then

$$\rho(\mathrm{Sym}^2(\sigma))(\Phi_p^{-1}) \simeq \begin{pmatrix} \gamma_p^2 & 0 & 0 \\ 0 & p^{d_3+1} & 0 \\ 0 & 0 & \delta_p^2 \end{pmatrix}$$

Then we can write the finite part of the L-function as

$$L^{\mathrm{coh}}(\tau\times\Pi, s) = \prod_p \frac{1}{\det\left(\mathrm{Id} - \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix}\right) \otimes \begin{pmatrix} \gamma_p^2 & 0 & 0 \\ 0 & p^{d_3+1} & 0 \\ 0 & 0 & \delta_p^2 \end{pmatrix} p^{-s})}$$

Here it becomes clear that this is the motivic L-function of the motive $M(\tau\times\Pi)$. Here the representation r of the dual group is the tensor product of the two tautological representations.

The local Euler-factor is of degree 6 it can be expressed in terms of the eigenvalues a_p, c_p and is given by

$$\big[\big(1+\left(-a_pc_p^2+2a_pp^{-1+h}\right)p^{-s}+(a_p^2p^{-2+2h}+c_p^4p^{-1+k}-4c_p^2p^{-2+h+k}+2p^{-3+2h+k})p^{-2s}+ (a_pc_p^2p^{-3+2h+k}+2a_pp^{-4+3h+k})p^{-3s}+p^{-6+2(2h+k)}p^{-4s}\big)\big)*(1-a_pp^{h-1}p^{-s}+p^{k+2h-3}p^{-2s})\big]^{-1}$$

Our motives $M(f), M(g)$ have Hodge types $\{(d_1+1,0),(0,d_1+1),(d_3+1,0),(0,d_3+1)\}$ and therefore we get for the Hodge type of $M(\tau\times\Pi)$

$$\{(d_1+2d_3+3,0),(d_1+d_3+2,d_3+1),(d_1+1,2d_3+2),(2d_3+2,d_1+1),(d_3+1,d_1+d_3+2),(0,d_1+2d_3+3)\}$$

it is pure of weight d_1+2d_3+3.

We reorder these Hodge type according to the size of the second component and get

$$\{(w,0),(w-a,a),(w-b,b),(b,w-b),(a,w-a),(0,w)\},$$

where now $0\le a\le b\le \frac{w}{2}$.

From the the Hodge type or from representation-theoretic considerations we get a Γ factor at infinity which is (if I am not mistaken)

$$L_\infty(\tau\times\Pi,s) = \frac{\Gamma(s)\Gamma(s-a)\Gamma(s-b)}{(2\pi)^{3s}}$$

Again we put

$$\Lambda^{\text{coh}}(\tau \times \Pi, s) = L_\infty(\tau \times \Pi, s) L^{\text{coh}}(\tau \times \Pi, s).$$

This function satisfies a functional equation:

$$\Lambda^{\text{coh}}(\tau \times \Pi, s) = \Lambda^{\text{coh}}(\tau \times \Pi, w+1-s)$$

Once we accept this functional equation then we have fast algorithms to compute the values $\Lambda^{\text{coh}}(\tau \times \Pi, s_0)$ at given argument s_0 up to very high precision.

(For classical modular forms f of weight k we have the following formula

$$\Lambda(f,s) = \sum_{n=1}^{\infty} \big((\frac{1}{2\pi})^s \frac{a_n}{n^s} \Gamma(s, 2\pi n A) + (-1)^{\frac{k}{2}} (\frac{1}{2\pi})^{k-s} \frac{a_n}{n^{k-s}} \Gamma(s, 2\pi n/A) \big)$$

where $\Gamma(s, 2\pi n A)$ is the incomplete Γ function and where A is a strictly positive real number. The right hand side is independent of A (this gives a good test that the functional equation is really correct) and $A = 1$ is the best choice. The sum is rapidly converging, because the incomplete Γ goes rapidly to zero.)

I remember that Don Zagier once mentioned that we always have such a formula to compute values of L-functions, once we can guess the functional equation and this formula can be used to confirm the guess.

This has been done by Tim Dokchitser in his Note "Computing special values of motivic L-functions. Experiment. Math. 13 (2004), no. 2, 137–149. "

Finally we discuss the special values. We have the above list of Hodge types, recall that the Hodge types lists those pairs (p,q) with $p+q = w = d_1 + 2d_3 + 2$ for which $h^{p,q}(M) \neq 0$. The Deligne conjecture predicts that we have to look at pairs (p_c, q_c) for which $p_c + q_c = w, p_c > q_c$ for which $h^{p_c,q_c} \neq 0$ and for which $h^{\nu, w-\nu} = 0$ for all $q_c < \nu < p_c$. This is the critical interval $M_{\text{crit}} = [(p_c, q_c), (q_c, p_c)]$ of our motive. One should look at it as an interval on the line $p + q = w$.

We look at our Hodge types

$$\{(d_1{+}2d_3{+}3, 0), (d_1{+}d_3{+}2, d_3{+}1), (d_1{+}1, 2d_3{+}2), (2d_3{+}2, d_1{+}1), (d_3{+}1, d_1{+}d_3{+}2), (0, d_1{+}2d_3{+}3)\}$$

We have to find the interval we have to distinguish cases. The first case is

a)

$$d_1 < 2d_3 + 1$$

Now we have two possibilities for the critical interval, it is either

a1)

$$[(2d_3+2, d_1+1), (d_1+1, 2d_3+2)]$$

a2)

$$[(d_1+d_3+2, d_3+1), (d_3+1, d_1+d_3+2)]$$

depending on which one is smaller.

The second case is
b)

$$d_1 > 2d_3 + 1$$

In this case the critical interval is clearly

$$[(d_1 + 1, 2d_3 + 2), (2d_3 + 2, d_1 + 1)],$$

In the paper with Raghuram [Ha-Rag] we will prove that we can define a period $\Omega(\tau_f \times \Pi_f)$ which under our assumptions (f, g have coefficients in $\mathbb{Q}$) is unique up to a sign such that

$$\Omega(\tau_f \times \Pi_f)^{\epsilon(a)} \frac{\Lambda^{\mathrm{coh}}(\tau \times \Pi, a)}{\Lambda^{\mathrm{coh}}(\tau \times \Pi, a+1)} \in \mathbb{Q} \text{ provided } p_c \geq a+1 \text{ , } a \geq q_c + 1$$

From our data $[p_c, q_c]$ and the value of a we can reconstruct the coefficient system λ.

"Large" primes occuring in the denominator of these rational number should produce congruence between eigenvalues of Hecke operators on Siegel modular forms of genus three and certain expressions in eigenvalues on pairs of modular forms of genus one.

The computation of the period is somewhat delicate. We give a definition in [Ha-Rag] and the period is well defined up to a unit (under our special assumptions up to ± 1) But it is not clear from the abstract definition how - given explicit data, i.e. f, g -we can really compute a number with high precision which gives us the value of the period.

There is a way out. Recall that we compute ratios of special values $a, a+1$ where a runs through an interval $[p_c - 1, q_c + 1]$ of integers, this interval can be quite long. So we simply choose our period such that for $a_0 = p_c - 1$

$$\Omega^*(\tau_f \times \Pi_f)^{\epsilon(a_0)} \frac{\Lambda^{\mathrm{coh}}(\tau \times \Pi, a_0)}{\Lambda^{\mathrm{coh}}(\tau \times \Pi, a_0 + 1)} = 1.$$

The correct period differs from this one by a rational number, which will have some prime factors $\{p_1, p_2, \dots, p_r\}$ in it. Now we can start to verify the above rationality assertion for all a and we can compute these ratios as rational numbers.

Recall that we are interested in arguments a for which our ratio of L-values divided by the "correct" period has a "large" prime p in its factorization (in the denominator). Now it would be really bad luck, if this prime p would be (always) member of $\{p_1, p_2, \dots, p_r\}$.

Hence if we find large primes p in the denominator of the ratios

$$\Omega^*(\tau_f \times \Pi_f)^{\epsilon(a)} \frac{\Lambda^{\mathrm{coh}}(\tau \times \Pi, a)}{\Lambda^{\mathrm{coh}}(\tau \times \Pi, a+1)}$$

for some values of a then we can look for congruences $\mod p$ between different kinds of Siegel modular forms.

4.6 The Hecke operators on the boundary cohomology

We go back to the very general case that $G/\mathrm{Spec}(\mathbb{Z})$ is a Chevalley scheme and let $P \subset G$ be a maximal parabolic subgroup, here we assume that it is conjugate to its opposite. We assume that $T/\mathrm{Spec}(\mathbb{Z})$ is a maximal split torus and $T \subset B \subset P$. Let $\pi = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ be the set of simple positive roots, let $\{\gamma_1, \gamma_2, \dots, \gamma_r\}$ be the set of dominant fundamental weights. We have

$$2\frac{<\gamma_i, \alpha_j>}{<\alpha_i, \alpha_i>} = \delta_{ij},$$

the dominant weights are elements in $X^*(T) \otimes \mathbb{Q}$. We also consider the cocharacters $\{\chi_1, \chi_2, \dots, \chi_r\} \in X_*(T) \otimes \mathbb{Q}$, which form the dual basis to the α_i. If we identify $X_*(T) \otimes \mathbb{Q} = X^*(T) \otimes \mathbb{Q}$ via the canonical quadratic form, then $\chi_i = \frac{2\gamma_i}{<\alpha_i,\alpha_i>}$.

We choose a parabolic subgroup P, let α_{i_0} be the erased simple root. We consider the cuspidal (inner ?) cohomology of the boundary stratum attached to P and consider an isotypical subspace

$$H_!^{\bullet - l(\tilde{w})}(S^M, \mathcal{M}(\tilde{w} \cdot \lambda))(\sigma_f)) \subset H^\bullet(\partial_P(S), \mathcal{M}).$$

Actually we should take an induced module on the left hand side, but let us assume that we only look at unramified cohomology, i.e. $K_f = G(\hat{\mathbb{Z}})$. Then induction simply means that we restrict the action of $\mathcal{H}^M$ to the action of $\mathcal{H}^G$ on $H_!^{\bullet - l(w)}(S^M, \mathcal{M}(w \cdot \lambda))$. We want to derive a formula for a "cohomological" Hecke operator in $\mathcal{H}^G$ as a sum over "cohomological" Hecke operator in $\mathcal{H}^M$.

The algebra of Hecke operators is generated by local algebras $\mathcal{H}_p^G$ and these local algebras commute (under our assumption that everything is unramified, they are even commutative).

We fix a prime p. To get Hecke operators we start from cocharacters $\chi = \sum m_i \chi_i : G_m \to T$, where the $m_i \in \mathbb{Z}$. This provides an element $\chi(p) \in T(\mathbb{Q}_p)$, and hence a double coset $K_p \chi(p) K_p$ whose characteristic function is denoted by T_χ. By convolution this defines an operator (also denoted by T_χ) on the cohomology with 3.1.2 rational coefficients

$$T_\chi : H^\bullet(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda \otimes \mathbb{Q}) \to H^\bullet(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda \otimes \mathbb{Q}).$$

We have defined the modified operators, which act on the cohomology with integral coefficients

$$T_\chi^{\mathrm{coh}} \text{ " = " } p^{c(\chi,\lambda)} T_\chi : H^\bullet(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda) \to H^\bullet(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda).$$

(See chap.3.pdf 3.1.2)

We have a formula for the action of $\mathcal{T}_\chi$ on the unramified spherical functions. We consider unramified characters $\nu_p : T(\mathbb{Q}_p) \to \mathbb{C}^\times$. Since $T(\mathbb{Q}_p) = X_*(T) \otimes \mathbb{Q}_p^\times$ we have for the module of unramified characters

$$\mathrm{Hom}_{un}(T(\mathbb{Q}_p), \mathbb{C}^\times) = \mathrm{Hom}(X_*(T), \mathbb{C}^\times) = X^*(T) \otimes \mathbb{C}^\times$$

If we pick a $\chi \in X_*(T)$ and a $\nu_p = \in \mathrm{Hom}_{un}(T(\mathbb{Q}_p), \mathbb{C}^\times)\nu_p(\chi(p))$ We have the embedding $X^*(T) \hookrightarrow \mathrm{Hom}_{un}(T(\mathbb{Q}_p), \mathbb{C}^\times)$ which is given by $\gamma \mapsto |\gamma|_p =$

$(x \mapsto |\gamma(x)|_p)$. I want to distinguish carefully between the algebraic character and its absolute value. If we have a $\gamma \in X^*(T)$ and a $\chi \in X_*(T)$ then we put

$$|\gamma|_p(\chi(p)) =< \chi, \gamma >_p = p^{-<\chi,\gamma>}$$

Especially we have the half sum of positive roots $\rho_B^G \in X^*(T) \otimes \mathbb{Q}$ and the resulting character $|\rho_B^G|$.

We define the spherical function ψ_{ν_p} by

$$\psi_{\nu_p}(g) = \nu_p(bk) = \nu_p(b)$$

and this will be an eigenfunction for the convolution with a Hecke operator

$$T_\chi * \psi_{\nu_p} = T_\chi^\vee(\nu_p)\psi_{\nu_p}.$$

This spherical function differs from the spherical function in chap3.pdf 2.3.4 they are related by the formula

$$\psi_{\nu_p}(g) = \phi_{\nu_p - |\rho_B^G|_p}(g)$$

We write a formula for $T_\chi^\vee(\nu_p)$ for the case that $\chi = \chi_i$ is one of our basis cocharacters χ_i. We look at the orbit of χ_i under the Weyl group, let W_i be the stabilizer of χ_i in W, then

$$T_\chi^\vee(\nu_p) = p^{<\chi_i,\rho_B^G>} \sum_{W/W_i} < w\chi_i, \nu_p - |\rho_B^G|_p > + \delta(\chi_i),$$

where $\delta(\chi_i)$ is a positive integer. It is zero if for all positive roots α we have $< \chi_i, \alpha > \in \{0, 1\}$, i.e. the coefficient of the root α_i in any positive root is always ≤ 1. (This extra term comes bla bla)

If we now have an isotypical submodule $H_!^\bullet(S^G, \mathcal{M}_\lambda)(\pi_f), \pi_f = \otimes_p \pi_p$, and $\pi_p = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \nu_p$ (algebraic induction) then our above formula says

$$T_{\chi_i}^{\mathrm{coh}}(\pi_f) = p^{<\chi_i,\lambda+\rho_B^G>}(\sum_{W/W_i} < w\chi_i, \nu_p - |\rho_B^G|_p >) + p^{<\chi_i,\lambda+\rho_B^G>}\delta(\chi_i). \quad (243)$$

The exponent $< \chi_i, \lambda + \rho_B^G >= c(\chi, \lambda)$, the δ is equal to zero because of our assumption.

Now we ask for a formula for the Hecke operator on T_χ^{coh} on an isotypical piece $H_!^{\bullet - l(\tilde{w})}(S^M, \mathcal{M}(\tilde{w}\cdot\lambda))(\sigma_f))$ in the cohomology of some boundary stratum. We assume that $\sigma_p = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{M(\mathbb{Q}_p)} \nu_p$. The Weyl group W_M acts on W/W_i from the left, let us choose a set of representatives $\{\dots, v, \dots\}$ for this action. Then the sum becomes

$$p^{<\chi_i,\lambda+\rho_B^G>}\Big(\sum_{v \in W_M \backslash W/W_i} \sum_{wv \in W_M/W_{M,i}} < wv\chi_i, \nu_p - |\rho_B^M|_p >) + p^{<\chi_i,\lambda+\rho_B^G>}\delta(\chi_i)\Big) \quad (244)$$

We want to transform this into a sum over Hecke operators acting on $H_!^{\bullet - l(\tilde{w})}(S^M, \mathcal{M}(\tilde{w}\cdot\lambda))(\sigma_f))$ we write

$$p^{<\chi_i,\lambda+\rho_B^G>}\Big(\sum_{v\in W_M\backslash W/W_i}\ \sum_{wv\in W_M/W_{M,i}} < wv\chi_i, \nu_p - |\rho_B^M|_p - |\rho_P|_p >\Big) + p^{<\chi_i,\lambda+\rho_B^G>}\delta(\chi_i) \tag{245}$$

The character $\rho_P = f_P\gamma_{i_0}$ is invariant under the action of W_M we can pull this factor in front

$$p^{<\chi_i,\lambda+\rho_B^G>}\Big(\sum_{v\in W_M\backslash W/W_i} p^{f_P<v\chi_i,\gamma_{i_0}>} \sum_{w\in W_M/W_{M,i}} < wv\chi_i, \nu_p - |\rho_B^M|_p >\Big) + p^{<\chi_i,\lambda+\rho_B^G>}\delta(\chi_i) \tag{246}$$

For a given $v \in W_M\backslash W$ the inner sum is the value of a Hecke operator on the cohomology $H_!^{\bullet-l(\tilde{w})}(S^M, \mathcal{M}(\tilde{w}\cdot\lambda))(\sigma_f))$ times a correcting factor. To compute this correcting factor we write

$$\tilde{w}(\lambda+\rho_B^G) = \tilde{\mu}^{(1)}_{\tilde{w},\lambda} + b(\tilde{w},\lambda)\gamma_{i_0} \tag{247}$$

Note that this expression is - as it must be- independent of the representative v. If we want to compute the correcting factor we have to choose the representative $v_k = w_k v, w_k \in W_M$ such that $v_k\chi_i$ is in the positive chamber with respect to the given Borel subgroup in M, i.e.

$$< v_k\chi_i, \alpha_\nu >\geq 0 \text{ for all } \nu \neq i_0 \tag{248}$$

this is certainly true if v_k is a Kostant representative.

Then the correcting factor becomes

$$p^{<v_k\chi_i,\tilde{\mu}^{(1)}_{\tilde{w},\lambda}-b(\tilde{w},\lambda)\gamma_{i_0}>} \tag{249}$$

(note the minus sign!) and hence we get

$$p^{<v_k\chi_i,\tilde{\mu}^{(1)}_{\tilde{w},\lambda}-b(\tilde{w},\lambda)\gamma_{i_0}>} \sum_{w\in W_M/W_{v,i}} < wv_k\chi_i, \nu_p - |\rho_B^M|_p >= T^{M,\mathrm{coh}}_{v_k\chi}(\sigma_f) \tag{250}$$

We get for our eigenvalue (??? wo ist das f_P geblieben???????)

$$\sum_{v_k\in W_M\backslash W/W_i} p^{<\chi_i,\lambda+\rho_B^G>-<v_k\chi_i,\tilde{\mu}^{(1)}_{\tilde{w},\lambda}-b(\tilde{w},\lambda)\gamma_{i_0}>} T_{v_k\chi_i}(\sigma_f)\Big) + p^{<\chi_i,\lambda+\rho_B^G>}\delta(\chi_i) \tag{251}$$

and this is equal to

$$\sum_{v_k\in W_M\backslash W/W_i} p^{<\chi_i,\lambda+\rho_B^G-v_k^{-1}(\tilde{\mu}^{(1)}_{\tilde{w},\lambda}-b(\tilde{w},\lambda)\gamma_{i_0})>}) T_{v_k\chi_i}(\sigma_f)\Big) + p^{<\chi_i,\lambda+\rho_B^G>}\delta(\chi_i) \tag{252}$$

We can still write this differently, we have

$$\tilde{\mu}^{(1)}_{\tilde{w},\lambda} - b(\tilde{w},\lambda)\gamma_{i_0} = \tilde{\mu}^{(1)}_{\tilde{w},\lambda} + b(\tilde{w},\lambda)\gamma_{i_0} - 2b(\tilde{w},\lambda)\gamma_{i_0} = \tilde{w}(\lambda+\rho_B^G) - 2b(\tilde{w},\lambda)\gamma_{i_0} \tag{253}$$

and then (252) becomes

$$\sum_{v_k\in W_M\backslash W/W_i} p^{<\chi_i,\lambda+\rho_B^G - v_k^{-1}(\tilde{w}(\lambda+\rho_B^G)-2b(\tilde{w},\lambda)\gamma_{i_0})>)}(T_{v_k\chi_i}(\sigma_f)) + p^{<\chi_i,\lambda+\rho_B^G>}\delta(\chi_i) \tag{254}$$

The factor in front is equal to one if $w=\tilde{w}$ and otherwise the exponent is a strictly positive number. Hence we get

$$T^{G,\mathrm{coh}}_{\chi_i}(\mathrm{Ind}(\sigma_p)) = T^{M,\mathrm{coh}}_{\tilde{w}\chi_i}(\sigma_p) + \qquad \text{Hecke-ind}$$

$$\sum_{w\in W^P/W_i, w\neq\tilde{w}} p^{<\chi_i,(\lambda+\rho_B^G)-w^{-1}(\tilde{w}(\lambda+\rho_B^G)>-2b(\tilde{w},\lambda)\gamma_{i_0})>} T^{M,\mathrm{coh}}_{w\chi_i}(\sigma_p) + p^{<\chi_i,\lambda>}\delta(\chi_i).$$

Let us call the first summand on the right hand side the "main" term. We observe that for $w\neq\tilde{w}$ the exponent $<\chi_i,(\lambda+\rho_B^G)-w^{-1}\tilde{w}(\lambda+\rho_B^G)>>0$ and if λ is regular this is also true for $<\chi_i,\lambda>$. This tells us that the eigenvalue $T^{M,\mathrm{coh}}_{\chi_i}(\mathrm{Ind}(\sigma_p))$ is a p-adic unit if and only if $T^{M,\mathrm{coh}}_{\tilde{w}\chi}(\sigma_p)$ is a p-adic unit, provided λ is regular or $\delta(\chi_i)=0$.

(For the special case $G=GSp_2/\mathrm{Spec}(\mathbb{Z})$ and P the Siegel parabolic this yields the formulae in 3.1.2.1 in "Eisenstein Kohomologie...". The formula for $T_{p,\beta}$ is wrong, I overlooked the term $p^{<\chi_i,\lambda>}\delta(\chi_i)$. This was discovered by Gerard, the congruences for the second Hecke operator became wrong.)

4.7 The general philosophy

Now we can formulate how the general form of a Ramunujan-type congruence should look like. We start from an isotypical subspace $H^\bullet(S^M,\mathcal{M}(w\cdot\lambda_R))(\sigma_f)$ where $R=\mathbb{Z}[1/N]$ where N is a suitable integer. Let $I_{\sigma_f}\subset\mathcal{H}_R^M$ be the annihilator of σ_f. Then the quotient $\mathcal{H}_R^M/I_{\sigma_f}=R(\sigma_f)$ is an order in an algebraic number field $\mathbb{Q}(\sigma_f)$. We consider the second constant term of the Eisenstein series evaluated at $s_w=0$ and assume that it is of the form

$$a(\sigma_f)\mathrm{Mot}(\sigma_f)$$

where $a(\sigma_f)\in\mathbb{Q}(\sigma)$ and where $\mathrm{Mot}(\sigma_f)$ has some kind of an interpretation as an element in some $\mathrm{Ext}^1_{\mathcal{MM}}$. Now we assume that a "large" prime $\mathfrak{l}\subset R(\sigma_f)$ divides the denominator of $a(\sigma_f)$. We assume that σ_ℓ is ordinary at $\mathfrak{l}$, i.e. $T^{M,\mathrm{coh}}_{\chi_i}(\sigma_\ell)\notin\mathfrak{l}$ for all i (some i_0 ?).

Then we can hope for an isotypical component Π_f for the Hecke algebra $\mathcal{H}_R^G$ in the cohomology $H^\bullet(S^G,\mathcal{M}_\lambda)(\Pi_f)$, we consider the order $\mathcal{H}_R^G/I_{\Pi_f}=R(\Pi_f)$, we expect to find a prime $\mathfrak{l}_1\subset R(\Pi_f)$ and an isomorphism between the completions

$$\Phi: R(\Pi_f)_{\mathfrak{l}_1}\xrightarrow{\sim} R(\sigma_f)_{\mathfrak{l}}$$

such that for all primes p

$$\Phi(T^G_{\chi_i}(\Pi_p)) \equiv T^{G,\mathrm{coh}}_{\chi_i}(\mathrm{Ind}(\sigma_p)) \mod \mathfrak{l}.$$

We consider the case where our modular forms f, g have rational coefficients, i.e. are of weight $12, 16, 18, 20, 22, 26$ this means that the values for d_1, d_3 are $10, 14, 16, 18, 20, 24$. Following a notation in representation theory we put

$$w \cdot \lambda = w(\lambda + \rho) - \rho = d_1(w \cdot \lambda)\gamma^M_{\alpha_1} + d_3(w \cdot \lambda)\gamma^M_{\alpha_3} + 1/2(-6 - n_2)\gamma_{\alpha_2}.$$

Given d_1, d_3 a value a in the upper half of the above range, we solve the equations

$$d_1(w_1 \cdot \lambda) = d_1,\ d_3(w_1 \cdot \lambda) = d_3,\ \frac{n_2}{2} + \frac{d_1}{2} + d_3 + 2 = a \qquad (case1)$$

$$d_1(w_2 \cdot \lambda) = d_1,\ d_3(w_2 \cdot \lambda) = d_3,\ \frac{n_1}{2} + \frac{d_1}{2} + d_3 + 3 = a \qquad (case2)$$

We introduce the number

$$\mathbf{w} = d_1 + 2d_3 + 3$$

and observe that $\frac{d_1}{2} + d_3 + 2 = \frac{\mathbf{w}+1}{2}$ is the reflection point of the functional equation. We rewrite our equations a little bit. In ($case1$)

$$\begin{gathered} k_1 - 4 = d_1 - 2 = n_2 + 2n_3 \\ k_3 - 4 = d_3 - 2 = n_1 + n_2 + n_3 \\ 2a - \mathbf{w} - 1 = n_2 \end{gathered}$$

and in (case2)

$$\begin{gathered} k_1 - 6 = d_1 - 4 = n_1 + 2n_2 + 2n_3 \\ k_3 - 4 = d_3 - 2 = n_3 \\ 2a - \mathbf{w} - 3 = n_1 \end{gathered}$$

As it turns out that for our restricted choice of f, g we never have solutions in ($case2$).

This gives us a unique highest weight $\lambda = \lambda(d_1, d_3, a)$ and a space of holomorphic modular cusp forms $S_{n_1,n_2,4+n_3}$ in which we should look for a cusp form satisfying congruences.

I want to give the precise form for the expected congruences. We choose the Hecke operator T_{χ_3}, this is the operator whose eigenvalues are the traces of the Frobenius, it has also the property that $< \chi_3, \alpha > \in \{-1, 0, 1\}$ for all roots α, and if we identify $X_*(T)_\mathbb{Q} = X_*(T)_\mathbb{Q}$ then $\chi_3 = \gamma_3$.

The Weyl group W is the semidirect product of S_3 and $(\mathbb{Z}/2\mathbb{Z})^3$ and is of order 48. The stabilizer W_3 of χ_3 is the subgroup S_3, this is the Weyl group of A_2. We have to study the double cosets

$$W_M \backslash W / W_3 = W^P / W_3.$$

The quotient W/W_3 has cardinality 8, on this quotient we have the action of W_M, this is the group generated by the reflections s_1, s_3 and hence is of order 4.

It is clear that we have two orbits of length 2 and one orbit of length 4. Hence the sum in (Hecke-ind) has three terms.

The orbit of length 4 gives us the "main" term in our formula (Hecke-ind) and $T^{M,\mathrm{coh}}_{\tilde{w}\chi_i}(\sigma_p) = a_p(f)a_p(g)$, where of course the two factors are the eigenvalues of f, g respectively.

The two other orbits correspond to the Kostant representatives $e = (\mathrm{Id}, \Theta_P,$ they are fixed by s_1, hence the W_M orbits are given by $\{(e, s_3), (\Theta_P, s_3\Theta_P)\}$. This means that for choice of w we have $T^{M,\mathrm{coh}}_{w^{-1}\tilde{w}\chi_i}(\sigma_p) = a_p(g)$, it remains to compute the factor in front. For $w = e$ or $w = \Theta_P$ this factor is

$$p^{<(\mathrm{Id}-\tilde{w}^{-1}w)\chi_3,\lambda+\rho>}$$

Our element $\tilde{w}$ is one of the two Kostant representatives w_1, w_2 on p. 1. Then $\tilde{w}^{-1}\Theta_P$ is equal to the the corresponding elements v_1, v_2. We get

$$< (\mathrm{Id} - w_1^{-1})\chi_3, \lambda + \rho >= n_2 + n_3 + 2 \qquad < (\mathrm{Id} - v_1^{-1})\chi_3, \lambda + \rho >= n_3 + 1$$
$$< (\mathrm{Id} - w_2^{-1})\chi_3, \lambda + \rho >= n_1 + n_2 + n_3 + 3 \quad < (\mathrm{Id} - v_2^{-1})\chi_3, \lambda + \rho >= n_2 + n_3 + 2$$

Hence we expect:

We choose triple d_1, d_3, a and a pair of eigenforms f, g with weight $d_1 + 2 = k_1, d_3 + 2 = k_3$. Let λ solve the appropriate equations $(case1), (case2)$. If a prime $\mathfrak{l}$ divides the **denominator** *of*

$$\Omega(\tau_f \times \Pi_f)^{\epsilon(a)} \frac{\Lambda^{coh}(\tau \times \Pi, a)}{\Lambda^{coh}(\tau \times \Pi, a+1)}$$

then we find an isotopical subspace $H^6_!(S^G, \mathcal{M}_\lambda)(\tilde{\Pi}_f)$ and a congruence

$$T^G_{\chi_3}(\tilde{\Pi}_p) \equiv a_p(g)(p^{n_3+1} + a_p(f) + p^{n_2+n_3+2}) \mod \mathfrak{l}$$

in $(case1)$ and

$$T^G_{\chi_3}(\tilde{\Pi}_p) \equiv a_p(g)(p^{n_2+n_3+2} + a_p(f) + p^{n_1+n_2+n_3+3}) \mod \mathfrak{l}$$

in $(case2)$

We compare to TABLE 1. in [BFG]: We have

$$(k_1, k_3) = (m_2, m_1)$$

and

$$r_1 = n_2 + n_3 + 2, r_2 = n_3 + 1 \text{ in } (case1),$$
$$r_1 = n_1 + n_2 + n_3 + 3, r_2 = n_2 + n_3 + 2 \text{ in } (case2).$$

Recall that we are interested in the special value $a+1$, we can say in (case1)

$$a + 1 = \frac{n_2 + 1}{2} + \frac{\mathbf{w}}{2} + 1 = \frac{r_1 - r_2 + \mathbf{w}}{2} + 1$$

and in $(case2)$

$$a + 1 = \frac{n_2 + 1}{2} + \frac{\mathbf{w}}{2} + 1 = \frac{r_1 - r_2 + \mathbf{w}}{2} + 1$$

Now I checked against TABLE1 in [BFG] and Anton's tables and the data match perfectly. We even see some "small" primes providing congruences. We see a 17^2 occuring in the case f of weight 12 and g of weight 18. We observe that both forms are ordinary at 17.

Remark: In our special case the expression for $T^{G,\mathrm{coh}}_{\chi_i}(\mathrm{Ind}(\sigma_p))$ is a sum of three terms, the term in the middle $a_p(g)a_p(f)$ has weight $\frac{d_1+d_2}{2}+1$ the first term has a lower weight the third term has a higher weight. The difference of the weights of the first and third term is up to a shift our evaluation point a. This means: The closer these two weights get, the closer a comes to the center of the L function.

We go to $g=4$. In this case our group $M=\mathrm{Gl}_4\cdot\mathbb{G}_m$. We choose a highest weight $\lambda=n_1\gamma_1+n_2\gamma_2+n_1\gamma_3+n_4\gamma_4$, the central character is trivial. It seems that the interesting Kostant representatives are

$$w'=s_4s_3s_2s_4 \text{ and } w=s_4s_3s_2s_4s_3s_1 \tag{255}$$

We get

$$\begin{aligned} w'(\lambda+\rho_B^G)&=(2+n_1+n_2)(\gamma_1^M+\gamma_3^M)+(3+n_1+2n_4)\gamma_2^M+\tfrac{1}{2}(1+n_1)\gamma_4,\\ w(\lambda+\rho_B^G)&=(2+n_1+n_2)(\gamma_1^M+\gamma_3^M)+(3+n_1+2n_4)\gamma_2^M+\tfrac{1}{2}(-1-n_1)\gamma_4 \end{aligned} \tag{256}$$

We see that $\mathcal{M}_\lambda(w\cdot\lambda)$ is self dual, this is the reason why we have chosen $n_1=n_3$. As usual we define numbers $d_1=d_3,d_2$ by

$$d_1+1=d_3+1=2+n_1+n_2, d_2+1=3+n_1+2n_4 \tag{257}$$

The dimension of $\mathcal{S}^G_{K_f}$ is 20, we look at our fundamental exact sequence

$$\begin{array}{cccc} H^5(\mathcal{S}^M_{K^M_f},\mathcal{M}_\lambda(w'\cdot\lambda)) & \xrightarrow{\delta} H^{10}_c(\mathcal{S}^G_{K_f},\mathcal{M}_\lambda) & \to H^{10}(\mathcal{S}^G_{K_f},\mathcal{M}_\lambda) & \to H^4(\mathcal{S}^M_{K^M_f},\mathcal{M}_\lambda(w\cdot\lambda)) \\ \uparrow & & & \uparrow \\ H^5_!(\mathcal{S}^M_{K^M_f},\mathcal{M}_\lambda(w'\cdot\lambda))(\sigma'_f) & & & H^4_!(\mathcal{S}^M_{K^M_f},\mathcal{M}_\lambda(w\cdot\lambda))(\sigma_f) \end{array} \tag{258}$$

This is the constellation where we can hope for extensions of mixed Tate motives. The difference of the weights of w' and w is two, which seems to be too big. But the cohomology of $\mathcal{S}^M_{K^M_f}$ is concentrated in degree 4 and 5, so we get boundary cohomology in degree 9 and 10.

We have to compute the second constant term. To do this we have to study the representation of the group ${}^LM$ on the Lie-algebra $\mathfrak{u}_P^\vee$. The Dynkin diagram for the Langlands dual group ${}^LG$ is

$$\alpha_1^\vee \quad - \quad \alpha_2^\vee \quad - \quad \alpha_3^\vee \quad >= \quad \alpha_4^\vee,$$

and we get ${}^LM$ if we erase $\alpha_4^\vee$. The representation of ${}^LM$ on $\mathfrak{u}_P^\vee$ decomposes into two irreducible representations, the first one has highest weight

$$\eta_1=\alpha_1^\vee+\alpha_2^\vee+\alpha_3^\vee+\alpha_4^\vee$$

and is up to a twist the tautological representation. The second one has highest weight

$$\eta_2 = \alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee + 2\alpha_4^\vee$$

and is (again up to a twist) the Λ^2 of the tautological representation. It is of dimension 6. We recall formula (100) in chap3.pdf. The number a in this formula takes the values 1,2 and we get

$$\begin{array}{c} \frac{\mathbf{w_1}}{2} =< \eta_1, \tilde{\mu}^{(1)} >= \frac{7}{2} + \frac{3}{2}n_1 + n_2 + n_4 = d_1 + \frac{1}{2}d_2 + \frac{3}{2} \\ \frac{\mathbf{w_2}}{2} =< \eta_2, \tilde{\mu}^{(1)} >= 5 + 2n_1 + n_2 + 2n_4 = d_1 + d_2 + 3 \end{array} \tag{259}$$

This implies that the second constant term is

$$\begin{array}{c} \frac{1}{\Omega(\sigma_f)} \frac{\Lambda^{\text{coh}}(\sigma_f, r_{\eta_1}, 4+2n_1+n_2+n_4)}{\Lambda^{\text{coh}}(\sigma_f, r_{\eta_1}, 5+2n_1+n_2+n_4)} \frac{\Lambda^{\text{coh}}(\sigma_f, r_{\eta_2}, 6+3n_1+n_2+2n_4)}{\Lambda^{\text{coh}}(\sigma_f, r_{\eta_2}, 7+3n_1+n_2+2n_4)} \\ = \\ \frac{1}{\Omega(\sigma_f)} \frac{\Lambda^{\text{coh}}(\sigma_f, r_{\eta_1}, \frac{\mathbf{w_1}}{2}+\frac{1}{2}(1+n_1))}{\Lambda^{\text{coh}}(\sigma_f, r_{\eta_1}, \frac{\mathbf{w_1}}{2}+\frac{1}{2}(1+n_1)+1)} \frac{\Lambda^{\text{coh}}(\sigma_f, r_{\eta_2}, \frac{\mathbf{w_2}}{2}+1+n_1)}{\Lambda^{\text{coh}}(\sigma_f, r_{\eta_2}, \frac{\mathbf{w_2}}{2}+2+n_1)} \end{array} \tag{260}$$

Since assume that we are in the unramified case the two isotypical subspaces σ_f' resp. σ_f in (258) provide Tate motives $\mathbb{Z}(-\frac{\mathbf{w_1}}{2} + \frac{1}{2}(n_1+1))$ resp. $\mathbb{Z}(-\frac{\mathbf{w_1}}{2} - \frac{1}{2}(n_1+1))$. Hence our usual construction of Anderson motives will provide elements

$$\mathcal{X}(\sigma_f) \in \text{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-1-n_1), \mathbb{Z}) \tag{261}$$

Since we want non torsion classes, we assume n_1 even. This implies that d_2 must be even and if we give ourselves $d_1 \geq 1, d_2 \geq 2$ and even, then we see that for our given d_1, d_2 and given Kostant representative $w = s_4s_3s_2s_4s_3s_1$ we find a dominant $\lambda = n_1\gamma_1 + n_2\gamma_2 + n_1\gamma_3 + n_4\gamma_4$ with

$$w(\lambda+\rho) = (d_1+1)(\gamma_1^M + \gamma_3^M) + (d_2+1)\gamma_2^M - \frac{1}{2}(1+n_1)\gamma_4$$

if and only if $n_1 \in [0, \min(d_1-1, d_2-2)]$ and even.

Again we consult [Ha-Rag] and find that the miracle happens again: The numbers $\frac{\mathbf{w_1}}{2} + \frac{1}{2}(1+n_1) + 1$ run through the critical arguments, for $n_1 = 0$ the number $\frac{\mathbf{w_1}}{2} + \frac{1}{2}$ is the smallest critical argument to the right from the central argument for the functional equation ($= \frac{\mathbf{w_1}}{2}$).

Hence we know that the factor in front

$$\frac{1}{\Omega(\sigma_f)} \frac{\Lambda^{\text{coh}}(\sigma_f, r_{\eta_1}, \frac{\mathbf{w_1}}{2} + \frac{1}{2}(1+n_1))}{\Lambda^{\text{coh}}(\sigma_f, r_{\eta_1}, \frac{\mathbf{w_1}}{2} + \frac{1}{2}(1+n_1)+1)} \tag{262}$$

is an algebraic number in F. The period $\Omega(\sigma_f)$ is locally well defined up to a unit and hence we can speak of the prime decomposition of this algebraic number. Hence we may apply the principles outlined in 2.1. and ask whether "large" primes $\mathfrak{l}$ which divide the denominator of the expression in (262) create eigenclasses in $H^{10}(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda)$ whose eigenvalues are congruent to the eigenvalues of σ_f modulo $\mathfrak{l}$.

In our heuristic reasoning we encounter new difficulties, before we discuss these problems I want to give the precise form of these congruences in the sense of (**??**,(Hecke-Ind)). The following considerations hold for arbitrary even value of g.

We choose the cocharacter $\chi_g : t \to T$ which satisfies $< \chi, \alpha_i >= 0$ for $i < g$ and $< \chi_g, \alpha_g >= 1$. (This means the first g entries on the diagonal are equal to t the other entries are equal to 1.) The stabilizer of this character in the Weyl group is $S_4 = W_M$ the Weyl group of M. Then we can represent the cosets W/W_M by the 2^g elements which exchange some of the $e_i \to f_i, f_i \to -e_i$ and leave the others fixed. So we can say that W/W_M is equal to the set of subsets of $\{1, 2, \ldots, g\}$. The Weyl group W_M also acts from the left on this coset space and acts transitively transitively on the set of subsets of a fixed cardinality h. Therefore the number of orbits is $g + 1$.

4.8 $g = 4$

We go back to the case $g = 4$, our cocharacter is χ_4 and our parabolic subgroup is the Siegel parabolic subgroup, i.e. $i_0 = 4$. Let w_P be the longest Kostant representative. We have the choices $v_k = e, v_k = s_4, v_k = s_4 s_3 s_2 s_4 s_3 s_1, w_P s_4, w_P$, they are Kostant representatives and hence they satisfy (248). We choose $\tilde{w} = s_4 s_3 s_2 s_4 s_3 s_1$.

We investigate the expressions

$$< \chi_4, \lambda + \rho_B^G - v_k^{-1}\Big(\tilde{w}(\lambda + \rho_B^G) - 2b(\tilde{w}, \lambda)\gamma_4\Big) > \tag{263}$$

these will give us the exponents in the powers of p which enter in the sum. To do this we have to write

$$\lambda + \rho_B^G - v_k^{-1}\Big(\tilde{w}(\lambda + \rho_B^G) - 2b(\tilde{w}, \lambda)\gamma_4\Big) = \sum m_j \alpha_j \tag{264}$$

and then

$$< \chi_4, \lambda + \rho_B^G - v_k^{-1}\Big(\tilde{w}(\lambda + \rho_B^G) - 2b(\tilde{w}, \lambda)\gamma_4\Big) >= m_4 \tag{265}$$

Perhaps it it even simpler to rewrite this in the terms of the μ-s. We observe that $< \chi_4, \tilde{\mu}^{(1)}_{e,\lambda} >= 0$ and hence

$$m_4 =< \chi_4, b(e, \lambda)\gamma_4 - v_k^{-1}\Big(\tilde{\mu}^{(1)}_{\tilde{w},\lambda} - b(\tilde{w}, \lambda)\gamma_4\Big) >= 2b(e, \lambda)- < \chi_4, v_k^{-1}\Big(\tilde{\mu}^{(1)}_{\tilde{w},\lambda} - b(\tilde{w}, \lambda)\gamma_4\Big) > \tag{266}$$

If we choose $v_k = e$ or $v_k = w_P$ then $v_k^{-1}\tilde{\mu}^{(1)}_{\tilde{w},\lambda} = \tilde{\mu}^{(1)}_{\tilde{w},\lambda}$ and hence $< \chi_4, v_k^{-1}\tilde{\mu}^{(1)}_{\tilde{w},\lambda} >= 0$, so

$$2b(e, \lambda)- < \chi_4, v_k^{-1}\Big(\tilde{\mu}^{(1)}_{\tilde{w},\lambda} - b(\tilde{w}, \lambda)\gamma_4\Big) >= 2b(e, \lambda) \pm b(\tilde{w}, \lambda) \tag{267}$$

These numbers are easy to compute and equal to

$$5 + 2n_1 + n_2 + 2n_4 \pm (1 + n_1) \tag{268}$$

Now we consider the two choices $v_k = s_4, w_P s_4$. In this case we get for the exponents

$$\frac{3}{2} + \frac{n_1}{2} + n_4 \pm \frac{1}{2}(1+n_1) \tag{269}$$

or more precisely we get

$$m_4 = < \chi_4, b(e,\lambda)\gamma_4 - v_k^{-1}\left(\tilde{\mu}^{(1)}_{\tilde{w},\lambda} - b(\tilde{w},\lambda)\gamma_4\right) > = \begin{cases} 1+n_4 & \text{if } v_k = s_4 \\ 2+n_1+n_4 & \text{if } v_k = w_P s_4 \end{cases} \tag{270}$$

Finally we choose $v_k = s_4 s_3 s_2 s_4 s_3 s_1 = \tilde{w}$. In this case the two μ contributions cancel and one also checks easily that $< \chi_4, v_k^{-1}\gamma_4 >= 0$. Hence the exponent is zero. Since χ_4 is miniscule we get $\delta(\chi_4) = 0$. We conclude that formula (254) yields

The cocharacters $\chi_4, s_4\chi_4, \tilde{w}\chi_4, w_P s_4, w_p$ define conjugacy classes of cocharacters for the group $M = \mathrm{Gl}_4 \cdot \mathbb{G}_m$. Since we assumed for simplicity that the central character of $\mathcal{M}_\lambda$ is trivial we can divide by the factor $\mathbb{G}_m$ and consider these cocharacters as homomorphisms from $\mathbb{G}_m$ to the standard maximal torus of $M = \mathrm{Gl}_4$. The conjugacy classes of of these five cocharacters are $\chi_4, \chi_3, \chi_2, \chi_1$ in the notation of chap3.pdf 3.1.4. and χ_0 which is the trivial character. We observe that $T^{M,\mathrm{coh}}_{\chi_4} = T^{M,\mathrm{coh}}_{\chi_0} = 1$ and therefore we get

$$\begin{array}{rc} T^{G,\mathrm{coh}}_{\chi_4}(\mathrm{Ind}(\sigma_p)) = & (p^{4+n_1+n_2+2n_4} + p^{6+3n_1+n_2+2n_4}) \\ + & p^{1+n_4} T^{M,\mathrm{coh}}_{\chi_3}(\sigma_p) + p^{2+n_1+n_4} T^{M,\mathrm{coh}}_{\chi_1}(\sigma_p) \\ + & T^{M,\mathrm{coh}}_{\chi_2}(\sigma_p) \end{array} \tag{271}$$

Therefore we can express the eigenvalues of the above Hecke operators at a prime p in terms of the Satake parameter $\omega_p = \{\omega_{1,p}, \omega_{2,p}, \omega_{3,p}, \omega_{4,p}\}$ of π_p. We get

$$T^{M,\mathrm{coh}}_{\chi_\nu}(\sigma_p) = \sum_{I:\#I=\nu} p^{<\chi_\nu, \tilde{\mu}^{(1)}_{\tilde{w},\lambda}>} \omega^*_{I,p} \tag{272}$$

This has now the same shape as the expressions which we have seen before. The numbers $T^{M;\mathrm{coh}}_{\tilde{w}\chi_i}(\sigma_f)$ are algebraic integers. We have exactly one term which does not have a strictly positive power of p in front of it. Therefore we may ask whether for a "large" prime $\mathfrak{l} \subset \mathcal{O}_F$, which divides the denominator of

$$\frac{1}{\Omega(\sigma_f)} \frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_1}, 4+2n_1+n_2+n_4)}{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_1}, 5+2n_1+n_2+n_4)}$$

"creates" an isomorphism class class Π_f with $H^{10}(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda)(\Pi_f)) \neq 0$ such that

$$T^{G,\mathrm{coh}}_{\chi_4}(\Pi_p) \equiv T^{G,\mathrm{coh}}(\mathrm{Ind}(\sigma_p)) \mod \mathfrak{l} \text{ for all primes } p \tag{273}$$

This is in perfect analogy to the cases $g = 2, 3$ where the congruences have been verified experimentally.

But we may have a problem. We still have the "motivic" factor

$$\frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 6+3n_1+n_2+2n_4)}{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 7+3n_1+n_2+2n_4)}. \tag{274}$$

In our previous cases this was a ratio

$$\frac{\zeta(\dots)}{\zeta(\dots+1)} \tag{275}$$

and this had an interpretation as an extension class in the Betti-de-Rham realization.

We now **assume** that the analogous computations to the computation in [Ha1], 4.2. and SecOPs.pdf work and especially that the secondary operator is non zero and is given by a "simple" rational number. Then

$$\mathcal{X}_{B-de-Rham}(\sigma_f) \in \mathrm{Ext}^1_{B-dRh}(\mathbb{Z}(-1-n_1), \mathbb{Z}(0)) = i\mathbb{R} \tag{276}$$

is essentially equal to this motivic factor

$$\frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 6+3n_1+n_2+2n_4)}{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 7+3n_1+n_2+2n_4)}. \tag{277}$$

This may challenge our belief that there are no exotic Tate motives, because otherwise we must have a relation

$$\frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 6+3n_1+n_2+2n_4)}{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 7+3n_1+n_2+2n_4)} \sim \zeta'(-n_1) \tag{278}$$

where $\sim$ means equal up to an algebraic number. This is hard to believe!

Of course some computations have to be checked. Especially we have to check whether, in analogy with the case $g=2$, the secondary operator on the cohomology of the relevant Harish-Chandra modules is non zero and has a "reasonable" value. If this is so, then we can say:

If (278) is not true then we can construct a mixed Tate motive $\mathcal{X}(\sigma_f)$ whose extension class in $\mathrm{Ext}^1_{B-deRh}(\mathbb{Z}(-1-n_1), \mathbb{Z}(0))$ is not in the rational line through $\zeta'(-n_1)/i\pi$

This does not destroy our hope for congruences. We may ask for the image of

$$\mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-1-n_1), \mathbb{Z}(0)) \xrightarrow{\mathrm{BdR}} \mathrm{Ext}^1_{B-dRh}(\mathbb{Z}(-1-n_1), \mathbb{Z}(0)).$$

If our construction works then it seems to be plausible that this image may generate even an infinite dimensional $\mathbb{Q}$-vector space. But perhaps there is some reason that its image is not infinitely divisible. Assuming this we can ask the question about congruences formulated above.

In principle we can check these questions experimentally. For the congruences Bergström and friends should extend their computations to $g=4$. More serious is the question whether (278) is true. If we find an algebraic number β such that

$$\frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 6+3n_1+n_2+2n_4)}{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 7+3n_1+n_2+2n_4)} = \beta\zeta'(-n_1)$$

up to a very high order of precision (high with respect to the height of β), then this does not prove that (278) is true, but it makes us almost sure. If we do not find such a number then it is very likely that (278) is false.

4.9 Non regular coefficients again

So far we always assumed that $n_1 > 0$. We expect that in this case the Eisenstein intertwining operator is holomorphic at $s = 0$. The second term in the constant term of the Eisenstein series is a product of two terms and the second factor is a ratio of Λ^2 -L values

$$\frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 5 + 2n_1 + n_2 + 2n_4 + n_1 + 1 + s)}{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 5 + 2n_1 + n_2 + 2n_4 + n_1 + 2 + s)} \tag{279}$$

where

$$\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, s) = \prod_p \prod_{I:\#I=2} \frac{1}{1 - p^{\frac{\mathbf{w}_2}{2}} \omega^*_{I,p} p^{-s}} = \prod_p \prod_{I:\#I=2} \frac{1}{1 - \tilde{\omega}_{I,p} p^{-s}} \tag{280}$$

and $\frac{\mathbf{w}_2}{2} =< \eta_2, \tilde{\mu}^{(1)}_{\tilde{w},\lambda} = 5 + 2n_1 + n_2 + 2n_4$.

Now the situation becomes very unclear. We have to evaluate at $s = \frac{\mathbf{w}_2}{2} + 1 + n_1$. Our estimates for the $\tilde{\omega}_{I,p}$ imply that $\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, s)$ is holomorphic at this argument if $n_1 > 0$. But if $n_1 = 0$ then we may have a first order pole. Actually we do not know whether such a pole is a first order pole, we only know from Langlands see **??** that the expression in (260) has at most a first order pole. But let us assume that we have a first order pole. This pole cancels if

$$\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_1}, 4 + n_2 + n_4) = 0 \tag{281}$$

This vanishing may be a rare event, but it can happen that for our given σ_f the L-function in the first factor satisfies the functional equation

$$\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_1}, 8 + 2n_2 + 2n_4 - s) = -\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_1}, s) \tag{282}$$

and then (281) is forced.

The situation is now analogous to the situation in section 4.3 and we may ask whether the minus sign in the functional equation implies that we have a submodule $\mathcal{SK}(\sigma_f) \subset H^{10}_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda)$ which is a direct sum of copies of $J(\sigma_f)$ and which provides a motive which is isomorphic to $\mathbb{M}(\sigma_f, r_{\eta_2})$.

More precisely we can define $\mathcal{SK}^\bullet(\sigma_f)$ as the image of the tautological map

$$\mathrm{Hom}_{\mathcal{H}^G_{K_f}}(J(\sigma_f), H^\bullet_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda)) \otimes J(\sigma_f) \xrightarrow{\mathrm{taut}} H^\bullet_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda) \tag{283}$$

For any prime ideal $\mathfrak{l}$ we have an action of the Galois action on

$$H^\bullet_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda \otimes F_{\mathfrak{l}})$$

which commutes with action of the Hecke algebra. (see the remark at the beginning of section 4 this induces an action on $\mathcal{SK}^\bullet(\sigma_f) \otimes F_{\mathfrak{l}}$ and therefore we get an action of the Galois group on

$$W_{\mathfrak{l}}(\sigma_f) = \mathrm{Hom}_{\mathcal{H}^G_{K_f}}(J(\sigma_f), H^\bullet_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda) \otimes F_{\mathfrak{l}})$$

such that the tautological map becomes an isomorphism of Galois$\times$ Hecke modules.

We still have the congruence relations and they tell us that for all primes p the eigenvalues of the Frobenius Φ_p^{-1} on $W_{\mathfrak{l}}(\sigma_f)$ have to be taken from the list

$$\mathcal{L}_p(\sigma_f) = \{p^{a(\nu)} p^{<\chi_\nu, \tilde{\mu}^{(1)}_{\tilde{w},\lambda}>} \omega^*_{I,p}\} \tag{284}$$

of summands occurring in the formulas (271) and (272).

This implies that we can have an non trivial $\mathcal{SK}^q(\sigma_f) \subset H^q_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda) \otimes F_{\mathfrak{l}})$ only if we have some summands in our list $\mathcal{L}_p(\sigma_f)$ which satisfy

$$|p^{a(\nu)} p^{<\chi_\nu, \tilde{\mu}^{(1)}_{\tilde{w},\lambda}>} \omega^*_{I,p}| = p^{\frac{q}{2} + + n_2 + 2n_4} = p^{<\chi_2, \tilde{\mu}^{(1)}_{\tilde{w},\lambda}>} \tag{285}$$

In analogy to what we have seen earlier we should choose $q = 10$. If $\mathcal{SK}^q(\sigma_f) \neq 0$ then we get that the members $p^{<\chi_2, \tilde{\mu}^{(1)}_{\tilde{w},\lambda}>} \omega^*_{I,p} \in \mathcal{L}_p(\sigma_f)$, which are also eigenvalues for Φ_p^{-1} must have absolute value $p^{<\chi_2, \tilde{\mu}^{(1)}_{\tilde{w},\lambda}>}$ and hence we get $|\omega^*_{I,p}| = 1$ for those values of I. If we assume that actually all $\omega^*_{I,p} \in \mathcal{L}_p(\sigma_f)$ with $\#I = 2$ occur as Frobenius eigenvalue, then we get the Ramanujan conjecture. Actually it seems to be plausible that each eigenvalue in the sublist where $\#I = 2$ occurs with multiplicity one. then we get that $\dim(W_{\mathfrak{l}}(\sigma_f)) = 6$. In this case we have found the motive $\mathbb{M}(\sigma_f, r_2)$ in side the cohomology $H^q_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda) \otimes F)$.

If $\Lambda^{\mathrm{coh}}(\sigma_f, r_1, 4 + n_2 + n_4 + s)$ has a first order zero at $s = 0$ then we can construct Anderson mixed motives (as in section 4.3), i.e. extensions

$$\begin{aligned} \mathcal{Y}(\sigma_f) &\in \mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-k), \mathcal{SK}(\sigma_f)) \\ \mathcal{Y}'(\sigma'_f) &\in \mathrm{Ext}^1_{\mathcal{MM}}(\mathcal{SK}(\sigma_f), \mathbb{Z}(-k+1)) \end{aligned} \tag{286}$$

where now $k = 5 + n_2 + n_4$, and these two extension come with a canonical "integral" to a biextension $(\widetilde{\mathcal{Y}'(\sigma'_f), \mathcal{Y}(\sigma_f)})$ and a computation like the one in SecOPs.pdf should yield

$$i[\widetilde{\mathcal{Y}'(\sigma'_f), \mathcal{Y}(\sigma_f)}] \sim \frac{\Lambda^{\mathrm{coh},\prime}(\sigma_f, r_1, 4 + n_2 + n_4)}{\Omega(\sigma_f)^\epsilon \Lambda^{\mathrm{coh}}(\sigma_f, r_1, 5 + n_2 + n_4)} \mathrm{Res}_{s=0} \frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 6 + n_2 + 2n_4 + s)}{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 7 + n_2 + 2n_4 + s)} \tag{287}$$

This formula gives us a strong hint that we always should have $\mathcal{SK}^{10}(\sigma_f) \neq 0$, because otherwise

$$i[\widetilde{\mathcal{Y}'(\sigma'_f), \mathcal{Y}(\sigma_f)}] \in \mathrm{Ext}^1_{B-dRh}(\mathbb{Z}(-k), \mathbb{Z}(-k+1))$$

and this last group is hypothetically $\log(\mathbb{Q}^\times_{>0})$ and this is again hard to believe.

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