Young Women in Topology

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Spectral flow, index and applications to measured foliations

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1. Topology and analysis: the spectral flow

The *spectral flow* of a continuos path $(D_t)_{t\in[0,1]}$ of bounded selfadjoint Fredholm operators on a Hilbert space H is defined (Atiyah–Lusztig) as

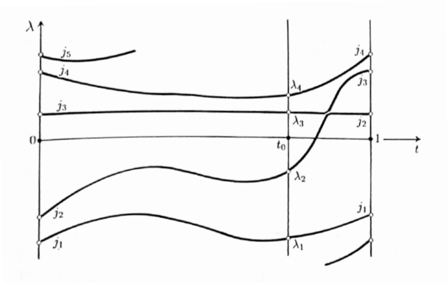


Fig. 17.1 Spectrum of a family with spectral flow = 1

$\mathrm{sf}(D_t)_{t\in[0,1]} :=$ *the net number of eigenvalues (counting multiplicity) changing sign from the start of the path to its end.*

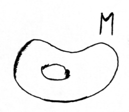
- The spectral flow is an homotopy invariant
- $\mathcal{F}^{sa} := \{T\colon H\to H$ bounded selfadjoint Fredholm operators$\}$ has three connected components $\mathcal{F}^{sa} = \mathcal{F}^{sa}_+ \cup \mathcal{F}^{sa}_- \cup \mathcal{F}^{sa}_*$: $\mathcal{F}^{sa}_\pm$ of essentially positive/negative operators, are contractible. The nontrivial $\mathcal{F}^{sa}_*$ is a classifying space for K^1 and *the spectral flow realizes the isomorphism* $\pi_1(\mathcal{F}^{sa}_*) = [S^1, \mathcal{F}^{sa}_*] \simeq K^1(S^1) = \mathbb{Z}$

Equivalent *analytic definition of spectral flow* given by Phillips: Fredholmness $\Rightarrow \exists 0 = t_0 < \cdots < t_n = 1$ and $a_1, a_2, \dots a_n$ positive so that the spectral projection $t \to P_i(t) = \chi_{[-a_1,+a_i]}(D_t)$ is continuos and finite rank on $[t_{i-1}, t_i]$

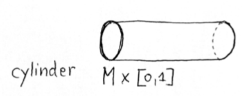
$$\mathrm{sf}(D_t)_{t\in[0,1]} = \sum_{k=1}^{n} \left(\dim P_i^+(t_i) - \dim P_i^+(t_{i-1})\right) \qquad (1)$$

Spectral flow and index

$(D_t)_{t\in[0,1]}$ continuos path of Dirac-type operator on a **closed** manifold M^{2l+1}, then $\mathrm{sf}(D_t)$ can be defined as above (the spectrum spec D_t is discrete).
The spectral flow is related to the index on the cylinder by the classical equality



$$\mathrm{sf}(D_t)_{t\in[0,1]} = \mathrm{ind}_{APS}\left(\tfrac{\partial}{\partial t} + D_t\right)$$



where the operator $\frac{\partial}{\partial t} + D_t$ on the cylinder has Atiyah–Patodi–Singer boundary conditions.
- proved by Robbin–Salamon via axiomatic approach.
- for Dirac operators, it follows for example from variational formula of the *eta-invariant* [Me].

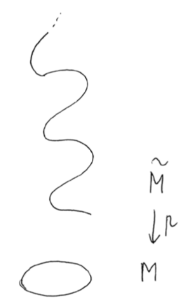
Definition 1 *The eta-invariant of a Dirac operator D is* $\eta(D) := \frac{1}{\sqrt{\pi}}\int_0^\infty \mathrm{tr}(De^{-tD^2})\frac{dt}{\sqrt{t}}$

$(D_t)_{t\in[0,1]}$ path of Dirac operators, for example corresponding to a path of metrics on M. The variational formula $\quad \eta(D_1) - \eta(D_0) = \dim\operatorname{Ker} D_1 - \dim\operatorname{Ker} D_0 + \int_0^1 F(t)ds + 2\,\mathrm{sf}(D_t)_{t\in[0,1]}$

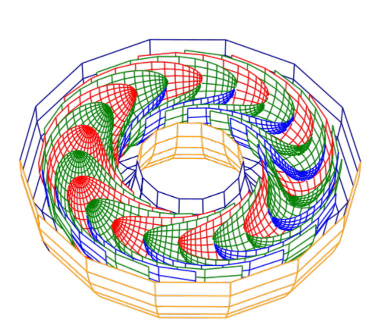
where $F(t) = \lim_{s\to 0}\left(s^{\frac12}\,\mathrm{Tr}(\dot{D}_t e^{-sD_t^2})\right)$, is actually equivalent to the equality spectral flow = index.

The signature operator on a closed manifold M^{2l+1} is $D^{sign} = d\tau + \tau d$, where $\tau\phi := i^{l+1+k(k+1)} * \phi$, $*$=Hodge star. $(D^{sign})^2 = \Delta \Rightarrow$ by Hodge theory $\operatorname{Ker} D^{sign} \simeq H^*_{dR}(M)$ so that a path of metrics g_t on M does not produce spectral flow for the corresponding path D_t^{sign}
Consequence: let $\alpha, \beta\colon \pi_1(M) \to U(n)$ be two representations, and D_α^{sign}, D_β^{sign} be the operators twisted by the flat bundles associated with α, β, then: *the rho-invariant* $\rho_{\alpha-\beta}(D^{sign}) := \eta(D_\alpha^{sign}) - \eta(D_\beta^{sign})$ *does not depend on the metric on* M (Atiyah–Patodi–Singer).

2. Von Neumann invariants: Galois coverings & measured foliations



Here the geometric operators may have *zero in the continuos spectrum*, so the usual definitions of index/ spectral flow do not apply. Yet they are Breuer–Fredholm, affiliated to a semifinite von Neumann algebra.
a. $\tilde{M} \to M$ Galois Γ-covering. A Γ-invariant Dirac operator $\tilde{D}$ acting on $\tilde{E} \to \tilde{M}$ is affiliated to $\mathcal{N} = \mathcal{B}_\Gamma(L^2(\tilde{M}, \tilde{E}))$
b. $(M, \mathcal{F})$ foliated manifold with holonomy invariant transverse measure Λ. A tangential Dirac operator D is affiliated to the von Neumann algebra of the foliation $\mathcal{N} = \mathcal{W}^*(\mathcal{F})$.



a. On $\mathcal{B}_\Gamma(L^2(\tilde{M}, \tilde{E}))$ Atiyah's L^2-trace tr_Γ gives a natural notion of Γ-dimension. $\operatorname{Ker}\tilde{D}$ and $\operatorname{Ker}\tilde{D}^*$ have finite tr_Γ-dimension $\Rightarrow \tilde{D}$ is tr_Γ-Breuer–Fredholm with Γ-index $\mathrm{ind}_\Gamma \tilde{D} := \mathrm{tr}_\Gamma P_{\ker\tilde{D}} - \mathrm{tr}_\Gamma P_{\operatorname{Ker}\tilde{D}^*}$
b. The measure Λ gives a semifinite trace tr_Λ on the von Neumann algebra $\mathcal{N} = \mathcal{W}^*(\mathcal{F})$. Connes' measured index is defined $\mathrm{ind}_\Lambda D = \dim_\Lambda \operatorname{Ker} D - \dim_\Lambda \operatorname{Ker} D^*$.
These are both examples of the following general Breuer–Fredholm theory.

Index and spectral flow in the semifinite context

Let $\mathcal{N}$ be a von Neumann algebra $\mathcal{N} \subset \mathcal{B}(H)$, endowed with a faithful normal, semifinite trace τ. $K(\mathcal{N})$ = ideal generated by projections of finite trace. $\pi\colon \mathcal{N} \to \mathcal{N}/K(\mathcal{N}) = \mathcal{Q}(\mathcal{N})$ projection to the Calkin algebra.

Definition 2 *A closed, densely defined operator D on H is* affiliated to $\mathcal{N}$ *if its bounded transform $F_D = D(1+D^*D)^{-\frac12} \in \mathcal{N}$. An unbounded operator D on H is* Breuer–Fredholm in $\mathcal{N}$ *if it is closed, densely defined, affiliated to $\mathcal{N}$, and $\pi(F_D) \in \mathcal{Q}(\mathcal{N})$ is invertible.*

The *index of a Breuer–Fredholm operator* D is defined by $\operatorname{ind} D := \tau(\chi_{\{0\}}(D^*D)) - \tau(\chi_{\{0\}}(DD^*))$.
Phillips' definition (1) can be extended to define a *real valued spectral flow for paths $(D_t)_{t\in[0,1]}$ of Breuer–Fredholm affiliated* operators with $t \to F_{D_t}$ continuous [Ph]

$$\mathrm{sf}(D_t) := \sum_{i=1}^{n} \mathrm{ec}(P_{i-1}P_i)$$

$ec(PQ) := \operatorname{ind}(PQ\colon QH \to PH)$, $P_t = \chi_{[0,+\infty)}(D_t)$, and $0 = t_0 < \cdots < t_n = 1$ so that $\pi(\chi_{[0,+\infty)}(D_t))$ is splitted so that continuos on $t \in [t_{i-1}, t_i]$.
Question: *prove the relation spectral flow = index in this context*
If D is a odd Dirac operator from geometric situations **a.**, **b.**, it has a well defined von Neumann eta-invariant $\eta_\tau(D)$ [CG], appearing in the relevant index formulæ for the boundary case [Rm, An].
Question: *can one prove variational formulas for von Neumann eta-invariants?*
Question: *what are the geometric consequences on measured foliations?*

3. Our results

• The equality index = spectral flow on semifinite von Neumann algebras

Let $\mathcal{N}$ be a von Neumann algebra, $\mathcal{N} \subset \mathcal{B}(H)$, endowed with a faithful normal, semifinite trace τ.

Theorem 3 *Let $(D_u)_{u\in[0,1]}$ be a path of selfadjoint operators affiliated to $\mathcal{N}$, with common domain and resolvents in $K(\mathcal{N})$. We assume that D_u depends continuously on u as a bounded operator from $H(D_0)$ to H (with respect to the operator norm).
Furthermore we assume that the endpoints D_0, D_1 are invertible. Then*

$$\mathrm{sf}((D_t)_{u\in[0,1]}) = \operatorname{ind}(\partial_u + D_u)\ .$$

When endpoints are not invertible: we consider abstract Atiyah–Patodi–Singer boundary conditions. Define the unbounded operator $(\partial_u + D_u)^{APS}$ on $L^2([0,1], H)$ as the closure of $\partial_u + D_u$ with domain

$$\{f \in C^\infty([0,1], H(D_0)) \mid P_0 f(0) = 0, (1-P_1)f(1) = 0\}$$

Proposition 4 *The operator $\mathcal{D}^{APS}$ is selfadjoint with resolvents in $K(B(L^2(I)) \otimes \mathcal{N})$. In particular it is affiliated to $B(L^2(I)) \otimes \mathcal{N}$ and Breuer–Fredholm.*

Theorem 5 *Let $(D_u)_{u\in[0,1]}$ be a path of selfadjoint operators with common domain and with resolvents in $K(\mathcal{N})$. We assume that D_u depends continuously on u as a bounded operator from $H(D_0)$ to H. Furthermore we assume that the path is constant near each of the endpoints. Then*

$$\mathrm{sf}((D_u)_{u\in[0,1]}) = \operatorname{ind}((\partial_u + D_u)^{APS})\ .$$

• Applications to geometric operators on a foliated manifold

Let $(M, \mathcal{F})$ be a closed manifold, foliated by an integrable distribution $T\mathcal{F} \subset TM$ of odd dimension $p = 2l+1$. Assume $\mathcal{F}$ is oriented, and *assume there exists a holonomy invariant transverse measure* Λ.

Consider $E := \Lambda T^*\mathcal{F} \otimes \mathbb{C}$, and let τ be the leafwise chirality grading, $\tau\phi := i^{l+1+k(k+1)} * \phi$, $\phi \in C^\infty(L_x, \Lambda^k T^*\mathcal{F}_{|L_x})$ (where $*$ is the leafwise Hodge star operator). The *leafwise odd signature operator* D^{sign} is defined on $\Omega^*_{tang}(M) = C^\infty_{tang}(M, E)$ by $D^{sign} = \tau d + d\tau$.
If $(g_u)_{u\in[0,1]}$ is a path of leafwise Riemannian metrics depending smoothly on the parameter: we get a path of chirality operators τ_u, and a path of signature operators D_u^{sign}, correspondingly.

Proposition 6 *The spectral flow of the path $(D_u^{sign})_{u\in[0,1]}$ is zero.*

$(M, \partial M, \mathcal{F}^{2l})$ foliated manifold with boundary, with a holonomy invariant transverse measure Λ (and foliation transverse to the boundary).

Definition [An] The analytic Λ-signature is defined to be the measured L^2-index $\sigma_{\Lambda,an}(M, \partial M) := \operatorname{ind}_{L^2,\Lambda}(D^{sign,+}) = \operatorname{ind}((D^{sign,+})^{APS}) + \mathrm{tr}_\Lambda(P_{\operatorname{Ker} D^\partial})$, where D^∂ is the odd signature operator induced on the boundary. It coincides with mesured Hodge- and de Rham-signature [An].

Proposition 7 $\sigma_{\Lambda,an}(M, \partial M)$ *does not depend on the metric on* M.

(all results are joint work with Charlotte Wahl)

4. Methods of proofs

- The equality index = spectral flow, Theorem 3, is proved using properties of spectral flow and index (homotopy invariance, additivity w.r.t. concatenation of paths and to direct sum) to reduce the statement to simpler paths. Some basic ideas come from the noncommutative case [LP].
- From the abstract setting of theorems 3 and 5 to the geometric situation of foliations a new phenomenon appears: the metric, and thus the von Neumann algebra, may depend on the parameter. Therefore we have to trivialize the path of Hilbert fields.
- The vanishing of the spectral flow, Prop 6 for a path of signature operators uses integral formulas. The conclusion is based on a beautiful lemma of Cheeger–Gromov [CG] which translates the cohomological nature of the kernel of D^{sign} into the following analytic property

$$\lim_{s\to\infty}\int_0^1 \sqrt{s}\,\mathrm{tr}_\Lambda\left(\dot{D}_u^{sign} e^{-s(D_u^{sign})^2}\right) du = 0 \qquad (2)$$

A very direct proof of Prop. 6 could be given if one could show that the projection onto the *positive part* of the spectrum of D_u^{sign} depends continuously on u: such a proof is not known.
- To conclude that the measured analytic signature of $(M, \partial M\mathcal{F})$ does not depend on the metric: taken $(g_u)_{u\in[1,2]}$ we prove a *gluing formula* $\operatorname{ind}((D_1^{sign,+})^{APS}) - \operatorname{ind}((D_2^{sign,+})^{APS}) = \operatorname{ind}((\partial_u + D_u^\partial)^{APS})$. Then the result follows from Theorem 5 and by the homotopy invariance of $\mathrm{tr}_\Lambda(P_{\operatorname{Ker} D^\partial})$, in [HL].

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