Prof. Dr. Daniel Huybrechts
Dr. Gebhard Martin

Winter term 2020/21

Algebraic Geometry 1 – Tutorial Week 1

• Recall the Zariski topology on $\mathrm{Spec}(A)$ for a ring A. Recall the localizations $A_{\mathfrak{p}}$ and A_a for $\mathfrak{p} \in \mathrm{Spec}(A)$ and $a \in A$.

• Discuss an example in which $s \in \mathcal{O}_X(U)$, $U \subset X = \mathrm{Spec}(A)$, is not of the form $s(\mathfrak{p}) = \frac{a}{b} \in A_{\mathfrak{p}}$, $a, b \in A$ (i.e. where one really has to restrict to smaller open neighborhoods of any point $\mathfrak{p} \in U$).

• Let $\mathcal{C}$ be a category, for example the category of sets (Sets) or abelian groups (Ab), and let $\varphi_1, \varphi_2 \colon M \rightrightarrows N$ be two morphisms in $\mathcal{C}$. A morphism $\varphi \colon K \to M$ with $\varphi_1 \circ \varphi = \varphi_2 \circ \varphi$ is called an *equalizer* of (φ_1, φ_2) if for all $\psi \colon P \to M$ in $\mathcal{C}$ with $\varphi_1 \circ \psi = \varphi_2 \circ \psi$ there exists a unique morphism $\tilde{\psi} \colon P \to K$ with $\varphi \circ \tilde{\psi} = \psi$. If an equalizer exists it is unique up to unique isomorphism.

(i) Show that in $\mathcal{C} = (\mathit{Sets})$ the subset $K := \{x \in M \mid \varphi_1(x) = \varphi_2(x)\} \subset M$ is an equalizer.

(ii) Show that in $\mathcal{C} = (\mathit{Ab})$ the kernel $\mathrm{Ker}(\varphi_1 - \varphi_2) \subset M$ is an equalizer.

• Recall the following example, which has been discussed in class: Let X be a topological space. For any open subset $U \subset X$ set $\mathcal{C}(U) := \{f \colon U \to \mathbb{R} \text{ continuous}\}$ and for an inclusion of open subsets $V \subset U$ let $\rho_{UV} \colon \mathcal{C}(U) \to \mathcal{C}(V)$ be the restriction map $\rho_{UV}(f) := f|_V$. This is a sheaf! Discuss $\rho_{VW} \circ \rho_{UV} = \rho_{UW} \colon \mathcal{C}(U) \to \mathcal{C}(W)$ for open subsets $W \subset V \subset U$ and the sheaf property.

• Let X be a topological space and G an abelian group. For an open subset $U \subset X$ let $\mathbf{G}(U)$ be the group of constant maps $f \colon U \to G$ and define again $\rho_{UV} \colon \mathbf{G}(U) \to \mathbf{G}(V)$ as the natural restriction maps. In other words, $\mathbf{G}(U) = G$ for any open subset $\emptyset \neq U \subset X$. This is not a sheaf! In order for this to make sense one has to define what $\mathbf{G}(\emptyset)$ is. There are two options (i) $\mathbf{G}(\emptyset) = G$ or (ii) $\mathbf{G}(\emptyset) = \{0\}$. None of the two works. In (i) consider the empty cover $\emptyset = \bigcup_{i \in I = \emptyset} U_i$ and in (ii) a disjoint union $U_1 \sqcup U_2$.

• Continuation: What happens if instead one considers the groups $\underline{G}(U)$ of locally constant functions (i.e. continuous functions $U \to G$, where G is endowed with the discrete topology)? This is a sheaf.

• Discuss (but not too much) that for a sheaf $\mathcal{F}$ one always automatically has $\mathcal{F}(\emptyset) = \{*\}$ (where $\{*\}$ is the terminal object in the category, e.g. $\mathcal{F}(\emptyset) = \{0\}$ if $\mathcal{F}$ is a sheaf of abelian groups).