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Algebraic Geometry I – Retry Exam 26.3.2021

Ruled surfaces

Throughout we assume that k is an algebraically closed field. Denote by $\mathbb{P}^n$ the projective space $\mathbb{P}^n_k$ and similarly $\mathbb{A}^n$ denotes $\mathbb{A}^n_k$.

1. As a warm up we shall look at trivial vector bundles and trivial projective bundles.

Exercise 1. Let X be a projective integral scheme over k and let $Y_n := \mathbb{A}^n \times_k X$, $n > 0$ with its natural (second) projection $\pi_n \colon Y_n \longrightarrow X$.

(i) Can π_n be proper?

(ii) Show that $\pi_n \colon Y_n \longrightarrow X$ is a vector bundle and determine the associated locally free sheaf.

(iii) Describe the ring $H^0(Y_n, \mathcal{O}_{Y_n})$. Is it a field?

(iv) Show that $Y_n \times_X Y_m \cong Y_{n+m}$.

(v) Assume we define in analogy $Z_n := \mathbb{P}^n \times_k X \longrightarrow X$. For which n and m does $Z_n \times_X Z_m \cong Z_{n+m}$ hold?

Solution

(i) The morphism π_n is never proper. Indeed, if it were, then so would be $\mathbb{A}^n = Y_n \times_X x \longrightarrow x = \operatorname{Spec} k$ for every closed point $x \in X$. But $\mathbb{A}^n$ is not proper over $\operatorname{Spec} k$.

(ii) Essentially by definition Y_n is the vector bundle associated with the locally free sheaf $\mathcal{O}_X^{\oplus n}$.

(iii) In particular, we have $H^0(Y_n, \mathcal{O}_{Y_n}) \cong H^0(X, S^*(\mathcal{O}_X^{\oplus n})) \cong k[x_1, \ldots, x_n]$ which is not a field.

(iv) By the standard rules for fibre products $Y_n \times_X Y_m \cong (\mathbb{A}^n \times_k X) \times_X (\mathbb{A}^m \times_k X) \cong (\mathbb{A}^n \times_k \mathbb{A}^m) \times_k X \cong \mathbb{A}^{n+m} \times_k X \cong Y_{n+m}$.

(v) This never holds. Indeed, in this case the fibre of $Z_n \times_X Z_m \longrightarrow X$ (say over a closed point) is of the form $\mathbb{P}^n \times_k \mathbb{P}^m$, which is not isomorphic to any projective space if $n, m > 0$.

2. A *ruled surface* is a projective surface S isomorphic (over k) to a projective bundle $\pi \colon \mathbb{P}(\mathcal{E}) \longrightarrow C$ associated with a locally free sheaf $\mathcal{E}$ of rank two on a regular projective curve C.

Exercise 2. (i) With the above notation, show that there exists an embedding $\operatorname{Pic}(C) \times \mathbb{Z} \hookrightarrow \operatorname{Pic}(\mathbb{P}(\mathcal{E}))$ given by $(\mathcal{L}, n) \longmapsto \mathcal{L}_n := \pi^*\mathcal{L} \otimes \mathcal{O}_\pi(n)$.

(In fact, one can show that this is an isomorphism. You may use this isomorphism in the rest of the exam.)

(ii) Given $\mathcal{L}_n \in \operatorname{Pic}(\mathbb{P}(\mathcal{E}))$, show that n is the degree of $\mathcal{L}_n$ restricted to any fiber of π and that $\pi_*\mathcal{L}_n \cong \mathcal{L} \otimes S^n(\mathcal{E})$ for $n \geq 0$.

(iii) Prove that $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{F})$ over C if and only if $\mathcal{E} \cong \mathcal{F} \otimes \mathcal{L}$ for some invertible sheaf $\mathcal{L}$ on C.

(iv) Show that there exists an isomorphism $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E}^*)$ compatible with the projections to C.

(v) Show that every ruled surface over C is birational to $\mathbb{P}^1 \times C$.

Solution

(i) It suffices to show that the map is injective. If $\mathcal{L}_n \cong \mathcal{O}$, then also its restriction to any (closed) fibre is trivial. But the restriction is $\mathcal{O}_{\mathbb{P}^1}(n)$ and hence $n = 0$. Since $\pi_*\pi^*\mathcal{L} \cong \mathcal{L}$ by the projection formula, we also get $\mathcal{L} \cong \mathcal{O}$.

(ii) Since $\pi^*\mathcal{L}$ has degree zero on every fiber and $\mathcal{O}_\pi(n)$ has degree n, the invertible sheaf $\mathcal{L}_n$ has degree n on every fiber. Now, $\pi_*\mathcal{L}_n \cong \mathcal{L} \otimes S^n(\mathcal{E})$ is just the projection formula and Exercise 79.

(iii) We know from the lecture that $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{F})$ if $\mathcal{E}$ and $\mathcal{F}$ differ by an invertible sheaf. Conversely, assume that the two ruled surfaces are isomorphic via $f : \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}(\mathcal{F})$ and let $\pi' : \mathbb{P}(\mathcal{F}) \longrightarrow C$ be the natural map. Write $f^*\mathcal{O}_{\pi'}(1) = \pi^*\mathcal{L} \otimes \mathcal{O}_\pi(n)$ for some $\mathcal{L} \in \mathrm{Pic}(C)$ and $n \in \mathbb{Z}$. Since f is an isomorphism over C and $\mathcal{O}_{\pi'}(1)$ has degree one on every fiber, so does $f^*\mathcal{O}_{\pi'}(1)$, hence $n = 1$. Therefore, we have

$$\mathcal{L} \otimes \mathcal{E} = \pi_*(f^*\mathcal{O}_{\pi'}(1)) = (\pi_* \circ (f^{-1})_*)\mathcal{O}_{\pi'}(1) = \pi'_*\mathcal{O}_{\pi'}(1) = \mathcal{F}.$$

(iv) The dual $\mathcal{E}^*$ of a locally free sheaf $\mathcal{E}$ of rank two is isomorphic to $\mathcal{E} \otimes \det(\mathcal{E})^*$, where $\det(\mathcal{E}) \coloneqq \bigwedge^2 \mathcal{E} \in \mathrm{Pic}(C)$. Hence, $\mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{E}^*)$ by (iii).

(v) Given a ruled surface $\pi : \mathbb{P}(\mathcal{E}) \longrightarrow C$ choose a non-empty open subscheme $U \subset C$ such that $\mathcal{E}|_C \cong \mathcal{O}_U^{\oplus 2}$. Then, $\mathbb{P}(\mathcal{E})|_U \cong \mathbb{P}(\mathcal{O}_U^{\oplus 2}) \cong \mathbb{P}^1 \times U$.

3. In the following we shall study *Hirzebruch surfaces*. By definition the n-th Hirzebruch surface is the surface $\mathbb{F}_n \coloneqq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ viewed as a projective bundle over $\mathbb{P}^1$.

Exercise 3. (i) Show that any ruled surface over $\mathbb{P}^1$ is isomorphic (over $\mathbb{P}^1$) to $\mathbb{F}_n$ for a uniquely determined $n \geq 0$.

(ii) Assume there exists a short exact sequence $0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(1) \longrightarrow 0$ on $\mathbb{P}^1$. Show that either $\mathbb{P}(\mathcal{E}) \cong \mathbb{F}_0$ or $\mathbb{P}(\mathcal{E}) \cong \mathbb{F}_2$.

(iii) Show that $\mathcal{O} \oplus \mathcal{O}(n)$, $n \geq 0$, on $\mathbb{P}^1$ can be globally generated and deduce from this a closed embedding $\mathbb{F}_n \hookrightarrow \mathbb{P}^2 \times_k \mathbb{P}^1$ compatible with the projection to $\mathbb{P}^1$.

(iv) Use the embedding in (iii) to consider $\mathbb{F}_n \subset \mathbb{P}^2 \times_k \mathbb{P}^1$ as an effective divisor in $\mathbb{P}^2 \times_k \mathbb{P}^1$ and show that $\mathcal{O}(\mathbb{F}_n) \cong \mathcal{O}(1) \boxtimes \mathcal{O}(n) \in \mathrm{Pic}(\mathbb{P}^2 \times_k \mathbb{P}^1)$.

(Recall that $\mathrm{Pic}(\mathbb{P}^2 \times_k \mathbb{P}^1) \cong \mathbb{Z}^{\oplus 2}$ is freely generated by the pull-backs of $\mathcal{O}(1)$ via the two projections p_1 and p_2 and that $\mathcal{O}(a) \boxtimes \mathcal{O}(b) \coloneqq p_1^*\mathcal{O}(a) \otimes p_2\mathcal{O}(b)$.)

Solution

(i) We recall from Exercise 65 that $\mathcal{E} \cong \mathcal{O}(a) \oplus \mathcal{O}(b)$ for certain $a, b \in \mathbb{Z}$ which are uniquely determined up to permutation. By part (iii) of the previous exercise, we have $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(b-a))$ and after possibly interchanging a and b, we may assume $b - a \geq 0$ and thus $\mathbb{P}(\mathcal{E}) \cong \mathbb{F}_{b-a}$. Next, we have to show that $\mathcal{O} \oplus \mathcal{O}(n)$ and $\mathcal{O} \oplus \mathcal{O}(m)$ differ by an invertible sheaf if and only if $n = m$. Thus, assume that $\mathcal{O}(a) \oplus \mathcal{O}(n+a) \cong \mathcal{O} \oplus \mathcal{O}(m)$ for some $a \geq 0$. Hence, $\{a, n+a\} = \{0, m\}$ and, since $n + a \geq a \geq 0$ and $m \geq 0$, this implies $a = 0$. Therefore, $n = m$.

(ii) Recall again that $\mathcal{E} \cong \mathcal{O}(a) \oplus \mathcal{O}(b)$ for some a, b. Since $\det(\mathcal{E}) \cong \mathcal{O}(-1) \otimes \mathcal{O}(1) \cong \mathcal{O}$, we have $a + b = 0$. Then observe that $h^0(\mathcal{E}) = 2$, which leaves only the possibilities $\mathcal{E} \cong \mathcal{O} \oplus \mathcal{O}$ or $\mathcal{E} \cong \mathcal{O}(-1) \oplus \mathcal{O}(1)$. Hence, $\mathbb{P}(\mathcal{E}) \cong \mathbb{F}_0$ or $\cong \mathbb{F}_2$ as claimed.

(iii) We use that $\mathcal{O}(n)$ on $\mathbb{P}^1$ is globally generated by $x_0^n, x_1^n \in H^0(\mathbb{P}^1, \mathcal{O}(n))$ for $n \geq 0$. This yields a surjection $\mathcal{O}^{\oplus 3} \twoheadrightarrow \mathcal{O} \oplus \mathcal{O}(n)$ and, therefore, a closed immersion $\mathbb{F}_n \hookrightarrow \mathbb{P}(\mathcal{O}^{\oplus 3}) \cong \mathbb{P}^2 \times_k \mathbb{P}^1$.

(iv) Write $\mathcal{O}(\mathbb{F}_n) \cong \mathcal{O}(a) \boxtimes \mathcal{O}(b)$. Restricting to the fibre of the projection to $\mathbb{P}^1$ shows that $a = -1$. Now tensor the short exact sequence $0 \longrightarrow \mathcal{O}(-\mathbb{F}_n) \longrightarrow \mathcal{O}_{\mathbb{P}^2 \times_k \mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{F}_n} \longrightarrow 0$ with $\mathcal{O}(1) \boxtimes \mathcal{O}$ and take the direct image under the projection to $\mathbb{P}^1$. Using $\mathcal{O}(-\mathbb{F}_n) = \mathcal{O}(-1) \boxtimes \mathcal{O}(-b)$ this yields the short exact sequence

$$0 \longrightarrow \mathcal{O}(-b) \longrightarrow \mathcal{O}^{\oplus 3} \longrightarrow \mathcal{O} \oplus \mathcal{O}(n) \longrightarrow 0$$

and taking determinants yields $-b + n = 0$, i.e. $b = n$ as claimed.

Now, we classify the Hirzebruch surfaces that can be realized as hypersurfaces in $\mathbb{P}^3$.

Exercise 4. (i) Compute the dimensions $h^i(\mathbb{F}_n, \mathcal{O}_{\mathbb{F}_n})$.

(ii) Discuss which Hirzebruch surfaces could possibly be isomorphic to hypersurfaces in $\mathbb{P}^3$.

Solution

(i) Since $\mathcal{O}(\mathbb{F}_n) \cong \mathcal{O}(1) \boxtimes \mathcal{O}(n) \in \operatorname{Pic}(\mathbb{P}^2 \times_k \mathbb{P}^1)$, we have as before a short exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \boxtimes \mathcal{O}(-n) \longrightarrow \mathcal{O}_{\mathbb{P}^2 \times_k \mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{F}_n} \longrightarrow 0.$$

By the Künneth formula (Exercise 68), we have $h^i(\mathbb{P}^2 \times_k \mathbb{P}^1, \mathcal{O}(-1) \boxtimes \mathcal{O}(-n)) = 0$ for $i = 0, 1, 2$. Hence, the long exact sequence in cohomology associated to the above short exact sequence shows that $h^i(\mathbb{F}_n, \mathcal{O}_{\mathbb{F}_n}) = h^i(\mathbb{P}^2 \times_k \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^2 \times_k \mathbb{P}^1})$. The latter can again be calculated using the Künneth formula:

$$h^i(\mathbb{F}_n, \mathcal{O}_{\mathbb{F}_n}) = h^i(\mathbb{P}^2 \times_k \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^2 \times_k \mathbb{P}^1}) = \begin{cases} 1 \text{ if } i = 0 \\ 0 \text{ else .} \end{cases}$$

(ii) Clearly, $\mathbb{F}_0 \cong \mathbb{P}^1 \times_k \mathbb{P}^1$ can be embedded into $\mathbb{P}^3$ via the Segre embedding $([x_0 : x_1], [y_0 : y_1]) \longmapsto [x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1]$, thus realizing $\mathbb{F}_0$ as a quadric hypersurface in $\mathbb{P}^3$. It turns out that this is the only Hirzebruch surface that embeds into $\mathbb{P}^3$.

A hyperplane X in $\mathbb{P}^3$ is not a Hirzebruch surface, since $X \cong \mathbb{P}^2$ and thus $\operatorname{Pic}(X) = \mathbb{Z}$. A hypersurface X in $\mathbb{P}^3$ of degree $d \geq 4$ has $h^2(X, \mathcal{O}_X) \geq 1$ by the long exact sequence in cohomology associated to

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

hence $X \not\cong \mathbb{F}_n$ by part (i).

Therefore, it suffices to show that cubic hypersurfaces in $\mathbb{P}^3$ are not ruled and that the only ruled quadric hypersurface in $\mathbb{P}^3$ is $\mathbb{F}_0$. For the latter observe that since k is algebraically closed, after coordinate change any quadratic equation $F \in k[x_1, x_1, x_2, x_3]$ can be written as $F = \sum a_i x_i^2$ (standard fact from linear algebra) and even better as $F = x_0^2 + \cdots + x_i^2$ for some $i \leq 3$. If $i = 3$, then $V_+(F) \cong \mathbb{F}_0$. For $i = 0$ or $i = 1$, the hypersurface $V_+(F)$ is not irreducible and, therefore, not isomorphic to any $\mathbb{F}_n$. Finally, for $i = 2$, the stalk of $\mathcal{O}_{V_+(F)}$ at the point $[0 : 0 : 0 : 1]$ is isomorphic to the localization of $k[X_1, X_1, X_2]/(X_0^2 + X_1^2 + X_2^2)$ which is not regular unlike any local ring $\mathcal{O}_{\mathbb{F}_n, x}$.

The case of cubic hypersurfaces is a little trickier. No point reduction if you did not come up with anything on this case. Essentially this follows from things we observed in the first exam that cubic surfaces contain many (namely 27) lines that yield to classes in $\operatorname{Pic}(S)$ which forces $\operatorname{Pic}(\mathrm{S}) > 2$. This is in contradiction to $\operatorname{Pic}(\mathbb{F}_n) \cong \mathbb{Z}^{\oplus 2}$.

4. Instead of vector bundles and projective bundles over $\mathbb{P}^1$ we now consider them over a curve of genus one.

Consider a cubic curve $C = V_+(F) \subset \mathbb{P}^2$, $F \in k[x_0, x_1, x_2]_3$ which we assume to be integral and regular in codimension one (a smooth cubic curve). For every closed point $x \in C$ there exists a unique non-split short exact sequence of the form

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E}_{(x)} \longrightarrow \mathcal{O}_C(x) \longrightarrow 0 \ . \tag{1}$$

For this one needs to recall that $H^0(C, \mathcal{O}_C(-x)) = 0$ and the fact that $H^1(C, \mathcal{O}_C(-x)) \cong \operatorname{Ext}^1_{\mathcal{O}_C}(\mathcal{O}_C(x), \mathcal{O}_C)$ parametrizes all extensions (split or non-split) of the type (1). More generally, for two invertible sheaves $\mathcal{L}_1, \mathcal{L}_2$ the group $\operatorname{Ext}^1(\mathcal{L}_2, \mathcal{L}_1) \cong H^1(C, \mathcal{L}_2^* \otimes \mathcal{L}_1)$ parametrizes all extensions of the form

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{L}_2 \longrightarrow 0 \ . \tag{2}$$

The class $\xi \in \operatorname{Ext}^1(\mathcal{L}_2, \mathcal{L}_1)$ corresponding to (2) is zero if and only if (2) splits.

Exercise 5. (i) Show that every locally free sheaf $\mathcal{E}$ of rank two on C can be written as an extension

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}_2 \longrightarrow 0$$

with $\mathcal{L}_1, \mathcal{L}_2 \in \operatorname{Pic}(C)$.

(ii) Show that for any two closed points $x, y \in C$ the sheaf $\mathcal{E}_{(x)}(-y) := \mathcal{E}_{(x)} \otimes \mathcal{O}_C(-y)$ has no global sections.

(iii) Compute the dimensions $h^0(C, \mathcal{E}_{(x)})$ and $h^1(C, \mathcal{E}_{(x)})$.

(iv) Show that $\mathcal{E}_{(x)}$ is the unique locally free sheaf of rank two with determinant $\det(\mathcal{E}_{(x)}) \coloneqq \bigwedge^2 \mathcal{E}_{(x)}$ isomorphic to $\mathcal{O}_C(x)$, which is not a direct sum of invertible sheaves. Recall that $H^1(C, \mathcal{L}) = 0$ for any $\mathcal{L} \in \mathrm{Pic}(C)$ with $\deg(\mathcal{L}) > 0$.

Solution

(i) Since C is projective, there exists an ample invertible sheaf $\mathcal{L}$ for which $\mathcal{E} \otimes \mathcal{L}^n$ is globally generated for $n \gg 0$. However, if $\mathcal{E} \otimes \mathcal{L}^n$ can be written as an extension of invertible sheaves, then tensoring with $\mathcal{L}^{-n}$ yields the desired representation for $\mathcal{E}$. Thus, we may assume that $\mathcal{E}$ itself is globally generated and, in particular and enough for our purposes, that $\mathcal{E}$ has a non-trivial global section. Picking one such global section yields a short exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0\ .$$

If $\mathcal{F}$ is torsion free or, equivalently, invertible, then we are done. If not let $\mathcal{L}_1 \coloneqq \ker(\mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{L}_2 \coloneqq \mathcal{F}/\mathrm{Torsion}(\mathcal{F}))$.

(ii) Tensor (1) with $\mathcal{O}_C(-y)$ and use that for $x \neq y$ we have $H^0(\mathcal{O}_C(-y)) = 0 = H^0(\mathcal{O}_C(x-y))$. If $x = y$, observe that if $H^0(C, \mathcal{E}_{(x)}(-x)) \longrightarrow H^0(C, \mathcal{O}_C)$ is not zero if and only if (1) tensored with $\mathcal{O}_C(-x)$ splits, which in turn would imply that (1) splits.

(iii) We use the long exact cohomology sequence associated with (1) which reads

$$0 \longrightarrow k \cong H^0(\mathcal{O}_C) \longrightarrow H^0(\mathcal{E}_{(x)}) \longrightarrow H^0(C, \mathcal{O}_C(x)) \xrightarrow{\delta} k \cong H^1(\mathcal{O}_C) \longrightarrow H^1(\mathcal{E}_{(x)}) \longrightarrow H^1(\mathcal{O}_C(x)) \longrightarrow 0$$

We know that $H^0(\mathcal{O}_C(x)) \cong k$ and by the Riemann–Roch formula $H^1(\mathcal{O}_C(x)) = 0$. It suffices to show that $\delta \neq 0$ which immediately yields $h^0(\mathcal{E}_{(x)}) = 1$ and $h^1(\mathcal{E}_{(x)}) = 0$. If $\delta = 0$, then the unique section of $\mathcal{O}_C(x)$ lifts to a section of $\mathcal{E}_{(x)}$. Then, either $\mathcal{E}_{(x)}$ can be written in two different ways as an extension (1), which would would contradict the uniqueness, or the section of $\mathcal{E}_{(x)}$ can be extended to a map $\mathcal{O}_C(x) \longrightarrow \mathcal{E}_{(x)}$ which would split the original sequence.

(iv) Assume $\mathcal{E}$ is locally free of rank two with determinant $\mathcal{O}_C(x)$. According to (i) we can write $\mathcal{E}$ as an extension

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}_2 \longrightarrow 0$$

of two invertible sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$. The additivity of χ and the Riemann–Roch formula for invertible sheaves shows that $\chi(\mathcal{E}) = \chi(\mathcal{L}_1) + \chi(\mathcal{L}_2) = \deg(\mathcal{L}_1) + \deg(\mathcal{L}_2) = \deg(\mathcal{L}_1 \otimes \mathcal{L}_2) = \deg(\det(\mathcal{E})) = 1$. Hence, $H^0(\mathcal{E}) \neq 0$ and any choice of a non-zero global section yields a short exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0\ .$$

If $\mathcal{F}$ is torsion free or, equivalently, locally free (we are on a smooth curve C), then it is invertible and in fact isomorphic to $\mathcal{O}_C(x)$, as $\mathcal{O}_C(x) \cong \det(\mathcal{E}) \cong \mathcal{O}_C \otimes \mathcal{F}$. In this case, either $\mathcal{E} \cong \mathcal{E}_{(x)}$ or $\mathcal{E}$ is isomorphic to a direct sum of invertible sheaves which is excluded. If $\mathcal{F}$ has non-trivial torsion, then there exists a short exact sequence

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}_2 \longrightarrow 0$$

with $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}(C)$ and $\deg(\mathcal{L}_1) > 0$. However, $\mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{L}_2, \mathcal{L}_1) \cong H^1(C, \mathcal{L}_1 \otimes \mathcal{L}_2^*)$ and $\deg(\mathcal{L}_1 \otimes \mathcal{L}_2^*) = \deg(\mathcal{L}_1) + \deg(\mathcal{L}_2^*) = \deg(\mathcal{L}_1) + \deg(\mathcal{L}_1(-x)) > 0$. Now use that $H^1(C, \mathcal{L}) = 0$ for any invertible sheaf with $\deg(\mathcal{L}) > 0$, which would lead to $\mathcal{E} \cong \mathcal{L}_1 \oplus \mathcal{L}_2$ which is excluded by assumption.