Prof. Dr. Daniel Huybrechts Winter term 2020/21
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Algebraic Geometry I – Retry Exam 26.3.2021

Begin: 09:00 **End:** 12:00

Technical details:

– Upload a scan of your solutions (max. 500kb per page) until 12:00 to the eCampus folder 'Exams/Exam submission'. Solutions uploaded any later than 12:00 will not be considered which results in you failing the exam.

– The exam will be posted at 9:00 on the usual website `https://www.math.uni-bonn.de/people/gmartin/AlgebraicGeometryWS2020.htmpl` and will also be available in the folder 'Exams' in eCampus (where this document can be found already now).

– Please sign the declaration of honor below and scan it together with your solutions. Without it your solutions will not be considered.

– If at all possible, upload your solution in one(!) pdf document with the declaration of honor as the first page. Name the file **firstname.name.pdf**. Please, write your name on each sheet.

– In case of technical problems (but only then), you can also send the scan to gmartin@math.uni-bonn.de (until 12:00 am).

– Allow at least 30-45min for scanning and uploading your solutions.

Guidelines:

– The exam consists of applying your knowledge of algebraic geometry to fill in the missing arguments in the text below. In particular, reading and understanding the arguments that are given is part of the task.

– It is recommended to work through the exercises in the given order.

– The exam will only lead to pass or fail. It will not be graded. A complete solution will be posted on eCampus, so that you can find out yourself how well you did in the exam.

– All arguments have to be justified. Apart from standard material from commutative algebra, you have to deduce everything by only using results explicitly stated or used, either in the lectures, on the exercise sheets or in the text below.

– You may use available resources to solve the exercise. You are allowed to consult the notes of the class, the exercise sheets, online books, etc. **You are not allowed** to contact any other person during the exam (by email, phone, social media, etc.) or to discuss your answers with anyone before successfully uploading them to eCampus.

I hereby swear that I completed the examination detailed above completely on my own and without any impermissible external assistance or through the use of non-permitted aids. I am aware that cheating during the execution of an examination (as detailed in §63 Para 5 of the Higher Education Act NRW) is a violation of the legal regulations for examinations and an administrative offense. The submission of false affirmation in lieu of an oath is a criminal offense.

Signed at: **Date:**

Name: **Student ID:** **Signature:**

Ruled surfaces

Throughout we assume that k is an algebraically closed field. Denote by $\mathbb{P}^n$ the projective space $\mathbb{P}^n_k$ and similarly $\mathbb{A}^n$ denotes $\mathbb{A}^n_k$.

1. As a warm up we shall look at trivial vector bundles and trivial projective bundles.

Exercise 1. Let X be a projective integral scheme over k and let $Y_n := \mathbb{A}^n \times_k X$, $n > 0$ with its natural (second) projection $\pi_n \colon Y_n \longrightarrow X$.

(i) Can π_n be proper?

(ii) Show that $\pi_n \colon Y_n \longrightarrow X$ is a vector bundle and determine the associated locally free sheaf.

(iii) Describe the ring $H^0(Y_n, \mathcal{O}_{Y_n})$. Is it a field?

(iv) Show that $Y_n \times_X Y_m \cong Y_{n+m}$.

(v) Assume we define in analogy $Z_n := \mathbb{P}^n \times_k X \longrightarrow X$. For which n and m does $Z_n \times_X Z_m \cong Z_{n+m}$ hold?

2. A *ruled surface* is a projective surface S isomorphic (over k) to a projective bundle $\pi \colon \mathbb{P}(\mathcal{E}) \longrightarrow C$ associated with a locally free sheaf $\mathcal{E}$ of rank two on a regular projective curve C.

Exercise 2. (i) With the above notation, show that there exists an embedding $\mathrm{Pic}(C) \times \mathbb{Z} \hookrightarrow \mathrm{Pic}(\mathbb{P}(\mathcal{E}))$ given by $(\mathcal{L}, n) \longmapsto \mathcal{L}_n := \pi^*\mathcal{L} \otimes \mathcal{O}_\pi(n)$.

(In fact, one can show that this is an isomorphism. You may use this isomorphism in the rest of the exam.)

(ii) Given $\mathcal{L}_n \in \mathrm{Pic}(\mathbb{P}(\mathcal{E}))$, show that n is the degree of $\mathcal{L}_n$ restricted to any fiber of π and that $\pi_*\mathcal{L}_n \cong \mathcal{L} \otimes S^n(\mathcal{E})$ for $n \geq 0$.

(iii) Prove that $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{F})$ over C if and only if $\mathcal{E} \cong \mathcal{F} \otimes \mathcal{L}$ for some invertible sheaf $\mathcal{L}$ on C.

(iv) Show that there exists an isomorphism $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E}^*)$ compatible with the projections to C.

(v) Show that every ruled surface over C is birational to $\mathbb{P}^1 \times C$.

3. In the following we shall study *Hirzebruch surfaces*. By definition the n-th Hirzebruch surface is the surface $\mathbb{F}_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ viewed as a projective bundle over $\mathbb{P}^1$.

Exercise 3. (i) Show that any ruled surface over $\mathbb{P}^1$ is isomorphic (over $\mathbb{P}^1$) to $\mathbb{F}_n$ for a uniquely determined $n \geq 0$.

(ii) Assume there exists a short exact sequence $0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(1) \longrightarrow 0$ on $\mathbb{P}^1$. Show that either $\mathbb{P}(\mathcal{E}) \cong \mathbb{F}_0$ or $\mathbb{P}(\mathcal{E}) \cong \mathbb{F}_2$.

(iii) Show that $\mathcal{O} \oplus \mathcal{O}(n)$, $n \geq 0$, on $\mathbb{P}^1$ can be globally generated and deduce from this a closed embedding $\mathbb{F}_n \hookrightarrow \mathbb{P}^2 \times_k \mathbb{P}^1$ compatible with the projection to $\mathbb{P}^1$.

(iv) Use the embedding in (iii) to consider $\mathbb{F}_n \subset \mathbb{P}^2 \times_k \mathbb{P}^1$ as an effective divisor in $\mathbb{P}^2 \times_k \mathbb{P}^1$ and show that $\mathcal{O}(\mathbb{F}_n) \cong \mathcal{O}(1) \boxtimes \mathcal{O}(n) \in \mathrm{Pic}(\mathbb{P}^2 \times_k \mathbb{P}^1)$.

(Recall that $\mathrm{Pic}(\mathbb{P}^2 \times_k \mathbb{P}^1) \cong \mathbb{Z}^{\oplus 2}$ is freely generated by the pull-backs of $\mathcal{O}(1)$ via the two projections p_1 and p_2 and that $\mathcal{O}(a) \boxtimes \mathcal{O}(b) := p_1^*\mathcal{O}(a) \otimes p_2\mathcal{O}(b)$.)

Now, we classify the Hirzebruch surfaces that can be realized as hypersurfaces in $\mathbb{P}^3$.

Exercise 4. (i) Compute the dimensions $h^i(\mathbb{F}_n, \mathcal{O}_{\mathbb{F}_n})$.

(ii) Discuss which Hirzebruch surfaces could possibly be isomorphic to hypersurfaces in $\mathbb{P}^3$.

4. Instead of vector bundles and projective bundles over $\mathbb{P}^1$ we now consider them over a curve of genus one.

Consider a cubic curve $C = V_+(F) \subset \mathbb{P}^2$, $F \in k[x_0, x_1, x_2]_3$ which we assume to be integral and regular in codimension one (a smooth cubic curve). For every closed point $x \in C$ there exists a unique non-split short exact sequence of the form

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E}_{(x)} \longrightarrow \mathcal{O}_C(x) \longrightarrow 0 \ . \tag{1}$$

For this one needs to recall that $H^0(C, \mathcal{O}_C(-x)) = 0$ and the fact that $H^1(C, \mathcal{O}_C(-x)) \cong \mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{O}_C(x), \mathcal{O}_C)$ parametrizes all extensions (split or non-split) of the type (1). More generally, for two invertible sheaves $\mathcal{L}_1, \mathcal{L}_2$ the group $\mathrm{Ext}^1(\mathcal{L}_2, \mathcal{L}_1) \cong H^1(C, \mathcal{L}_2^* \otimes \mathcal{L}_1)$ parametrizes all extensions of the form

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{L}_2 \longrightarrow 0 \ . \tag{2}$$

The class $\xi \in \mathrm{Ext}^1(\mathcal{L}_2, \mathcal{L}_1)$ corresponding to (2) is zero if and only if (2) splits.

Exercise 5. (i) Show that every locally free sheaf $\mathcal{E}$ of rank two on C can be written as an extension

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}_2 \longrightarrow 0$$

with $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}(C)$.

(ii) Show that for any two closed points $x, y \in C$ the sheaf $\mathcal{E}_{(x)}(-y) := \mathcal{E}_{(x)} \otimes \mathcal{O}_C(-y)$ has no global sections.

(iii) Compute the dimensions $h^0(C, \mathcal{E}_{(x)})$ and $h^1(C, \mathcal{E}_{(x)})$.

(iv) Show that $\mathcal{E}_{(x)}$ is the unique locally free sheaf of rank two with determinant $\det(\mathcal{E}_{(x)}) := \bigwedge^2 \mathcal{E}_{(x)}$ isomorphic to $\mathcal{O}_C(x)$, which is not a direct sum of invertible sheaves. Recall that $H^1(C, \mathcal{L}) = 0$ for any $\mathcal{L} \in \mathrm{Pic}(C)$ with $\deg(\mathcal{L}) > 0$.