Prof. Dr. Daniel Huybrechts
Dr. Gebhard Martin

Winter term 2020/21

Algebraic Geometry I – Exam 25.2.2021

Begin: 09:00 **End:** 12:00

Technical details:

– Upload a scan of your solutions (max. 500kb per page) until 12:00 to the eCampus folder 'Exams/Exam submission'. Solutions uploaded any later than 12:00 will not be considered which results in you failing the exam.

– The exam will be posted at 9:00 on the usual website `https://www.math.uni-bonn.de/people/gmartin/AlgebraicGeometryWS2020.htmpl` and will also be available in the folder 'Exams' in eCampus (where this document can be found already now).

– Please sign the declaration of honor below and scan it together with your solutions. Without it your solutions will not be considered.

– If at all possible, upload your solution in one(!) pdf document with the declaration of honor as the first page. Name the file **firstname.name.pdf**. Please, write your name on each sheet.

– In case of technical problems (but only then), you can also send the scan to gmartin@math.uni-bonn.de (until 12:00 am).

– Allow at least 30-45min for scanning and uploading your solutions.

Guidelines:

– The exam consists of applying your knowledge of algebraic geometry to fill in the missing arguments in the text below. In particular, reading and understanding the arguments that are given is part of the task.

– It is recommended to work through the exercises in the given order.

– The exam will only lead to pass or fail. It will not be graded. A complete solution will be posted on eCampus, so that you can find out yourself how well you did in the exam.

– All arguments have to be justified. Apart from standard material from commutative algebra, you have to deduce everything by only using results explicitly stated or used, either in the lectures, on the exercise sheets or in the text below.

– You may use available resources to solve the exercise. You are allowed to consult the notes of the class, the exercise sheets, online books, etc. **You are not allowed** to contact any other person during the exam (by email, phone, social media, etc.) or to discuss your answers with anyone before successfully uploading them to eCampus.

I hereby swear that I completed the examination detailed above completely on my own and without any impermissible external assistance or through the use of non-permitted aids. I am aware that cheating during the execution of an examination (as detailed in §63 Para 5 of the Higher Education Act NRW) is a violation of the legal regulations for examinations and an administrative offense. The submission of false affirmation in lieu of an oath is a criminal offense.

Signed at: **Date:**

Name: **Student ID:** **Signature:**

Surfaces in $\mathbb{P}^3_k$

We assume that k is an algebraically closed field and denote by $\mathbb{P}^n$ the projective space $\mathbb{P}^n_k$.

1. A *surface in* $\mathbb{P}^3$ is the zero set

$$S := V_+(F)$$

of a homogenous polynomial $0 \neq F \in H^0(\mathbb{P}^3, \mathcal{O}(d))$ of degree $d > 0$.

Exercise 1. (i) Consider the properties 'integral' and 'non-reduced'. Give explicit examples of surfaces $S \subset \mathbb{P}^3$ with these properties.

(ii) For fixed degree d, how many irreducible components can S maximally have? Give an explicit example for each d realizing the maximum.

(iii) Assume S is integral. Determine the transcendence degree (over k) of its function field $K(S)$.

(iv) We know that $H^0(S, \mathcal{O}_S) \cong k$ for integral S. Does this still hold true when S is not integral?

(v) Describe explicitly an open affine cover of S.

Solution An example of a non-reduced surface of degree is given by $V_+(x_0^d)$. The surface $V_+(x_0) \cong \mathbb{P}^2$ is integral. The maximal number of irreducible component is d which is realized by the union $S = V_+(x_0 - \lambda_i x_1)$ of d hyperplanes. Here, $\lambda_1, \ldots, \lambda_d \in k$ are pairwise distinct. Indeed, if $S_i \subset S$ is an irreducible component then $S_i = V_+(F_i)$ for an irreducible polynomial F_i which divides F. Clearly, there are at most d pairwise distinct (up to scaling) F_i dividing F.

For (iii), use that $K(S)$ equals the function fields of $S \cap D_+(x_0) \cong V(\bar{F}) \subset \mathbb{A}^3_k$, where $\bar{F} \in k[y_1 := \frac{x_1}{x_0}, y_2 := \frac{x_2}{x_0}, y_3 := \frac{x_3}{x_0}]$ is the polynomial $F(1, y_1, y_2, y_3)$. Hence, $K(S) \cong Q(k[y_1, y_2, y_3]/(\bar{F}))$ whose transcendence degree equals the dimension of $k[y_1, y_2, y_3]/(\bar{F})$ which is two. The argument is valid only in the case that $S \cap D_+(x_0) \neq \emptyset$, i.e. the case $F = x_0$ has to be excluded for which the assertion is clear.

In fact, for any S restriction defines an isomorphism $H^0(\mathbb{P}^3, \mathcal{O}) \xrightarrow{\sim} H^0(S, \mathcal{O}_S)$, for the cokernel of the map is contained in $H^1(\mathbb{P}^3, \mathcal{O}(-d)) = 0$. This answers (iv).

Clearly, the intersection $S \cap D_+(x_i) = V_+(F) \cap D_+(x_i) \subset \mathbb{A}^3$ is affine and since $\mathbb{P}^3 = \bigcup_{i=0}^3 D_+(x_i)$, an open affine cover is given by $S = \bigcup_{i=0}^3 (S \cap D_+(x_i))$.

We denote by $\mathcal{O}_S(i)$ the restriction of $\mathcal{O}(i)$ to $S \subset \mathbb{P}^3$ and call the invertible sheaf $\omega_S := \mathcal{O}_S(d-4)$ the *canonical sheaf* of S.

Exercise 2. (i) Prove that $H^1(S, \mathcal{O}_S) = 0 = H^1(S, \omega_S)$.

(ii) Prove that $h^2(S, \mathcal{O}_S) = h^0(S, \omega_S)$ and express this number in terms of d.

(iii) Construct an isomorphism $H^2(S, \omega_S) \cong k$.

If at the end of the exam you still have time, you may want to prove that in fact

$$h^i(S, \mathcal{O}_S(n)) = h^{2-i}(S, \omega_S \otimes \mathcal{O}_S(-n))$$

for all $n \in \mathbb{Z}$. This is generalized by Serre duality which for a locally free sheaf $\mathcal{F}$ on S and its dual $\mathcal{F}^* := \mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{O}_S)$ asserts the existence of an isomorphism of k-vector spaces

$$H^i(S, \mathcal{F}) \cong H^{2-i}(S, \mathcal{F}^* \otimes \omega_S)^*.$$

Solution For (i) we use the exact sequence

$$0 \longrightarrow \mathcal{O}(-d) \longrightarrow \mathcal{O} \longrightarrow i_*\mathcal{O}_S \longrightarrow 0 \tag{1}$$

for the inclusion $i\colon S \hookrightarrow \mathbb{P}^3$. Taking cohomology yields a long exact sequence

$$\cdots \longrightarrow H^1(\mathbb{P}^3, \mathcal{O}) \longrightarrow H^1(S, \mathcal{O}_S) \longrightarrow H^2(\mathbb{P}^3, \mathcal{O}(-d)) \longrightarrow \cdots,$$

where the outer terms vanish since $H^i(\mathbb{P}^n, \mathcal{O}(m)) = 0$ for all $0 < i < n$ and all m. Hence, also $H^1(S, \mathcal{O}_S) = 0$. Similarly, the long exact cohomology sequence associated with (1) after taking tensor product with $\mathcal{O}(d-4)$ yields the second vanishing $H^1(S, \omega_S) = 0$.

For (ii), use again the long exact cohomology sequence associated with (1) to deduce the exact sequence

$$\cdots \longrightarrow H^2(\mathbb{P}^3, \mathcal{O}) \longrightarrow H^2(S, \mathcal{O}_S) \longrightarrow H^3(\mathbb{P}^3, \mathcal{O}(-d)) \longrightarrow H^3(\mathbb{P}^3, \mathcal{O}) \longrightarrow \cdots$$

Now $H^2(\mathbb{P}^3, \mathcal{O}) = 0$ and $H^3(\mathbb{P}^3, \mathcal{O}(m)) = 0$ for $m > -4$ and, therefore, $H^2(S, \mathcal{O}_S) \cong H^3(\mathbb{P}^3, \mathcal{O}(-d))$. The latter space is zero for $d < 4$ and dual to the $\binom{d-1}{3}$-dimensional space $H^0(\mathbb{P}^3, \mathcal{O}(d-4))$. To determine $h^0(S, \omega_S)$ use the isomorphism $H^0(\mathbb{P}^3, \mathcal{O}(d-4)) \cong H^0(S, \omega_S)$, which follows from the same long exact cohomology sequence and the vanishings $H^0(\mathbb{P}^3, \mathcal{O}(-4)) = 0 = H^1(\mathbb{P}^3, \mathcal{O}(-4))$.

Eventually, to prove (iii) we tensor (1) with $\mathcal{O}(d-4)$. The long exact cohomology sequence then yields the exact sequence

$$\cdots \longrightarrow H^2(\mathbb{P}^3, \mathcal{O}(d-4)) \longrightarrow H^2(S, \omega_S) \longrightarrow H^3(\mathbb{P}^3, \mathcal{O}(-4)) \longrightarrow H^3(\mathbb{P}^3, \mathcal{O}(d)),$$

where the outer terms vanish and $H^3(\mathbb{P}^3, \mathcal{O}(-4))$ is dual to $H^0(\mathbb{P}^3, \mathcal{O})$ which in turn is naturally isomorphic to k.

2. Consider the linear coordinates x_0, x_1, x_2, x_3 on $\mathbb{P}^3$ and the induced morphism

$$\xi\colon \mathcal{O} \longrightarrow \mathcal{O}(1)^{\oplus 4}, \lambda \longmapsto \bigoplus \lambda x_i$$

of coherent sheaves on $\mathbb{P}^3$. Observe that if $U := H^0(\mathbb{P}^3, \mathcal{O}(1))$, then ξ can be interpreted as the map $\mathcal{O} \longrightarrow \mathcal{O}(1) \otimes U^*$[1] corresponding to the global section in

$$H^0(\mathbb{P}^3, \mathcal{O}(1) \otimes U^*) \cong U \otimes U^* \cong \mathrm{End}(U)$$

given by id_U. We denote the sheaf cokernel of ξ by $\mathcal{T}$ (the *tangent sheaf* of $\mathbb{P}^3$).

For $S = V_+(F)$ as before, consider the partial derivative $\partial_i F \in H^0(\mathbb{P}^3, \mathcal{O}(d-1))$. Then the map $(t_0, t_1, t_2, t_3) \longmapsto \sum t_i \partial_i F$ defines a map

$$\eta\colon \mathcal{O}(1)^{\oplus 4} \longrightarrow \mathcal{O}(d).$$

Exercise 3. (i) Show that ξ is injective.

(ii) For any given point $x \in \mathbb{P}^3$ corresponding to a line $\ell \subset U^*$ describe a natural isomorphism of the fibre of $\mathcal{T}$ at x with $\mathrm{Hom}(\ell, U^*/\ell)$. (Aside: One can show that $\mathrm{Hom}(\ell, U^*/\ell)$ is also naturally isomorphic to the Zariski tangent space at x, which is why $\mathcal{T}$ is called the tangent bundle.)

(iii) Show that $\mathcal{T}$ is locally free (of rank three).

(iv) Show that the restriction of η to S factors uniquely through the restriction $\mathcal{O}_S(1)^{\oplus 4} \twoheadrightarrow \mathcal{T}|_S$ of ξ to S. (Use the well-known Euler identity $\sum x_i \partial_i F = d \cdot F$.)

(v) Under which condition on F is the induced map $\mathcal{T}|_S \longrightarrow \mathcal{O}_S(d)$ surjective? Write down one example for F of degree $d = 3$ where surjectivity holds (under appropriate assumptions on the characteristic of k).

[1] For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ the tensor product $\mathcal{F} \otimes U$ is by definition the tensor product of $\mathcal{F}$ as a sheaf over the constant sheaf associated with k with the constant sheaf associated with U.

Solution The composition of ξ with the first projection is the map $\mathcal{O} \longrightarrow \mathcal{O}(1)$ given by the section $x_0 \in H^0(\mathbb{P}^3, \mathcal{O}(1))$, which is non-zero. Then use that on an integral scheme any non-zero morphism $\mathcal{L}_1 \longrightarrow \mathcal{L}_2$ between two invertible sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$ is injective. This proves (i).

To prove (ii) recall that the fibre of $\mathcal{O}(1)$ at a point $x \in \mathbb{P}^3$ corresponding to a line $\ell \subset U^*$ is naturally isomorphic to the dual ℓ^*. By the abstract description above the fibre of ξ at x is then given by $k \longrightarrow \ell^* \otimes U^* \cong \mathrm{Hom}(\ell, U^*)$ which sends $1 \in k$ to the natural embedding $\ell \hookrightarrow U^*$. Clearly, its cokernel is $\mathrm{Hom}(\ell, U^*/\ell)$. For the identification with the Zariski tangent space, recall first that the Zariski tangent space at a point x is the vector space $T_x = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$, so that for the last part one needs to describe an isomorphism $(\mathfrak{m}_x/\mathfrak{m}_x^2)^* \cong \mathrm{Hom}(\ell, U^*/\ell)$ or, equivalently, an isomorphism $\mathfrak{m}_x/\mathfrak{m}_x^2 \cong (k \cdot \lambda) \otimes \ker(\lambda)^*$, where λ is any non-zero element in ℓ viewed as a linear form $U \longrightarrow k$. Eventually use that $\mathfrak{m}_x$ is the homogeneous localization of the homogeneous maximal ideal in S^*U corresponding to x which is generated by $\mathrm{Ker}(\lambda) \subset U$.

On each of the standard open subsets $D_+(x_i) \cong \mathrm{Spec}(k[x_0/x_i, \ldots, x_3/x_i])$ the sheaf $\mathcal{T}$ corresponds to a module T_i over $A_i := k[x_0/x_i, \ldots, x_3/x_i]$. The polynomial ring $A_i \cong k[y_1, y_2, y_3]$ is regular and, therefore, all Zariski tangent spaces are of the same dimension three. Hence, T_i is an A_i-module with fibres all of the same dimension three. From commutative algebra we know that this implies that all stalks $\mathcal{T}_x$ are free $\mathcal{O}_{\mathbb{P}^3,x}$-modules or rank three. This is enough to conclude that $\mathcal{T}$ is locally free of rank three, cf. Exercise 46 (iii)

For (iv) observe that the short exact sequence $0 \longrightarrow \mathcal{O} \xrightarrow{\xi} \mathcal{O}(1)^{\oplus 4} \longrightarrow \mathcal{T} \longrightarrow 0$ remains exact when restricted to S and thus yields a short exact sequence $0 \longrightarrow \mathcal{O}_S \xrightarrow{\xi|_S} \mathcal{O}_S(1)^{\oplus 4} \longrightarrow \mathcal{T}|_S \longrightarrow 0$. Applying $\mathrm{Hom}(\ ,\mathcal{O}_S(d))$ to it yields the exact sequence

$$0 \longrightarrow \mathrm{Hom}(\mathcal{T}|_S, \mathcal{O}_S(d)) \longrightarrow \mathrm{Hom}(\mathcal{O}_S(1)^{\oplus 4}, \mathcal{O}(d)) \longrightarrow \mathrm{Hom}(\mathcal{O}_S, \mathcal{O}_S(d)) \longrightarrow \cdots$$

By the Euler identity, the image of $\eta|_S \in \mathrm{Hom}(\mathcal{O}_S(1)^{\oplus 4}, \mathcal{O}(d))$ in $\mathrm{Hom}(\mathcal{O}_S, \mathcal{O}_S(d))$ is the restriction of $d \cdot F$ to S which, of course, vanishes. Hence, $\eta|_S$ can be written in a unique way as the composition of $\mathcal{O}_S(1)^{\oplus 4} \twoheadrightarrow \mathcal{T}|_S$ and a morphism $\mathcal{T}|_S \longrightarrow \mathcal{O}_S(d)$.

The map $\mathcal{T}|_S \longrightarrow \mathcal{O}_S(d)$ is surjective if and only if $\eta|_S$ is surjective, which is equivalent to $(\partial_i F)|_S \in H^0(S, \mathcal{O}_S(d-1))$ generating the invertible sheaf $\mathcal{O}_S(d-1)$ or, still equivalent, to $\bigcap V_+(\partial_i F) \cap S = \emptyset$. This is satisfied e.g. for $F = x_0^3 + x_1^3 + x_2^3 + x_3^3$ assuming $\mathrm{char}(k) \neq 3$, because then $\bigcap V_+(\partial_i F) = \bigcap V_+(x_i^2) = \emptyset$.

Assume that the assumption in (v) above holds and define the *tangent bundle* $\mathcal{T}_S$ of S as the kernel of $\mathcal{T}|_S \longrightarrow \mathcal{O}_S(d)$. Thus, there exists a short exact sequence

$$0 \longrightarrow \mathcal{T}_S \longrightarrow \mathcal{T}|_S \longrightarrow \mathcal{O}_S(d) \longrightarrow 0\ .$$

Recall that for a locally free sheaf $\mathcal{F}$ of rank r the determinant $\det(\mathcal{F})$ is defined as $\bigwedge^r(\mathcal{F})$.

Exercise 4. (i) Show that there exists an isomorphism of invertible sheaves

$$\omega_S \cong \det(\mathcal{T}_S)^*.$$

(ii) If S is described as $\mathrm{Proj}(A = k[x_0, \ldots, x_3]/(F))$, how do you describe $\mathcal{T}_S$ as $\widetilde{M}$ for a graded module over A?

(iii) Let now S be the image of the Segre embedding $\psi\colon \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. Compute the pull-back of $\mathcal{T}_S$ and of ω_S to $\mathbb{P}^1$ under the composition $\varphi := \psi \circ \Delta\colon \mathbb{P}^1 \hookrightarrow S$.

Solution (i) Taking determinants of the two short exact sequences

$$0 \longrightarrow \mathcal{T}_S \longrightarrow \mathcal{T}|_S \longrightarrow \mathcal{O}_S(d) \longrightarrow 0 \ \text{ and } \ 0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(1)^{\oplus 4} \longrightarrow \mathcal{T}|_S \longrightarrow 0 \tag{2}$$

yields

$$\det(\mathcal{T}_S) \otimes \mathcal{O}_S(d) \cong \det(\mathcal{T}|_S) \cong \det(\mathcal{O}_S(1)^{\oplus 4}) \cong \mathcal{O}_S(4),$$

which immediately yields $\omega_S \cong \mathcal{O}_S(d-4) \cong \det(\mathcal{T}_S)^*$.

(ii) Abbreviate $B := k[x_0, \ldots, x_3]$. Then, by definition, $\mathcal{T} \cong \widetilde{N}$ with $N := \mathrm{coker}\left(B \longrightarrow B(1)^{\oplus 4},\ b \longmapsto \oplus b x_i\right)$ and we can take $M := \ker\left(N \otimes_B A \longrightarrow A(d), \oplus \lambda_i \longmapsto \sum \partial_i F \lambda_i\right)$.

The Segre embedding is induced by the complete linear system $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1) \boxtimes \mathcal{O}(1))$. Hence, $\psi^* \mathcal{O}_S(1) \cong \mathcal{O}(1) \boxtimes \mathcal{O}(1)$ and, therefore, $(\psi \circ \Delta)^* \mathcal{O}_S(1) \cong \Delta^*(\mathcal{O}(1) \boxtimes \mathcal{O}(1)) \cong \mathcal{O}(2)$. Taking appropriate powers yields $(\psi \circ \Delta)^* \omega_S \cong \mathcal{O}(2d-8) \cong$

$\mathcal{O}(-4)$, because $S = V_+(F \coloneqq x_0 \cdot x_1 - x_2 \cdot x_3)$ is a surface of degree $d = 2$. In particular, $\varphi^*\mathcal{T}_S$ is a locally free sheaf of rank two with determinant $\mathcal{O}(4)$. In other words, writing $\varphi^*\mathcal{T}_S \cong \mathcal{O}(a) \oplus \mathcal{O}(b)$, we know already $a + b = 4$ and we may assume $a \leq b$. On the other hand, pulling back both sequences in (2), tensoring both sequences with $\mathcal{O}(-3)$ (on $\mathbb{P}^1$!) shows $h^0(\mathbb{P}^1, (\varphi^*\mathcal{T}|_S)(-3)) = 2$ and $H^1(\mathbb{P}^1, (\varphi^*\mathcal{T}|_S)(-3)) = 0$. This implies $h^1(\mathbb{P}^1, (\varphi^*\mathcal{T}_S)(-3)) \leq h^0(\mathbb{P}^1, \mathcal{O}(1)) = 2$, i.e. $a-3 \geq -3$. Hence, only the following options for (a, b) are possible: $(a, b) = (0, 4)$, $(a, b) = (1, 3)$, or $(a, b) = (2, 2)$. Twisting the pull back of (2) with $\mathcal{O}(-2)$ (on $\mathbb{P}^1$!) and using that $(\partial_i F) \colon H^0(\mathbb{P}^1, \mathcal{O})^{\oplus 4} \twoheadrightarrow H^0(\mathbb{P}^1, \mathcal{O}(2))$ is surjective, implies $H^1(\mathbb{P}^1, (\varphi^*\mathcal{T}_S)(-2)) = 0$ which excludes the case $(a, b) = (0, 4)$. Suppose $(a, b) = (1, 3)$, then $h^1(\mathbb{P}^1, (\varphi^*\mathcal{T}_S)(-3)) = 1$ and since $H^1(\mathbb{P}^1, \mathcal{O}(1)) = 0$, the pullback of the first exact sequence tensored with $\mathcal{O}(-3)$ yields $h^1(\mathbb{P}^1, (\varphi^*\mathcal{T}|_S)(-3)) \leq 1$. On the other hand, the pullback of the second exact sequence tensored with $\mathcal{O}(-3)$ together with $h^1(\mathbb{P}^1, \mathcal{O}(-3)) = 2$ and $h^1(\mathbb{P}^1, \mathcal{O}(-2)^{\oplus 4}) = 4$ shows $h^1(\mathbb{P}^1, (\varphi^*\mathcal{T}|_S)(-3) \geq 2$. Contradiction. Hence, $(a, b) = (2, 2)$.

3. We shall now restrict to cubic surfaces $S = V_+(F)$, i.e. F is homogenous of degree three, and to avoid complications we assume $\mathrm{char}(k) \neq 2, 3$. A special example is the Fermat cubic surface $S_0 = V_+(F_0)$ with $F_0 = x_0^3 + x_1^3 + x_2^3 + x_3^3$.

We are interested in lines $\ell \subset \mathbb{P}^3$ contained in cubic surfaces. Here, by definition, a line $\ell \subset \mathbb{P}^3$ is a closed subscheme of the form $V_+(\lambda_1, \lambda_2)$, where $\lambda_1, \lambda_2 \in H^0(\mathbb{P}^3, \mathcal{O}(1))$ are linearly independent. In particular, any line is isomorphic to $\mathbb{P}^1$ (over k).

Exercise 5. (i) Show that every line $\ell \subset \mathbb{P}^3$ is contained in some cubic surface $S \subset \mathbb{P}^3$. Describe the space of cubic surfaces containing a fixed line.

(ii) How many distinct lines $\ell \subset \mathbb{P}^3$ can you describe in S_0? Describe a *triangle* in S_0, i.e. a hyperplane section $S_0 \cap \mathbb{P}^2$ that is a union of lines.

(iii) Assume a line ℓ is contained in a factorial cubic surface $S \subset \mathbb{P}^3$ and consider ℓ as a Weil divisor on S with associated invertible sheaf $\mathcal{O}(\ell)$. Then $\deg(\mathcal{O}(\ell)|_\ell) = -1$. Try to show this under suitable assumptions on ℓ and S. Deduce that $\mathcal{O}(\ell)$ is not isomorphic to any $\mathcal{O}_S(n)$.

(iv) Are the invertible sheaves $\mathcal{O}(\ell) \in \mathrm{Pic}(S_0)$ associated with the lines you found in (ii) linearly independent in $\mathrm{Pic}(S_0)$?

Solution (i) Fix a line $\ell \subset \mathbb{P}^3$ and consider the associated exact sequence $0 \longrightarrow \mathcal{I}_\ell \longrightarrow \mathcal{O} \longrightarrow i_*\mathcal{O}_\ell \longrightarrow 0$. Tensoring with $\mathcal{O}(3)$ and taking global section shows that the space $H^0(\mathbb{P}^3, \mathcal{I}_\ell(3))$ of homogenous cubic polynomials vanishing on ℓ is the kernel of the restriction map $r \colon H^0(\mathbb{P}^3, \mathcal{O}(3)) \twoheadrightarrow H^0(\mathbb{P}^1, \mathcal{O}(3))$. The latter is of dimension four, while the former is of dimension 20. Hence, $h^0(\mathbb{P}^3, \mathcal{I}_\ell(3)) \geq 16$ and in fact equality holds, as r is surjective. The linear system of all cubics containing a line is thus $\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{I}_\ell(3)) \cong \mathbb{P}^{15}$.

For (ii), observe that the following lines are contained in S_0: $V_+(x_0 + \xi_1 x_1, x_2 + \xi_2 x_3)$, $V_+(x_0 + \xi_3 x_2, x_1 + \xi_4 x_3)$, and $V_+(x_0 + \xi_5 x_3, x_1 + \xi_6 x_2)$, where $\xi_i^3 = -1$. This yields 27 distinct lines contained in S_0 which is in fact the maximum number (for regular cubic surfaces). The intersection $S_0 \cap V_+(x_2 + x_3) \subset V_+(x_2 + x_3) \cong \mathbb{P}^2$ is described by the equation $x_0^3 + x_1^3$. Since $V_+(x_0^3 + x_1^3) \subset \mathbb{P}^2$ is the union of the three lines $V_+(x_0 + \xi x_1)$ with $\xi^3 = -1$, we have found a triangle in S_0.

(iii) If $\mathbb{P}^2 \cap S$ is a union $\ell \cup Q \subset \mathbb{P}^2$ with ℓ not contained in Q. Then $1 = \deg(\mathcal{O}(1)|_\ell) = \deg(\mathcal{O}(\mathbb{P}^2)|_\ell) = \deg(\mathcal{O}(Q)|_\ell) + \deg(\mathcal{O}(\ell)|_\ell) = 2 + \deg(\mathcal{O}(\ell)|_\ell)$, which proves the claim. (Note that the last equality is obvious for the invertible sheaf $\mathcal{O}_{\mathbb{P}^2}(Q)$ on $\mathbb{P}^2$ associated with $Q \subset \mathbb{P}^2$, which is $\mathcal{O}_{\mathbb{P}^2}(2)$. Here we need it for the invertible sheaf $\mathcal{O}_S(Q)$ on S associated with $Q \subset S$, which needs an extra argument: $\deg(\mathcal{O}(Q)|_\ell) = h^0(\ell, \mathcal{O}(Q)|_\ell) - 1 = h^0(\ell, \mathcal{O}_\ell) + h^0(\ell, \mathcal{O}_{Q \cap \ell}) - 1$, which is the same for $\mathcal{O}_S(Q)$ and $\mathcal{O}_{\mathbb{P}^2}(Q) \cong \mathcal{O}_{\mathbb{P}^2}(2)$.) Suppose $\mathcal{O}(\ell) \cong \mathcal{O}_S(n)$ for some n. Since $H^0(S, \mathcal{O}(\ell)) \neq 0$, we must have $n > 0$. On the other hand, $\deg(\mathcal{O}(\ell)|_\ell) = -1$, but $\mathcal{O}_S(n)|_\ell$ is clearly effective and even ample. Contradiction.

(iv) Whenever $\mathbb{P}^2 \cap S$ is a triangle, i.e. consists of three lines ℓ_1, ℓ_2, ℓ_3, $\mathcal{O}(\ell_1) \otimes \mathcal{O}(\ell_2) \otimes \mathcal{O}(\ell_3) \cong \mathcal{O}_S(1)$. Applying this to two triangles yields a linear relation between the six involved lines. Explicitly, intersecting S_0 with the two planes $V_+(x_2 + x_3)$ and $V_+(x_0 + x_1)$ yields six lines $\ell_i \coloneqq V_+(x_0 + \xi_i x_1, x_2 + x_3)$ and $\ell_i' \coloneqq V_+(x_0 + x_1, x_2 + \xi_i x_3)$ with $\mathcal{O}(\ell_1) \otimes \mathcal{O}(\ell_2) \otimes \mathcal{O}(\ell_3) \cong \mathcal{O}(\ell_1') \otimes \mathcal{O}(\ell_2') \otimes \mathcal{O}(\ell_3')$.