Prof. Dr. Daniel Huybrechts
Dr. Gebhard Martin

Winter term 2020/21

Algebraic Geometry I – Exam 25.2.2021

Begin: 09:00 **End:** 12:00

Technical details:

– Upload a scan of your solutions (max. 500kb per page) until 12:00 to the eCampus folder 'Exams/Exam submission'. Solutions uploaded any later than 12:00 will not be considered which results in you failing the exam.

– The exam will be posted at 9:00 on the usual website `https://www.math.uni-bonn.de/people/gmartin/AlgebraicGeometryWS2020.htmpl` and will also be available in the folder 'Exams' in eCampus (where this document can be found already now).

– Please sign the declaration of honor below and scan it together with your solutions. Without it your solutions will not be considered.

– If at all possible, upload your solution in one(!) pdf document with the declaration of honor as the first page. Name the file **firstname.name.pdf**. Please, write your name on each sheet.

– In case of technical problems (but only then), you can also send the scan to gmartin@math.uni-bonn.de (until 12:00 am).

– Allow at least 30-45min for scanning and uploading your solutions.

Guidelines:

– The exam consists of applying your knowledge of algebraic geometry to fill in the missing arguments in the text below. In particular, reading and understanding the arguments that are given is part of the task.

– It is recommended to work through the exercises in the given order.

– The exam will only lead to pass or fail. It will not be graded. A complete solution will be posted on eCampus, so that you can find out yourself how well you did in the exam.

– All arguments have to be justified. Apart from standard material from commutative algebra, you have to deduce everything by only using results explicitly stated or used, either in the lectures, on the exercise sheets or in the text below.

– You may use available resources to solve the exercise. You are allowed to consult the notes of the class, the exercise sheets, online books, etc. **You are not allowed** to contact any other person during the exam (by email, phone, social media, etc.) or to discuss your answers with anyone before successfully uploading them to eCampus.

I hereby swear that I completed the examination detailed above completely on my own and without any impermissible external assistance or through the use of non-permitted aids. I am aware that cheating during the execution of an examination (as detailed in §63 Para 5 of the Higher Education Act NRW) is a violation of the legal regulations for examinations and an administrative offense. The submission of false affirmation in lieu of an oath is a criminal offense.

Signed at: **Date:**

Name: **Student ID:** **Signature:**

Surfaces in $\mathbb{P}^3_k$

We assume that k is an algebraically closed field and denote by $\mathbb{P}^n$ the projective space $\mathbb{P}^n_k$.

1. A *surface in* $\mathbb{P}^3$ is the zero set

$$S := V_+(F)$$

of a homogenous polynomial $0 \neq F \in H^0(\mathbb{P}^3, \mathcal{O}(d))$ of degree $d > 0$.

Exercise 1. (i) Consider the properties 'integral' and 'non-reduced'. Give explicit examples of surfaces $S \subset \mathbb{P}^3$ with these properties.

(ii) For fixed degree d, how many irreducible components can S maximally have? Give an explicit example for each d realizing the maximum.

(iii) Assume S is integral. Determine the transcendence degree (over k) of its function field $K(S)$.

(iv) We know that $H^0(S, \mathcal{O}_S) \cong k$ for integral S. Does this still hold true when S is not integral?

(v) Describe explicitly an open affine cover of S.

We denote by $\mathcal{O}_S(i)$ the restriction of $\mathcal{O}(i)$ to $S \subset \mathbb{P}^3$ and call the invertible sheaf $\omega_S := \mathcal{O}_S(d-4)$ the *canonical sheaf* of S.

Exercise 2. (i) Prove that $H^1(S, \mathcal{O}_S) = 0 = H^1(S, \omega_S)$.

(ii) Prove that $h^2(S, \mathcal{O}_S) = h^0(S, \omega_S)$ and express this number in terms of d.

(iii) Construct an isomorphism $H^2(S, \omega_S) \cong k$.

If at the end of the exam you still have time, you may want to prove that in fact

$$h^i(S, \mathcal{O}_S(n)) = h^{2-i}(S, \omega_S \otimes \mathcal{O}_S(-n))$$

for all $n \in \mathbb{Z}$. This is generalized by Serre duality which for a locally free sheaf $\mathcal{F}$ on S and its dual $\mathcal{F}^* := \mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{O}_S)$ asserts the existence of an isomorphism of k-vector spaces

$$H^i(S, \mathcal{F}) \cong H^{2-i}(S, \mathcal{F}^* \otimes \omega_S)^*.$$

2. Consider the linear coordinates x_0, x_1, x_2, x_3 on $\mathbb{P}^3$ and the induced morphism

$$\xi \colon \mathcal{O} \longrightarrow \mathcal{O}(1)^{\oplus 4}, \lambda \longmapsto \bigoplus \lambda x_i$$

of coherent sheaves on $\mathbb{P}^3$. Observe that if $U := H^0(\mathbb{P}^3, \mathcal{O}(1))$, then ξ can be interpreted as the map $\mathcal{O} \longrightarrow \mathcal{O}(1) \otimes U^*$[1] corresponding to the global section in

$$H^0(\mathbb{P}^3, \mathcal{O}(1) \otimes U^*) \cong U \otimes U^* \cong \mathrm{End}(U)$$

given by id_U. We denote the sheaf cokernel of ξ by $\mathcal{T}$ (the *tangent sheaf* of $\mathbb{P}^3$).

For $S = V_+(F)$ as before, consider the partial derivative $\partial_i F \in H^0(\mathbb{P}^3, \mathcal{O}(d-1))$. Then the map $(t_0, t_1, t_2, t_3) \longmapsto \sum t_i \partial_i F$ defines a map

$$\eta \colon \mathcal{O}(1)^{\oplus 4} \longrightarrow \mathcal{O}(d).$$

[1] For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ the tensor product $\mathcal{F} \otimes U$ is by definition the tensor product of $\mathcal{F}$ as a sheaf over the constant sheaf associated with k with the constant sheaf associated with U.

Exercise 3. (i) Show that ξ is injective.

(ii) For any given point $x \in \mathbb{P}^3$ corresponding to a line $\ell \subset U^*$ describe a natural isomorphism of the fibre of $\mathcal{T}$ at x with $\mathrm{Hom}(\ell, U^*/\ell)$. (Aside: One can show that $\mathrm{Hom}(\ell, U^*/\ell)$ is also naturally isomorphic to the Zariski tangent space at x, which is why $\mathcal{T}$ is called the tangent bundle.)

(iii) Show that $\mathcal{T}$ is locally free (of rank three).

(iv) Show that the restriction of η to S factors uniquely through the restriction $\mathcal{O}_S(1)^{\oplus 4} \twoheadrightarrow \mathcal{T}|_S$ of ξ to S. (Use the well-known Euler identity $\sum x_i \partial_i F = d \cdot F$.)

(v) Under which condition on F is the induced map $\mathcal{T}|_S \longrightarrow \mathcal{O}_S(d)$ surjective? Write down one example for F of degree $d = 3$ where surjectivity holds (under appropriate assumptions on the characteristic of k).

Assume that the assumption in (v) above holds and define the *tangent bundle* $\mathcal{T}_S$ of S as the kernel of $\mathcal{T}|_S \longrightarrow \mathcal{O}_S(d)$. Thus, there exists a short exact sequence

$$0 \longrightarrow \mathcal{T}_S \longrightarrow \mathcal{T}|_S \longrightarrow \mathcal{O}_S(d) \longrightarrow 0\,.$$

Recall that for a locally free sheaf $\mathcal{F}$ of rank r the determinant $\det(\mathcal{F})$ is defined as $\bigwedge^r(\mathcal{F})$.

Exercise 4. (i) Show that there exists an isomorphism of invertible sheaves

$$\omega_S \cong \det(\mathcal{T}_S)^*.$$

(ii) If S is described as $\mathrm{Proj}(A = k[x_0, \ldots, x_3]/(F))$, how do you describe $\mathcal{T}_S$ as $\widetilde{M}$ for a graded module over A?

(iii) Let now S be the image of the Segre embedding $\psi\colon \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. Compute the pull-back of $\mathcal{T}_S$ and of ω_S to $\mathbb{P}^1$ under the composition $\varphi := \psi \circ \Delta\colon \mathbb{P}^1 \hookrightarrow S$.

3. We shall now restrict to cubic surfaces $S = V_+(F)$, i.e. F is homogenous of degree three, and to avoid complications we assume $\mathrm{char}(k) \neq 2, 3$. A special example is the Fermat cubic surface $S_0 = V_+(F_0)$ with $F_0 = x_0^3 + x_1^3 + x_2^3 + x_3^3$.
We are interested in lines $\ell \subset \mathbb{P}^3$ contained in cubic surfaces. Here, by definition, a line $\ell \subset \mathbb{P}^3$ is a closed subscheme of the form $V_+(\lambda_1, \lambda_2)$, where $\lambda_1, \lambda_2 \in H^0(\mathbb{P}^3, \mathcal{O}(1))$ are linearly independent. In particular, any line is isomorphic to $\mathbb{P}^1$ (over k).

Exercise 5. (i) Show that every line $\ell \subset \mathbb{P}^3$ is contained in some cubic surface $S \subset \mathbb{P}^3$. Describe the space of cubic surfaces containing a fixed line.

(ii) How many distinct lines $\ell \subset \mathbb{P}^3$ can you describe in S_0? Describe a *triangle* in S_0, i.e. a hyperplane section $S_0 \cap \mathbb{P}^2$ that is a union of lines.

(iii) Assume a line ℓ is contained in a factorial cubic surface $S \subset \mathbb{P}^3$ and consider ℓ as a Weil divisor on S with associated invertible sheaf $\mathcal{O}(\ell)$. Then $\deg(\mathcal{O}(\ell)|_\ell) = -1$. Try to show this under suitable assumptions on ℓ and S. Deduce that $\mathcal{O}(\ell)$ is not isomorphic to any $\mathcal{O}_S(n)$.

(iv) Are the invertible sheaves $\mathcal{O}(\ell) \in \mathrm{Pic}(S_0)$ associated with the lines you found in (ii) linearly independent in $\mathrm{Pic}(S_0)$?