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Winter term 2020/21

Exercises, Algebraic Geometry I – Week 9

Exercise 55. *Additivity of the Euler characteristic* (4 points)
Let X be a projective scheme over a field k. Recall that by Serre's theorem for every coherent sheaf $\mathcal{F}$ on X the k-vector spaces $H^i(X, \mathcal{F})$ are finite-dimensional k-vector spaces. Define the *Euler characteristic* of $\mathcal{F}$ as

$$\chi(X, \mathcal{F}) \coloneqq \sum_{i=0}^{\infty} (-1)^i \dim_k H^i(X, \mathcal{F}).$$

Show that for a short exact sequence of coherent sheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ one has

$$\chi(X, \mathcal{G}) = \chi(X, \mathcal{F}) + \chi(X, \mathcal{H}).$$

(We assume that $H^i(X, \mathcal{F}) = 0$ for $i \gg 0$ for every $\mathcal{F}$, which we have not proved yet. This is a result of Grothendieck, see [Hartshorne, III.Thm. 2.7] which holds for sheaves of abelian groups on noetherian topological spaces and is not exceedingly hard to prove.)

Exercise 56. *Global regular functions on projective schemes* (3 points)
Let X be a projective scheme over a field k. Then $H^0(X, \mathcal{O}_X)$ is a finite-dimensional vector space by Serre's theorem. Show that $H^0(X, \mathcal{O}_X)$ is a finite field extension of k if X is integral. In particular, $H^0(X, \mathcal{O}_X) = k$ if k is algebraically closed.

Exercise 57. *Arithmetic genus* (3 points)
The arithmetic genus of a projective scheme X of dimension n over a field k is defined as

$$p_a(X) \coloneqq (-1)^n (\chi(X, \mathcal{O}_X) - 1),$$

So if X is an integral curve and $k = \bar{k}$, i.e. $n = 1$, then $p_a(X) = \dim_k H^1(X, \mathcal{O}_X)$. Show that for $X \subset \mathbb{P}^2_k$ given by a polynomial of degree d, $p_a(X) = (d-1)(d-2)/2$.

Exercise 58. *Products of* Proj (5 points)
Let $B = \bigoplus_{d \geq 0} B_d$ and $C = \bigoplus_{d \geq 0} C_d$ be two graded rings with $A \coloneqq B_0 \cong C_0$. Consider $B \times_A C \coloneqq \bigoplus_{d \geq 0} B_d \otimes_A C_d$ and the schemes $X \coloneqq \mathrm{Proj}(B)$ and $Y \coloneqq \mathrm{Proj}(C)$.

(i) Show that $X \times_{\mathrm{Spec}(A)} Y \cong \mathrm{Proj}(B \times_A C)$.

(ii) Prove that under this isomorphism $\mathcal{O}(1)$ on $\mathrm{Proj}(B \times_A C)$ is isomorphic to $p_1^* \mathcal{O}_X(1) \otimes p_2^* \mathcal{O}_Y(1)$, where p_1 and p_2 are the two projections from $X \times_{\mathrm{Spec}(A)} Y$.

Due Friday 14 January 2021.

The last exercise is not strictly necessary for the understanding of the lectures at this point.

Exercise 59. *Extending coherent sheaves* (5 extra points)
Consider the following statement:

$(*)$ If X is a Noetherian scheme, $i : U \hookrightarrow X$ is an open subscheme of X, $\mathcal{F}$ is a coherent sheaf on U, and $\mathcal{G}$ is a quasi-coherent sheaf on X such that $\mathcal{F} \subset \mathcal{G}|_U$, then there exists a coherent subsheaf $\mathcal{F}' \subset \mathcal{G}$ such that $\mathcal{F}'|_U = \mathcal{F}$.

(i) Prove that every quasi-coherent sheaf on a Noetherian affine scheme is the union of its coherent subsheaves.

(Here, we say that a sheaf of abelian groups $\mathcal{F}$ on a topological space X is a union of subsheaves of abelian groups $\mathcal{F}_\alpha$ if for every open $U \subset X$ the group $\mathcal{F}(U)$ is the union of its subgroups $\mathcal{F}_\alpha(U)$.)

(ii) Show that $(*)$ holds if X is affine.

(iii) Show that $(*)$ holds.

(iv) Note that, in the setting of $(*)$, $i_*\mathcal{F}$ is quasi-coherent and $\mathcal{F} = (i_*\mathcal{F})|_U$, so $\mathcal{F}$ always extends to some coherent subsheaf of $i_*\mathcal{F}$. Use this to show that if X is a projective scheme over a field k (or, more generally, over a Noetherian ring), then $\mathcal{F}$ admits a resolution by locally free sheaves $\mathcal{L}_i$ of finite rank on U:

$$\ldots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0$$

In fact, the $\mathcal{L}_i$ can be assumed to be direct sums of line bundles on U.