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Winter term 2020/21

Exercises, Algebraic Geometry I – Week 8

Exercise 48. *Restriction of sections on affine schemes* (2 points)
Let X be the affine scheme $\mathrm{Spec}(A)$ and $\mathcal{F}$ a quasi-coherent sheaf on X. Prove the following assertions:

(i) Assume $0 = s|_{D(a)} \in \Gamma(D(a), \mathcal{F})$ for some $s \in \Gamma(X, \mathcal{F})$ and $a \in A$. Then there exists $n > 0$ such that $a^n \cdot s = 0$ in $\Gamma(X, \mathcal{F})$.

(ii) Let $t \in \Gamma(D(a), \mathcal{F})$. Then there exists $n > 0$ and $s \in \Gamma(X, \mathcal{F})$ such that $s|_{D(a)} = a^n \cdot t$ in $\Gamma(D(a), \mathcal{F})$.

Exercise 49. *Direct images of (quasi)-coherent sheaves* (4 points)
Find examples of a scheme morphism $f\colon X \to Y$ and a coherent sheaf $\mathcal{F}$ such that $f_*\mathcal{F}$ is not coherent, although $\mathcal{F}$ is. Is f_* or f^* compatible with tensor products?

Exercise 50. *Veronese embedding* (4 points)
Let $B = \bigoplus_{i \geq 0} B_i$ be a graded ring. Then $B^{(d)}$ denotes the graded ring defined by $B^{(d)} := \bigoplus_{i \geq 0} B_i^{(d)}$ with $B_i^{(d)} := B_{di}$.

(i) Show that there exists an isomorphism $\varphi\colon \mathrm{Proj}(B) \xrightarrow{\sim} \mathrm{Proj}(B^{(d)})$ with $\varphi^*\mathcal{O}(1) \cong \mathcal{O}(d)$.

(ii) Consider the case $B = k[x_0, \ldots, x_n]$. Show that the surjection $k[y_0, \ldots, y_N] \twoheadrightarrow B^{(d)}$ mapping y_i to the i-th monomial of degree d in the variables x_i defines a closed embedding
$$\mathbb{P}^n_k \cong \mathrm{Proj}(B) \cong \mathrm{Proj}(B^{(d)}) \hookrightarrow \mathbb{P}^N_k.$$
What is N? Show that for k algebraically closed the morphism is given on closed points by $[\lambda_0 : \cdots : \lambda_n] \mapsto [\lambda_0^d : \cdots : \lambda^I : \cdots : \lambda_n^d]$ with λ^I running through all monomials of degree d.

Exercise 51. *Ideal sheaf of diagonal* (4 points)
Let X be a separated k-scheme and $\mathcal{I} := \mathcal{I}_\Delta$ be the ideal sheaf of its diagonal $\Delta \subset X \times_k X$. Show that there is an exact sequence
$$0 \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_{X \times_k X}/\mathcal{I}^2 \to \mathcal{O}_\Delta \to 0.$$
Prove that $\mathcal{I}/\mathcal{I}^2$ can be naturally viewed as the direct image under the diagonal morphism $\Delta\colon X \to X \times_k X$ of a sheaf, called the *cotangent sheaf* $\Omega_{X/k}$, on X. Prove that $\Omega_{X/k}$ is (locally) free for $X = \mathbb{A}^n_k$.

Due Friday 08 January 2021.

Exercise 52. *The irrelevant ideal* (3 points)
Let $B = \bigoplus_{d\geq 0} B_d$ be a graded ring and $\mathfrak{a} \subset B$ a homogeneous ideal. Show that the following conditions are equivalent.

(i) $V_+(\mathfrak{a}) = \emptyset$.

(ii) $B_+ \subset \sqrt{\mathfrak{a}}$.

(iii) If $\mathfrak{a} = (a_i)$ with a_i homogeneous, then $\bigcup D_+(a_i) = \mathrm{Proj}(B)$.

Exercise 53. *Examples of* Proj (4 points)
Describe the following schemes:

(i) $\mathrm{Proj}(\mathbb{Z}[X])$.

(ii) $\mathbb{P}^1_k$ for $k = \bar{k}$.

(iii) $\mathbb{P}^2_k \setminus D_+(x^2 + y^2 - z^2)$.

(iv) $\mathrm{Proj}(k[x,y]/(x^2, y^2))$.

Exercise 54. *Tensor products of quasi-coherent sheaves* (4 points)
Let A be a ring and M, N be two A-modules. Show that $(M \otimes_A N)^\sim \cong \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$ on $\mathrm{Spec}(A)$. Is there an analogous version in the graded case, i.e. for sheaves on $\mathrm{Proj}(B)$?

Reflex test[1]

1. In which sense is a (pre-)sheaf a functor?
2. Let $\mathrm{Spec}(A) \subset X$ be a open and affine and $x = \mathfrak{p} \in \mathrm{Spec}(A)$. Express $\mathcal{O}_{X,x}$ in terms of A.
3. Is $f^{-1}\colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ exact?
4. What is an example of a ringed spaces that is not a locally ringed space?
5. What are birth year and place of Grothendieck?
6. What is the link between ring homomorphisms $A \to B$ and scheme morphisms $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$?
7. $R^i f_*$ is the derived functor of what functor? Is it left or right exact?
8. Let $A = k[[t]]$ be the ring of formal power series over a field. Describe the topological space $\mathrm{Spec}(A)$.
9. What is the relation between injective and flabby sheaves?
10. Do you know an example of an affine morphism that is not proper or closed?
11. Is the disjoint union of two (affine) schemes again a(n affine) scheme?
12. What is the difference between connected and irreducible?
13. Is a reduced scheme integral?
14. What is an example of a property of a morphism that is not preserved under base change?
15. What is your favorite example of a morphism $X \to Y$ with X irreducible and X_y not irreducible for some $y \in Y$.

[1] These quick exercises are meant to test your reflexes. These are the things you should know instantaneously and without much thinking. If you do not get them right the first time, try again later. Do not hand in solutions for them.

16. Let $B = \bigoplus_{i\geq 0} B_i$ be a graded algebra over a field $k = B_0$. If B is finite-dimensional as a k-vector space, what is $\mathrm{Proj}(B)$?
17. Can a non-reduced scheme have a generic point?
18. For a graded ring B, is the nilradical a homogeneous ideal?
19. Is the scheme $\mathrm{Spec}(\bar{\mathbb{Q}} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}})$ connected?
20. Let $X = \mathrm{Spec}(\mathbb{Z})$ and $p \in \mathbb{Z}$ be a prime number. What is the stalk $\mathcal{O}_{X,(p)}$ of $\mathcal{O}_X$ at $(p) \in X$? What is the stalk $\mathcal{O}_{X,(0)}$ at the generic point?
21. Let $X = \mathrm{Spec}(\mathbb{Z}[x])$. Describe the fibre product $X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{F}_p)$.
22. Let $X = \mathrm{Spec}(k[x])$, $Y = \mathrm{Spec}(k[y])$ and $f\colon X \to Y$ be induced by the map $y \mapsto x^2$. Describe the fibre of f over the closed point (y).
23. Let $X = \mathrm{Spec}(k[x,y]/(xy))$ and $o \in X$ be the point corresponding to the maximal ideal $\mathfrak{m} = (x,y)$. What is the dimension of the tangent space T_o of X at o?
24. Let $X = \mathrm{Spec}(k[x])$. Consider the two projections $p_1, p_2\colon X \times X \to X$. Let $a \in X$ be a point, such that $p_1(a) = p_2(a)$. Is it true that a lies in the image of the diagonal morphism $\Delta\colon X \to X \times X$?
25. Let $X = \mathrm{Spec}(k[x,y]/(xy^2))$. Describe the reduction X_{red}. Describe the normalization of the reduction $\widetilde{X_{\mathrm{red}}}$.
26. Let k be a field, $X = \mathrm{Spec}(k[x])$ and $f\colon X \to \mathrm{Spec}(k)$ be the natural morphism. Describe $f_*\mathcal{O}_X$? Same question for $X = \mathrm{Proj}(k[x,y])$.
27. Consider the ideal $I = (x) \subset k[x]$ and the corresponding ideal sheaf $\widetilde{I}$ on $\mathrm{Spec}(k[x])$. Is $\widetilde{I}$ locally free? Same question for $I = (x,y) \subset k[x,y]$.
28. If for a graded algebra $B = \bigoplus_{i\geq 0} B_i$ over a field $k = B_0$ we have $\mathrm{Proj}(B) = \emptyset$, is it true that B is a finite-dimensional k-vector space?
29. Let X be the complement of a closed point in $\mathrm{Spec}(\mathbb{C}[x,y])$. Is X affine?
30. Consider the ideal $\mathfrak{a} = (x,y) \subset \mathbb{C}[x,y]$ and the corresponding ideal sheaf $\tilde{\mathfrak{a}}$ on $X = \mathrm{Spec}(\mathbb{C}[x,y])$. Compute the dimension $\dim(\widetilde{\mathfrak{a}}_p \otimes_{\mathcal{O}_{X,p}} \mathbb{C})$ for all closed points $p \in X$.
31. Let G be an abelian group. What is the difference between the constant presheaf $\mathbf{G}$ and the constant sheaf $\underline{G}$ on the scheme $\mathrm{Spec}(k[x,y]/(x, y^2-1))$?
32. Find an example of a quasi-finite morphism of schemes that is not finite.
33. What is an example of a surjection of sheaves $\mathcal{F} \twoheadrightarrow \mathcal{G}$ for which $\Gamma(X,\mathcal{F}) \to \Gamma(X,\mathcal{G})$ is not surjective.
34. For which of the following constructions for sheaves the natural pre-sheaf needs to be sheafified: $\mathrm{Ker}(\varphi)$, $f_*\mathcal{F}$, $\mathrm{Coker}(\varphi)$, $\mathrm{Im}(\varphi)$, $\mathcal{F} \otimes \mathcal{G}$.
35. For a graded ring B is the natural inclusion $\mathrm{Proj}(B) \hookrightarrow \mathrm{Spec}(B)$ a morphism of schemes?
36. Is $\mathcal{O}_X$ a flasque sheaf?
37. Is the projection $\mathbb{A}^2_k \to \mathbb{A}^1_k$ on the x-axis closed?
38. Let X be a scheme of finite type over k. Are all local rings $\mathcal{O}_{X,x}$ finite type k-algebras?
39. What does adjunction say for f_* and f^*?
40. Describe a sheaf (of $\mathcal{O}_X$-modules) on $X = \mathbb{A}^1_k$ that is not quasi-coherent.
41. Consider the inclusion $U \subset X$ of an open subscheme as a morphism of schemes and decide whether it is (in general) separated, proper, finite type, quasi-finite, finite, quasi-compact, Noetherian, locally Noetherian with irreducible, integral, connected, reduced, fibres. The same for a closed immersion $Z \subset X$.

Happy holidays!