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Winter term 2020/21

Exercises, Algebraic Geometry I – Week 7

Exercise 42. *Adjoint functors $f^* \dashv f_*$* (4 points)
Consider a morphism of ringed spaces $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. Show that f^* is left adjoint to f_*, i.e. for all $\mathcal{F} \in \mathrm{Mod}(X, \mathcal{O}_X)$ and $\mathcal{G} \in \mathrm{Mod}(Y, \mathcal{O}_Y)$ there exists an isomorphism (functorial in $\mathcal{F}$ and $\mathcal{G}$):

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$

Exercise 43. *$M \mapsto \tilde{M}$ and adjunction* (4 points)
Let X be the affine scheme $\mathrm{Spec}(A)$ and consider an A-module M and a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules. Show that (A-mod) $\to \mathrm{Mod}(X, \mathcal{O}_X)$, $M \mapsto \tilde{M}$ is left adjoint to $\mathrm{Mod}(X, \mathcal{O}_X) \to$ (A-mod), $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$, i.e. that there exists a functorial (in M and $\mathcal{F}$) isomorphism

$$\mathrm{Hom}_A(M, \Gamma(X, \mathcal{F})) \cong \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F}).$$

Exercise 44. *f^* and $\otimes$ are only right exact* (5 points)
Let $(X, \mathcal{O}_X)$ be a ringed space and consider $\mathcal{F}, \mathcal{G} \in \mathrm{Mod}(X, \mathcal{O}_X)$. Show that for all $x \in X$ there exists a natural isomorphism

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x.$$

Prove that $\mathcal{F} \otimes_{\mathcal{O}_X} (\ )\colon \mathrm{Mod}(X, \mathcal{O}_X) \to \mathrm{Mod}(X, \mathcal{O}_X)$ and $f^*\colon \mathrm{Mod}(Y, \mathcal{O}_Y) \to \mathrm{Mod}(X, \mathcal{O}_X)$ for a morphism of ringed spaces $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ are both right exact functors. Describe examples showing that in general they are not left exact.

Exercise 45. *Projection formula* (4 points)
Consider a morphism of ringed spaces $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $\mathcal{F} \in \mathrm{Mod}(X, \mathcal{O}_X)$ and $\mathcal{G} \in \mathrm{Mod}(Y, \mathcal{O}_Y)$. Suppose $\mathcal{G}$ is locally free of finite rank. Show that there exists a natural isomorphism

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \cong f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}.$$

Please turn over

Due Monday 21 December 2020.

Exercise 46. *Fibre dimension* (4 points)
Let X be a noetherian scheme and let $\mathcal{F}$ be a coherent sheaf on X. We will consider the function

$$\varphi(x) := \dim_{k(x)} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x),$$

where $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ is the residue field of the point $x \in X$. Use Nakayama lemma to prove the following statements.

(i) The function φ is upper semi-continuous, i.e. for any $n \in \mathbb{Z}$ the set $\{x \in X \mid \varphi(x) \geq n\}$ is closed.

(ii) If $\mathcal{F}$ is locally free and X is connected, then φ is a constant function.

(iii) Conversely, if X is reduced and φ is constant, then $\mathcal{F}$ is locally free.

The last exercise is not strictly necessary for the understanding of the lectures at this point.

Exercise 47. *Local Ext-sheaves* (5 extra points)
Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}, \mathcal{G} \in \mathrm{Mod}(X, \mathcal{O}_X)$. Similarly to Exercise 1, one can show that the presheaf of $\mathcal{O}_X$-modules

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) : U \mapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$$

is a sheaf. Consider the functor

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -) : \mathrm{Mod}(X, \mathcal{O}_X) \to \mathrm{Mod}(X, \mathcal{O}_X)$$

(i) Show that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -)$ is a left exact functor. Deduce that its right derived functors $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, -) := R^i\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -)$ exist.

(ii) Show that if $\mathcal{I} \in \mathrm{Mod}(X, \mathcal{O}_X)$ is injective, then so is $\mathcal{I}|_U \in \mathrm{Mod}(U, \mathcal{O}_U)$ for every open $U \subset X$. Deduce that

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})|_U \cong \mathcal{E}xt^i_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

(iii) Consider an exact sequence

$$\ldots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0,$$

where the $\mathcal{L}_i$ are locally free sheaves of finite rank on $(X, \mathcal{O}_X)$. Show that $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \cong H^i(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}_\bullet, \mathcal{G}))$.

(Warning: Note that this does not imply that one can define derived functors $\mathcal{E}xt^i_{\mathcal{O}_X}(-, \mathcal{G})$ of the contravariant functor $\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{G})$, since $\mathrm{Mod}(X, \mathcal{O}_X)$ usually does not admit enough projective objects.)

(iv) Assume that X is a Noetherian scheme and that $\mathcal{F}$ is coherent. Show that for every point $x \in X$, one has

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x),$$

where on the right hand side we consider the i-th derived functor of $\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, -) : \mathrm{Mod}(\mathcal{O}_{X,x}) \to \mathrm{Mod}(\mathcal{O}_{X,x})$.

(Notation: Sometimes, one writes $\mathcal{E}xt^i_X(\mathcal{F}, \mathcal{G})$ resp. $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})$ if the sheaf $\mathcal{O}_X$ resp. the ringed space $(X, \mathcal{O}_X)$ is clear from the context.)

Information from the Fachschaft:

This year's Math Christmas party will take place at Thursday, the 17.12. starting 18 ct. online via zoom. All current information can be found on https://fsmath.uni-bonn.de/events-detail/events/virtual-christmas-party.html. Swing by!