Exercise 35. Proper and separated morphisms (5 points)
Decide which of the following morphisms are proper and which are separated. Here $k$ is any field.

(i) $\text{Spec}(\mathbb{Q}) \to \text{Spec}(\mathbb{Z})$,

(ii) $V(xy - 1) \subset A^2_k \to A^1_k$ (first projection).

(iii) $A^n_k \to \text{Spec}(k)$,

(iv) Let $X$ resp. $Y$ be the schemes obtained by glueing $A^1_k$ with itself over the open set $D(t)$ via $k[t, t^{-1}] \to k[t, t^{-1}]$, $t \mapsto t$ resp. $t \mapsto t^{-1}$. Consider the natural morphisms $X, Y \to \text{Spec}(k)$ and $A^1_k \to X, Y$.

Exercise 36. A non-affine open subscheme of an affine scheme (4 points)
Let $k$ be a field and consider the affine plane $A^2_k = \text{Spec} k[x, y]$. Let $0 \in A^2_k$ be the closed point corresponding to the maximal ideal $(x, y)$ and let $U := A^2_k \setminus 0$.

(i) Show that restriction of sections induces an isomorphism $H^0(A^2_k, \mathcal{O}_{A^2_k}) \cong H^0(U, \mathcal{O}_U)$.

(ii) Deduce that $U$ is not an affine scheme.

(iii) Show that $\hat{H}^1(\{D(x), D(y)\}, \mathcal{O}_U) \cong \bigoplus_{i,j < 0} \langle x^i y^j \rangle$.

Exercise 37. Properties of morphisms (4 points)
Verify the following assertions.

(i) Show that a morphism $f : X \to Y$ of schemes which is surjective, of finite type, and quasi-finite, need not be finite.

(ii) Show that ‘quasi-finite’ and ‘injective’ are not preserved under base-change.

(iii) Show that ‘being an open/closed immersion’ is preserved under base-change.

(iv) Show that ‘having reduced/integral/connected fibres’ is not preserved under base-change.

Exercise 38. Intersections of affine open subschemes (3 points)
Let $U, V \subset X$ be two affine open subschemes of a scheme $X$.

(i) Show that the intersection $U \cap V$ need not be affine.

(ii) Prove that $U \cap V$ is affine if $X$ is separated (i.e. if $X \to \text{Spec}(\mathbb{Z})$ is separated).

(Hint: First, show that $X \times_X \times_Z (U \times V) \cong U \cap V$, where $X$ is embedded into $X \times X$ via the diagonal.)
Exercise 39. The image of a proper scheme is proper (3 points)
Let \( f : X \to Y \) be a morphism of \( S \)-schemes. Suppose that \( Y \to S \) is separated.

(i) Show that the graph \( \Gamma_f : X \to X \times_S Y \) is a closed immersion.

(ii) Let \( Z \subset X \) be a closed subscheme that is proper over \( S \). Show that \( f(Z) \subset Y \) is closed.

Exercise 40. Morphisms into separated schemes (4 points)
Consider schemes \( X \) and \( Y \) over a base scheme \( S \), i.e. together with morphisms \( \pi_X : X \to S \) and \( \pi_Y : Y \to S \). Assume that \( X \) is reduced (or even stronger integral) and that \( Y \to S \) is separated.

Show that two morphisms \( f_1, f_2 : X \to Y \) over \( S \) (i.e. such that \( \pi_X = \pi_Y \circ f_i \)) that coincide on a dense open subset \( U \subset X \) are actually equal. (Hint: Consider the composition of the graph \( X \to X \times_S Y \) of \( f \) with \( (g, \text{id}) : X \times_S Y \to Y \times_S Y \).) Give counterexamples if one of the hypotheses is dropped.

The last exercise is not strictly necessary for the understanding of the lectures at this point.

Exercise 41. Base change as a functor (3 extra points)
Consider a morphism of schemes \( S \to T \). Every \( S \)-scheme \( X \), i.e. every morphism \( X \to S \), yields by composition with \( S \to T \) a \( T \)-scheme \( X \to S \to T \) which we shall denote \( T X \).

Conversely, to every \( T \)-scheme \( Y \to T \) base change defines an \( S \)-scheme \( Y_S := S \times_T Y \to S \).

(i) Show that this defines two functors
\[
T( ) : (\text{Sch}/S) \to (\text{Sch}/T), \ X \mapsto T X \quad \text{and} \quad ( )_S : (\text{Sch}/T) \to (\text{Sch}/S), \ Y \mapsto Y_S
\]
which are adjoint to each other. More precisely, \( T( ) \) is left adjoint to \( ( )_S \), i.e. \( T( ) \dashv ( )_S \).

(ii) For any \( S \)-scheme \( X \) consider the functor
\[
\text{Res}_{S/T}(X) : (\text{Sch}/T)^{\text{op}} \to (\text{Sets})
\]
\[
Y \mapsto \text{Mor}_S(Y_S, X).
\]
If this functor is representable by a \( T \)-scheme, which will be denoted by \( \text{Res}_{S/T}(X) \), it is called the Weil restriction and satisfies \( h_{\text{Res}_{S/T}(X)} \cong \text{Res}_{S/T}(X) \).

One can show that the Weil restriction exists for finite field extensions \( S = \text{Spec}(K) \to T = \text{Spec}(k) \) and \( X = \text{Spec}(A) \), where \( A \) is a finite type \( K \)-algebra. In this case, one writes \( \text{Res}_{K/k}(X) \) for the Weil restriction.

Set \( S := \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}) \). Show that the rational points of \( S \) can be described as
\[
S(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \bigg| a, b \in \mathbb{R}, \ a^2 + b^2 \neq 0 \right\}.
\]

(Remark: Usually, i.e. for general \( S \to T \) and \( X \), the Weil restriction does not exist.)

Information from the Fachschaft:
This year’s Math Christmas party will take place at Thursday, the 17.12., starting 18 ct. online via zoom. All current information can be found on https://fsmath.uni-bonn.de/events-detail/events/virtual-christmas-party.html. Swing by!