Prof. Dr. Daniel Huybrechts
Dr. Gebhard Martin

Winter term 2020/21

Exercises, Algebraic Geometry I – Week 6

Exercise 35. *Proper and separated morphisms* (5 points)
Decide which of the following morphisms are proper and which are separated. Here k is any field.

(i) $\mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z})$,

(ii) $V(xy-1) \subset \mathbb{A}^2_k \to \mathbb{A}^1_k$ (first projection).

(iii) $\mathbb{A}^n_k \to \mathrm{Spec}(k)$,

(iv) Let X resp. Y be the schemes obtained by glueing $\mathbb{A}^1_k$ with itself over the the open set $D(t)$ via $k[t,t^{-1}] \to k[t,t^{-1}]$, $t \mapsto t$ resp. $t \mapsto t^{-1}$. Consider the natural morphisms $X, Y \to \mathrm{Spec}(k)$ and $\mathbb{A}^1_k \to X, Y$.

Exercise 36. *A non-affine open subscheme of an affine scheme* (4 points)
Let k be a field and consider the affine plane $\mathbb{A}^2_k = \mathrm{Spec}\ k[x,y]$. Let $0 \in \mathbb{A}^2_k$ be the closed point corresponding to the maximal ideal (x,y) and let $U := \mathbb{A}^2_k \setminus 0$.

(i) Show that restriction of sections induces an isomorphism $H^0(\mathbb{A}^2_k, \mathcal{O}_{\mathbb{A}^2_k}) \xrightarrow{\sim} H^0(U, \mathcal{O}_U)$.

(ii) Deduce that U is not an affine scheme.

(iii) Show that $\check{H}^1(\{D(x), D(y)\}, \mathcal{O}_U) \cong \bigoplus_{i,j<0} \langle x^i y^j \rangle$.

Exercise 37. *Properties of morphisms* (4 points)
Verify the following assertions.

(i) Show that a morphism $f\colon X \to Y$ of schemes which is surjective, of finite type, and quasi-finite, need not be finite.

(ii) Show that 'quasi-finite' and 'injective' are not preserved under base-change.

(iii) Show that 'being an open/closed immersion' is preserved under base-change.

(iv) Show that 'having reduced/integral/connected fibres' is not preserved under base-change.

Exercise 38. *Intersections of affine open subschemes* (3 points)
Let $U, V \subset X$ be two affine open subschemes of a scheme X.

(i) Show that the intersection $U \cap V$ need not be affine.

(ii) Prove that $U \cap V$ is affine if X is separated (i.e. if $X \to \mathrm{Spec}(\mathbb{Z})$ is separated).
(Hint: First, show that $X \times_{X \times_{\mathbb{Z}} X} (U \times V) \cong U \cap V$, where X is embedded into $X \times X$ via the diagonal.)

Please turn over

Due Friday 11 December 2020.

Exercise 39. *The image of a proper scheme is proper* (3 points)
Let $f\colon X \to Y$ be a morphism of S-schemes. Suppose that $Y \to S$ is separated.

(i) Show that the graph $\Gamma_f\colon X \to X \times_S Y$ is a closed immersion.

(ii) Let $Z \subset X$ be a closed subscheme that is proper over S. Show that $f(Z) \subset Y$ is closed.

Exercise 40. *Morphisms into separated schemes* (4 points)
Consider schemes X and Y over a base scheme S, i.e. together with morphisms $\pi_X\colon X \to S$ and $\pi_Y\colon Y \to S$. Assume that X is reduced (or even stronger integral) and that $Y \to S$ is separated. Show that two morphisms

$$f_1, f_2\colon X \to Y$$

over S (i.e. such that $\pi_X = \pi_Y \circ f_i$) that coincide on a dense open subset $U \subset X$ are actually equal. (*Hint*: Consider the composition of the graph $X \to X \times_S Y$ of f with $(g, \mathrm{id})\colon X \times_S Y \to Y \times_S Y$.) Give counterexamples if one of the hypotheses is dropped.

The last exercise is not strictly necessary for the understanding of the lectures at this point.

Exercise 41. *Base change as a functor* (3 extra points)
Consider a morphism of schemes $S \to T$. Every S-scheme X, i.e. every morphism $X \to S$, yields by composition with $S \to T$ a T-scheme $X \to S \to T$ which we shall denote ${}_TX$. Conversely, to every T-scheme $Y \to T$ base change defines an S-scheme $Y_S := S \times_T Y \to S$.

(i) Show that this defines two functors

$${}_T(\ )\colon (Sch/S) \to (Sch/T),\ X \mapsto {}_TX \text{ and } (\ )_S\colon (Sch/T) \to (Sch/S),\ Y \mapsto Y_S$$

which are adjoint to each other. More precisely, ${}_T(\ )$ is left adjoint to $(\ )_S$, i.e. ${}_T(\ ) \dashv (\ )_S$.

(ii) For any S-scheme X consider the functor

$$\begin{aligned}\mathbf{Res}_{S/T}(X)\colon \quad (Sch/T)^{\mathrm{op}} &\to (Sets)\\ Y &\mapsto \mathrm{Mor}_S(Y_S, X).\end{aligned}$$

If this functor is representable by a T-scheme, which will be denoted by $\mathrm{Res}_{S/T}(X)$, it is called the *Weil restriction* and satisfies $h_{\mathrm{Res}_{S/T}(X)} \cong \mathbf{Res}_{S/T}(X)$.

One can show that the Weil restriction exists for finite field extensions $S = \mathrm{Spec}(K) \to T = \mathrm{Spec}(k)$ and $X = \mathrm{Spec}(A)$, where A is a finite type K-algebra. In this case, one writes $\mathrm{Res}_{K/k}(X)$ for the Weil restriction.

Set $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{A}^1_{\mathbb{C}} \setminus 0)$. Show that the rational points of $\mathbb{S}$ can be described as

$$\mathbb{S}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{R},\ a^2 + b^2 \neq 0 \right\}.$$

(Remark: Usually, i.e. for general $S \to T$ and X, the Weil restriction does not exist.)

Information from the Fachschaft:

This year's Math Christmas party will take place at Thursday, the 17.12. starting 18 ct. online via zoom. All current information can be found on https://fsmath.uni-bonn.de/events-detail/events/virtual-christmas-party.html. Swing by!