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Winter term 2020/21

Exercises, Algebraic Geometry I – Week 5

Exercise 28. *Reduced schemes and reduction of schemes* (4 points)
Let X be a scheme. The *reduction* of X is a reduced scheme X_{red} together with a morphism $\iota : X_{\text{red}} \to X$ such that every morphism $Z \to X$ from a reduced scheme Z factors uniquely through ι.

(i) Show that X is reduced if and only if all local rings $\mathcal{O}_{X,x}$ are reduced.

(ii) Show that if $X = \text{Spec}(A)$ is affine, then $\text{Spec}(A/\mathfrak{N}(A))$, where $\mathfrak{N}(A)$ is the nilradical of A, together with the morphism ι induced by the quotient map $A \to A/\mathfrak{N}(A)$ is the reduction of X.

(iii) Show that every scheme admits a (necessarily unique) reduction. Show that ι is a homeomorphism of topological spaces.

Exercise 29. *Distinguished open sets* (3 points)
Recall that the open sets $D(a) \subset \text{Spec}(A)$ of all prime ideals not containing $a \in A$ form a basis of the topology. Define similar sets $X_a \subset X$ for any scheme X and any $a \in \Gamma(X, \mathcal{O}_X)$. More precisely, let $X_a \subset X$ be the set of points $x \in X$ such that the stalk $a_x \in \mathcal{O}_{X,x}$ is not contained in the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ (or, equivalently, that the image of a in the residue field $k(x)$ is non-trivial).

(i) Prove that X_a is an open subset.

(ii) Give an example of a scheme X such that the X_a do not form a basis of the topology.

Exercise 30. *Fibres* (3 points)
Consider the subscheme $X \subset \mathbb{A}^2_{\mathbb{Z}} := \text{Spec}(\mathbb{Z}[X,Y])$ given by $XY^2 - m$, for some $m \in \mathbb{Z}$. Study the fibres of $X \to \text{Spec}(\mathbb{Z})$. Which ones are irreducible?

Exercise 31. *Integral and irreducible fibres* (4 points)
Find examples for the following phenomena:

(i) Show that there exist morphisms $X \to Y$ with Y integral and such that all fibres X_y are irreducible without X being irreducible.

(ii) Show that there exist fields and morphisms $X \to \text{Spec}(k[x])$ with X integral, the generic fibre X_η non-empty and integral but no closed fibre integral.

(iii) Show that there exist morphisms $X \to \text{Spec}(\mathbb{Q}[x])$ with X integral and infinitely many irreducible and infinitely many reducible closed fibres. What happens for the geometric closed fibres in your example?

Please turn over

Due Friday 4 December 2020.

Exercise 32. *Points under base change* (3 points)
Consider the natural morphism $\mathbb{A}^2_{\overline{\mathbb{Q}}} \to \mathbb{A}^2_{\mathbb{Q}}$ and determine the images of the following points:

(i) $(x - \sqrt{2}, y - \sqrt{2})$,

(ii) $(x - \sqrt{2}, y - \sqrt{3})$,

(iii) $(\sqrt{2}x - \sqrt{3}y)$.

Exercise 33. *Morphisms into separated schemes* (5 points)
Consider schemes X and Y over a base scheme S, i.e. together with morphisms $\pi_X \colon X \to S$ and $\pi_Y \colon Y \to S$. Assume that X is reduced and that $Y \to S$ is separated. Show that two morphisms

$$f_1, f_2 \colon X \to Y$$

over S (i.e. such that $\pi_X = \pi_Y \circ f_i$) that coincide on a dense open subset $U \subset X$ are actually equal. (*Hint*: Consider the composition of the graph $X \to X \times_S Y$ of f_2 with $(f_1, \mathrm{id}) \colon X \times_S Y \to Y \times_S Y$.) Give counterexamples if one of the hypotheses is dropped.

The last exercise is not strictly necessary for the understanding of the lectures at this point.

Exercise 34. *Functors of points* (4 extra points)
We consider the category (Sch/S) of all S-schemes over a fixed scheme S. Thus, the objects of (Sch/S) consists of a scheme X together with a morphism $\pi_X \colon X \to S$. Morphisms $(X, \pi_X) \to (Y, \pi_Y)$ in (Sch/S) are morphisms of schemes $f \colon X \to Y$ with $\pi_Y \circ f = \pi_X$.

Consider the functor of points $h_X \colon (Sch/S)^{\mathrm{op}} \to (Sets)$ that for a fixed S-scheme $X \to S$ maps any S-scheme $Y \to S$ to the set of morphisms $\mathrm{Mor}_S(Y, X)$ of S-schemes $Y \to X$.

(i) Define the notion of fibre product functor $h_X \times h_Y \colon (Sch/S)^{\mathrm{op}} \to (Sets)$ and show that this functor is isomorphic to $h_{X \times_S Y}$, i.e.

$$h_{X \times_S Y} \cong h_X \times h_Y.$$

Compare this to the fact that the underlying topological space $|X \times_S Y|$ of $X \times_S Y$ is usually different from $|X| \times_{|S|} |Y|$ (example!).

(ii) For $S = \mathrm{Spec}(k)$ and a field extension K/k, show that $h_X(\mathrm{Spec}(K)) = X(K)$, see Exercise 17.

(iii) Find examples of morphisms $X \to Y$ (of S-schemes) which are not determined by their underlying continuous maps. Observe, however, that $X \to Y$ is determined by $h_X \to h_Y$.