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Winter term 2020/21

Exercises, Algebraic Geometry I – Week 4

Exercise 21. *Cohomology under closed embeddings* (4 points)
Let X be a topological space and $i\colon Z \hookrightarrow X$ the inclusion of a closed subset. Prove the following assertions.

(i) The direct image functor $i_*\colon \mathrm{Sh}(Z) \to \mathrm{Sh}(X)$ is exact.

(ii) The direct image $i_*\mathcal{I}$ of a flasque sheaf is flasque again.

(iii) For any sheaf $\mathcal{F} \in \mathrm{Sh}(Z)$, there exists an isomorphism $H^i(Z, \mathcal{F}) \cong H^i(X, i_*\mathcal{F})$ for all i.

Which of these assertions fails if instead $i\colon U \hookrightarrow X$ is the inclusion of an open subset?

Exercise 22. *Cohomology with support* (4 points)
Let X be a topological space and $i\colon Z \hookrightarrow X$ the inclusion of a closed subset. For $\mathcal{F} \in \mathrm{Sh}(X)$, recall the definition of the subgroup $\Gamma_Z(X, \mathcal{F}) \subset \Gamma(X, \mathcal{F})$ of sections of $\mathcal{F}$ with support in Z.

(i) Show that $\Gamma_Z(X, -) : \mathrm{Sh}(X) \to (Ab)$ is a left-exact functor. Its right-derived functors are denoted by $H^i_Z(X, -)$ and are called *cohomology groups of X with support in Z*, and coefficients in a given sheaf.

(ii) Show that $H^i_Z(X, \mathcal{F}) = 0$ for all $i > 0$ and all flasque sheaves $\mathcal{F}$.

(iii) Let $U = X \setminus Z$. Show that for any $\mathcal{F} \in \mathrm{Sh}(X)$, there is a long exact sequence of cohomology groups

$$0 \to H^0_Z(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}|_U) \to H^1_Z(X, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \dots$$

Exercise 23. *Normalization* (4 points)
A scheme X is *normal* if all its local rings $\mathcal{O}_{X,x}$ are integrally closed domains. The *normalization* of an integral scheme X is an irreducible normal scheme $\tilde{X}$ together with a dominant morphism $\nu : \tilde{X} \to X$ such that every dominant morphism $Z \to X$ from an irreducible normal scheme Z factors uniquely through ν.

(i) Let $X = \mathrm{Spec}(A)$ for an integral domain A, let $\tilde{X} = \mathrm{Spec}(\tilde{A})$, where $\tilde{A}$ is the integral closure of A in its field of fractions $K(X)$, and let $\nu : \tilde{X} \to X$ be the morphism induced by the inclusion $A \subset \tilde{A}$. Show that $\tilde{X}$ together with ν is the normalization of X.

(ii) Show that every integral scheme admits a (necessarily unique) normalization.

Please turn over

Due Friday 27 November 2020.

Exercise 24. *Direct image* (3 points)
Consider the map $f\colon S^1 \to S^1$, $z \mapsto z^2$. Show that $R^1 f_* \underline{\mathbb{Z}} = 0$ and prove $(f_*\underline{\mathbb{Z}})_x \cong \mathbb{Z} \oplus \mathbb{Z}$ for all $x \in S^1$. Is $f_*\underline{\mathbb{Z}} \cong \underline{\mathbb{Z}}^{\oplus 2}$?

Exercise 25. *Morphisms to affine schemes* (3 points)
Let $(f, f^\sharp)\colon X \to \operatorname{Spec}(A)$ be a morphism of schemes. Taking global sections of $f^\sharp\colon \mathcal{O}_{\operatorname{Spec}(A)} \to f_*\mathcal{O}_X$ yields a homomorphism of rings $A \to \Gamma(X, \mathcal{O}_X)$. Show that this defines a bijection

$$\operatorname{Mor}_{(Sch)}(X, \operatorname{Spec}(A)) \cong \operatorname{Mor}_{(Rings)}(A, \Gamma(X, \mathcal{O}_X)).$$

Exercise 26. *Schemes are T_0-spaces* (3 points)
Let X be a scheme. Prove the following assertions.

(i) If X is irreducible and consists of at least two points, then X is not Hausdorff.

(ii) Show that X is a T_0-space, i.e. for any two distinct points $x, y \in X$ there exists an open subset $U \subset X$ containing exactly one of the two points.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 27. *Injective resolutions of groups and modules* (5 extra points)
Let A be a ring and let I be an A-module.

(i) Show that I is injective if for any ideal $\mathfrak{a} \subset A$ the induced map

$$\operatorname{Hom}_A(A, I) \to \operatorname{Hom}_A(\mathfrak{a}, I)$$

is surjective.

(ii) Show that any divisible group G (i.e. $g \mapsto ng$ is surjective for all $n > 0$) is an injective object in (Ab). In particular, $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective.

(iii) Show that $I(G) = \prod_{J(G)} \mathbb{Q}/\mathbb{Z}$ is a divisible (and hence injective) group. Here, the index set $J(G)$ is the set $\operatorname{Hom}_{(Ab)}(G, \mathbb{Q}/\mathbb{Z})$.

(iv) Show that the natural map $G \to I(G)$, $g \mapsto (f(g))_f$ is injective. (For $g \in G$ pick a non-trivial homomorphism $\langle g \rangle \to \mathbb{Q}/\mathbb{Z}$ and use the injectivity of $\mathbb{Q}/\mathbb{Z}$ to extend it to a homomorphism $G \to \mathbb{Q}/\mathbb{Z}$.) Conclude that the category (Ab) of abelian groups has enough injectives.

(v) Check that the same argument shows that the category of A-modules has enough injectives for any ring.