Exercises, Algebraic Geometry I – Week 3

Exercise 14. Flasque resolutions (4 points)
Let $X$ be a topological space and let $\mathcal{F}$ be a sheaf on $X$. Consider the presheaf
$$\mathcal{F}_0 : U \mapsto \{ s : U \to \biguplus_{x \in X} \mathcal{F}_x \mid s(x) \in \mathcal{F}_x \}.$$  
Show that $\mathcal{F}_0$ is a flasque sheaf and deduce from this that every sheaf $\mathcal{F}$ admits a flasque resolution $0 \to \mathcal{F} \to \mathcal{F}_0 \to \mathcal{F}_1 \to \ldots$.

Exercise 15. Universality of $\delta$-functors (5 points)
Let $(H^i) : A \to B$ be a $\delta$-functor. Show that if $H^1$ is erasable and $(\tilde{H}^i) : A \to B$ is another $\delta$-functor with $H^0 = \tilde{H}^0$, then for any object $A \in \text{Ob}(A)$ there exists a unique map $H^1(A) \to \tilde{H}^1(A)$ which is compatible with $\delta$. (This is the key step towards the proof of Grothendieck’s theorem that $\delta$-functors with erasable $H^i$, $i > 0$, are universal.)

Exercise 16. Direct image under point inclusion (3 points)
Let $x \in X$ be an arbitrary point of a topological space $X$. Is the direct image $(i_x)_* : \text{Sh}(\{x\}) \to \text{Sh}(X)$ associated with the inclusion $i_x : \{x\} \to X$ exact? Give a proof or a counterexample.

Exercise 17. Rational points (3 points)
Let $(X, \mathcal{O}_X)$ be an affine scheme and let $x \in X$ with residue field $k(x) := \mathcal{O}_X(x)/\mathfrak{m}_x$.

(i) Show that, for a field $K$, to give a morphism of affine schemes $(\text{Spec}(K), \mathcal{O}_{\text{Spec}(K)}) \to (X, \mathcal{O}_X)$ with image $x$ is equivalent to giving a field inclusion $k(x) \hookrightarrow K$.

(ii) If $(X, \mathcal{O}_X)$ is an affine $k$-scheme for some field $k$, i.e. a morphism of schemes $(X, \mathcal{O}_X) \to (\text{Spec}(k), \mathcal{O}_{\text{Spec}(k)})$ is fixed, show that every residue field $k(x)$ is naturally a field extension $k \subset k(x)$. A point $x \in X$ is rational if this extension is bijective, i.e. $k = k(x)$. The set of rational points is denoted by $X(k)$. Show that $X(k)$ can be described as the set of $k$-morphisms $\text{Spec}(k) \to (X, \mathcal{O}_X)$, i.e. morphisms such that the composition

$$(\text{Spec}(k), \mathcal{O}_{\text{Spec}(k)}) \to (X, \mathcal{O}_X) \to (\text{Spec}(k), \mathcal{O}_{\text{Spec}(k)})$$

is the identity of schemes.

Due Friday 20 November 2020.
Exercise 18. Zariski tangent space (4 points)
Let \((X, \mathcal{O}_X)\) be an affine scheme. For a point \(x \in X\) the quotient \(\mathfrak{m}_x/\mathfrak{m}_x^2\) is considered as vector space over the residue field \(k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x\). The Zariski tangent space \(T_x\) of \(X\) at \(x \in X\) is defined as the dual of this vector space, i.e.

\[
T_x := (\mathfrak{m}_x/\mathfrak{m}_x^2)^* = \text{Hom}_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x)).
\]

Assume \((X, \mathcal{O}_X)\) is an affine \(k\)-scheme (see previous exercise) and denote the ring of dual numbers \(k[t]/(t^2)\) by \(k[\varepsilon]\).

Show that giving a morphism \((\text{Spec}(k[\varepsilon], \mathcal{O}) \to (X, \mathcal{O}_X)\) that is compatible with the morphisms to \((\text{Spec}(k), \mathcal{O})\) is equivalent to giving a rational point \(x \in X\) (see previous exercise) and an element \(v \in T_x\).

Exercise 19. Extension by zero (4 points)
Consider a topological space \(X\) with a closed subset \(i: Y \hookrightarrow X\) and its open complement \(U := X \setminus Y\). For a sheaf \(G\) on \(U\) we denote by \(j_!G\) the extension by zero, i.e. the sheaf associated to the presheaf defined by \((j_!G)(V) = G(V)\) if \(V \subset U\) and \(= 0\) otherwise. For a sheaf \(F\) on \(X\) we denote by \(F|_Y\) and \(F|_U\) its restriction (i.e. the inverse image under the inclusion) to the sets \(Y\) and \(U\), respectively.

(i) Show that for any sheaf \(F\) on \(X\) there exists a short exact sequence

\[
0 \to j_!(F|_U) \to F \to i_* (F|_Y) \to 0.
\]

(ii) Consider \(X = \mathbb{A}^1_k\) as an affine algebraic set (or as an affine scheme). Let \(x \neq y \in \mathbb{A}^1_k\) be (closed) points and let \(U := X \setminus \{x, y\}\). Show that \(H^1(X, \mathbb{Z}_U) \neq 0\), where \(\mathbb{Z}_U := j_!(\mathbb{Z})\).

Exercise 20. Cohomology of the circle (5 points + 1 extra point)
Let \(S^1\) be the circle with the usual topology and \(C\) the sheaf of continuous real-valued functions on \(S^1\).

(i) Show that \(\mathbb{Z} \subset H^1(S^1, \mathbb{Z})\).

(ii) (1 extra point) Show that \(\mathbb{Z} = H^1(S^1, \mathbb{Z})\).

(iii) Show that \(H^1(S^1, C) = 0\).