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Winter term 2020/21

Exercises, Algebraic Geometry I – Week 3

Exercise 14. *Flasque resolutions* (4 points)
Let X be a topological space and let $\mathcal{F}$ be a sheaf on X. Consider the presheaf

$$\mathcal{F}_0 \colon U \mapsto \{s \colon U \to \coprod_{x \in X} \mathcal{F}_x \mid s(x) \in \mathcal{F}_x\}.$$

Show that $\mathcal{F}_0$ is a flasque sheaf and deduce from this that every sheaf $\mathcal{F}$ admits a flasque resolution $0 \to \mathcal{F} \to \mathcal{F}_0 \to \mathcal{F}_1 \to \ldots$.

Exercise 15. *Universality of δ-functors* (5 points)
Let $(H^i) \colon \mathcal{A} \to \mathcal{B}$ be a δ-functor. Show that if H^1 is erasable and $(\tilde{H}^i) \colon \mathcal{A} \to \mathcal{B}$ is another δ-functor with $H^0 = \tilde{H}^0$, then for any object $A \in \mathrm{Ob}(\mathcal{A})$ there exists a unique map $H^1(A) \to \tilde{H}^1(A)$ which is compatible with δ. (This is the key step towards the proof of Grothendieck's theorem that δ-functors with erasable H^i, $i > 0$, are universal.)

Exercise 16. *Direct image under point inclusion* (3 points)
Let $x \in X$ be an arbitrary point of a topological space X. Is the direct image $(i_x)_* \colon \mathrm{Sh}(\{x\}) \to \mathrm{Sh}(X)$ associated with the inclusion $i_x \colon \{x\} \hookrightarrow X$ exact? Give a proof or a counterexample.

Exercise 17. *Rational points* (3 points)
Let $(X, \mathcal{O}_X)$ be an affine scheme and let $x \in X$ with residue field $k(x) \coloneqq \mathcal{O}_{X,x}/\mathfrak{m}_x$.

(i) Show that, for a field K, to give a morphism of affine schemes $(\mathrm{Spec}(K), \mathcal{O}_{\mathrm{Spec}(K)}) \to (X, \mathcal{O}_X)$ with image x is equivalent to giving a field inclusion $k(x) \hookrightarrow K$.

(ii) If $(X, \mathcal{O}_X)$ is an affine k-scheme for some field k, i.e. a morphism of schemes

$$(X, \mathcal{O}_X) \to (\mathrm{Spec}(k), \mathcal{O}_{\mathrm{Spec}(k)})$$

is fixed, show that every residue field $k(x)$ is naturally a field extension $k \subset k(x)$. A point $x \in X$ is *rational* if this extension is bijective, i.e. $k = k(x)$. The *set of rational points* is denoted by $X(k)$. Show that $X(k)$ can be described as the set of k-morphisms $\mathrm{Spec}(k) \to (X, \mathcal{O}_X)$, i.e. morphisms such that the composition

$$(\mathrm{Spec}(k), \mathcal{O}_{\mathrm{Spec}(k)}) \to (X, \mathcal{O}_X) \to (\mathrm{Spec}(k), \mathcal{O}_{\mathrm{Spec}(k)})$$

is the identity of schemes.

Due Friday 20 November 2020.

Exercise 18. *Zariski tangent space* (4 points)
Let $(X, \mathcal{O}_X)$ be an affine scheme. For a point $x \in X$ the quotient $\mathfrak{m}_x/\mathfrak{m}_x^2$ is considered as vector space over the residue field $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$. The *Zariski tangent space* T_x of X at $x \in X$ is defined as the dual of this vector space, i.e.

$$T_x := (\mathfrak{m}_x/\mathfrak{m}_x^2)^* = \mathrm{Hom}_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x)).$$

Assume $(X, \mathcal{O}_X)$ is an affine k-scheme (see previous exercise) and denote the *ring of dual numbers* $k[t]/(t^2)$ by $k[\varepsilon]$.

Show that giving a morphism $(\mathrm{Spec}(k[\varepsilon], \mathcal{O}) \to (X, \mathcal{O}_X)$ that is compatible with the morphisms to $(\mathrm{Spec}(k), \mathcal{O})$ is equivalent to giving a rational point $x \in X$ (see previous exercise) and an element $v \in T_x$.

Exercise 19. *Extension by zero* (4 points)
Consider a topological space X with a closed subset $i\colon Y \hookrightarrow X$ and its open complement $U := X \backslash Y$. For a sheaf $\mathcal{G}$ on U we denote by $j_!\mathcal{G}$ the *extension by zero*, i.e. the sheaf associated to the presheaf defined by $(j_!\mathcal{G})(V) = \mathcal{G}(V)$ if $V \subset U$ and $= 0$ otherwise. For a sheaf $\mathcal{F}$ on X we denote by $\mathcal{F}|_Y$ and $\mathcal{F}|_U$ its restriction (i.e. the inverse image under the inclusion) to the sets Y and U, respectively.

(i) Show that for any sheaf $\mathcal{F}$ on X there exists a short exact sequence

$$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Y) \to 0.$$

(ii) Consider $X = \mathbb{A}_k^1$ as an affine algebraic set (or as an affine scheme). Let $x \neq y \in \mathbb{A}_k^1$ be (closed) points and let $U := X \setminus \{x, y\}$. Show that $H^1(X, \mathbb{Z}_U) \neq 0$, where $\mathbb{Z}_U := j_!(\underline{\mathbb{Z}})$.

Exercise 20. *Cohomology of the circle* (5 points + 1 extra point)
Let S^1 be the circle with the usual topology and $\mathcal{C}$ the sheaf of continuous real-valued functions on S^1.

(i) Show that $\mathbb{Z} \subset H^1(S^1, \underline{\mathbb{Z}})$.

(ii) (1 extra point) Show that $\mathbb{Z} = H^1(S^1, \underline{\mathbb{Z}})$.

(iii) Show that $H^1(S^1, \mathcal{C}) = 0$.