Prof. Dr. Daniel Huybrechts
Dr. Gebhard Martin

Winter term 2020/21

Exercises, Algebraic Geometry I – Week 2

Exercise 8. *Gluing of sheaves* (4 points)
Let X be a topological space and let $X = \bigcup U_i$ be an open covering. We use the shorthand $U_{ij} \coloneqq U_i \cap U_j$ and $U_{ijk} \coloneqq U_i \cap U_j \cap U_k$.
Consider sheaves $\mathcal{F}_i$ on U_i and isomorphisms (*gluings*) $\varphi_{ij} \colon \mathcal{F}_i|_{U_{ij}} \xrightarrow{\sim} \mathcal{F}_j|_{U_{ij}}$. Show that if the *cocycle condition* $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on U_{ijk} is satisified, then there exists a sheaf $\mathcal{F}$ on X together with isomorphisms $\varphi_i \colon \mathcal{F}|_{U_i} \cong \mathcal{F}_i$ such that $\varphi_{ij} \circ \varphi_i = \varphi_j$ on U_{ij}. The sheaf $(\mathcal{F}, \varphi_i)$ is unique up to unique isomorphism.

Exercise 9. *Direct and inverse image are adjoint* (5 points)
Let $f \colon X \to Y$ be a continuous map. Show that $f_* \colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is right adjoint to $f^{-1} \colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ (one writes $f^{-1} \dashv f_*$), i.e. for all $\mathcal{F} \in \mathrm{Sh}(X)$ and $\mathcal{G} \in \mathrm{Sh}(Y)$, there exists an isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathcal{G}, f_*\mathcal{F})$$

which is functorial in $\mathcal{F}$ and $\mathcal{G}$. Show that, in particular, there exist natural homomorphisms

$$\mathcal{G} \to f_* f^{-1}\mathcal{G} \text{ and } f^{-1} f_* \mathcal{F} \to \mathcal{F}.$$

Verify also that for the composition of two continuous maps $f \colon X \to Y$ and $g \colon Y \to Z$ one has $(g \circ f)_* = g_* \circ f_*$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Exercise 10. *Local rings of continuous functions* (4 points)
Let X be a topological space and let $\mathcal{C}$ be the sheaf of continuous functions on X. Consider for a point $x \in X$ the stalk $\mathcal{C}_x$. Show that the map $\mathcal{C}_x \to \mathbb{R}$, $f \mapsto f(x)$ is well defined and that $\mathcal{C}_x$ is a local ring with maximal ideal $\mathfrak{m}_x \coloneqq \{f \in \mathcal{C}_x \mid f(x) = 0\}$. Describe similar situations involving differentiable or holomorphic functions.

Exercise 11. *Direct sum of sheaves* (3 points)
Let $\mathcal{F}$, $\mathcal{G}$ be two sheaves of abelian groups on a topological space X. Show that $\mathcal{F} \oplus \mathcal{G} \colon U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$ defines a sheaf.

Exercise 12. *Functor of sections is left-exact* (4 points)
Let $U \subset X$ be an open set. Prove that $\Gamma(U, \ ) \colon \mathrm{Sh}(X) \to (Ab)$ is a left exact functor, i.e. for any short exact sequence of sheaves $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ the sequence $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is exact. (*Warning:* But usually $\mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is not surjective, i.e. $\Gamma(U, \ )$ is not exact.)

Please turn over

Due Friday 13 November 2020.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 13. *Functor of points and the Yoneda lemma* (4 extra points)
Let $\mathcal{C}$ be a category with sets of morphisms between two objects X, Y denoted $\mathrm{Mor}(X, Y)$. Then every object X in $\mathcal{C}$ induces a functor

$$h_X : \mathcal{C}^{\mathrm{op}} \to (Sets),\ Y \mapsto h_X(Y) := \mathrm{Mor}(Y, X).$$

Observe that $h_X(X)$ contains a distinguished element.

(i) Consider the three categories $\mathcal{C} := (Top)$ (of topological spaces); $\mathcal{C} := (Ab)$ (of abelian groups); $\mathcal{C} := (Rings)$ (of rings) and denote for each object X in $\mathcal{C}$ by $|X|$ the underlying set (the set of points). Show that in all three cases there exists an object Z in $\mathcal{C}$ such that for all X the set of points $|X|$ can be recovered as $|X| = h_X(Z)$.

(ii) Consider the category of affine schemes $\mathcal{C} := (AffSch)$. Does there exist an object as in (i) in this case?

(iii) For an arbitrary category $\mathcal{C}$, denote by $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, (Sets))$ the category of functors $\mathcal{C}^{\mathrm{op}} \to (Sets)$ and consider the functor

$$\begin{aligned} h\colon \quad \mathcal{C} &\to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, (Sets)) \\ X &\mapsto h_X. \end{aligned}$$

The Yoneda lemma then asserts that h is a fully faithful embedding, in other words h defines an equivalence of categories between $\mathcal{C}$ and a full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, (Sets))$. Spell out what this means and try to prove it. Check Vakil's notes on the subject (or any other source). Objects in the image of h (or, more precisely, objects isomorphic to objects in the image) are called *representable functors*.