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Winter term 2020/21

Exercises, Algebraic Geometry I – Week 13

Exercise 78. *Line bundles over the projective space* (5 points)
Consider the projective space $\mathbb{P}^n_k = \mathrm{Proj}(k[x_0, \dots, x_n])$ over a field k (or, alternatively, over any ring) and the line bundle $\mathbb{V}(\mathcal{O}(1))$ associated with the invertible sheaf $\mathcal{O}(1)$. Let $\mathbb{V}(\mathcal{O}(1))_0 \subseteq \mathbb{V}(\mathcal{O}(1))$ be the *zero section*, i.e., the image of the section of $\mathbb{V}(\mathcal{O}(1)) \to \mathbb{P}^n_k$ corresponding to $0 \in H^0(\mathbb{P}^n_k, \mathcal{O}(-1))$.

(i) Show that there exists an isomorphism $\mathbb{V}(\mathcal{O}(1)) \setminus \mathbb{V}(\mathcal{O}(1))_0 \cong \mathbb{A}^{n+1}_k \setminus \{(0)\}$.

(ii) Describe the resulting projection $\mathrm{Spec}(k[x_0, \dots, x_n]) \setminus \{0\} \to \mathrm{Proj}(k[x_0, \dots, x_n])$ in terms of prime ideals.

(iii) Show that every quasi-projective scheme over k admits a line bundle for which the complement of the zero section is quasi-affine (i.e. an open subset of an affine k-scheme).

(In the next term we will prove the Jouanolou trick that says that there exists an affine scheme $Y \to \mathbb{P}^n$ which Zariski locally is of the form $U \times \mathbb{A}^n \to U$ (without being a vector bundle)).

Exercise 79. *Projective bundles* (4 points)
Consider a locally free sheaf $\mathcal{E}$ of rank $r+1$ on a scheme X and its symmetric algebra $\mathcal{S} := S^*(\mathcal{E}) = \bigoplus S^d(\mathcal{E})$. Show that for the projection $\pi\colon \mathbb{P}(\mathcal{E}) \to X$ there exist isomorphisms

$$\mathcal{E} \cong \pi_*\mathcal{O}_\pi(1) \text{ and } S^d(\mathcal{E}) \cong \pi_*\mathcal{O}_\pi(d).$$

Exercise 80. *Splitting principle* (3 points)
Under the assumptions of the previous exercise, show that there exists a short exact sequence

$$0 \longrightarrow \mathcal{O}_\pi(-1) \longrightarrow \pi^*\mathcal{E}^* \longrightarrow \mathcal{F}_1^* \longrightarrow 0$$

with $\mathcal{F}_1$ locally free. Use this to prove the existence of morphisms

$$g\colon P_r \longrightarrow P_{r-1} \longrightarrow \cdots \longrightarrow P_0 = X\ ,$$

for which each $P_i \to P_{i-1}$ is a $\mathbb{P}^i$-bundle and the pull-back of $\mathcal{E}$ to P_r admits a filtration $0 = \mathcal{F}_{r+1} \subset \mathcal{F}_r \subset \cdots \subset \mathcal{F}_0 = g^*\mathcal{E}$ for which every quotient $\mathcal{F}_i/\mathcal{F}_{i+1}$ is an invertible sheaf.

Exercise 81. *A blow-up does not determine its center* (5 points)
Let X be a noetherian scheme and let $\mathcal{I}$ be a coherent sheaf of ideals on X. Prove the following assertions:

(i) For every $d > 0$, there is an isomorphism $\mathrm{Bl}_{V(\mathcal{I})}(X) \cong \mathrm{Bl}_{V(\mathcal{I}^d)}(X)$ over X.

(ii) If $\mathcal{J}$ is an invertible coherent sheaf of ideals on X, then there is an isomorphism $\mathrm{Bl}_{V(\mathcal{I})}(X) \cong \mathrm{Bl}_{V(\mathcal{I}\cdot\mathcal{J})}(X)$ over X.

(iii) Give an example of X and $\mathcal{I}$ such that $\mathrm{Bl}_{V(\mathcal{I})}(X) \not\cong \mathrm{Bl}_{V(\sqrt{\mathcal{I}})}(X)$, where $\sqrt{\mathcal{I}}$ is the radical of $\mathcal{I}$.