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Winter term 2020/21

Exercises, Algebraic Geometry I – Week 12

Exercise 72. *Gonality of a curve* (3 points)
Let X be an integral, projective, normal curve over an algebraically closed field k. Show that there exists a finite morphism $f\colon X \to \mathbb{P}^1_k$ of degree $\leq g(X)+1$. In other words, the gonality of a curve X is bounded by $\mathrm{gon}(X) \leq g(X)+1$. Here, $g(X)$ is the genus of X, i.e. $g(X) \coloneqq \dim_k H^1(X, \mathcal{O}_X) = p_a(X)$, see Exercise 57.

Exercise 73. *Rational functions with prescribed poles* (4 points)
Let X be an integral, projective, normal curve over an algebraically closed field k. Prove the following assertions.

(i) For every closed point $x \in X$ there exists a rational function $f \in K(X)$ with a pole at x (of some order $n > 0$) and no pole anywhere else.

(ii) For given closed points $x_1, \ldots, x_r \in X$ there exists a rational function $f \in K(X)$ with poles at all x_i (of some order $n_i > 0$) and no pole anywhere else.

Exercise 74. *Inflection points of plane cubics* (3 points)
Let $X \subset \mathbb{P}^2_k$, $k = \bar{k}$ be a normal cubic with an inflection point $x_0 \in X$ leading to a group structure on the set of closed points of X with x_0 as the origin. Show that any other inflection point $x \in X$ is of order three. Is the converse also true?

Exercise 75. *Euler–Poincaré characteristic of invertible sheaves* (4 points)
Let X be an integral, projective, normal curve over an algebraically closed field k. Let $\mathcal{L}$ be an invertible sheaf on X.

(i) Show that, for every closed point $x \in X$, one has $\chi(\mathcal{L}(x)) = 1 + \chi(\mathcal{L})$.

(ii) Deduce the Riemann–Roch theorem for invertible sheaves on curves:

$$\chi(\mathcal{L}) = \deg(\mathcal{L}) + 1 - g(X).$$

(In particular, $\chi(\mathcal{L})$ only depends on the degree of $\mathcal{L}$.)

Exercise 76. *Morphisms from projective spaces* (4 points)
Let $f\colon \mathbb{P}^n_k \to X$ be a morphism of projective k-schemes. Show that either the image of f consists of a single point or that f is quasi-finite (and in fact finite).

Due Friday 29 January 2021. This is the last sheet that counts towards the necessary 50% needed for the exam.

The last exercise is not strictly necessary for the understanding of the lectures at this point.

Exercise 77. *The algebraic first Cousin problem* (4 extra points)
Let $X = \mathbb{P}^1_k$, where k is an algebraically closed field. Let $\mathcal{K}_X$ be the constant sheaf associated to the function field $K(X)$ of X.

(i) For every closed point $x \in X$ set $I_x := K(X)/\mathcal{O}_{X,x}$, considered as a skyscraper sheaf on X with support $\{x\}$. Show that

$$\mathcal{K}_X/\mathcal{O}_X \cong \bigoplus_{x \in X \text{ closed}} I_x$$

(ii) Show that

$$0 \to \mathcal{O}_X \to \mathcal{K}_X \to \mathcal{K}_X/\mathcal{O}_X \to 0$$

is a flabby resolution of $\mathcal{O}_X$.

(iii) Use (ii) to show that $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$.