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Winter term 2020/21

Exercises, Algebraic Geometry I – Week 11

Exercise 66. *Ample invertible sheaves* (3 points)
Let X be a noetherian scheme.

(i) Show that if $\mathcal{L}$ and $\mathcal{M}$ are two invertible sheaves on X such that $\mathcal{L}$ is ample, then $\mathcal{L}^n \otimes \mathcal{M}$ is ample for $n \gg 0$. Conclude that any invertible sheaf $\mathcal{M}$ on X is isomorphic to some $\mathcal{L}_1 \otimes \mathcal{L}_2^*$ with $\mathcal{L}_1$ and $\mathcal{L}_2$ ample if there exists an ample invertible sheaf on X at all.

(ii) Is the tensor product $\mathcal{L}_1 \otimes \mathcal{L}_2$ of two ample (resp. very ample) invertible sheaves again ample (resp. very ample)?

Exercise 67. *Base locus* (4 points)
Let X be a projective integral scheme over an algebraically closed field k and let $\mathcal{L}$ be an invertible sheaf on X. A point $x \in X$ is a base point of a linear system $V \subset H^0(X, \mathcal{L})$ (or of $\mathbb{P}(V) \subset |\mathcal{L}| := \mathbb{P}(H^0(X, \mathcal{L}))$) if $s_x \in \mathfrak{m}_x \cdot \mathcal{L}$ for all $s \in V$. Thus, $\mathcal{L}$ is globally generated if and only if $|\mathcal{L}|$ has no base points.

(i) Prove that the base locus $\mathrm{Bs}(V) \subset X$, i.e. the set of all base points, is closed.

(ii) Assume that X is factorial. Show that for any $\mathcal{L}$ with $|\mathcal{L}| \neq \emptyset$ there exists an effective (Cartier) divisor D such that the base locus of the complete linear system given by $\mathcal{L}(-D) := \mathcal{L} \otimes \mathcal{O}_X(-D)$ is of codimension ≥ 2 and such that $\mathcal{L}(-D) \hookrightarrow \mathcal{L}$ induces an isomorphism $H^0(X, \mathcal{L}(-D)) \xrightarrow{\sim} H^0(X, \mathcal{L})$.

(iii) Determine the base locus of the *Hesse pencil*

$$V := \{f_{t_0,t_1} := t_0(x_0^3 + x_1^3 + x_2^3) + t_1 x_0 x_1 x_2\} \subset H^0(\mathbb{P}_k^2, \mathcal{O}(3)).$$

Exercise 68. *Ample invertible sheaves on the quadric surface* (5 points)
Consider the quadric $\mathbb{P}_k^1 \times \mathbb{P}_k^1 \cong Q = V_+(x_0x_3 - x_1x_2) \subset \mathbb{P}_k^3$ and use that $\mathrm{Pic}(Q) \cong \mathbb{Z} \oplus \mathbb{Z}$, i.e., every invertible sheaf on Q is isomorphic to a unique $\mathcal{O}(a, b) := p_1^*\mathcal{O}(a) \otimes p_2^*\mathcal{O}(b)$.

(i) Determine all ample invertible sheaves on Q. Are they all very ample?

(ii) Quite generally, if X and Y are two quasi-compact and separated schemes over k, $\mathcal{F}$ is a quasi-coherent sheaf on X, and $\mathcal{G}$ is a quasi-coherent sheaf on Y, then one can define the box product sheaf $\mathcal{F} \boxtimes \mathcal{G} := p_1^*\mathcal{F} \otimes p_2^*\mathcal{G}$ on $X \times Y$ and show that there is a *Künneth isomorphism*

$$H^n(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) \cong \bigoplus_{i+j=n} H^i(X, \mathcal{F}) \otimes_k H^j(Y, \mathcal{G}).$$

This isomorphism can be used to determine the cohomology groups of all invertible sheaves on Q. Calculate as many cohomology groups $H^n(Q, \mathcal{O}(a, b))$ as you can without using the above Künneth isomorphism.

Due Friday 29 January 2021.

Exercise 69. *Projection from a linear subspace* (3 points)
Describe the linear system $V \subset H^0(\mathbb{P}^n_k, \mathcal{O}(1))$ that yields the linear projection from the plane $\mathbb{P}^2_k \subset \mathbb{P}^n_k$ defined by $x_{i>2} = 0$ onto $\mathbb{P}^{n-3}_k \subset \mathbb{P}^n_k$ defined by $x_0 = x_1 = x_2 = 0$.

Exercise 70. *Base locus of powers* (3 points)
Let X be a projective scheme over an algebraically closed field k and let $\mathcal{L} \in \mathrm{Pic}(X)$. Show that the base locus $\mathrm{Bs}(\mathcal{L}) \coloneqq \mathrm{Bs}(H^0(X, \mathcal{L}))$ of $\mathcal{L}$ contains the base locus $\mathrm{Bs}(\mathcal{L}^n)$ of any power $\mathcal{L}^n$, where $n > 0$. Is it true that for $n > m$ one necessarily has $\mathrm{Bs}(\mathcal{L}^n) \subset \mathrm{Bs}(\mathcal{L}^m)$?

The last exercise is not strictly necessary for the understanding of the lectures at this point.

Exercise 71. *The Grothendieck group of a scheme* (5 extra points)
Let X be a noetherian scheme. The *Grothendieck group* $K_0(X)$ of X is defined as the quotient of the free abelian group generated by all coherent sheaves on X by the subgroup generated by the expressions $\mathcal{F} - \mathcal{F}' - \mathcal{F}''$, whenever there is an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of coherent sheaves on X.

(i) If X is integral and $\mathcal{F}$ is a coherent sheaf on X, we define the *rank* of $\mathcal{F}$ as $\mathrm{rank}(\mathcal{F}) := \dim_{\mathcal{O}_{X,\eta}}(\mathcal{F}_\eta)$, where η is the generic point of X. Show that $\mathrm{rank}(-)$ defines a surjective homomorphism from $K_0(X)$ to $\mathbb{Z}$.

(ii) Let $Y \subset X$ be a closed subscheme and let $\mathcal{F}$ be a coherent sheaf on X with support on Y. Show that $\mathcal{F}$ admits a finite filtration by coherent sheaves

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n = \mathcal{F}$$

such that each $\mathcal{F}_i/\mathcal{F}_{i-1}$ is the pushforward of a coherent sheaf on Y.

(iii) Let $\iota : Y \hookrightarrow X$ be a closed immersion. Show that there is an exact sequence

$$K_0(Y) \overset{\alpha}{\to} K_0(X) \overset{\beta}{\to} K_0(X - Y) \to 0,$$

where α is induced by ι_* and β is induced by $(-)|_{X-Y}$.
(Hints: First, note that $\beta \circ \alpha = 0$, so that β induces a homomorphism

$$\overline{\beta} : K_0(X)/\alpha(K_0(Y)) \to K_0(X - Y).$$

Then, use Exercise 59 and Part (ii) of the current exercise to construct an inverse to $\overline{\beta}$.)

(iv) Let k be a field. Calculate $K_0(\mathrm{Spec}\ k), K_0(\mathbb{A}^1_k)$, and $K_0(\mathbb{P}^1_k)$.