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Summer term 2021

Algebraic Geometry II – Retry Exam 09.09.2021

Begin: 09:00 **End:** 12:00

Technical details:

– Upload a scan of your solutions (max. 500kb per page) until 12:00 to the eCampus folder 'Exams/Exam submission'. Solutions uploaded any later than 12:00 will not be considered which results in you failing the exam.

– The exam will be posted at 9:00 on the usual website `https://www.math.uni-bonn.de/people/gmartin/AlgebraicGeometrySS2021.htmpl` and will also be available in the folder 'Exams' in eCampus (where this document can be found already now).

– Please sign the declaration of honor below and scan it together with your solutions. Without it your solutions will not be considered.

– If at all possible, upload your solution in one(!) pdf document with the declaration of honor as the first page. Name the file **firstname.name.pdf**. Please, write your name on each sheet.

– In case of technical problems (but only then), you can also send the scan to gmartin@math.uni-bonn.de (until 12:00 am).

– Allow at least 30-45min for scanning and uploading your solutions.

Guidelines:

– The exam consists of applying your knowledge of algebraic geometry to fill in the missing arguments in the text below. In particular, reading and understanding the arguments that are given is part of the task.

– It is recommended to work through the exercises in the given order.

– The exam will only lead to pass or fail. It will not be graded. A complete solution will be posted on eCampus, so that you can find out yourself how well you did in the exam.

– All arguments have to be justified. Apart from standard material from commutative algebra, you have to deduce everything by only using results explicitly stated or used, either in the lectures, on the exercise sheets or in the text below.

– You may use available resources to solve the exercise. You are allowed to consult the notes of the class, the exercise sheets, online books, etc. **You are not allowed** to contact any other person during the exam (by email, phone, social media, etc.) or to discuss your answers with anyone before successfully uploading them to eCampus.

I hereby swear that I completed the examination detailed above completely on my own and without any impermissible external assistance or through the use of non-permitted aids. I am aware that cheating during the execution of an examination (as detailed in §63 Para 5 of the Higher Education Act NRW) is a violation of the legal regulations for examinations and an administrative offense. The submission of false affirmation in lieu of an oath is a criminal offense.

Signed at: **Date:**

Name: **Student ID:** **Signature:**

1. Spreading

Exercise 1. Consider the following homogeneous polynomial $F = x_0^{10} + x_1^{10} + x_2^{10}$ in three variables.

(i) Define a scheme $\mathcal{X}$ together with a flat morphism $\pi\colon \mathcal{X} \longrightarrow \mathrm{Spec}(\mathbb{Z})$ such that the fibre over the generic point is $V_+(F) \subset \mathbb{P}^2_{\mathbb{Q}}$.

(ii) Find a minimal positive integer N such that the restriction of π to the open subset $\mathrm{Spec}(\mathbb{Z}[1/N]) \subset \mathrm{Spec}(\mathbb{Z})$ is a smooth morphism.

(iii) Are there fibres of π that are non-reduced or reducible?

Solution (i) Define $\mathcal{X} := V_+(F) \subseteq \mathbb{P}^2_{\mathbb{Z}}$ and $\pi : \mathcal{X} \longrightarrow \mathbb{P}^2_{\mathbb{Z}} \longrightarrow \mathrm{Spec}(\mathbb{Z})$. Clearly, the generic fibre of π is $V_+(F) \subseteq \mathbb{P}^2_{\mathbb{Q}}$. Since the target of π is reduced and the Hilbert polynomial of its fibers is constant, π is flat.

(ii) The partial derivates of F are $10x_i^9$. Hence, by the Jacobian criterion for smoothness, the fiber $\mathcal{X}_k$ of π over a field k of characteristic p is smooth if and only if $p \neq 2, 5$. Thus, $N = 10$.

(iii) If $p \notin \{2, 5\}$, then $\mathcal{X}_k$ is smooth and connected (being a hypersurface in $\mathbb{P}^2_k$), hence irreducible and reduced. If $p \in \{2, 5\}$, then $F = (x_0^{10/p} + x_1^{10/p} + x_2^{10/p})^p$, hence $\mathcal{X}_k$ is non-reduced. In fact, F is the p-th power of a polynomial defining a smooth curve (by the same argument as in (i)), hence $\mathcal{X}_k$ is irreducible. Summarizing, there are non-reduced fibers of π, but all fibers are irreducible.

2. Curves on surfaces

Exercise 2. Let C be a smooth projective irreducible curve over an algebraically closed field k and let $S = C \times C$ with the two projections $p_i\colon S \longrightarrow C$, $i = 1, 2$.

(i) Show that $p_1^* \oplus p_2^*\colon \mathrm{Pic}(C) \oplus \mathrm{Pic}(C) \longrightarrow \mathrm{Pic}(S)$ is injective.

(ii) Fix a closed point $x \in C$ and consider the invertible sheaves $\mathcal{O}_S(a, b) := p_1^*\mathcal{O}_C(ax) \otimes p_2^*\mathcal{O}_C(bx)$. Decide for which a and b the sheaf $\mathcal{O}_S(a, b)$ is ample.

(iii) Consider the diagonal $C \cong \Delta \subset S$ and determine the self-intersection number $(\Delta.\Delta)$. Is $\mathcal{O}_S(\Delta)$ contained in the image of $p_1^* \oplus p_2^*$?

(iv) Compute the Hilbert polynomial of the curve Δ with respect to an ample invertible sheaf of the form $\mathcal{O}_S(a, b)|_\Delta$. Is it equal to the Hilbert polynomial of the sheaf $\mathcal{O}_S(\Delta)$ with respect to the ample invertible sheaf $\mathcal{O}_S(a, b)$?

Solution Fix a closed point $x \in C$, let $\iota_1 : C \times \{x\} \longrightarrow S$ and $\iota_2 : \{x\} \times C \longrightarrow S$ be the two embeddings.

(i) Let $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}(C)$. Note that $p_i \circ \iota_i = \mathrm{id}$ while $p_1 \circ \iota_2$ and $p_2 \circ \iota_1$ factor through $\{x\} \longrightarrow C$. Hence,

$$\iota_i^*(p_1^*\mathcal{L}_1 \otimes p_2^*\mathcal{L}_2) = \mathcal{L}_i. \tag{1}$$

Thus, if $p_1^*\mathcal{L}_1 \otimes p_2^*\mathcal{L}_2 = \mathcal{O}_S$, then $\mathcal{L}_i = \iota_i^*\mathcal{O}_S = \mathcal{O}_C$, so $p_1^* \oplus p_2^*$ is injective.

(ii) The restriction of an ample sheaf to a closed subscheme is still ample and an invertible sheaf on a curve is ample if and only if it has positive degree. Applying this to fibers of the two projections yields that if $\mathcal{O}_S(a, b)$ is ample, then $a, b > 0$. Conversely, if $a, b > 0$, choose $n \gg 0$ such that $\mathcal{O}_C(anx)$ and $\mathcal{O}_C(bnx)$ are very ample and let $f_i\colon C \longrightarrow \mathbb{P}^{N_i}$ be the two induced closed embeddings. Then, composing $f_1 \times f_2$ with the Segre embedding yields a closed embedding $f\colon S \longrightarrow \mathbb{P}^N$ such that $f^*\mathcal{O}(1) = \mathcal{O}_S(an, bn)$, hence $\mathcal{O}_S(a, b)$ is ample.

(iii) Recall that $\mathcal{O}_\Delta(-\Delta) \cong \omega_C$, hence we have $(\Delta, \Delta) = \deg \mathcal{O}_\Delta(\Delta) = \deg \omega_C^* = 2 - 2g$, where g is the genus of C. Assume $\mathcal{O}_S(\Delta) = p_1^*\mathcal{L}_1 \otimes p_2^*\mathcal{L}_2$. Let $y \neq x \in C$ be another closed point. The restriction of $\mathcal{O}_S(\Delta)$ to $C \times \{x\}$ is $\mathcal{O}_C(x)$ while the restriction of $\mathcal{O}_S(\Delta)$ to $C \times \{y\}$ is $\mathcal{O}_C(y)$. So, by (1), we have $\mathcal{O}_C(x) = \mathcal{O}_C(y)$, that is, x and y are linearly equivalent. Since $x \neq y$, this implies $C \cong \mathbb{P}^1$. Conversely, we know that $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = p_1^*\mathrm{Pic}(\mathbb{P}^1) \oplus p_2^*\mathrm{Pic}(\mathbb{P}^1)$, so $\mathcal{O}(\Delta)$ is contained in the image of $p_1^* \oplus p_2^*$ in this case.

(iv) The two maps $\Delta \longrightarrow S \xrightarrow{p_i} C$ for $i = 1, 2$ are isomorphisms, hence $\deg(\mathcal{O}_\Delta(a, b)) = a + b$. By the Riemann-Roch formula, we obtain

$$\begin{aligned} P(\Delta, \mathcal{O}_\Delta)(m) &= \chi(\Delta, \mathcal{O}_\Delta(am, bm)) \\ &= am + bm - g + 1, \end{aligned}$$

where g is the genus of C. Since this polynomial has degree 1 in m, it cannot be equal to the Hilbert polynomial of $\mathcal{O}_S(\Delta)$ with respect to any ample invertible sheaf, since the latter has degree 2 in m.

3. Higher direct images

Exercise 3. In the situation of Exercise 2, let $\iota\colon \tilde{S} \longrightarrow S$ be the blow-up of the point $(x,x) \in S$ and consider the composition $f\colon \tilde{S} \xrightarrow{\ \iota\ } S \xrightarrow{\ p_1\ } C$

(i) Describe the maximal open subsets in S and C over which ι resp. f are flat. Answer the same question for 'flat' replaced by 'smooth'.

(ii) Determine the direct image sheaves $R^i f_* \mathcal{O}_{\tilde{S}}$. Do they satisfy base change? In other words, for which points $y \in C$ and for which i is the natural map

$$R^i f_* \mathcal{O}_{\tilde{S}} \otimes k(y) \longrightarrow H^i(\tilde{S}_y, \mathcal{O}_{\tilde{S}_y})$$

an isomorphism?

(iii) Describe a coherent sheaf $\mathcal{F}$ on $\tilde{S}$ that is not f-flat and such that $f(\mathrm{Supp}(\mathcal{F})) = C$. Determine and compare the functions $y \longmapsto h^i(\tilde{S}_y, \mathcal{F}_y)$ and $y \longmapsto \dim R^i f_* \mathcal{F} \otimes k(y)$ (say on k-rational points $y \in C$).

Solution

(i) First, recall that ι is an isomorphism away from (x,x) and $\iota^{-1}(x,x) \cong \mathbb{P}^1$. Since fiber dimension is constant in flat families, the maximal open subset in S over which ι is flat is $S \setminus \{(x,x)\}$ and this is also the maximal open subset over which ι is smooth.

By the above, the fibers of f are the same as the fibers of p_1 away from x. The fiber $f^{-1}(x)$ contains two irreducible components, namely the exceptional curve for ι and the strict transform of $p_1^{-1}(x)$.

Now, C and S are smooth, hence so is $\tilde{S}$, and therefore f is flat by miracle flatness, that is, the maximal open subset in C over which f is flat is C itself. Since $\iota_* \mathcal{O}_{\tilde{S}} = \mathcal{O}_S$ and $(p_1)_* \mathcal{O}_S = \mathcal{O}_C$, we have $f_* \mathcal{O}_{\tilde{S}} = \mathcal{O}_C$, hence the fibers of f are connected. Thus, by the previous paragraph, the fiber $f^{-1}(x)$ is not smooth, hence f is not smooth over x. In particular, the maximal open subset over which f is smooth is $C \setminus \{x\}$.

(ii) Since the fibers of f are 1-dimensional, we have $R^i f_* \mathcal{O}_{\tilde{S}} = 0 = H^i(\tilde{S}_y, \mathcal{O}_{\tilde{S}_y})$ for $i \geq 2$. As explained in (i), we have $f_* \mathcal{O}_{\tilde{S}} = \mathcal{O}_C$. We claim that $R^1 f_* \mathcal{O}_{\tilde{S}} = (R^1 p_1)_* \mathcal{O}_S = \mathcal{O}_C \otimes H^1(C, \mathcal{O}_C)$, so that in particular $R^1 f_* \mathcal{O}_{\tilde{S}}$ is locally free and hence $f_* \mathcal{O}_{\tilde{S}}$ satisfies base change (alternatively, you can check directly that $f_* \mathcal{O}_{\tilde{S}}$ satisfies base change).

We have an exact sequence (which can be derived, for example, from the Leray spectral sequence)

$$0 \longrightarrow R^1 p_{1*}(\iota_* \mathcal{O}_{\tilde{S}}) \longrightarrow R^1 f_* \mathcal{O}_{\tilde{S}} \longrightarrow p_{1*}(R^1 \iota_* \mathcal{O}_{\tilde{S}}) \longrightarrow 0.$$

As explained in (i), we have $\iota_* \mathcal{O}_{\tilde{S}} = \mathcal{O}_S$ and from the lecture we know that $R^1 \iota_* \mathcal{O}_{\tilde{S}} = 0$. Thus, the exact sequence shows that $R^1 f_* \mathcal{O}_{\tilde{S}} \cong (R^1 p_1)_* \mathcal{O}_S$. The latter is isomorphic to $\mathcal{O}_C \otimes H^1(C, \mathcal{O}_C)$ by base change (see Exercise 59 (ii)).

(iii) We take $\mathcal{F} = \mathcal{O}_{\tilde{S}} \oplus k(y,y)$ for some point $x \neq y \in C$. Then, $\mathcal{F}$ is not f-flat, since $\mathcal{F}_{(y,y)} = \mathcal{O}_{\tilde{S},(y,y)} \oplus k(y,y)$ is not a flat $\mathcal{O}_{C,y}$-module. We have $\mathrm{Supp}(\mathcal{F}) = \tilde{S}$, hence $f(\mathrm{Supp}(\mathcal{F})) = C$. Now, we calculate $f_* \mathcal{F} = f_* \mathcal{O}_{\tilde{S}} \oplus f_* k(y,y) = \mathcal{O}_C \oplus k(y)$ and $\mathcal{F}_y = \mathcal{O}_{\tilde{S},y} \oplus k(y)$. Hence, $\mathcal{F}$ also satisfies base change at y.

4. Étale morphisms

We study morphisms $f_i\colon X_i \longrightarrow Y_i$, $i = 1, 2$, between finite type k-schemes, where k is an algebraically closed field of characteristic 0.

Exercise 4. (i) Decide whether with f_i smooth (flat, unramified, or étale) also the product $f \coloneqq f_1 \times f_2\colon X_1 \times_k X_2 \longrightarrow Y_1 \times_k Y_2$ of the f_i is smooth (flat, unramified, resp. étale). What about the composition $p_1 \circ f$ with the first projection $p_1\colon Y_1 \times_k Y_2 \longrightarrow Y_1$?

(ii) Assume the f_i are dominant morphisms between smooth integral k-schemes of the same dimension. Describe the ramification divisor R_f in terms of R_{f_i}. Describe $\omega_{X_1 \times_k X_2}$ in terms of $\omega_{Y_1 \times_k Y_2}$ and R_f.

(iii) Assume X and Y are smooth projective curves. Assume further that there exists an isomorphism $F\colon X \times_k \mathbb{P}^1 \longrightarrow Y \times_k \mathbb{P}^1$. Are X and Y isomorphic?

(iv) How many étale morphisms $\mathbb{P}^n \times_k \mathbb{P}^m \longrightarrow \mathbb{P}^N \times_k \mathbb{P}^M$ of degree at least 2 are there? (Hint: First, show that such a morphism can only exist if $\{n, m\} = \{N, M\}$.)

Solution

(i) We can write f as $X_1 \times_k X_2 \longrightarrow Y_1 \times_k X_2 \longrightarrow Y_1 \times_k Y_2$, where the first arrow is the base change of f_1 along the first projection $Y_1 \times X_2 \longrightarrow Y_1$ and the second arrow is the base change of f_2 along the second projection $Y_1 \times Y_2 \longrightarrow Y_2$. Since being smooth (flat, unramified, or étale) is stable under base change and composition, f satisfies these properties if the f_i do.

The morphism $p_1 \circ f$ is flat if f_i is, since $p_1\colon Y_1 \times_k Y_2 \longrightarrow Y_1$ is flat. The other properties are usually not inherited by $p_1 \circ f$: Indeed, take $X_1 = Y_1 = \operatorname{Spec} k$ and $X_2 = Y_2 = \operatorname{Spec} k[x]/(x^2)$ with $f_2 = \mathrm{id}$. Then, the f_i are étale but $p_1 \circ f_2 = p_1\colon \operatorname{Spec} k[x]/(x^2) \longrightarrow \operatorname{Spec} k$ is neither smooth nor unramified (let alone étale).

(ii) There is a natural isomorphism $\Omega_{X_1 \times X_2} \cong p_1^*\Omega_{X_1} \oplus p_2^*\Omega_{X_2}$ and similarly for $Y_1 \times Y_2$. Hence, the cotangent sequence for $X_1 \times_k X_2 \longrightarrow Y_1 \times_k Y_2 \longrightarrow \operatorname{Spec} k$ is isomorphic to

$$p_1^* f_1^* \Omega_{Y_1} \oplus p_2^* f_2^* \Omega_{Y_2} \longrightarrow p_1^*\Omega_{X_1} \oplus p_2^*\Omega_{X_2} \longrightarrow \Omega_{X_1 \times X_2 / Y_1 \times Y_2} \longrightarrow 0.$$

Thus, $R_f = R_{f_1} \times X_2 + X_1 \times R_{f_2}$. By definition of R_f, we have $\omega_{X_1 \times X_2} = f^*\omega_{Y_1 \times Y_2} \otimes \mathcal{O}(R_f)$.

(iii) Yes. First, note that we have the equality of genera $g(X) = 1 - \chi(\mathcal{O}_{X \times \mathbb{P}^1}) = 1 - \chi(\mathcal{O}_{Y \times \mathbb{P}^1}) = g(Y)$, hence we may assume that X and Y are curves of positive genus.

Consider the composition $f\colon X \times \mathbb{P}^1 \xrightarrow{F} Y \times \mathbb{P}^1 \longrightarrow Y$. Since $g(Y) > 0$, the fibers of p_1 are contracted by f, hence, by rigidity (Exercise 54), the morphism f factors through a morphism $g : X \longrightarrow Y$. As F is an isomorphism, the morphism g satisfies

$$g_*\mathcal{O}_X = g_* p_{1*} \mathcal{O}_{X \times \mathbb{P}^1} = f_*\mathcal{O}_{X \times \mathbb{P}^1} = p_{1*}\mathcal{O}_{Y \times \mathbb{P}} = \mathcal{O}_Y.$$

By Exercise 49, this implies that g is an isomorphism.

(iv) There are none. Indeed, assume that there exists an étale morphism $f\colon \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^N \times \mathbb{P}^M$. Note that f is proper and étale, hence finite and surjective. Therefore, $n + m = N + M$.

First, assume that $N \neq 0$ and that $n > N$. Then, every morphism from $\mathbb{P}^n$ to $\mathbb{P}^N$ is constant, hence, by rigidity (Exercise 54), there exists a morphism $g\colon \mathbb{P}^m \longrightarrow \mathbb{P}^N$ such that $p_1 \circ f = g \circ p_2$. Since $p_1 \circ f$ is smooth and p_2 is smooth and surjective, g is smooth. In particular, g is surjective, hence $m \geq N$. But $m > N$ is impossible, for then g would be constant hence $N = 0$. Thus, $m = N$ and therefore $n = M$, i.e., $\{n, m\} = \{N, M\}$. The same argument applies if $m > N$.

Next, assume that $N = 0$. Then we are looking at an étale morphism $f\colon \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^N$. Since f is étale, $f^*\mathcal{O}(-N - 1) = f^*\omega_{\mathbb{P}^N} = \omega_{\mathbb{P}^n \times \mathbb{P}^m} = \mathcal{O}(-n-1, -m-1)$. Assume $n, m \neq 0$ so that $n, m < N$. Then $\operatorname{Pic}(\mathbb{P}^n \times \mathbb{P}^m) = \mathbb{Z} \oplus \mathbb{Z}$, so both $-n-1$ and $-m-1$ would have to be divisible by $-N-1$, which is absurd. Therefore, either $n = 0$ or $m = 0$ and thus $\{n, m\} = \{N, M\}$.

Now, since f is étale, we have

$$f^*\mathcal{O}(N+1, M+1) = f^*\omega^*_{\mathbb{P}^N \times \mathbb{P}^M} = \omega^*_{\mathbb{P}^n \times \mathbb{P}^m} = \mathcal{O}(n+1, m+1)$$

and since $\{N, M\} = \{n, m\}$, the induced pullback of global sections is not only injective (since f is dominant) but in fact an isomorphism. In particular, f factors the closed immersion of $\mathbb{P}^n \times \mathbb{P}^m$ into projective space given by the complete linear system of global sections of $\mathcal{O}(n+1, m+1)$. Hence, f is injective. Since f is also étale, it is an isomorphism.